Electromagnetic energy and antennas

Mats Gustafsson

Electrical and Information Technology, Lund University, Sweden

Acknowledgments

- The Swedish Research Council.
- Swedish Foundation for Strategic Research (SSF).

Collaboration with:

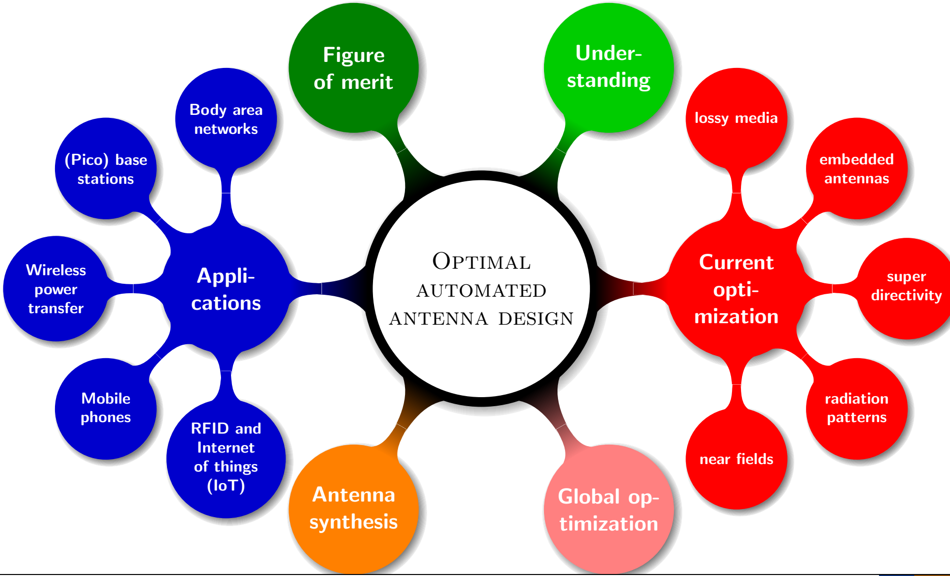
- Marius Cismasu, Lund University
- Doruk Tayli, Lund University
- Sven Nordebo, Linnæus University
- Lars Jonsson, KTH



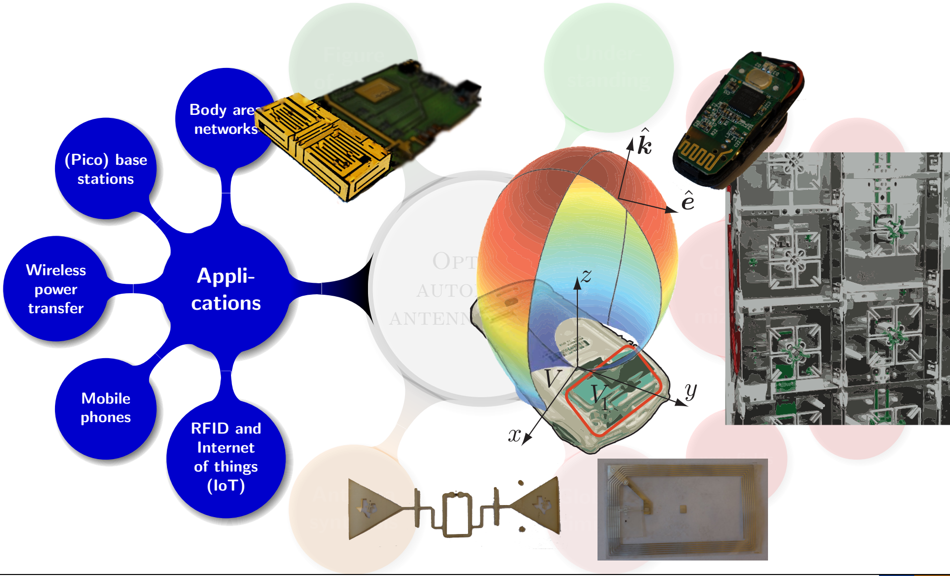

VETENSKAPSRÅDET
THE SWEDISH RESEARCH COUNCIL

SWEDISH FOUNDATION for STRATEGIC RESEARCH

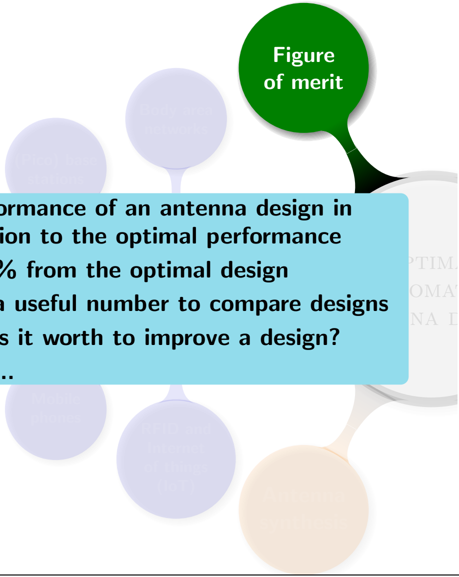
Optimal (automated) antenna design



Optimal (automated) antenna design

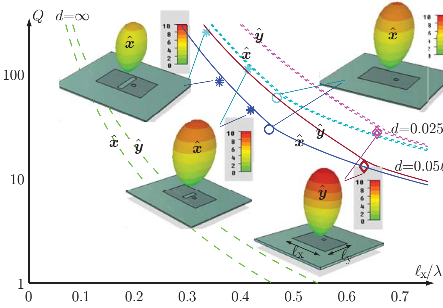


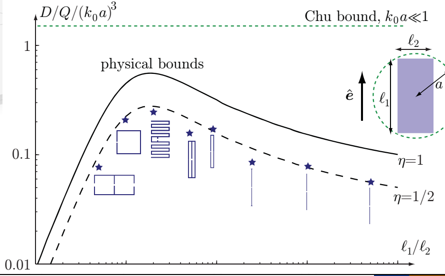
Optimal (automated) antenna design



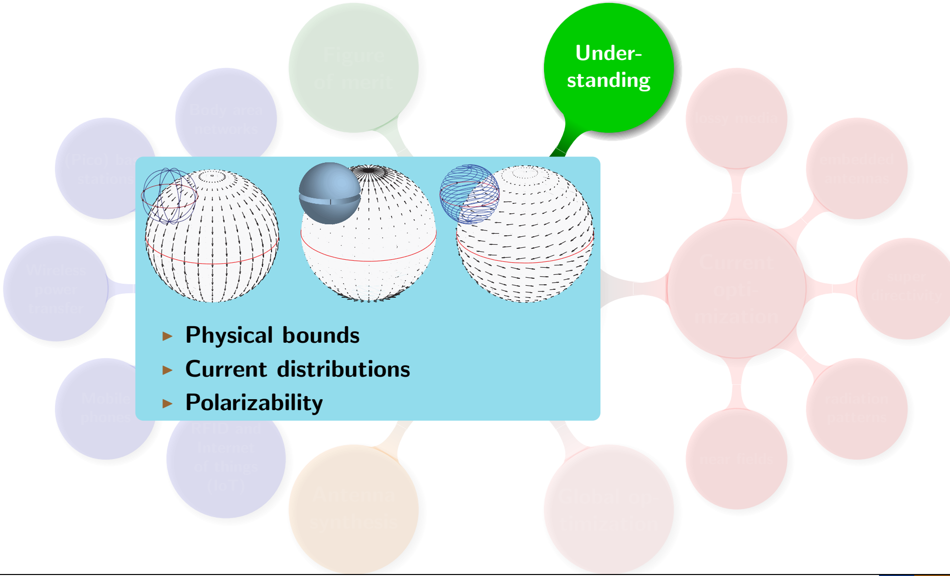
Performance of an antenna design in relation to the optimal performance

- % from the optimal design
- a useful number to compare designs
- is it worth to improve a design?
- ...



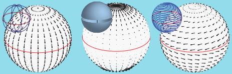


Optimal (automated) antenna design

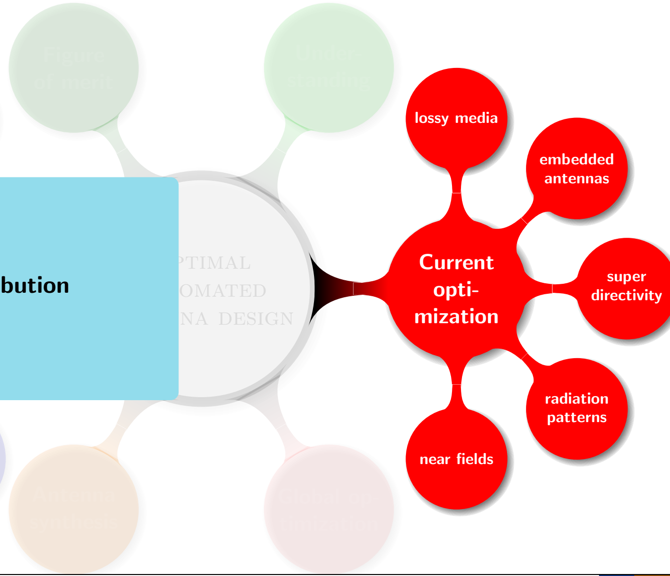


- **Physical bounds**
- **Current distributions**
- **Polarizability**

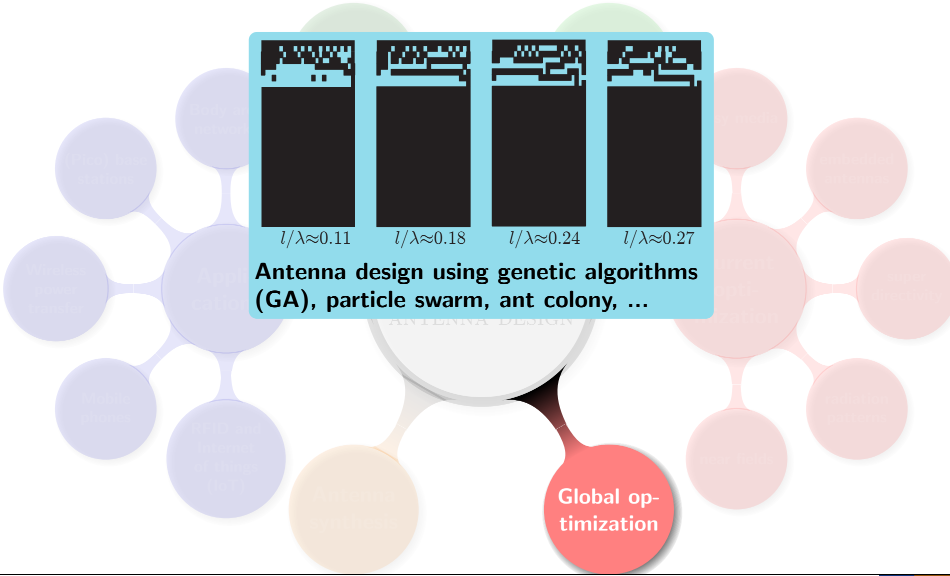
Optimal (automated) antenna design



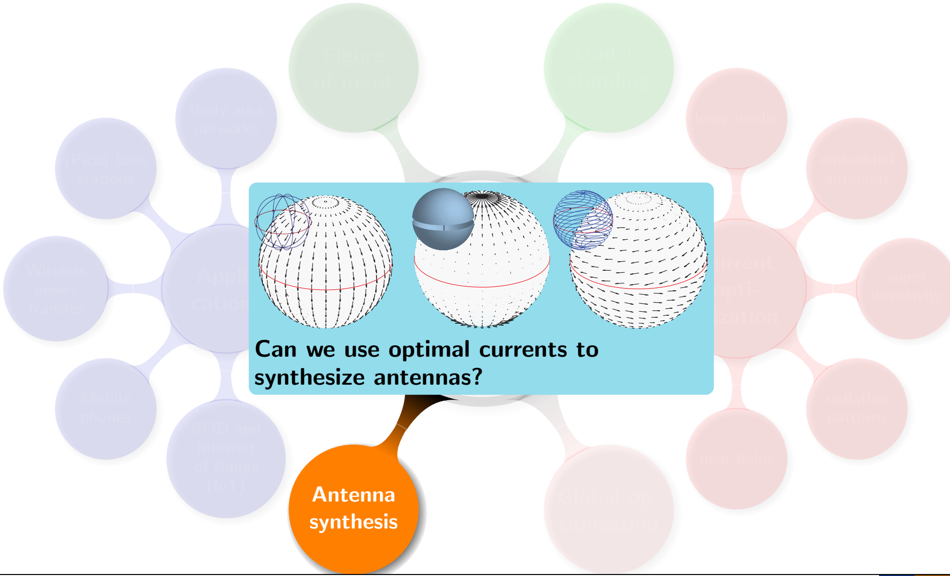
- **Optimal current distribution**
- **Physical bounds**
- **Convex optimization**
- **...**



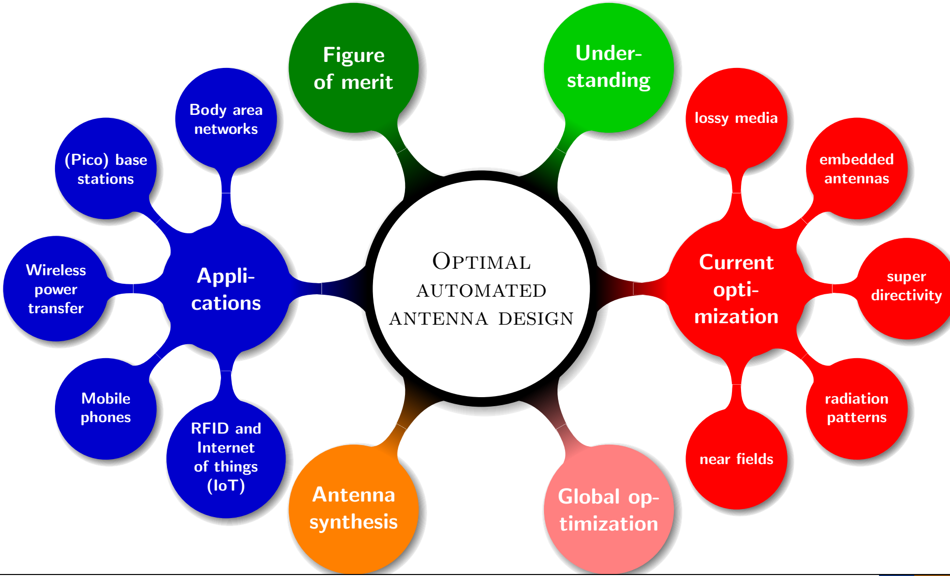
Optimal (automated) antenna design



Optimal (automated) antenna design



Optimal (automated) antenna design



Optimization of antenna currents: examples

Gain over Q

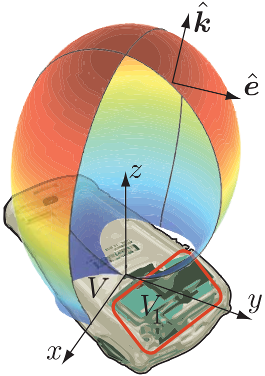
minimize Stored energy
subject to Radiation intensity $= P_0$

Q for superdirectivity $D \geq D_0$.

minimize Stored energy
subject to Radiation intensity $= D_0 P_{\text{rad}}/(4\pi)$
Radiated power $\leq P_{\text{rad}}$

Embedded structures

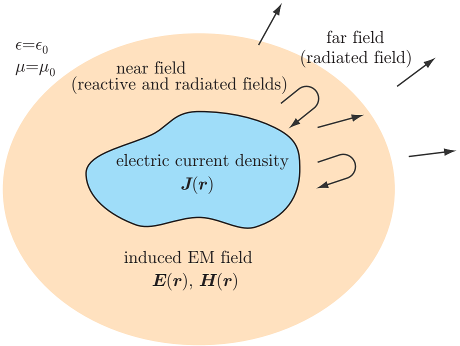
minimize Stored energy
subject to Radiation intensity $= P_0$
Correct induced currents



Need to:

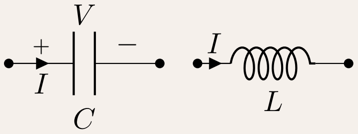
1. Express the *stored energy* in the current density $\boldsymbol{J}$.
2. Solve the optimization problems.

What is (stored) EM energy?



- Time average energy density $\epsilon_0|\boldsymbol{E}|^2/4$ and $\mu_0|\boldsymbol{H}|^2/4$.
- What is stored and radiated?
- How can we express the (stored) energy in the current density?
- Here, currents in free space.

Lumped elements

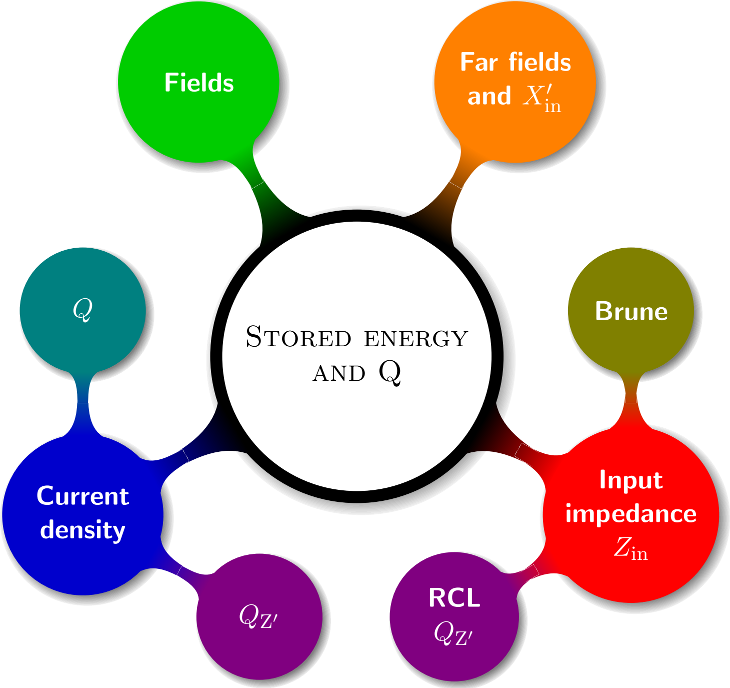


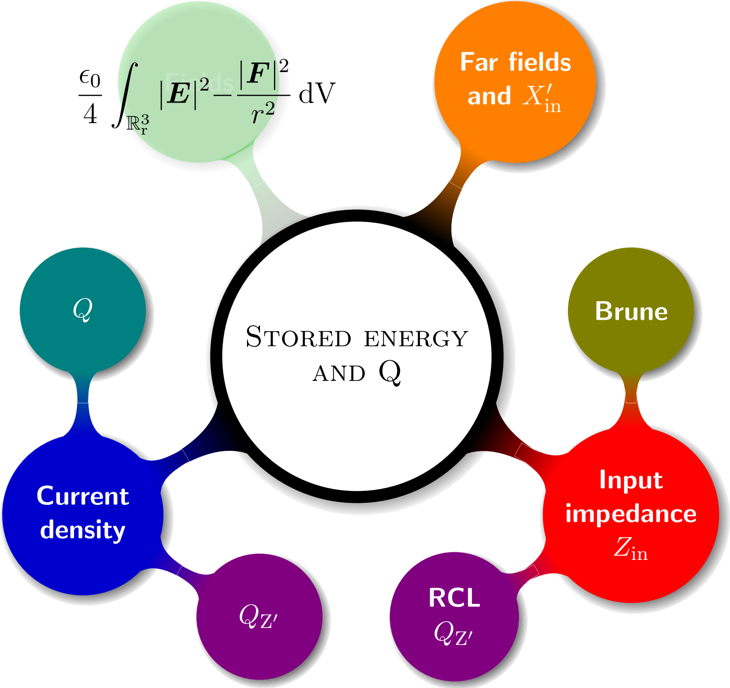
Time average stored energy in capacitors

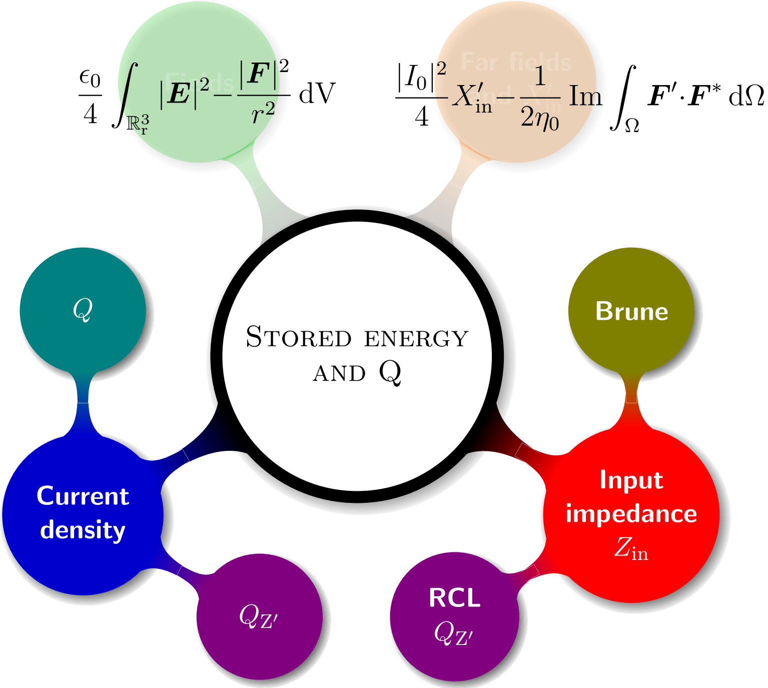
$$W^{(\mathrm{E})} = \frac{C|V|^2}{4} = \frac{|I|^2}{4\omega^2 C}$$

and in inductors

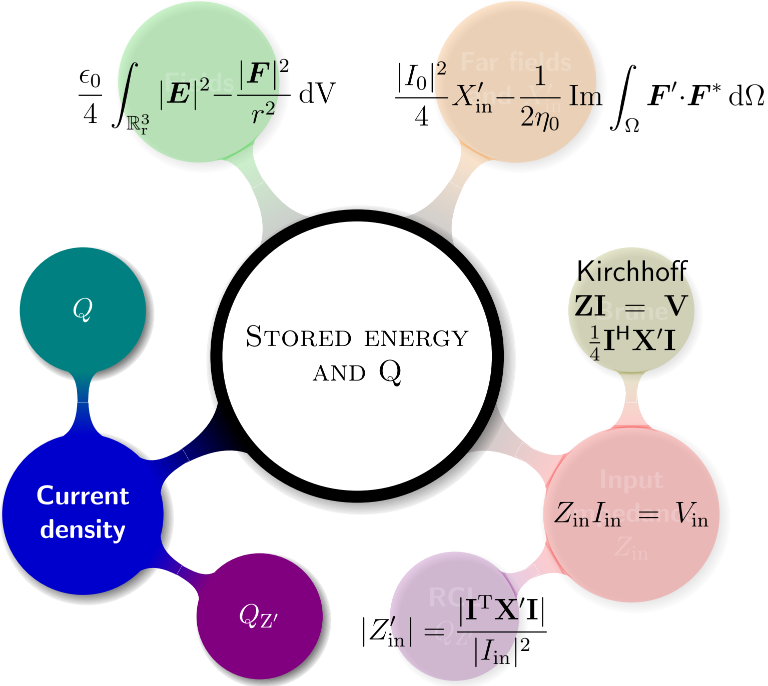
$$W^{(\mathrm{M})} = \frac{L|I|^2}{4}$$


Stored energy and Q
Fields
Far fields and X'_{in}
Brune
Input impedance Z_{in}
RCL $Q_{Z'}$
$Q_{Z'}$
Current density
Q

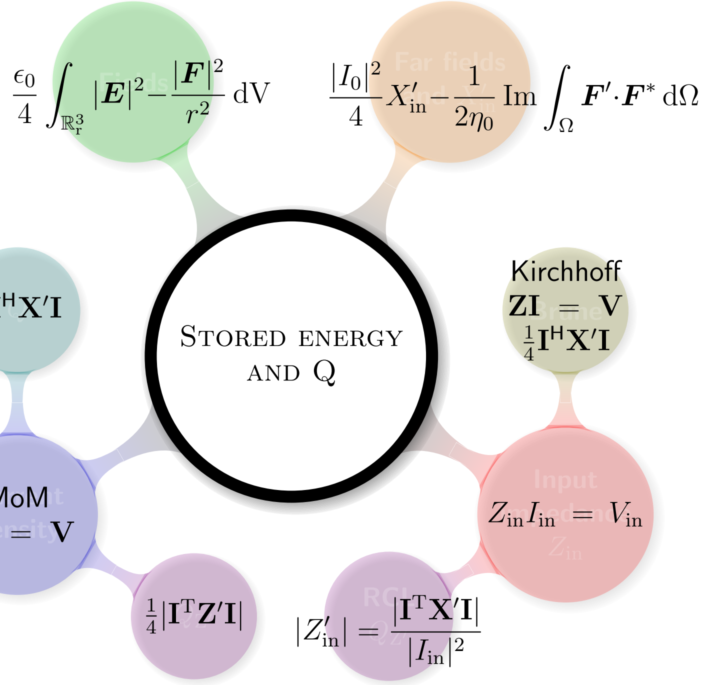

$$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$$
Far fields and X'_{in}
Stored energy and Q
Q
Brune
Current density
Input impedance Z_{in}
$Q_{\mathrm{Z}'}$
RCL $Q_{\mathrm{Z}'}$

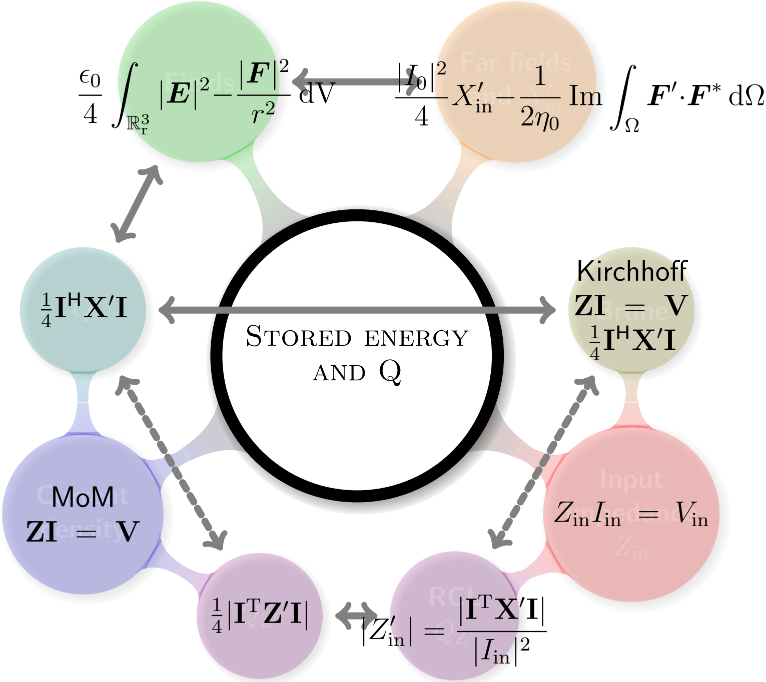


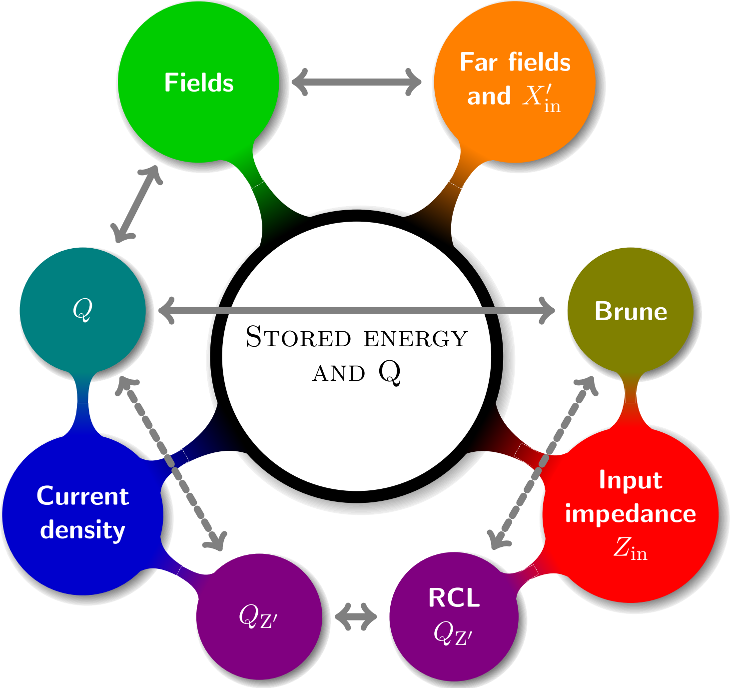
$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}}|\boldsymbol{E}|^2-\frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
Far fields
$\frac{|I_0|^2}{4}X'_{\mathrm{in}}-\frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega}\boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
STORED ENERGY AND Q
Q
Brune
Current density
Input impedance Z_{in}
$Q_{\mathrm{Z}'}$
RCL $Q_{\mathrm{Z}'}$



$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
$\frac{|I_0|^2}{4}X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega}\boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
Stored energy and Q
Q
Kirchhoff
$\mathbf{ZI}=\mathbf{V}$
$\frac{1}{4}\mathbf{I}^{\mathrm{H}}\mathbf{X}'\mathbf{I}$
Current density
$Z_{\mathrm{in}}I_{\mathrm{in}}=V_{\mathrm{in}}$
$Q_{Z'}$
$|Z'_{\mathrm{in}}| = \frac{|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$


STORED ENERGY AND Q
$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
$\frac{|I_0|^2}{4} X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega} \boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
MoM
$\mathbf{ZI} = \mathbf{V}$
$\frac{1}{4}|\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}|$
Kirchhoff
$\mathbf{ZI} = \mathbf{V}$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
$Z_{\mathrm{in}} I_{\mathrm{in}} = V_{\mathrm{in}}$
$|Z'_{\mathrm{in}}| = \frac{|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$


$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_r}|\boldsymbol{E}|^2-\frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
$\frac{|I_0|^2}{4}X'_{\mathrm{in}}-\frac{1}{2\eta_0}\,\mathrm{Im}\int_\Omega \boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
Stored energy and Q
Kirchhoff
$\mathbf{ZI}=\mathbf{V}$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
MoM
$\mathbf{ZI}=\mathbf{V}$
$Z_{\mathrm{in}}I_{\mathrm{in}}=V_{\mathrm{in}}$
$\frac{1}{4}|\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}|$
$|Z'_{\mathrm{in}}|=\frac{|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$


Fields
Far fields and X'_{in}
Q
Brune
STORED ENERGY AND Q
Current density
Input impedance Z_{in}
$Q_{Z'}$
RCL $Q_{Z'}$

MoM for Q and $Q_{\mathrm{Z}'}$ (I)

Use a method of moments (MoM) formulation of the electric field integral equation (EFIE). Impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$

$$\frac{Z_{mn}}{\eta} = \mathrm{j} \int_S \int_S \left(k^2 \boldsymbol{\psi}_{m1} \cdot \boldsymbol{\psi}_{n2} - \nabla_1 \cdot \boldsymbol{\psi}_{m1} \nabla_2 \cdot \boldsymbol{\psi}_{n2}\right) \frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi k r_{12}} \,\mathrm{dS}_1 \,\mathrm{dS}_2$$

where $\boldsymbol{\psi}_{n1} = \boldsymbol{\psi}_n(\boldsymbol{r}_1)$, $\boldsymbol{\psi}_{n2} = \boldsymbol{\psi}_n(\boldsymbol{r}_2)$, $n = 1, ..., N$, and $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$.

The current density is $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} I_n \boldsymbol{\psi}_n(\boldsymbol{r})$ with the expansion coefficients determined from

$$\mathbf{Z}\mathbf{I} = \mathbf{V} \quad \text{or } \mathbf{I} = \mathbf{Z}^{-1}\mathbf{V} = \mathbf{Y}\mathbf{V}$$

where $\mathbf{V}$ is a column matrix with the excitation coefficients.

The input admittance is

$$Y_{\mathrm{in}} = 1/Z_{\mathrm{in}} = \mathbf{V}^{\mathrm{T}}\mathbf{Y}\mathbf{V}/V_{\mathrm{in}}^2$$

where $Z_{\mathrm{in}} = R_{\mathrm{in}} + \mathrm{j}X_{\mathrm{in}}$ is the input impedance.

MoM for Q and $Q_{Z'}$ (II)

Differentiate the MoM impedance matrix

$$\frac{k\,\partial Z_{mn}}{\eta\,\partial k} = \int_V\int_V \mathrm{j}\left(k^2\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2} + \nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\right)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi kr_{12}} + \left(k^2\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2} - \nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\right)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi}\,\mathrm{dS}_1\,\mathrm{dS}_2$$

Differentiated input admittance (frequency independent $\mathbf{V}$ (MoM))

$$V_{\mathrm{in}}^2 Y_{\mathrm{in}}' = (\mathbf{V}^{\mathrm{T}}\mathbf{Y}\mathbf{V})' = \mathbf{V}^{\mathrm{T}}\mathbf{Y}'\mathbf{V} = -\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}.$$

The stored energy determined from $\mathbf{X}' = \mathrm{Im}\,\mathbf{Z}'$

$$W_{\mathbf{X}'}^{(\mathrm{E})} + W_{\mathbf{X}'}^{(\mathrm{M})} = \frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$$

is identical to the stored energy expression (free space) introduced by Vandenbosch (IEEE-TAP 2010), see ▸ ?? and already considered by Harrington (1968,1971,1975).

Q and $Q_{\mathrm{Z}'}$ for free-space self-resonant antennas

Assume for simplicity a **self-resonant** antenna (circuit)

$$Q_{\mathrm{Z}'} = \frac{\omega|Z'_{\mathrm{in}}|}{2R_{\mathrm{in}}} = \frac{\omega|\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}|}{2\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

and using MoM with the stored energy by Vandenbosch

$$Q = \frac{2\omega\max\{W^{(\mathrm{E})}_{\mathbf{X}'}, W^{(\mathrm{M})}_{\mathbf{X}'}\}}{P_{\mathrm{d}}} = \frac{\omega\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}}{2\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

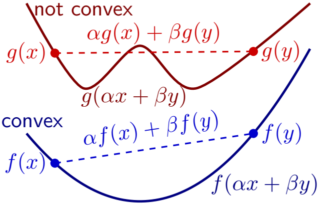
Transpose for $Q_{\mathrm{Z}'}$ and Hermitian transpose for Q

- $\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I} \geq 0$ for positive semidefinite matrices $\mathbf{X}'$.
- $|\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}| = 0$ for some $\mathbf{I}$ $(\mathrm{rank} > 1)$.

See also Capek+*etal.* IEEE-TAP 2014 for $Q_{\mathrm{Z}'}$ using $\mathbf{I}^{\mathsf{H}}$ and $\mathbf{I}'$ and ▸ ?? for stored energy and Q-factors in lumped circuits.

Convex optimization

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \ i = 1, ..., N_1 \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$



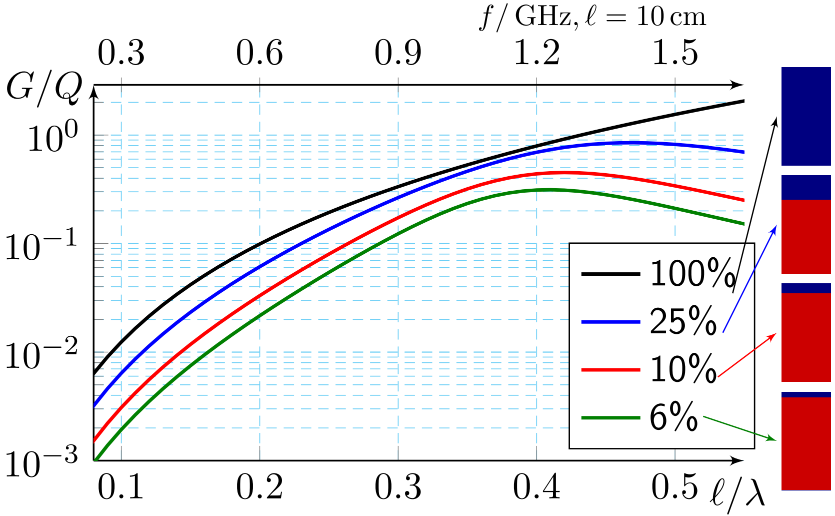
where $f_i(x)$ are convex, *i.e.*, $f_i(\alpha \mathbf{x} + \beta \mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$ for $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

Solved with efficient standard algorithms. No risk of getting trapped in a local minimum. A problem is 'solved' if formulated as a convex optimization problem.

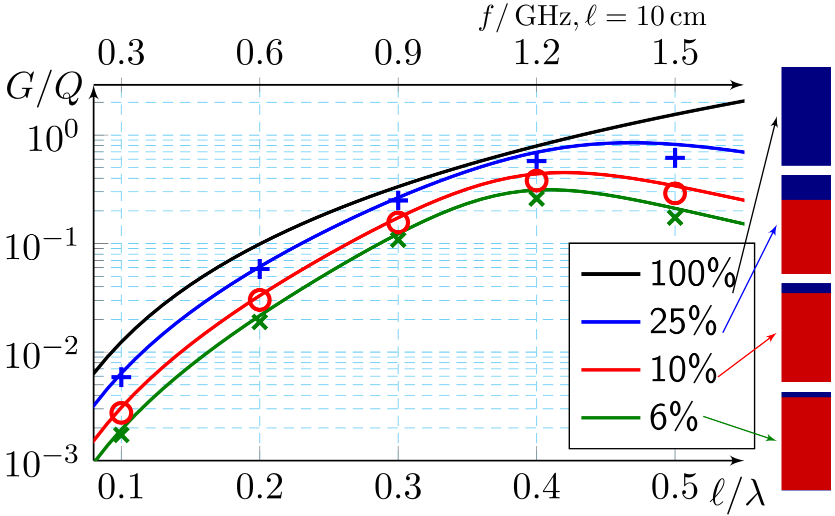
Antenna performance expressed in the current density $\boldsymbol{J}$, *e.g.*,

- Radiated field $\boldsymbol{F}(\hat{\boldsymbol{k}}) = -\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} \times \int_V \boldsymbol{J}(\boldsymbol{r}) \mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}} \, \mathrm{dV}$ is affine.
- Radiated power, stored electric and magnetic energies, and Ohmic losses are positive semi-definite quadratic forms in $\boldsymbol{J}$.

Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region



Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region



Why convex optimization?

Solved if formulated as a convex optimization problem.

Consider the G/Q problem

$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \end{aligned}$$

Many (optimization) algorithms can be used to solve this problem.

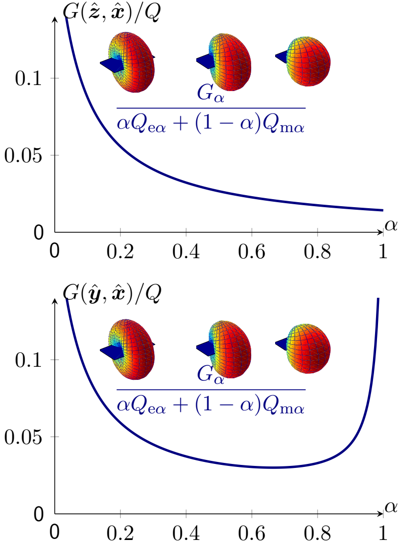
- Can *e.g.*, use any of the solvers included in CVX.
  - Very simple to use.
  - Good for small problems but less efficient for larger problems.
- A dedicated solver for quadratic programs.
  - More efficient for larger problems.
- Random search, eg genetic algorithms (GA), particle swarms,....
  - Very inefficient. Note you do not (should not) use (GA, ...) to solve *e.g.*, $\mathbf{A}\mathbf{x} = \mathbf{b}$ $(\min. ||\mathbf{A}\mathbf{x} - \mathbf{b}||)$.
- We also use a dual formulation
  - Computational efficient for large problems.
  - Illustrates dual problems and posteriori error estimates.

Why convex optimization: illustration

The upper bound on $G/Q|_{\mathrm{ub}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (blue) curve

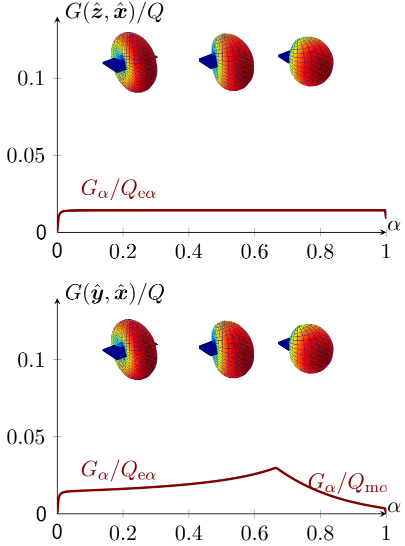
$$\left.\frac{G}{Q}\right|_{\mathrm{ub}} \leq \frac{G_\alpha}{\alpha Q_{\mathrm{e}\alpha} + (1-\alpha) Q_{\mathrm{m}\alpha}}$$

This is efficiently solved by golden section search and parabolic interpolation.



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

Why convex optimization: illustration



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

We also compute the actual G/Q for the current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{e\alpha}, Q_{m\alpha}\}} \leq \frac{G}{Q}\bigg|_{\mathrm{ub}}$$

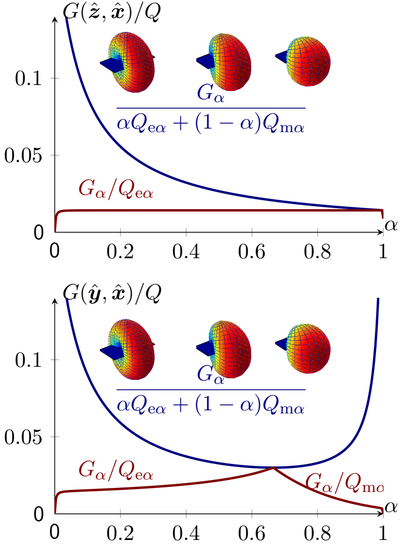
Why convex optimization: illustration

The upper bound on $G/Q|_{\mathrm{ub}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (blue) curve

$$\left.\frac{G}{Q}\right|_{\mathrm{ub}} \leq \frac{G_\alpha}{\alpha Q_{\mathrm{e}\alpha} + (1-\alpha) Q_{\mathrm{m}\alpha}}$$

This is efficiently solved by golden section search and parabolic interpolation.
We also compute the actual G/Q for the current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{\mathrm{e}\alpha}, Q_{\mathrm{m}\alpha}\}} \leq \left.\frac{G}{Q}\right|_{\mathrm{ub}}$$



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

Why: simple optimization formulations

Super directivity:

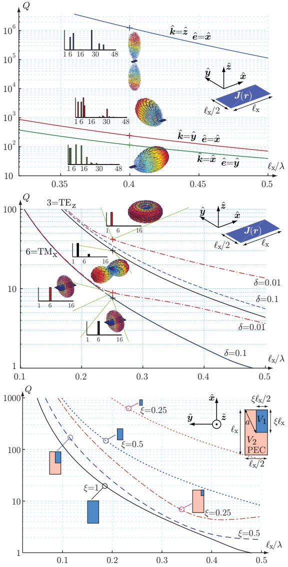
$$\begin{aligned} \text{minimize} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \\ & \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \leq 4\pi/(\eta_0 D_0) \end{aligned}$$

Prescribed far field:

$$\begin{aligned} \text{minimize} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \int_{\Omega} |\boldsymbol{F}(\hat{\boldsymbol{k}}) - \boldsymbol{F}_0(\hat{\boldsymbol{k}})|^2 \, \mathrm{d}\Omega_{\hat{\boldsymbol{k}}} < \delta \end{aligned}$$

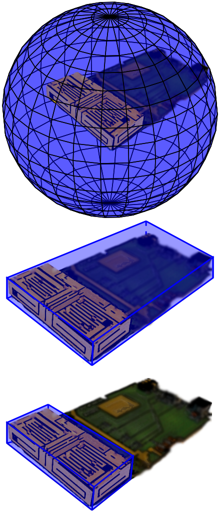
Embedded antennas:

$$\begin{aligned} \text{minimize} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \\ & \mathbf{I}_2 = \mathbf{C}\mathbf{I}_1 \end{aligned}$$



Summary

- Stored energy expressed in the current density, fields, and input impedance.
- Can we define stored energy for a Herglotz function?
- Energy in complex media?
- Optimization of the antenna structure (global optimization) and the antenna currents (convex optimization).
- Convex optimization for bounds and optimal currents: G/Q, superdirective, embedded, ...
- Closed form solutions for small antennas.



Initial results for efficiency, more realistic geometries (phones), SAR, MIMO. Investigating dielectrics, volume currents, magnetic currents, ...

Slides at:
http://www.eit.lth.se/staff/mats.gustafsson

References

Current optimization and physical bounds

- M. Gustafsson, M. Cismasu, B.L.G. Jonsson, *Physical bounds and optimal currents on antennas*, IEEE-TAP, 2012.
- M. Gustafsson, S. Nordebo, *Optimal antenna currents for Q, superdirectivity, and radiation patterns using convex optimization*, IEEE-TAP, 2013.
- M. Cismasu, M. Gustafsson, *Antenna Bandwidth Optimization with Single Frequency Simulation*, IEEE-TAP, 2014.

Stored energy expressed in the current density

- G.A.E. Vandenbosch, *Reactive energies, impedance, and Q factor of radiating structures*, IEEE-TAP, 2010.
- M. Gustafsson, B.L.G. Jonsson, *Stored Electromagnetic Energy and Antenna Q*, arXiv:1211.5521, 2012.
- G.A.E. Vandenbosch, *Radiators in time domain, part I, II*, IEEE-TAP, 2013.
- M. Capek, L. Jelinek, P. Hazdra, and J. Eichler, *The measurable Q factor and observable energies of radiating structures*, IEEE-TAP, 2014.
- M. Gustafsson, D. Tayli, M. Cismasu, *Q factors for antennas in dispersive media*, arXiv:1408.6834, 2014.

Convex optimization

- S. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.
- M. Grant, S. Boyd, CVX, http://cvxr.com/cvx/

See also: http://www.eit.lth.se/staff/mats.gustafsson

References

- S. R. Best. The radiation properties of electrically small folded spherical helix antennas. *IEEE Trans. Antennas Propagat.*, **52**(4), 953–960, 2004.
- S. R. Best, E. E. Altshuler, A. D. Yaghjian, J. M. McGinthy, and T. H. O'Donnell. An impedance-matched 2-element superdirective array. *Antennas and Wireless Propagation Letters, IEEE*, **7**, 302–305, 2008.
- S. P. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge Univ Pr, 2004.
- M. Capek, P. Hazdra, and J. Eichler. A method for the evaluation of radiation Q based on modal approach. *IEEE Trans. Antennas Propagat.*, **60**(10), 4556–4567, 2012.
- M. Capek, L. Jelinek, P. Hazdra, and J. Eichler. The measurable Q factor and observable energies of radiating structures. *IEEE Trans. Antennas Propagat.*, **62**(1), 311–318, Jan 2014.
- C. J. Carpenter. Electromagnetic energy and power in terms of charges and potentials instead of fields. *IEE Proc. A*, **136**(2), 55–65, 1989.
- J. Chalas, K. Sertel, and J. L. Volakis. Computation of the Q limits for arbitrary-shaped antennas using characteristic modes. In *Antennas and Propagation (APSURSI), 2011 IEEE International Symposium on*, pages 772–774. IEEE, 2011.
- L. J. Chu. Physical limitations of omnidirectional antennas. *J. Appl. Phys.*, **19**, 1163–1175, 1948.
- M. Cismasu and M. Gustafsson. Antenna bandwidth optimization with single frequency simulation. Technical Report LUTEDX/(TEAT-7227)/1–28/(2013), Lund University, Department of Electrical and Information Technology, P.O. Box 118, S-221 00 Lund, Sweden, 2013. http://www.eit.lth.se.
- R. E. Collin and S. Rothschild. Evaluation of antenna Q. *IEEE Trans. Antennas Propagat.*, **12**, 23–27, January 1964.
- H. D. Foltz and J. S. McLean. Limits on the radiation Q of electrically small antennas restricted to oblong bounding regions. In *IEEE Antennas and Propagation Society International Symposium*, volume 4, pages 2702–2705. IEEE, 1999.
- W. Geyi. A method for the evaluation of small antenna Q. *IEEE Trans. Antennas Propagat.*, **51**(8), 2124–2129, 2003.
- W. Geyi. Physical limitations of antenna. *IEEE Trans. Antennas Propagat.*, **51**(8), 2116–2123, August 2003.
- M. Gustafsson, M. Cismasu, and S. Nordebo. Absorption efficiency and physical bounds on antennas. *International Journal of Antennas and Propagation*, **2010**(Article ID 946746), 1–7, 2010.
- M. Gustafsson and S. Nordebo. Optimal antenna currents for Q, superdirectivity, and radiation patterns using convex optimization. *IEEE Trans. Antennas Propagat.*, **61**(3), 1109–1118, 2013.
- M. Gustafsson, C. Sohl, and G. Kristensson. Physical limitations on antennas of arbitrary shape. *Proc. R. Soc. A*, **463**, 2589–2607, 2007.
- M. Gustafsson, C. Sohl, and G. Kristensson. Illustrations of new physical bounds on linearly polarized antennas. *IEEE Trans. Antennas Propagat.*, **57**(5), 1319–1327, May 2009.
- M. Gustafsson, M. Cismasu, and B. L. G. Jonsson. Physical bounds and optimal currents on antennas. *IEEE Trans. Antennas Propagat.*, **60**(6), 2672–2681, 2012.
- M. Gustafsson and B. L. G. Jonsson. Stored electromagnetic energy and antenna Q. Technical Report LUTEDX/(TEAT-7222)/1–25/(2012), Lund University, Department of Electrical and Information Technology, P.O. Box 118, S-221 00 Lund, Sweden, 2012. http://www.eit.lth.se.
- M. Gustafsson and S. Nordebo. Bandwidth, Q factor, and resonance models of antennas. *Progress in Electromagnetics Research*, **62**, 1–20, 2006.
- T. V. Hansen, O. S. Kim, and O. Breinbjerg. Stored energy and quality factor of spherical wave functions–in relation to spherical antennas with material cores. *IEEE Trans. Antennas Propagat.*, **60**(3), 1281–1290, 2012.
- J. S. McLean. A re-examination of the fundamental limits on the radiation Q of electrically small antennas. *IEEE Trans. Antennas Propagat.*, **44**(5), 672–676, May 1996.
- C. Sohl and M. Gustafsson. A priori estimates on the partial realized gain of Ultra-Wideband (UWB) antennas. *Quart. J. Mech. Appl. Math.*, **61**(3), 415–430, 2008.
- J. C.-E. Sten, P. K. Koivisto, and A. Hujanen. Limitations for the radiation Q of a small antenna enclosed in a spheroidal volume: axial polarisation. *AEÜ Int. J. Electron. Commun.*, **55**(3), 198–204, 2001.
- H. L. Thal. New radiation Q limits for spherical wire antennas. *IEEE Trans. Antennas Propagat.*, **54**(10), 2757–2763, October 2006.
- H. L. Thal. Q Bounds for Arbitrary Small Antennas: A Circuit Approach. *IEEE Trans. Antennas Propagat.*, **60**(7), 3120–3128, 2012.
- G. A. E. Vandenbosch. Reactive energies, impedance, and Q factor of radiating structures. *IEEE Trans. Antennas Propagat.*, **58**(4), 1112–1127, 2010.
- G. A. E. Vandenbosch. Simple procedure to derive lower bounds for radiation Q of electrically small devices of arbitrary topology. *IEEE Trans. Antennas Propagat.*, **59**(6), 2217–2225, 2011.
- G. A. E. Vandenbosch. Radiators in time domain, part I: electric, magnetic, and radiated energies. *IEEE Trans. Antennas Propagat.*, **61**(8), 3995–4003, 2013.
- G. A. E. Vandenbosch. Radiators in time domain, part II: finite pulses, sinusoidal regime and Q factor. *IEEE Trans. Antennas Propagat.*, **61**(8), 4004–4012, 2013.
- J. Volakis, C. C. Chen, and K. Fujimoto. *Small Antennas: Miniaturization Techniques & Applications*. McGraw-Hill, New York, 2010.
- H. A. Wheeler. Fundamental limitations of small antennas. *Proc. IRE*, **35**(12), 1479–1484, 1947.
- A. D. Yaghjian, M. Gustafsson, and B. L. G. Jonsson. Minimum Q for lossy and lossless electrically small dipole antennas. *Progress In Electromagnetics Research*, **143**, 641–673, 2013.
- A. D. Yaghjian and H. R. Stuart. Lower bounds on the Q of electrically small dipole antennas. *IEEE Trans. Antennas Propagat.*, **58**(10), 3114–3121, 2010.
- A. D. Yaghjian and S. R. Best. Impedance, bandwidth, and Q of antennas. *IEEE Trans. Antennas Propagat.*, **53**(4), 1298–1324, 2005.