Herglotz functions and several variables

Recall: $n = 1$

Def: h is called Herglotz function

$:\Leftrightarrow$ $h: \mathbb{C}^+ \to \mathbb{C}^+$ analytic

Theorem (Integralrepresentation)

h Herglotz fct. $\Leftrightarrow$

$\exists\, a \geq 0,\ b \in \mathbb{R}$, μ pos. Borelmeasure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$

such that

$$h(z) = \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t) + a \cdot z + b$$

Remarks:
- "$\mu(t) = \operatorname{Im} h(t + i0)$"
- Note: no restriction on the measure (except growth property)
- continuation to the lower half-plane: $h(\bar{z}) := \overline{h(z)}$

Example

$$h_1(z) = \sum_{j=1}^{N} \frac{\mu_j}{x_j - z} \qquad (x_j \in \mathbb{R},\ \mu_j > 0)$$

$$h_2(z) = \begin{cases} i & z \in \mathbb{C}^+ \\ -i & z \in \mathbb{C}^- \end{cases}$$

(OBS: h_2 not analytic in $\mathbb{R}$
here μ is Lebesgue-measure)

$h_3(z) = \sqrt{z}$ (where $\sqrt{z} := |z|^{1/2} \cdot e^{i\frac{\operatorname{Arg} z}{2}}$ with $\operatorname{Arg} z \in (-\pi, \pi]$)

Definition: H Herglotz fct. in n variables $:\Leftrightarrow$

$H: (\mathbb{C}^+)^n \to \mathbb{C}^+$ analytic (i.e. $\operatorname{Im} H(z_1, \dots, z_n) \geq 0$ for $\operatorname{Im} z_j > 0$ $j = 1, \dots, n$)

Question: Characterization via integral representation?

Partial results in the literature:

① Vladimirov

Thm (on p. 263)

Let $H : (\mathbb{C}^+)^n \to \mathbb{C}$ be analytic. Then the following are equivalent:

(1) $\operatorname{Im} H(z_1, \dots, z_n) \geq 0$ (i.e. H Herglotz in n variables)

(2) "representation via Fourier transform"

(3) "representation for $\operatorname{Im} H(z_1, \dots, z_n) = \dots$"

(4)
$$H(z_1, \dots, z_n) = i \int_{\mathbb{R}^n} \left[\frac{2 i^n}{(2\pi)^n} \left(\frac{1}{z_1 - t_1} - \frac{1}{-i - t_1} \right) \cdot \ldots \cdot \left(\frac{1}{z_n - t_n} - \frac{1}{-i - t_n} \right) - \frac{1}{\pi^n |i - t_1|^2 \cdot \ldots \cdot |i - t_n|^2} \right] d\mu(t_1, \dots, t_n)$$
$$+ (a, z) + b$$

Here "$\mu = \operatorname{Im} H_+$" $\quad b = \operatorname{Re} f(i, \dots, i) \quad a_j = \lim_{y_j \to \infty} \frac{\operatorname{Im} H(iy)}{y_j}$

Comments

- we are interested in $(1) \Leftrightarrow (4)$
- For $n = 1$ this recovers the well-known integral representation
- note: In a_j there might be a typo (I guess: only $z_j = iy_j$)
- **Problems**
  - "only one direction"? Which measures μ can appear here? Examples show: not all
  - The proof uses a lot of earlier results and is hence neither direct or transparent.

② Korányi, Pukánsky (Transactions of the AMS, 1963)

Deals with functions from the poly disc $\mathbb{D}^n := \{(z_1,\dots,z_n) \in \mathbb{C}^n ; |z_j| = 1,\ j=1,\dots,n\}$ to the right half plane $\mathbb{C}_+$.

Theorem $\varphi : \mathbb{D}^n \to \mathbb{C}$ analytic. Then

$\operatorname{Re}\varphi(w_1,\dots,w_n) \geq 0$ for $|w_i| < 1$ $\iff$

$$\varphi(w_1,\dots,w_n) = \int_{\mathbb{T}^n} \left[2 \frac{u_1}{u_1 - w_1} \cdot \ldots \cdot \frac{u_n}{u_n - w_n} - 1 \right] d\sigma(u_1,\dots,u_n) + i \operatorname{Im}\varphi(0,\dots,0)$$

with σ pos. Borelmeasure on $\mathbb{T}^n$ such that

$$\int_{\mathbb{T}^n} u_1^{k_1} \cdot \ldots \cdot u_n^{k_n} \, d\sigma(u_1,\dots,u_n) = 0 \quad \text{except for } k_j \geq 0 \ \ j = 1,\dots,n \quad \text{or } k_j \leq 0 \ \ j=1,\dots,n$$

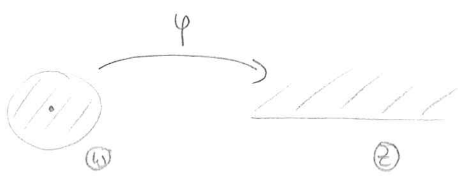
Comment: • $n = 1$ no condition on the measure!

Aim (not yet reached because of lack of time): Transform this result to Herglotz functions.

We demonstrate the idea for $n = 1$:

$\varphi(w) := i \dfrac{1+w}{1-w}$

$\varphi^{-1}(z) = \dfrac{z-i}{z+i}$



note $\varphi(1) = \infty$

$f : \mathbb{D}^n \to \mathbb{C}_+ \iff H(z) := i f(\varphi^{-1}(z_1),\dots,\varphi^{-1}(z_n))$ Herglotz

So for $n=1$: $\varphi(w) = \int\limits_{|u|=1} \left(2\frac{u}{u-w} - 1\right) d\sigma(u) + i \operatorname{Im} \varphi(0) : \mathbb{D} \to \mathbb{C}_+$

we have

$$h(z) = i\varphi(\varphi^{-1}(z)) = i\varphi\left(\frac{z-i}{z+i}\right)$$

$$= i \int\limits_{|u|=1} \left(2\frac{u}{u-\frac{z-i}{z+i}} - 1\right) d\sigma(u) + \operatorname{Re} h(0)$$

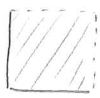
$$= i \int\limits_{\substack{|u|=1 \\ u\neq 1}} \left(2\frac{u}{u-\frac{z-i}{z+i}} - 1\right) d\sigma(u) + i \int\limits_{u=1} \left(2\frac{1}{1-\frac{z-i}{z+i}} - 1\right) d\sigma(u) + \operatorname{Re} h(0)$$

$$= \left[\begin{array}{l} u = \frac{t-i}{t+i} \\ d\sigma(u) = \frac{2}{t^2+1}\, d\tilde{\sigma}(t) \end{array}\right] = \underbrace{i \int\limits_{\mathbb{R}} \left(2\frac{\frac{t-i}{t+i}}{\frac{t-i}{t+i} - \frac{z-i}{z+i}} - 1\right) \frac{2}{t^2+1}\, d\tilde{\sigma}(t)} + z\cdot \underbrace{\sigma(0)}_{=:a} + \underbrace{\operatorname{Re} h(0)}_{\geq 0}$$

$$= \int\limits_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) \underbrace{2\, d\tilde{\sigma}(t)}_{=:d\mu(t)} + az + b \qquad \square$$

$n=2$ Note $\iint\limits_{|u_1|=|u_2|=1} \cdots \, d\sigma(u_1,u_2)$ gives after reparametrization $u_j = e^{i\xi_j}$

$$\iint\limits_{[0,2\pi)^2} \cdots \, d\sigma_1(\xi_1,\xi_2)$$



Hence the integral becomes

$$\iint\limits_{(0,2\pi)^2} \cdots \, d\sigma_1(\xi_1,\xi_2) + \int\limits_{(0,2\pi)} \cdots \, d\sigma_1(0,\xi_2) + \int\limits_{(0,2\pi)} \cdots \, d\sigma_1(\xi_1,0) + (\ldots)\, \sigma_1(0,0)$$

double integral becomes the integral term in Vladimirov

remaining terms have to become $(a;z)$ in Vladimirov (but we dont see this yet)

Ongoing work:

- identify (a,z) and describe more explicit
- translate the restrictive property of the measure

Further questions:

- integral identities (moments?)
- Operator representations