Passive systems, limitation examples.
– One and multi-dimensional cases

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Plan of the talk.

- Review 1D physical limitations. Key feature is that the Laplace transform g yields a **Herglotz** function.
- Define multi-dimensional passivity with respect to a cone Γ. Here g is holomorphic and have $\operatorname{Re} g > 0$.
- Passivity gives a representation theorem in higher dimensions.
- A holomorphic function with a norm-bound is an alternative method to obtain an integral relation. Multidimensional Kramers-Kronig relations.
- This is work in progress; Review of known tools.

Note: A function $h(z)$ is a **Herglotz function** if $h(z)$ is holomorphic for $\operatorname{Im} z > 0$ and $\operatorname{Im} h \geq 0$.

Refs: Vladimirov, Methods of the theory of Generalized Functions, 2002
Bernland, Luger, Gustafsson 2011.

Definition of linear system [Vladimirov 2002]

Input: $u(x) = (u_1(x), \ldots, u_N(x))$. **Output:** $f(x) = (f_1, \ldots, f_N)$.

- Linearity. If u_a generates f_a, and u_b generates f_b then $\alpha u_a + \beta u_b$ generates $\alpha f_a + \beta f_b$.
- Reality: If u is real, then f is real-valued.
- Continuity: If $u_j \to 0$ for all $j \in [1, N]$ in $\mathcal{E}'$ then $f_k \to 0$ in $\mathcal{D}'$ for all k.
- Translational invariance: If $f(x)$ is associated with $u(x)$ then for any translation $h \in \mathbb{R}^n$ to the original perturbed $u(x+h)$ there corresponds a response perturbation $f(x+h)$

There exists a unique $N \times N$ matrix $Z(x)$, with $Z_{jk} \in \mathcal{D}'(\mathbb{R}^n)$ such that $f = Z * u$.

$\mathcal{D}$ is smooth functions of compact support. $\mathcal{D}'$ is the space of generalized functions. $\mathcal{E}'$ is the space of generalized functions with compact support.

Passivity

Admittance-passivity

A function Z is *admittance passive* relative to the cone Γ if for any $\phi(x) \in \mathcal{D}^{\times N}$ then

$$\operatorname{Re}\int_{-\Gamma}(Z*\phi)\cdot\bar{\phi}\,\mathrm{d}x = \operatorname{Re}\int_{-\Gamma}\int(Z(x-y)\phi(y))\cdot\bar{\phi}(x)\,\mathrm{d}y\,\mathrm{d}x \geq 0$$

Theorem Every passive $Z*$ defines, via the formula

$$S* = (Z+I\delta)^{-1}*(Z-I\delta)*$$

an abstract scatting operator: $\operatorname{supp} Z \subset \Gamma$, $\int(S*\phi)\cdot S*\phi\,\mathrm{d}x \leq \int \phi\cdot\phi\,\mathrm{d}x$.

Remark 1: Scattering passivity 1d: $\int_{-\infty}^{T}|f|^2-|u|^2\,\mathrm{d}t > 0\ \forall T\in\mathbb{R}$.

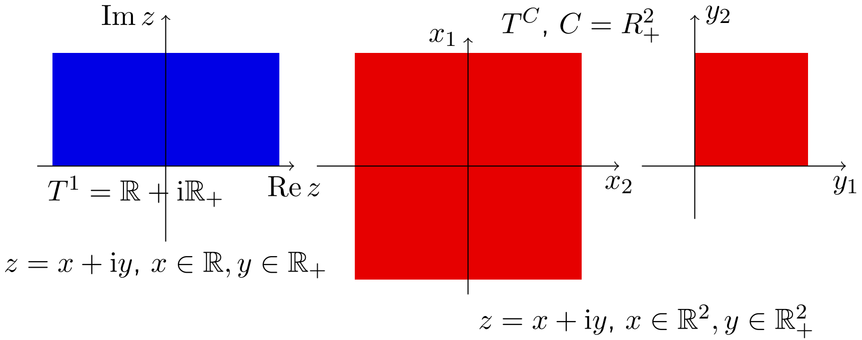
Remark 2:

Linear passive system without translational invariance are studied in Drozhzhinov 1981.

Spectral function is holomorphic

Theorem see Vladimirov 20.2.7

The Laplace transform $Z(z) = L[Z](z)$ (where $s = -\mathrm{i}z$) of a passive linear system matrix $Z(z)$ is holomorphic in for $z \in T^C$ where $T^C = \mathbb{R}^n + \mathrm{i}C$, $C = int\Gamma^*$, furthermore $\mathrm{Re}\, L(Z) \geq 0 \Rightarrow (L(Z)a + \overline{L(Z)}^T a) \cdot \bar{a} \geq 0$ in T^C.



Note in 1 dimension we have that $\mathrm{i}q(\zeta)$ is a Herglotz-function.
Here, if $s = \sigma - \mathrm{i}\omega = -\mathrm{i}z$, then $z = \omega + \mathrm{i}\sigma$.

Acute cones

Cone

- A cone $\Gamma \subset \mathbb{R}^n$, with vertex 0 is a set such that if $x \in \Gamma$, then $\lambda x \in \Gamma$ for all $\lambda > 0$.
- A cone is acute if the convex hull of Γ does not contain an integral straight line i.e. $x = x_0 + te \in \Gamma$ for $t \in (-\infty, \infty)$.
- The conjugate Γ^* to the cone $\Gamma \subset \mathbb{R}^n$ is the set

$$\Gamma^* = \{\xi \in \mathbb{R}^n : \xi \cdot x \geq 0, \text{ for all } x \in \Gamma\}$$

- Tubular neighbourhood: $T^C = \mathbb{R}^n + \mathrm{i}C \subset \mathbb{C}^n$, $C = \mathrm{int}\ \Gamma^*$. (Laplace transform domain).

Examples of acute cones, Vladimirov 4.4

$$\mathbb{R}^1_+, \quad \mathbb{R}^n_+ = \{x : x_1 > 0, x_2 > 0, \ldots x_n > 0\}, \ (\mathbb{R}^n_+)^* = \overline{\mathbb{R}^n_+},$$

$$V^+ = \{x = (x_0, \boldsymbol{x}) : x_0 > |\boldsymbol{x}|\} \subset \mathbb{R}^4, \ (V^+)^* = \overline{V^+}$$

$$P_n \subset \mathbb{R}^{n^2}, \text{ positive hermitian matrices}, \ P_n^* = P_n$$

Example 1: Extinction cross section

Optical theorem, and forward scattering

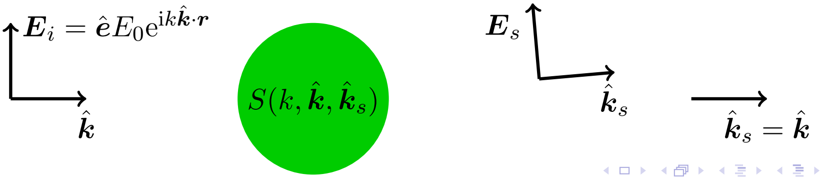
The extinction cross section $\sigma_e(\omega, \hat{\boldsymbol{k}})$ with $\omega = ck$, is the imaginary part of a Laplace transform of a linear passive operator. We have

$$0 \leq \sigma_e(\omega, \hat{\boldsymbol{k}}) = \frac{4\pi}{k} \operatorname{Im} \hat{\boldsymbol{e}}^* \cdot S(k, \hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}) \cdot \hat{\boldsymbol{e}} = \operatorname{Im} h_{\hat{\boldsymbol{k}}}(\omega).$$

Here $h_{\hat{\boldsymbol{k}}}$ is a **Herglotz** function. We have

$$h_{\hat{\boldsymbol{k}}}(\omega) \to \gamma(\hat{\boldsymbol{k}})k, \text{ as } \omega \to 0, \text{ and } h_{\hat{\boldsymbol{k}}}(\omega) \to 2\mathrm{i}A(\hat{\boldsymbol{k}}) \text{ as } \omega \to \infty.$$

$\gamma = \hat{\boldsymbol{e}}^* \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + \hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}^* \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot \hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}$, A is the projected area in direction $\hat{\boldsymbol{k}}$.



Asymptotic, representation ans sum-rule

For a Herglotz function h we have that

$$h(\omega) = \sum_n a_{2n-1}\omega^{2n-1} + o(\omega^{2N-1}),\ \omega\hat{\to}0 \tag{1}$$

$$h(\omega) = \sum_n b_{2n-1}\omega^{1-2n} + o(\omega^{1-2N}),\ \omega\hat{\to}\infty \tag{2}$$

and from a representation theorem we obtain the sum-rule:

$$\frac{2}{\pi}\int_0^\infty \frac{\operatorname{Im} h(\omega)}{\omega^{2n}}\,\mathrm{d}\omega = a_{2n-1} - b_{1-2n}$$

Compositions of Herglotz-functions are Herglotz function. Note that

$$h_\Delta(z) = \frac{1}{\pi}\ln\frac{z-\Delta}{z+\Delta},\ \operatorname{Im} z > 0$$

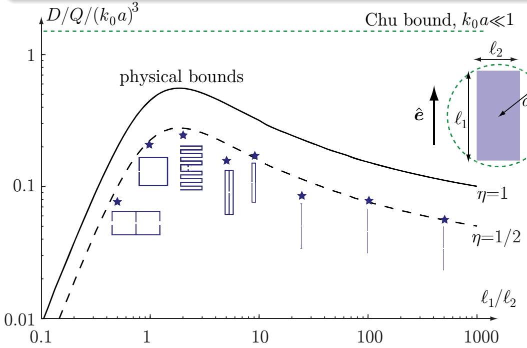
is a Herglotz function.

Bound on D/Q, Gustafsson etal 2007

Partial Directivity over antenna Quality factor

$$\int_0^\infty \frac{\sigma_e(k)}{k^2}\,\mathrm{d}k = \frac{\pi}{2}\gamma \;\Rightarrow\; \frac{D}{Q} \leq \frac{\eta k_0^3}{2\pi}\gamma,\;\; \eta \in [0,1].$$

where we used that $n = 1$, $a_1 = \gamma$, $b_{-1} = 0$ in the sum-rule theorem. Where we have used $\sigma_e \geq \sigma_a = \pi(1 - |\Gamma|^2)D/k^2$ and a first dominant resonance at $k = k_0$.



Ref:
Gustafsson etal 2009

Example 2: Limits on small and/or negative ε

Isotropic homogeneous $\varepsilon(t)$ [Gustafsson, Sjöberg 2010]

We have input $\boldsymbol{E}$ and output $\partial_t \boldsymbol{D}$ through:

$$\boldsymbol{F} = \partial_t \boldsymbol{D}(t) = \partial_t(\varepsilon_0 \varepsilon_\infty \boldsymbol{E}(t) + \varepsilon_0 \int_{\mathbb{R}} \chi(t-t')\boldsymbol{E}(t')\,\mathrm{d}t'),\ \varepsilon_\infty > 0$$

where $\chi = 0$ for $t < 0$.
The system is passive if $\int_{-\infty}^{T} \boldsymbol{E} \cdot \partial_t \boldsymbol{D}\,\mathrm{d}t \geq 0$, for all $T \in \mathbb{R}$.
Passivity yields that the holomorphic function in T^1:

$$g(z) = -\mathrm{i}z\varepsilon(z) = L[\partial_t(\varepsilon\cdot)](z),$$

satisfy $\mathrm{Re}\, g \geq 0$ and $g(z)$ is holomorphic for $\mathrm{Im}\, z > 0$.
Note: $h(z) = \mathrm{i}g(z)$ is a Herglotz function.

Fundamental bound on ε

Let $h_\varepsilon = \omega(\varepsilon - \varepsilon_m)/\varepsilon_0$.
Given a lossless ε and a desired goal value ε_m, then since

$$h_\Delta(z) = \frac{1}{\pi}\ln\frac{z-\Delta}{z+\Delta}, \quad h_\Delta(h_\varepsilon(z)) \sim \begin{cases} O(1) & z\hat{\to}0 \\ \frac{-2\omega_0\Delta}{\omega\pi(\varepsilon_\infty-\varepsilon_m)} & z\hat{\to}\infty \end{cases}$$

Thus (sum-rule)

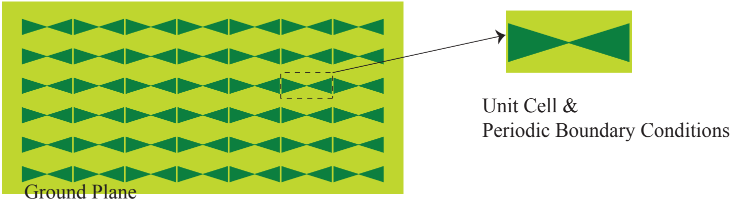
$$B \leq \min_{\omega\in B}\mathrm{Im}(h_\Delta(h_\varepsilon(\omega))) \leq \frac{1}{\omega_0}\int_{\omega_1}^{\omega_2}\mathrm{Im}(h_\Delta(h_\varepsilon(\omega)))\,\mathrm{d}\omega \leq \frac{\Delta}{(\varepsilon_\infty - \varepsilon_m)}$$

where $B = (\omega_2 - \omega_1)/\omega_0$. Selecting $\max_{\omega\in B}|h_\varepsilon| = \Delta$ gives [Gustafsson, Sjöberg 2010]

$$\max_{\omega\in B}|\varepsilon - \varepsilon_m| \geq \frac{B}{1+B/2}(\varepsilon_\infty - \varepsilon_m)\begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless} \end{cases}$$

Assumptions

- Unit-cell model – array is approximated with infinite periodic array
- The element is build of passive linear and time-invariant loss-less materials.
- Impedance bandwidth model: One band or multi-band, with wall-type reflection coefficient.
- For this study we focus on linear polarization, corresponding to the TE-mode (E-orthogonal to the surface normal)



Sum-rule result for Γ^{TE}. (Rozanov 2000)

Projection of Lowest Floquet mode is scattering passive, hence:

$$I(\theta) := \int_0^\infty \omega^{-2} \ln(|\Gamma^{TE}(\omega,\theta)|^{-1})\,\mathrm{d}\omega \leq q(\theta)$$

Sjöberg and Gustafsson, 2011 showed that

$$q(\theta) = \frac{\pi d}{c}(1 + \frac{\tilde{\gamma}}{2dA})\cos\theta \leq \frac{\pi d \mu_s}{c}\cos\theta$$

d-thickness, A-unit cell area, $\tilde{\gamma}$-function of polarizability tensor, μ_s, maximum relative static permeability.

Herglotz function $h(z) = -\mathrm{i}\ln(\Gamma^{TE}/B(z))$, where $B(z)$ is a Blaschke product that remove complex zeros z_n in the upper half-plane, where

$$B(z) = \left(\frac{z-\mathrm{i}}{z+\mathrm{i}}\right)^k \prod_{z_n \neq i} \frac{|z_n^2+1|}{z_n^2+1}\frac{z-z_n}{z-z_n^*}.$$

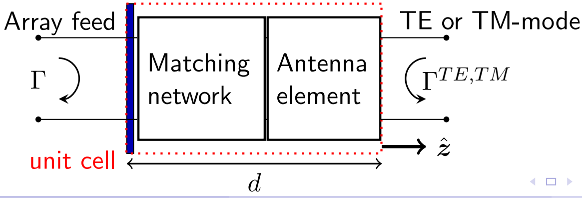
Sum-rule result [Jonsson etal 2013]

- For $\omega < \omega_G$ (grating lobe ansatz) we have $|\Gamma| = |\Gamma^{TE}|$
- Given M frequency bands $B_m := [\lambda_{-,m}, \lambda_{+,m}]$,
- Define $|\Gamma_m| := \max_{\lambda \in B_m, \theta \in [\theta_0, \theta_1]} |\Gamma(\lambda, \theta)|$.
- Clearly $\ln(|\Gamma(\lambda, \theta)|^{-1}) \geq \ln(|\Gamma_m|^{-1})$

Hence:

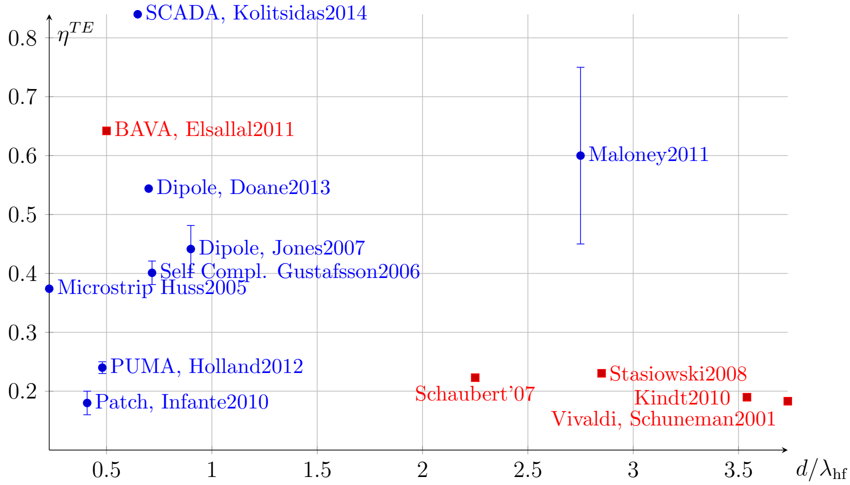
$$0 \leq \eta_M^{TE} := \frac{\sum_{m=1}^{M} \ln(|\Gamma_m|^{-1})(\lambda_{m,+} - \lambda_{m,-})}{2\pi^2 \mu_s d \cos\theta_1} \leq \eta_0 \leq 1$$

Here η_M^{TE} is the *Array Figure of Merit* for a M-band antenna.



See also:
Doane etal 2013

Performance indicator for published antennas



Ref: **J**, Kolitsidas, Hussain IEEE AWPL, 12(1) p1539-1542, 2013. Ref: Kolitsidas etal 2014

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The **Kramers-Kronig relation** for an analytic function $\chi = \chi_1 + \mathrm{i}\chi_2$

$$\chi_1(\omega) = \frac{1}{\pi} P \int_{\mathbb{R}} \frac{\chi_2(\xi)}{\xi - \omega} \, \mathrm{d}\omega$$
$$\chi_2(\omega) = -\frac{1}{\pi} P \int_{\mathbb{R}} \frac{\chi_1(\xi)}{\xi - \omega} \, \mathrm{d}\omega$$

The cone here is $\Gamma = R_+$, the Cauchy kernel is $\frac{1}{\xi - \omega' - \mathrm{i}\omega''}$.

Example for the analytical $\omega\varepsilon(\omega)$ yields (ε continuous, bounded)

$$\operatorname{Re} \varepsilon(\omega) = \varepsilon_\infty + \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|\xi - \omega| > \varepsilon} \frac{\operatorname{Im}(\varepsilon(\xi))}{\xi - \omega} \, \mathrm{d}\xi, \ \omega \in \mathbb{R}$$

where we have assumed that $\operatorname{Im} \varepsilon(\omega) \leq C/|\omega|$ as $\omega \to \pm\infty$.
[Ref: See e.g. Bernland etal 2011, and references there in]

Cauchy Kernels $\mathcal{K}_C$ [Vladimirov 10.2]

- Given a connected open cone in $\mathbb{R}^n$ with vertex 0, then

$$\mathcal{K}_C(z) = \int_{C^*} \mathrm{e}^{\mathrm{i}z\cdot\xi}\,\mathrm{d}\xi = F[\theta_{C^*}\mathrm{e}^{-y\cdot\xi}],\ z = x + \mathrm{i}u$$

- $\mathcal{K}_{\mathbb{R}^n_+}(z) = \frac{\mathrm{i}}{z_1\cdots z_n} \Rightarrow \mathcal{K}_1(x) = \frac{\mathrm{i}}{x+\mathrm{i}0} = \pi\delta(x) + \mathrm{i}P\frac{1}{x}$.
- $\mathcal{K}_{V^+}(z) = 2^n\pi^{(n-1)/2}\Gamma(\frac{n+1}{2})(-z^2)^{-\frac{n+1}{2}}$, $z \in T^{V^+}$, $z^2 = z_0^2 - z_1^2 - \cdots - z_n^2$.
- $$\mathcal{K}_{P^n}(Z) = \pi^{n(n-1)/2}\mathrm{i}^{n^2}\frac{1!\dots(n-1)!}{(\det Z)^n},\ Z \in T^{P_n},$$

Let $g = F[f]$. If $g \in L^2_s$, $\Leftrightarrow g(\xi)((1+|\xi|^2)^{s/2} \in L^2$, then $f \in (H_s, \|\cdot\|_s)$ (Sobolev-space). If $f \in H_s$, then the transform $f(z) = \int_{\mathbb{R}^n}\mathcal{K}_C(z-x')f(x')\,\mathrm{d}x'$ is called the **Cauchy-Bochner transform**. $z \in T^C \cup T^{-C}$.

Pair of Hilbert-transforms

Theorem II (V10.6) Generalized Kramers-Kronig relation

Let $f(z)$ be holomorphic in T^C, and $\sup_{y\in C}\|f(x+\mathrm{i}y)\|_s<\infty$. ($H^{(s)}$ Banach space). Let $f_+(x)$ be the boundary value of $f(z)$ as $y\to 0$, $y\in C$. The following statements are equivalent

- $f_+(x)$ is a boundary value in H_s from some function $f(z)$ in $H^{(s)}(C)$.
- f_+ is in H_s and

$$\operatorname{Re} f_+(x)=\frac{-2}{(2\pi)^n}\int_{\mathbb{R}^n}(\operatorname{Im} f_+)(x')(\operatorname{Im}\mathcal{K}_C)_+(x-x')\,\mathrm{d}x',$$

$$\operatorname{Im} f_+(x)=\frac{2}{(2\pi)^n}\int_{\mathbb{R}^n}(\operatorname{Re} f_+)(x')(\operatorname{Im}\mathcal{K}_C)_+(x-x')\,\mathrm{d}x',$$

- f_+ is in H_s and $\operatorname{supp} F^{-1}(f_+)\subset C^*$.

Note: $\operatorname{Re} f_+$ and $\operatorname{Im} f_+$ form a pair of Hilbert-transforms. Here

$$(\operatorname{Im}\mathcal{K}_C)_+(x)=\operatorname{Im}\left(\mathrm{i}^n\Gamma(n)\int_{S^{n-1}\cap C^*}\frac{\mathrm{d}\sigma}{[x\cdot\sigma+\mathrm{i}0]^n}\right)$$

Spatial dispersion

Let $\varepsilon(\omega, \boldsymbol{k})$ be analytic in $(\omega, \boldsymbol{k}) \in T^{V^+}$, and with boundary value $\varepsilon_+(\omega, k)$ in H_s for $(\omega, \boldsymbol{k}) \in \mathbb{R}^4$ then

$$\operatorname{Re} \varepsilon_+(\omega, \boldsymbol{k}) = \frac{-2}{(2\pi)^n} (\operatorname{Im} \mathcal{K}_{V^+})_+ * \operatorname{Im} \varepsilon_+ =$$

$$\frac{\Gamma(2)}{\pi^3} \int_{\mathbb{R}} \int_{\mathbb{R}^3} (\operatorname{Im} \mathcal{K}_{V^+})_+(\omega - \omega', \boldsymbol{k} - \boldsymbol{k}') \operatorname{Im} \varepsilon(\omega', \boldsymbol{k}') \, \mathrm{d}\omega \, \mathrm{d}V_k$$

Note:

1. An explicit form of $(\operatorname{Im} \mathcal{K}_{V^+})_+$, can be expressed in terms of the generalized functions $P^{(k)} \frac{1}{\sigma \cdot x}$ and $\delta^{(n)}(\sigma \cdot x)$.
2. Application to periodic structures.
3. Verify the expressions with respect to constants.

Application examples in $\mathbb{R}^2_+$. Transmission line

Cables and transmission lines have a domain $\mathbb{R}^2_+$, i.e. $(t,x) \in \mathbb{R}^2_+$.

Examples include the transmission line impedance $Z(x,t)$ at $(x,t) \in \mathbb{R}^2_+$.
If $Z(\omega,k)$ is in H_s **and** it is a boundary value of a holomorphic function in $T^{\mathbb{R}^2_+}$ we would get the relation

$$\operatorname{Re} Z(\omega,k) = \frac{-1}{2\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}(\operatorname{Im} K_{\mathbb{R}^2_+})(\omega-\omega',k-k')\operatorname{Im} Z(\omega',k')\,\mathrm{d}\omega\,\mathrm{d}k,$$

where

$$\operatorname{Im} \mathcal{K}_{\mathbb{R}^2_+}(\omega,k) = \pi\delta(\omega)P\frac{1}{k} + \pi\delta(k)P\frac{1}{\omega}$$

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Poisson Kernel

- $\mathcal{P}_C(x,y) = \frac{\mathcal{K}_C(x+\mathrm{i}y)}{\pi^n \mathcal{K}_C(\mathrm{i}y)}, \quad (x,y) \in T^C$
- $\mathcal{P}_{\mathbb{R}^n_+}(x,y) = \frac{y_1 \cdots y_n}{\pi^n |z_1|^2 \cdots |z_n|^2}$
- $\mathcal{P}_{V+}(x,y) = \frac{2^n \Gamma(\frac{n+1}{2})}{\pi^{\frac{n+3}{2}}} \frac{(y^2)^{\frac{n+1}{2}}}{|(x+\mathrm{i}y)^2|^{n+1}}$

Schwartz kernel

- $\mathcal{S}_C(z,z^0) = \frac{2\mathcal{K}_C(z)\mathcal{K}_C(-\overline{z^0})}{(2\pi)^n \mathcal{K}_C(z-\overline{z^0})} - \mathcal{P}_C$
- $$\mathcal{S}_{\mathbb{R}^n_+} = \frac{2\mathrm{i}^n}{(2\pi)^n}\left(\frac{1}{z_1} - \frac{1}{\overline{z_1^0}}\right)\cdots\left(\frac{1}{z_n} - \frac{1}{\overline{z_n^0}}\right) - \mathcal{P}_{\mathbb{R}^n_+}$$
- $\mathcal{S}_{V+}$ is also known explicitly.

Properties of Holomorphic functions with non-negative imaginary part

Let $u \in \mathcal{P}_+(T^C)$ then $0 \le u(x,y) = \operatorname{Im} f \in H_+(T^C)$ and $\mu = u(x,+0)$ is a non-negative tempered measure, and u have the representation:

$$u(x,y) = \int_{\mathbb{R}^n} \mathcal{P}_C(x-x',y)\mu(\mathrm{d}x') + v_C(y), \ (x,y) \in T^C$$

where $v_C > 0$ continuous, $v_C \to 0$, as $C \ni y \to 0$.

Note: 1) $u \in \mathcal{P}_+$ is pluriharmonic and positive functions, i.e. $\partial_{z_j}\partial_{\overline{z_k}}u = 0$.
H_+ functions are holomorphic with non-negative imaginary part.
2) If in addition $\int P_C(x-x')\mu(\mathrm{d}x') \in \mathcal{P}_+$, then $v_c = (a,y)$ and

$$f(z) = \mathrm{i}\int_{\mathbb{R}^n} \mathcal{S}_C(z-x';z^+-x')\mu(\mathrm{d}x') + (a,z) + b(z^0), \ z \in T^C$$

where $b \in \mathbb{R}$
3) For R_+ this is Herglotz-Nevanlinna representation theorem.

Generalizations

Favorite candidates

- Spatial dispersion: $\omega\varepsilon(\omega, \boldsymbol{k})$, $(\omega, \boldsymbol{k})$ dual variables to $(t, x) \in V^+$ the light cone.
- Transmission line impedance and reflection coefficients $Z(z, t)$, $\Gamma(z, t)$. Cone: R^2_+. [Transmission line, Cable]
- $\sigma_e(\omega, \hat{k})$, $\omega \in T^1$, $\hat{k} \in S^2$. $\hat{k}$ is direction, not Laplace/Fourier transform of space variable. $(t, \hat{k}) \in \mathbb{R}^+ \times S^2$ is no natural cone, its a subset of $\mathbb{R}^+ \times \mathbb{R}^3$, which is not acute. (Complexify?)
- $\Gamma(\omega, \theta, \phi)$, $(\omega, \theta, \phi) \subset T^3 = \mathbb{R}^3 + \mathrm{i}\mathbb{R}^3_+$. $(\theta, \phi) = (\theta, \phi) + \hat{u}_{\pi, 2\pi}$.
- Green's functions and the Resolvent of Self-Adjoint operators.

Acute cone is equivalent with that $int\Gamma^* \neq 0$. I.e. non-acute cones have empty dual cones.

Open questions

- Does a sum-rule for any n exist for multi-dimensional holomorphic functions with positive imaginary part.
- Given $Z(z,t)$ does $Z(z=0,\omega)$ obtain additional properties given that $Z(k,\omega) \in H_+(Z)$. Generalizations to scattering?
- Matrix-case: Scattering matrix.
- Implicit check if $g(z)$ is in H_+. (On boundary or Growth conditions).
- Does v_C the growth-term vanish if $u(x,y)$ have right asymptotic behavior?

Conclusions

- One-dimensional Herglotz-functions have remarkable properties which carry over to matrices.
- A function that is holomorphic and $\sup_{y \in C} \|f(x + \mathrm{i}y)\|_s < \infty$, have a pair of Hilbert transforms (Generalized Kramers-Kronig relation) [Cauchy Kernel]
- A passive linear system in a cone, have a Laplace-transformed kernel that is holomorphic and with positive real part. Thus we have a representation theorem with Poisson Kernel.