Passive approximation and optimization

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Outline

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  - Disciplined convex programming using CVX
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    *=simple proofs. **=more advanced proofs.
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    - Analytic interpolation and continuation of $\tan(\omega)$ and $\tan(-1/\omega)$ based on finite data
    - Approximation of metamaterials

A brief summary on convex optimization

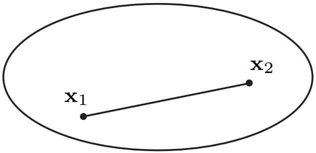
Main ref: S. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press, 2004.

Convex sets: A set $S \subset \mathbb{R}^n$ is *convex* if

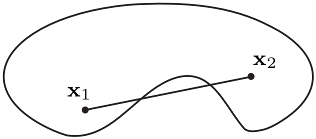
$$\mathbf{x}_1, \mathbf{x}_2 \in S \Rightarrow (1-t)\mathbf{x}_1 + t\mathbf{x}_2 \in S,$$

for all $0 < t < 1$. Geometrically this means that the straight line between any two points in a convex set remains in the set.

a) A convex set in $\mathbb{R}^2$



b) A non-convex set in $\mathbb{R}^2$



A brief summary on convex optimization

Convex combinations: Let $S \subset \mathbb{R}^n$ be an arbitrary set. A *convex combination* of the elements $\mathbf{x}_i \in S$ is a positive linear combination

$$\mathbf{z} = \sum_{i=1}^{m} t_i \mathbf{x}_i \quad \text{where} \quad t_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{m} t_i = 1.$$

If S is a convex set, it can be shown (by induction) that any convex combination of elements of S belongs to S.

Proof: Suppose that $\mathbf{x}_1$, $\mathbf{x_2}$ and $\mathbf{x_3}$ belong to the convex set S. Let

$$\mathbf{z} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + t_3\mathbf{x}_3,$$

where $t_1 + t_2 + t_3 = 1$ and $t_3 \neq 1$ (otherwise $\mathbf{z} = \mathbf{x}_3 \in S$). Then

$$\mathbf{z} = (1 - t_3) \underbrace{\left(\frac{t_1}{1-t_3}\mathbf{x}_1 + \frac{t_2}{1-t_3}\mathbf{x}_2\right)}_{\in S \text{ since } \frac{t_1}{1-t_3} + \frac{t_2}{1-t_3} = 1} + t_3\mathbf{x}_3 \in S.$$

A brief summary on convex optimization

Intersection of convex sets(*):

The intersection $\cap_\alpha S_\alpha$ of convex sets S_α is a convex set.

Convex hull(*):

Let $S \subset \mathbb{R}^n$ be an arbitrary set. The *convex hull* $\mathrm{conv}(S)$ of S is the set of all convex combinations of elements in S. It can be shown that $\mathrm{conv}(S)$ is the smallest convex set containing S, *i.e.*,

$$\mathrm{conv}(S) = \cap_\alpha T_\alpha$$

where the intersection is taken over all convex sets T_α containing S, *i.e.*, $S \subset T_\alpha$.

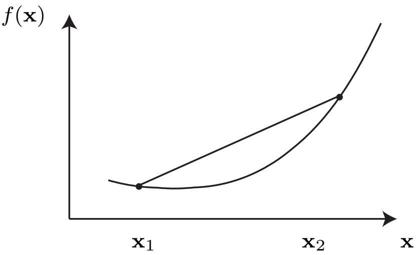
A brief summary on convex optimization

Convex functions: Let f be a function defined on the convex set $S \subset \mathbb{R}^n$. The function f is said to be a *convex function* if

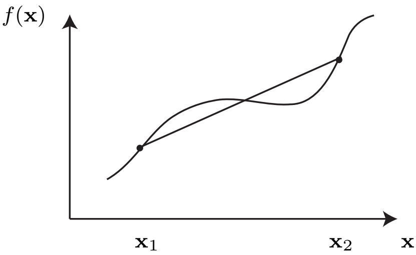
$$\mathbf{x}_1, \mathbf{x}_2 \in S \Rightarrow f((1-t)\mathbf{x}_1 + t\mathbf{x}_2) \leq (1-t)f(\mathbf{x}_1) + tf(\mathbf{x}_2), \quad (1)$$

for all $0 < t < 1$. If the function f satisfies a strict inequality in (1), it is said to be a *strictly convex function*. A function f is said to be *concave* if $-f$ is convex.

a) A convex function on $\mathbb{R}$



b) A non-convex function on $\mathbb{R}$



A brief summary on convex optimization

Linear functions(*):

A linear (or affine) function f is both convex and concave (but it is not strictly convex). In particular, if f is linear (or affine), both f and $-f$ are convex.

Positive linear combinations of convex functions(*):

Let f_1 and f_2 be convex functions defined on a convex set S. Then the positive linear combination

$$f = \alpha_1 f_1 + \alpha_2 f_2$$

(with $\alpha_1 > 0$ and $\alpha_2 > 0$) is a convex function. If any one of f_1 and f_2 is strictly convex, then the function f is strictly convex.

Differentiable convex functions(**):

Let f be a two times differentiable continuous function defined on the open convex set $S \subset \mathbb{R}^n$ ($f \in C^2(S)$). It can then be shown that f is a convex function if and only if the Hessian $\mathbf{H}_{ij}(\mathbf{x}) = \partial_{x_i}\partial_{x_j} f(\mathbf{x})$ is a positively semidefinite matrix for all $\mathbf{x} \in S$, *i.e.*,

$$\mathbf{H}(\mathbf{x}) \geq 0 \ \forall \mathbf{x} \in S \Leftrightarrow f \text{ convex.}$$

If the Hessian $\mathbf{H}(\mathbf{x})$ is positively definite for all $\mathbf{x} \in S$ then f is strictly convex, *i.e.*,

$$\mathbf{H}(\mathbf{x}) > 0 \ \forall \mathbf{x} \in S \Rightarrow f \text{ strictly convex,}$$

but the converse is not true (take *e.g.*, $f(x) = x^4$).

Quadratic forms(**):

Let

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} + c$$

be a quadratic form where $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ vectors, c a constant and $(\cdot)^{\mathrm{T}}$ denotes the transpose. The function f is convex if and only if the Hessian matrix $\mathbf{A}$ is positively semidefinite, *i.e.*,

$$\mathbf{A} \geq 0 \Leftrightarrow f \text{ convex.}$$

The function f is strictly convex if and only if the Hessian matrix $\mathbf{A}$ is positively definite, *i.e.*,

$$\mathbf{A} > 0 \Leftrightarrow f \text{ strictly convex.}$$

A brief summary on convex optimization

Level set(*):

Let g be a convex function defined on the convex set $S \subset \mathbb{R}^n$ and α an arbitrary real number. Then the level set

$$S_\alpha = \{\mathbf{x} \in S | g(\mathbf{x}) \leq \alpha\}$$

is a convex set.

Continuity(**):

Let f be a convex function defined on the convex set $S \subset \mathbb{R}^n$. Then f is a continuous function on the interior of S.

A brief summary on convex optimization

Convex optimization:

A convex optimization problem in general is a problem of the form

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in S, \end{array} \tag{2}$$

where f is a convex function defined on the convex set S.

It is common that the convex set S is given in the form

$$S = \{\mathbf{x} \in \mathbb{R}^n | g_i(\mathbf{x}) \leq 0\}$$

where $\{g_i(\mathbf{x})\}_{i=1}^M$ is a set of convex functions representing the *convex constraints*.

A local minimum is also a global minimum:

Below are given some definitions and a proof.

Consider the convex optimization problem (2) where f is a convex function defined on the convex set S.

- If $\mathbf{x} \in S$ it is called a *feasible point*.
- The function f has a *global minimum* at $\mathbf{x}$ if

$$f(\mathbf{x}) \leq f(\boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in S$$

- The function f has a *local minimum* at $\mathbf{x}$ if

$$f(\mathbf{x}) \leq f(\boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in S_r = \{\boldsymbol{\xi} \in S | \|\boldsymbol{\xi} - \mathbf{x}\| < r\}$$

  for *some* r-neighbourhood S_r with radius $r > 0$, and where $\| \cdot \|$ is a suitable norm.

A brief summary on convex optimization

Proof: Let $\mathbf{x} \in S$ be a local minimum and S_r a corresponding r-neighbourhood. Suppose now that $\mathbf{x}$ is not a global minimum, then there is a feasible $\mathbf{y} \in S$ with minimum value $f(\mathbf{y}) < f(\mathbf{x})$. Since $\mathbf{x}$ is a local minimum it follows that $\mathbf{y} \notin S_r$ and hence $\|\mathbf{y} - \mathbf{x}\| \geq r$. Define the point

$$\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y} \quad \text{where} \quad t = \frac{r}{2\|\mathbf{y}-\mathbf{x}\|} \leq \frac{1}{2},$$

so that

$$\|\mathbf{z} - \mathbf{x}\| = t\|\mathbf{y} - \mathbf{x}\| = \frac{r}{2},$$

and hence $\mathbf{z} \in S_r$. On the other hand, it follows by the convexity of f that

$$f(\mathbf{z}) = f((1-t)\mathbf{x} + t\mathbf{y}) \leq (1-t)f(\mathbf{x}) + tf(\mathbf{y}) < f(\mathbf{x}),$$

which contradicts the assumption that $\mathbf{x}$ is a local minimum.

A brief summary on convex optimization

If f is strictly convex then a minimum is also a unique minimum:

Proof: Let $\mathbf{x} \in S$ be a minimum so that $f(\mathbf{x}) \leq f(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in S$. Suppose that $\mathbf{y} \in S$ where $\mathbf{y} \neq \mathbf{x}$ is also a minimum so that $f(\mathbf{y}) = f(\mathbf{x})$. Let $\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y}$ where $0 < t < 1$ and hence $\mathbf{z} \in S$. Then by the strict convexity

$$f(\mathbf{z}) = f((1-t)\mathbf{x} + t\mathbf{y}) < (1-t)f(\mathbf{x}) + tf(\mathbf{y}) = f(\mathbf{x}),$$

which contradicts the assumption that $\mathbf{x}$ is a minimum.

A brief summary on convex optimization

The norm:

Any norm $f(\mathbf{x}) = \|\mathbf{x}\|$ is a convex function on $\mathbb{R}^n$ since by the triangle inequality

$$\begin{aligned} f((1-t)\mathbf{x}_1 + t\mathbf{x}_2) = \|(1-t)\mathbf{x}_1 + t\mathbf{x}_2\| \leq (1-t)\|\mathbf{x}_1\| + t\|\mathbf{x}_2\| \\ = (1-t)f(\mathbf{x}_1) + tf(\mathbf{x}_2), \end{aligned}$$

where $0 < t < 1$.

Comments: It follows from the properties of the norm that the minimum of $f(\mathbf{x}) = \|\mathbf{x}\|$ is the unique vector $\mathbf{x} = \mathbf{0}$. Note however that $f(\mathbf{x}) = \|\mathbf{x}\|$ is not a strictly convex function. To see this, choose *e.g.*, $\mathbf{x}_2 = \alpha\mathbf{x}_1$ where $\alpha > 1$ and verify that $f((1-t)\mathbf{x}_1 + t\mathbf{x}_2) = (1-t)f(\mathbf{x}_1) + tf(\mathbf{x}_2)$ for all $0 < t < 1$. Note however that the function $f(\mathbf{x}) = \|\mathbf{x}\|_2^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$ is strictly convex.

Norm of a linear (or affine) form:

The norm of a linear (or affine) form is a convex function on $\mathbb{R}^n$. Consider *e.g.*, the function $f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|$ where $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{x}$ an $n \times 1$ vector and $\mathbf{b}$ an $m \times 1$ vector. The function $f(\mathbf{x})$ is convex since by the triangle inequality

$$
\begin{aligned}
&f((1-t)\mathbf{x}_1 + t\mathbf{x}_2) = \|\mathbf{A}((1-t)\mathbf{x}_1 + t\mathbf{x}_2) - \mathbf{b}\| \\
&= \|(1-t)(\mathbf{A}\mathbf{x}_1 - b) + t(\mathbf{A}\mathbf{x}_2 - \mathbf{b})\| \leq (1-t)\|\mathbf{A}\mathbf{x}_1 - \mathbf{b}\| + t\|\mathbf{A}\mathbf{x}_2 - \mathbf{b}\| \\
&\qquad = (1-t)f(\mathbf{x}_1) + tf(\mathbf{x}_2),
\end{aligned}
$$

where $0 < t < 1$.

Norm of a linear (or affine) form:

Comments: Note that if the Euclidean norm $\|\cdot\|_2$ is used, if $m > n$ and if the rank of the matrix $\mathbf{A}$ is $\mathrm{rank}\{\mathbf{A}\} = n$, then the minimum of f (the least squares solution) is unique (even though the function f is not strictly convex).

It is evident that the well-posedness of the minimization of $f(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{b}\|$ depends on the properties of the matrix (or operator) $\mathbf{A}$. If the matrix $\mathbf{A}$ is not full rank and has a non-trivial null space, then the optimization problem can be regularized by adding a strictly convex penalty term as *e.g.*, $f(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{b}\|_2 + \alpha\|\mathbf{x}\|_2^2$ where $\alpha > 0$ (*cf.*, Tikhonov regularization). Now the new function f is strictly convex, and if α is sufficiently small the optimal solution will coincide with the pseudo-inverse.

Disciplined convex programming in Matlab:

As an example, consider the following convex optimization problem related to the normed error

$$\begin{array}{ll} \text{minimize} & \|\mathbf{Ax}-\mathbf{b}\|_p \\ \text{subject to} & \mathbf{Bx}\le\mathbf{c} \\ & \mathbf{Cx}=\mathbf{d}, \end{array} \qquad (3)$$

where the $n \times 1$ vector $\mathbf{x}$ is the unknown optimization variable and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given matrices and $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are given vectors of suitable dimensions. Here, $\|\cdot\|_p$ denotes that a p-norm is used where $p \geq 1$. The max-norm is denoted $\|\cdot\|_\infty$.

A brief summary on convex optimization

Disciplined convex programming in Matlab:

Once formulated on a standard form the problem (3) can be solved efficiently by using the CVX Matlab software for disciplined convex programming which is available via the link http://cvxr.com/cvx/. Here, the following lines in Matlab may be used

```
cvx_begin
        variable x(n)
        minimize(norm(A * x - b, p))
        subject to
        B * x <= c
        C * x == d
cvx_end
```

where n, p ($p \geq 1$ or $p = \mathtt{Inf}$), A, B, C, b, c and d have been defined in the preamble.

Herglotz functions

Analytic functions with the property

$$\operatorname{Im} h(\omega) \geq 0 \quad \text{for} \quad \omega \in \mathbb{C}_+ = \{\omega \in \mathbb{C} |\operatorname{Im} \omega > 0\}$$

Integral representation

$$h(\omega) = b_1\omega + \alpha + \int_{-\infty}^{\infty} \left(\frac{1}{\xi - \omega} - \frac{\xi}{1 + \xi^2} \right) \mathrm{d}\beta(\xi)$$

Here, $b_1 \geq 0$, $\alpha \in \mathbb{R}$

$\mathrm{d}\beta(\xi)$ is a positive Borel measure with $\int_{\mathbb{R}} \mathrm{d}\beta(\xi)/(1+\xi^2) < \infty$

Notation: $\mathrm{d}\beta(\xi) = \beta'(\xi)\,\mathrm{d}\xi = \frac{1}{\pi} \operatorname{Im} h(\xi + \mathrm{i}0)\,\mathrm{d}\xi \overset{\text{def}}{=} \frac{1}{\pi} \operatorname{Im} h(\xi)\,\mathrm{d}\xi$

Symmetric Herglotz functions

Conjugate symmetry and symmetric measure

$$h(\omega) = -h^*(-\omega^*) \quad \Leftrightarrow \quad \begin{array}{l} \operatorname{Im} h(-\xi) = \operatorname{Im} h(\xi) \\ \operatorname{Re} h(-\xi) = -\operatorname{Re} h(\xi) \end{array}$$

Integral representation for $\omega \in \mathbb{C}_+$

$$h(\omega) = b_1\omega + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\xi - \omega} \operatorname{Im} h(\xi)\, \mathrm{d}\xi$$

Distributional limit on the real line $\omega \in \mathbb{R}$

$$\frac{1}{\xi - \omega} \to \frac{1}{\xi - \omega} + \mathrm{i}\pi\delta(\xi - \omega) \quad \text{as} \quad \operatorname{Im}\omega \to 0+$$

$$\operatorname{Re} h(\omega) = b_1\omega + \frac{1}{\pi}\fint_{-\infty}^{\infty} \frac{1}{\xi - \omega} \operatorname{Im} h(\xi)\, \mathrm{d}\xi$$

where the integral is a Cauchy principal value integral

Symmetric Herglotz functions and related sum rules

Suppose that the following asymptotic properties are valid

$$h(\omega) = \begin{cases} a_{-1}\omega^{-1} + a_1\omega + \ldots + a_{2N_0-1}\omega^{2N_0-1} + o(\omega^{2N_0-1}) & \text{as } \omega \hat{\to} 0 \\ b_1\omega + b_{-1}\omega^{-1} + \ldots + b_{1-2N_\infty}\omega^{1-2N_\infty} + o(\omega^{1-2N_\infty}) & \text{as } \omega \hat{\to} \infty \end{cases}$$

then the following integral identities (sum rules) hold

$$\frac{2}{\pi}\int_{0+}^{\infty} \frac{\operatorname{Im} h(\xi)}{\xi^{2n}}\,\mathrm{d}\xi \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0+}\lim_{y\to 0+}\frac{2}{\pi}\int_{\varepsilon}^{1/\varepsilon}\frac{\operatorname{Im} h(\xi+\mathrm{i}y)}{\xi^{2n}}\,\mathrm{d}\xi = a_{2n-1} - b_{2n-1}$$

for $n = 1 - N_\infty, \ldots, N_0$

A general approximation problem

$$\begin{aligned} &\text{minimize} && \|h(\xi) - f(\xi)\|_\Omega \\ &\text{subject to} && f(\xi) \text{ continuous on } \Omega \\ & && h(\xi) \overset{\text{def}}{=} h(\xi + \mathrm{i}0) \exists \text{ continuous on } \Omega \end{aligned}$$

Approximation domain Ω is a closed and bounded subset of $\mathbb{R}$

$h(\omega)$ is a symmetric Herglotz function regular in a neighbourhood of $\mathbb{C}^+ \cup \Omega$

A non-trivial example

$-f(\xi)$ can be continued to a Herglotz function $h_0(\omega)$ which is analytic in a neighbourhood of $\mathbb{C}^+ \cup \Omega$

Large argument asymptotics: $h_0(\omega) = b_1^0 \omega + o(\omega)$ as $\omega \hat{\to} \infty$

Then $\sup_{\xi \in \Omega} |h(\xi) - f(\xi)| \geq b_1^0 \frac{1}{2} |\Omega|$

[1] M. Gustafsson and D. Sjöberg. Sum rules and physical bounds on passive metamaterials. New Journal of Physics, 12, 043046, 2010.

Derivation of sum rule and non-trivial bound

Following [1] above, employ the **auxiliary Herglotz function**

$$h_\Delta(z) = \frac{1}{\pi}\int_{-\Delta}^{\Delta} \frac{1}{\xi - z}\,\mathrm{d}\xi = \frac{1}{\pi}\ln\frac{z-\Delta}{z+\Delta} = \begin{cases} \mathrm{i} + o(1) & \text{as } z\hat{\to}0 \\ \dfrac{-2\Delta}{\pi z} + o(z^{-1}) & \text{as } z\hat{\to}\infty \end{cases}$$

where $\operatorname{Im} h_\Delta(z) \geq \frac{1}{2}$ for $|z| \leq \Delta$ and $\operatorname{Im} z \geq 0$

Composite Herglotz function $\widetilde{h}(\omega) = h_\Delta(h(\omega) + h_0(\omega))$

where $h(\omega) + h_0(\omega) = (b_1 + b_1^0)\omega + o(\omega)$ as $\omega\hat{\to}\infty$

$$\widetilde{h}(\omega) = \begin{cases} o(\omega^{-1}) & \text{as } \omega\hat{\to}0 \\ \dfrac{-2\Delta}{\pi(b_1 + b_1^0)}\omega^{-1} + o(\omega^{-1}) & \text{as } \omega\hat{\to}\infty \end{cases}$$

Sum rule for $n = 0$

$$\frac{2}{\pi}\int_0^\infty \operatorname{Im}\widetilde{h}(\xi)\,\mathrm{d}\xi = a_{-1} - b_{-1} = \frac{2\Delta}{\pi(b_1 + b_1^0)}$$

Derivation of sum rule and non-trivial bound

Sum rule for $n=0$

$$\frac{2}{\pi}\int_0^{\infty}\operatorname{Im}\widetilde{h}(\xi)\,\mathrm{d}\xi = a_{-1}-b_{-1}=\frac{2\Delta}{\pi(b_1+b_1^0)}$$

Let $\Delta=\sup_{\xi\in\Omega}|h(\xi)+h_0(\xi)|$, the following **integral inequalities** hold

$$\frac{1}{\pi}|\Omega|\le\frac{2}{\pi}\int_\Omega\underbrace{\operatorname{Im}\widetilde{h}(\xi)}_{\ge\frac{1}{2}}\mathrm{d}\xi\le\frac{2}{\pi}\int_0^{\infty}\operatorname{Im}\widetilde{h}(\xi)\,\mathrm{d}\xi=\frac{2}{\pi(b_1+b_1^0)}\sup_{\xi\in\Omega}|h(\xi)+h_0(\xi)|$$

or

$$\sup_{\xi\in\Omega}|h(\xi)+h_0(\xi)|\ge(b_1+b_1^0)\frac{1}{2}|\Omega|\quad\text{where}\quad|\Omega|=\int_\Omega\mathrm{d}\xi$$

Permittivity functions: If $h(\omega)=\omega\epsilon(\omega)$ and $h_0(\omega)=-f(\omega)=-\omega\epsilon_{\mathrm{t}}$ with $\epsilon_{\mathrm{t}}<0$, then $b_1=\epsilon_\infty$ and $b_1^0=-\epsilon_{\mathrm{t}}$ and

$$\sup_{\xi\in\Omega}|\epsilon(\xi)-\epsilon_{\mathrm{t}}|\ge\frac{(\epsilon_\infty-\epsilon_{\mathrm{t}})\frac{1}{2}B}{1+\frac{B}{2}}$$

where $\Omega=\omega_0[1-\frac{B}{2},1+\frac{B}{2}]$ and where ω_0 is the center frequency and $0<B<2$, see also [1].

Discretization

Discretization of positive measure on $\Omega_1 = [\omega_1, \omega_4]$**,** $\Delta\omega = \frac{\omega_4 - \omega_1}{N-1}$

$$\operatorname{Im} h(\omega) = \sum_{n=0}^{N-1} x_n \left[p(\omega - (n+n_1)\Delta\omega) + p(\omega + (n+n_1)\Delta\omega) \right], \quad x_n \geq 0$$

Triangular basis functions

$$p(\omega) = \begin{cases} 1 - \dfrac{|\omega|}{\Delta\omega} & |\omega| \leq \Delta\omega \\ 0 & |\omega| > \Delta\omega \end{cases}$$

Hilbert transform on approximation domain $\Omega = [\omega_2, \omega_3] \subset \Omega_1$

$$\operatorname{Re} h(\omega) = b_1\omega + \sum_{n=0}^{N-1} x_n \left[\hat{p}(\omega - (n+n_1)\Delta\omega) + \hat{p}(\omega + (n+n_1)\Delta\omega) \right]$$

$$\hat{p}(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - \omega} p(\xi) \, \mathrm{d}\xi$$
$$= \left(2\omega \ln|\omega| - (\omega - \Delta\omega) \ln|\omega - \Delta\omega| - (\omega + \Delta\omega) \ln|\omega + \Delta\omega| \right) / (\pi \Delta\omega)$$

Convex optimization

Approximation problem

$$\begin{aligned}
&\text{minimize} && \|h(\xi) - f(\xi)\|_\Omega \\
&\text{subject to} && f(\xi) \text{ continuous on } \Omega \\
& && h(\xi) \stackrel{\text{def}}{=} h(\xi + \mathrm{i}0) \exists \text{ continuous on } \Omega
\end{aligned}$$

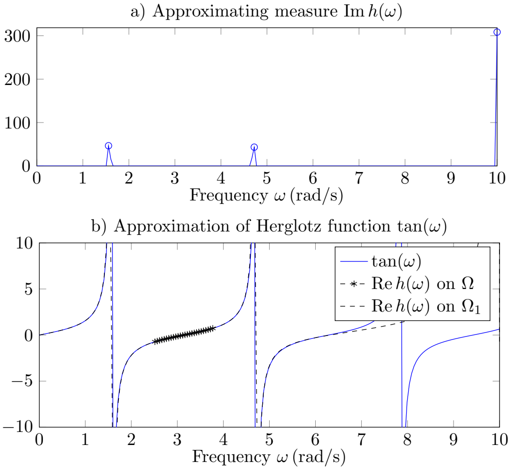
Disciplined convex programming in MATLAB [2]

```
A = H + 1i * I
cvx_begin
        variable x(N)
        minimize(norm(A * x - f, Inf))
        subject to
        x >= 0
cvx_end
```

[2] M. Grant and S. Boyd. CVX: A system for disciplined convex programming, release 2.0, ©2012 CVX Research, Inc., Austin, TX.

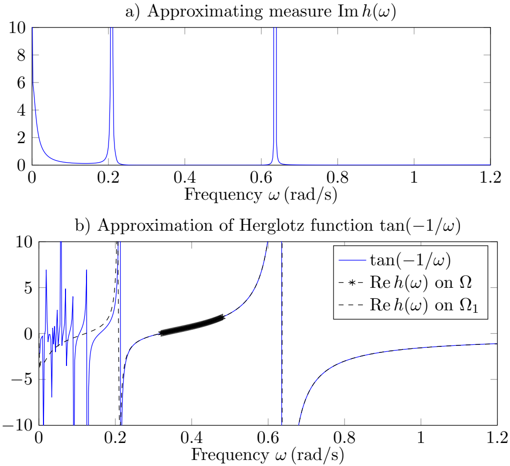
Approximation of $\tan(\omega)$

$$h(\xi) \stackrel{\text{def}}{=} h(\xi + \mathrm{i}0) = \tan(\xi) + \mathrm{i}\pi \sum_{l=-\infty}^{\infty} \delta(\xi - \frac{\pi}{2} + l\pi)$$



Approximation of $\tan(-1/\omega)$

$$h(\xi) \overset{\text{def}}{=} h(\xi + \mathrm{i}0) = \tan(-1/\xi) + \mathrm{i}\pi \sum_{l=-\infty}^{\infty} \frac{1}{(l\pi - \frac{\pi}{2})^2} \delta(\xi - \frac{1}{l\pi - \frac{\pi}{2}})$$



Approximation of metamaterial

Herglotz function $h(\omega) = \omega\epsilon(\omega)$ with $\epsilon_\infty = 1$ (and $\epsilon_{\mathrm{s}} \stackrel{\mathrm{def}}{=} \epsilon(0)$)

Metamaterial $f(\omega) = \omega\epsilon_{\mathrm{t}}$ for $\omega \in \Omega$ and where $\epsilon_{\mathrm{t}} = -1$

Convex optimization problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\sup_{\xi\in\Omega} \frac{1}{\xi}|h(\xi) - f(\xi)| \\ \text{subject to} & \displaystyle\frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\, h(\xi)}{\xi^2}\,\mathrm{d}\xi \leq \epsilon_{\mathrm{s}}^{\max} - \epsilon_\infty \end{array}$$

Discretization of sum rule

$$\frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\, h(\xi)}{\xi^2}\,\mathrm{d}\xi = \frac{2}{\pi\Delta\omega}\sum_{n=0}^{N-1} x_n \ln\frac{(n+n_1)^2}{(n+n_1-1)(n+n_1+1)}$$

assuming that $n_1 > 1$ where $\omega_1 = n_1\Delta\omega$ $(\mathrm{Im}\, h(\xi) = o(\xi))$

Approximation of metamaterial

Disciplined convex programming in MATLAB [2]

```
Omega;                % Approximation domain
A = H + 1i * I;       % Representation of Herglotz function
B;                    % Representation of sum rule
b = epss - epsinf;    % Static and optical responses
```

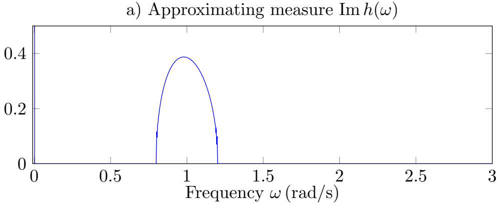
```
cvx_begin
        variable x(N)
        minimize(norm((epsinf * Omega + A * x - f)./Omega, Inf))
        subject to
        x >= 0
        B * x <= b
cvx_end
```

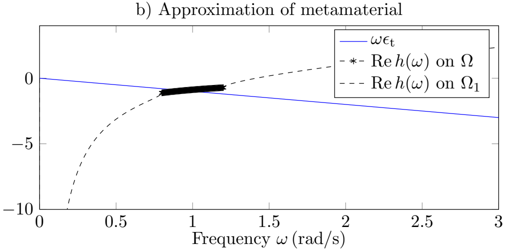
[2] M. Grant and S. Boyd. CVX: A system for disciplined convex programming, release 2.0, ©2012 CVX Research, Inc., Austin, TX.

Approximation of metamaterial

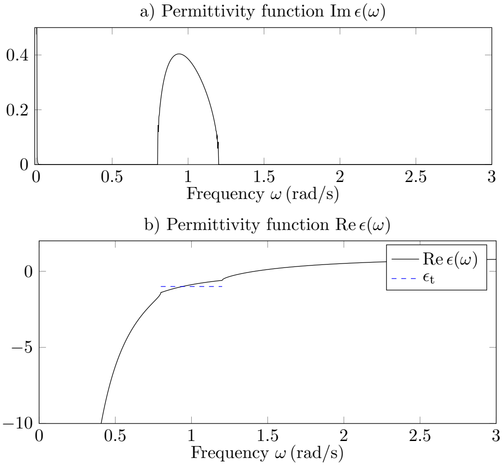
Herglotz function $h(\omega) = \omega\epsilon(\omega)$ with $\epsilon_\infty = 1$

Metamaterial $f(\omega) = \omega\epsilon_{\mathrm{t}}$ for $\omega \in \Omega$ and where $\epsilon_{\mathrm{t}} = -1$



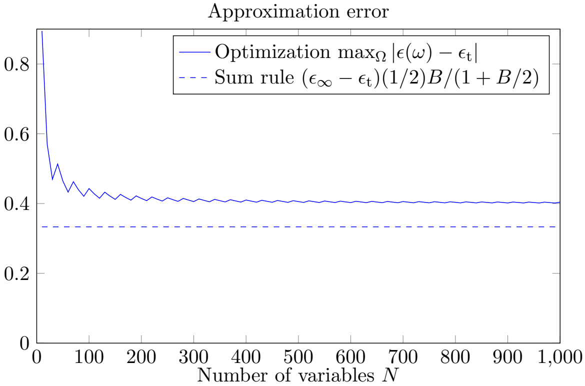


Approximation of metamaterial (permittivity)



Note that $\omega\epsilon(\omega) = -\frac{1}{\pi}\frac{1}{\omega} + \mathrm{i}\delta(\omega) = \omega\left(-\frac{1}{\pi}\frac{1}{\omega^2} - \mathrm{i}\delta'(\omega)\right)$

Convergence of approximation (metamaterial)



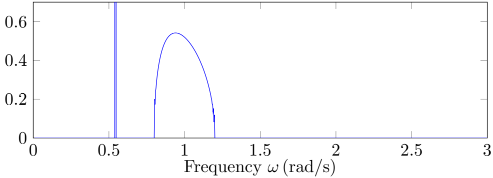
Here, the relative bandwidth is $B = 0.4$

Approximation of metamaterial with static permittivity

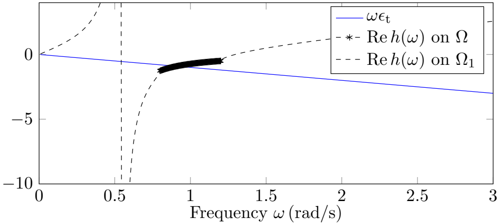
Herglotz function $h(\omega) = \omega\epsilon(\omega)$ with $\epsilon_\infty = 1$ and $\epsilon_{\mathrm{s}}^{\max} = 5$

Metamaterial $f(\omega) = \omega\epsilon_{\mathrm{t}}$ for $\omega \in \Omega$ and where $\epsilon_{\mathrm{t}} = -1$

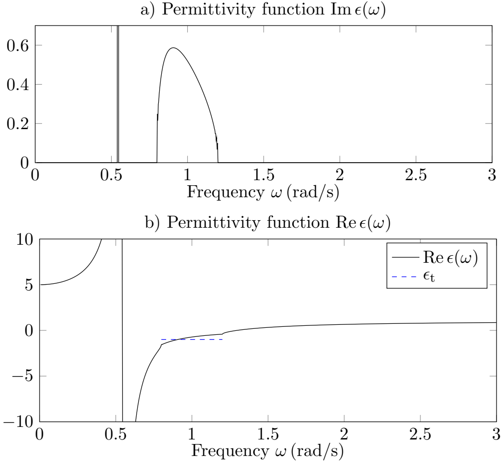
a) Approximating measure $\operatorname{Im} h(\omega)$



b) Approximation of metamaterial



Approximation of metamaterial (permittivity)



Note that $\omega\epsilon(\omega) = \omega\left(-\frac{2}{\pi}\frac{1}{(\omega-\omega_0)(\omega+\omega_0)} + \frac{\mathrm{i}}{\omega_0}\delta(\omega-\omega_0) - \frac{\mathrm{i}}{\omega_0}\delta(\omega+\omega_0)\right)$

Summary and future work

- A general approximation problem has been formulated based on the class of symmetric Herglotz functions.
- The approximation problem can be discretized and solved by using a disciplined convex programming in MATLAB which facilitates a straightforward addition of auxiliary convex constraints.
- Future work aims at
    - Rigorous mathematical formulation and proof of convergence
    - Further applications of dispersion analysis and optimization of passive materials and structures
    - Optimization of matrix valued Herglotz functions for complex materials