Complex Analysis in Several Variables

Ragnar Sigurdsson
University of Iceland

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2. Analytic (holomorphic) functions

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The differential of a function $f : X \to \mathbb{C}$ is:

$$d_a f = \frac{\partial f}{\partial x}(a)dx + \frac{\partial f}{\partial y}(a)dy = \frac{\partial f}{\partial z}(a)dz + \frac{\partial f}{\partial \bar{z}}(a)d\bar{z},$$

where $dz = dx + iy$, $d\bar{z} = dx - idy$, and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

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For functions of n complex variables $z = (z_1, \ldots, z_n)$, where $z_j = x_j + iy_j$, the differential is

$$d_a f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(a)dz_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}(a)d\bar{z}_j = \partial f + \bar{\partial} f.$$

Definition f is *analytic* if $d_a f$ is $\mathbb{C}$-linear at every point a in X, which means that the Cauchy-Riemann-equations are satisfied:

$$\frac{\partial f}{\partial \bar{z}_j}(a) = 0, \qquad j = 1, \dots, n.$$

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It is actually possible to give a weaker definition:

Theorem (Hartogs 1906): If $f : X \to \mathbb{C}$ is a function and for every point $a \in X$ the function

$$\zeta \mapsto f(a_1, \ldots, a_{j-1}, \zeta, a_{j+1}, \ldots, a_n)$$

is analytic in a neighbourhood of a_j, then $f \in \mathcal{O}(X)$

A Cauchy formula and power series:

Let

$$D = D(a_1, r_1) \times \cdots \times D(a_n, r_n)$$

be a polydisc and denote the distinguished boundary by

$$\partial_0 D = \partial D(a_1, r_1) \times \cdots \times \partial D(a_n, r_n)$$

If f is analytic in a neigbourhood of $\overline{D}$, then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n, \qquad z \in D.$$

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This gives a local power series expansion

$$f(z) = \sum_{\alpha} \partial^\alpha f(a)(z-a)^\alpha,$$

where

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}, \quad \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

3. Holomorphic maps

Let X be open in $\mathbb{C}^n$. A map $F = (F_1, \ldots, F_m) : X \to \mathbb{C}^m$ is said to be holomorphic if $F_j \in \mathcal{O}(X)$ for all j.

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If $m = n$ and F is bijective onto its image $Y = F(X)$, then we say that F is biholomorphic.

We say that two domains X and Y are biholomorphically equivalent if there exists a biholomorphic map on X with $Y = F(X)$

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This does not generalize to higher dimensions:

Theorem: If $n > 1$, then the unit ball

$$\mathbb{B}_n = \{z \in \mathbb{C}^n \, ; \ |z_1|^2 + \cdots + |z_n|^n < 1\}$$

in $\mathbb{C}^n$ is not biholomorphically equivalent to a polydisc.

4. Domains of holomorphy

With the aid of the Weierstrass product theorem it is possible to show that for every open X in $\mathbb{C}$ there exits $f \in \mathcal{O}(X)$, which *can not* be extended to a holomorphic function in a neighbourhood of a boundary point of X.

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Theorem (Hartogs 1906): Let X be an open subset of $\mathbb{C}^n$, $n > 1$, K be a compact subset of X such that $X \setminus K$ is connected. Then every function $f \in \mathcal{O}(X \setminus K)$ extends uniquely to a function $F \in \mathcal{O}(X)$.

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We have a little bit technical definition:

Definition: An open set X is said to be a *domain of holomorphy* if there *do not exist* non-empty open sets X_1 and X_2 with X_2 is connected, $X_2 \not\subset X$, and $X_1 \subset X \cap X_2$, such that for every $f \in \mathcal{O}(X)$ there exists $F \in \mathcal{O}(X_2)$ such that $f = F$ on X_1.

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There exist many equivalent characterizations of domains of holomorphy: (Oka, Cartan, Bremermann, and Norguet, c.a. 1937-1954).

5. Integral representations

The Cauchy-Fantappiè-Leray formula: Let ω be a domain in X with smoooth boundary $\partial\omega$ in X and assume that ω is defined by the function ϱ in the sense that

$$\omega = \{z \in X \,;\, \varrho(z) < 0\}$$

and the gradient

$$\varrho' = (\partial\varrho/\partial z_1, \ldots, \partial\varrho/\partial z_n)$$

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Then for every $f \in \mathcal{O}(X)$ we have

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial\omega} \frac{f(\zeta)\,\partial\varrho \wedge (\bar\partial\partial\varrho)^{n-1}}{\langle \varrho'(\zeta), \zeta - z\rangle^n}, \qquad z \in \omega.$$

6. Plurisubharmonic functions

A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is said to be plurisubharmonic if

$$\sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq 0.$$

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Examples:

(i) Convex functions on convex domains X.

(ii) $\varphi = \log |f|$, $f \in \mathcal{O}(X)$.

(iii) $\varphi = \log(|f_1|^{p_1} + \cdots + |f_m|^{p_m})$, $f_j \in \mathcal{O}(X)$.

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The class of $\mathcal{PSH}(X)$ of plurisubharmonic functions has many good properties which enables us to construct functions, e.g.,

$$\varphi = \sup\{\varphi_1, \ldots, \varphi_m\} \in \mathcal{PSH}(X), \qquad \text{if} \qquad \varphi_j \in \mathcal{PSH}(X)$$

7. The inhomogeneous Cauchy-Riemann equations

Assume that we have a function $g \in C^\infty(X)$ and that we want to modify g, so that it becomes holomorphic, i.e., we want to find $u \in C^\infty(X)$ such that

$$F = G - u.$$

Then u has to satisfy the inhomogeneous Cauchy-Riemann equations

$$\frac{\partial u}{\partial \bar{z}_j} = f_j, \qquad \text{with} \quad f_j = \frac{\partial G}{\partial \bar{z}_j}$$

Hörmander's existence theorem

Let X be a domain of holomorphy, $f_j \in C^1(X)$ be functions on X satisfying

$$\frac{\partial f_j}{\partial \bar{z}_k} = \frac{\partial f_k}{\partial \bar{z}_j}.$$

and let $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ be a *plurisubharmonic function*.

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and let $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ be a *plurisubharmonic function*. Then there exists a function u on X satisfying the Cauchy-Riemann equations

$$\frac{\partial u}{\partial \bar{z}_j} = f_j,$$

with the L^2-estimate of u in terms of $f = (f_1, \ldots, f_n)$,

$$\int_X |u|^2 (1 + |z|^2)^{-2} e^{-\varphi} \, d\lambda \leq \int_X |f|^2 e^{-\varphi} \, d\lambda$$