Passive approximation and optimization

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Outline

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  - Disciplined convex programming using CVX
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**Convex sets:** A set \( S \subset \mathbb{R}^n \) is **convex** if

\[
x_1, x_2 \in S \Rightarrow (1 - t)x_1 + tx_2 \in S,
\]

for all \( 0 < t < 1 \). Geometrically this means that the straight line between any two points in a convex set remains in the set.
Convex combinations: Let $S \subset \mathbb{R}^n$ be an arbitrary set. A **convex combination** of the elements $x_i \in S$ is a positive linear combination

$$z = \sum_{i=1}^{m} t_i x_i \quad \text{where} \quad t_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{m} t_i = 1.$$ 

If $S$ is a convex set, it can be shown (by induction) that any convex combination of elements of $S$ belongs to $S$.

**Proof:** Suppose that $x_1$, $x_2$ and $x_3$ belong to the convex set $S$. Let

$$z = t_1 x_1 + t_2 x_2 + t_3 x_3,$$

where $t_1 + t_2 + t_3 = 1$ and $t_3 \neq 1$ (otherwise $z = x_3 \in S$). Then

$$z = (1 - t_3) \left( \frac{t_1}{1 - t_3} x_1 + \frac{t_2}{1 - t_3} x_2 \right) + t_3 x_3 \in S \quad \text{since} \quad \frac{t_1}{1 - t_3} + \frac{t_2}{1 - t_3} = 1$$
Intersection of convex sets(*):

The intersection $\cap_\alpha S_\alpha$ of convex sets $S_\alpha$ is a convex set.

Convex hull(*):

Let $S \subset \mathbb{R}^n$ be an arbitrary set. The convex hull $\text{conv}(S)$ of $S$ is the set of all convex combinations of elements in $S$. It can be shown that $\text{conv}(S)$ is the smallest convex set containing $S$, i.e.,

$$\text{conv}(S) = \cap_\alpha T_\alpha$$

where the intersection is taken over all convex sets $T_\alpha$ containing $S$, i.e., $S \subset T_\alpha$. 
A brief summary on convex optimization

**Convex functions:** Let $f$ be a function defined on the convex set $S \subset \mathbb{R}^n$. The function $f$ is said to be a *convex function* if

\[ x_1, x_2 \in S \Rightarrow f((1 - t)x_1 + tx_2) \leq (1 - t)f(x_1) + tf(x_2), \quad (1) \]

for all $0 < t < 1$. If the function $f$ satisfies a strict inequality in (1), it is said to be a *strictly convex function*. A function $f$ is said to be *concave* if $-f$ is convex.

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a) A convex function on $\mathbb{R}$

b) A non-convex function on $\mathbb{R}$
A brief summary on convex optimization

Linear functions(*): 
A linear (or affine) function $f$ is both convex and concave (but it is not strictly convex). In particular, if $f$ is linear (or affine), both $f$ and $-f$ are convex.

Positive linear combinations of convex functions(*): 
Let $f_1$ and $f_2$ be convex functions defined on a convex set $S$. Then the positive linear combination 

$$f = \alpha_1 f_1 + \alpha_2 f_2$$

(with $\alpha_1 > 0$ and $\alpha_2 > 0$) is a convex function. If any one of $f_1$ and $f_2$ is strictly convex, then the function $f$ is strictly convex.
Differentiable convex functions(**):

Let $f$ be a two times differentiable continuous function defined on the open convex set $S \subset \mathbb{R}^n$ ($f \in C^2(S)$). It can then be shown that $f$ is a convex function if and only if the Hessian $H_{ij}(x) = \partial_{x_i} \partial_{x_j} f(x)$ is a positively semidefinite matrix for all $x \in S$, i.e.,

$$H(x) \succeq 0 \forall x \in S \iff f \text{ convex}.$$ 

If the Hessian $H(x)$ is positively definite for all $x \in S$ then $f$ is strictly convex, i.e.,

$$H(x) > 0 \forall x \in S \Rightarrow f \text{ strictly convex},$$

but the converse is not true (take e.g., $f(x) = x^4$).
Quadratic forms(**): 

Let 

\[ f(x) = \frac{1}{2} x^T Ax + b^T x + c \]

be a quadratic form where \( A \) is an \( n \times n \) matrix, \( x \) and \( b \) are \( n \times 1 \) vectors, \( c \) a constant and \((\cdot)^T\) denotes the transpose. The function \( f \) is convex if and only if the Hessian matrix \( A \) is positively semidefinite, \( i.e., \)

\[ A \geq 0 \iff f \text{ convex}. \]

The function \( f \) is strictly convex if and only if the Hessian matrix \( A \) is positively definite, \( i.e., \)

\[ A > 0 \iff f \text{ strictly convex}. \]
Level set(*): 

Let $g$ be a convex function defined on the convex set $S \subset \mathbb{R}^n$ and $\alpha$ an arbitrary real number. Then the level set

$$S_\alpha = \{ x \in S | g(x) \leq \alpha \}$$

is a convex set.

Continuity(**): 

Let $f$ be a convex function defined on the convex set $S \subset \mathbb{R}^n$. Then $f$ is a continuous function on the interior of $S$. 
Convex optimization:

A convex optimization problem in general is a problem of the form

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in S, \\
\end{align*}
\]

(2)

where \( f \) is a convex function defined on the convex set \( S \).

It is common that the convex set \( S \) is given in the form

\[
S = \{ x \in \mathbb{R}^n | g_i(x) \leq 0 \}
\]

where \( \{ g_i(x) \}_{i=1}^M \) is a set of convex functions representing the \emph{convex constraints}. 
A brief summary on convex optimization

A local minimum is also a global minimum:

Below are given some definitions and a proof.

Consider the convex optimization problem (2) where \( f \) is a convex function defined on the convex set \( S \).

- If \( x \in S \) it is called a feasible point.
- The function \( f \) has a global minimum at \( x \) if
  \[
  f(x) \leq f(\xi), \quad \forall \xi \in S
  \]
- The function \( f \) has a local minimum at \( x \) if
  \[
  f(x) \leq f(\xi), \quad \forall \xi \in S_r = \{ \xi \in S \mid \|\xi - x\| < r \}
  \]
  for some \( r \)-neighbourhood \( S_r \) with radius \( r > 0 \), and where \( \| \cdot \| \) is a suitable norm.
Proof: Let \( x \in S \) be a local minimum and \( S_r \) a corresponding \( r \)-neighbourhood. Suppose now that \( x \) is not a global minimum, then there is a feasible \( y \in S \) with minimum value \( f(y) < f(x) \). Since \( x \) is a local minimum it follows that \( y \not\in S_r \) and hence \( \|y - x\| \geq r \). Define the point

\[
z = (1 - t)x + ty \quad \text{where} \quad t = \frac{r}{2\|y - x\|} \leq \frac{1}{2},
\]

so that

\[
\|z - x\| = t\|y - x\| = \frac{r}{2},
\]

and hence \( z \in S_r \). On the other hand, it follows by the convexity of \( f \) that

\[
f(z) = f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y) < f(x),
\]

which contradicts the assumption that \( x \) is a local minimum.
If \( f \) is strictly convex then a minimum is also a unique minimum:

**Proof:** Let \( x \in S \) be a minimum so that \( f(x) \leq f(\xi) \) for all \( \xi \in S \). Suppose that \( y \in S \) where \( y \neq x \) is also a minimum so that \( f(y) = f(x) \). Let \( z = (1 - t)x + ty \) where \( 0 < t < 1 \) and hence \( z \in S \). Then by the strict convexity

\[
f(z) = f((1 - t)x + ty) < (1 - t)f(x) + tf(y) = f(x),
\]

which contradicts the assumption that \( x \) is a minimum.
A brief summary on convex optimization

The norm:

Any norm $f(x) = \|x\|$ is a convex function on $\mathbb{R}^n$ since by the triangle inequality

$$f((1 - t)x_1 + tx_2) = \|(1 - t)x_1 + tx_2\| \leq (1 - t)\|x_1\| + t\|x_2\|$$

$$= (1 - t)f(x_1) + tf(x_2),$$

where $0 < t < 1$.

Comments: It follows from the properties of the norm that the minimum of $f(x) = \|x\|$ is the unique vector $x = 0$. Note however that $f(x) = \|x\|$ is not a strictly convex function. To see this, choose e.g., $x_2 = \alpha x_1$ where $\alpha > 1$ and verify that

$$f((1 - t)x_1 + tx_2) = (1 - t)f(x_1) + tf(x_2)$$

for all $0 < t < 1$. Note however that the function $f(x) = \|x\|^2 = x^T x$ is strictly convex.
A brief summary on convex optimization

Norm of a linear (or affine) form:

The norm of a linear (or affine) form is a convex function on \( \mathbb{R}^n \). Consider e.g., the function \( f(x) = \|Ax - b\| \) where \( A \) is an \( m \times n \) matrix, \( x \) an \( n \times 1 \) vector and \( b \) an \( m \times 1 \) vector. The function \( f(x) \) is convex since by the triangle inequality

\[
\begin{align*}
    f((1 - t)x_1 + tx_2) &= \|A((1 - t)x_1 + tx_2) - b\| \\
    &= \|(1 - t)(Ax_1 - b) + t(Ax_2 - b)\| \\
    &\leq (1 - t)\|Ax_1 - b\| + t\|Ax_2 - b\| \\
    &= (1 - t)f(x_1) + tf(x_2),
\end{align*}
\]

where \( 0 < t < 1 \).
A brief summary on convex optimization

Norm of a linear (or affine) form:

**Comments:** Note that if the Euclidean norm $\| \cdot \|_2$ is used, if $m > n$ and if the rank of the matrix $A$ is $\text{rank}\{A\} = n$, then the minimum of $f$ (the least squares solution) is unique (even though the function $f$ is not strictly convex).

It is evident that the well-posedness of the minimization of $f(x) = \|Ax - b\|$ depends on the properties of the matrix (or operator) $A$. If the matrix $A$ is not full rank and has a non-trivial null space, then the optimization problem can be regularized by adding a strictly convex penalty term as e.g.,

$$f(x) = \|Ax - b\|_2 + \alpha \|x\|_2^2$$

where $\alpha > 0$ (cf., Tikhonov regularization). Now the new function $f$ is strictly convex, and if $\alpha$ is sufficiently small the optimal solution will coincide with the pseudo-inverse.
Disciplined convex programming in Matlab:

As an example, consider the following convex optimization problem related to the normed error

$$\text{minimize} \quad \|Ax - b\|_p$$

subject to

$$Bx \leq c$$

$$Cx = d,$$ (3)

where the $n \times 1$ vector $x$ is the unknown optimization variable and $A$, $B$ and $C$ are given matrices and $b$, $c$ and $d$ are given vectors of suitable dimensions. Here, $\| \cdot \|_p$ denotes that a $p$-norm is used where $p \geq 1$. The max-norm is denoted $\| \cdot \|_\infty$. 
Disciplined convex programming in Matlab:

Once formulated on a standard form the problem (3) can be solved efficiently by using the CVX Matlab software for disciplined convex programming which is available via the link http://cvxr.com/cvx/. Here, the following lines in Matlab may be used

```matlab
cvx_begin
variable x(n)
minimize(norm(A * x - b, p))
subject to
B * x <= c
C * x == d

cvx_end
```

where $n$, $p$ ($p \geq 1$ or $p = \text{Inf}$), $A$, $B$, $C$, $b$, $c$ and $d$ have been defined in the preamble.
Herglotz functions

Analytic functions with the property

\[ \text{Im } h(\omega) \geq 0 \quad \text{for} \quad \omega \in \mathbb{C}_+ = \{ \omega \in \mathbb{C} | \text{Im } \omega > 0 \} \]

Integral representation

\[ h(\omega) = b_1 \omega + \alpha + \int_{-\infty}^{\infty} \left( \frac{1}{\xi - \omega} - \frac{\xi}{1 + \xi^2} \right) d\beta(\xi) \]

Here, \( b_1 \geq 0, \alpha \in \mathbb{R} \)

\( d\beta(\xi) \) is a positive Borel measure with \( \int_{\mathbb{R}} d\beta(\xi)/(1 + \xi^2) < \infty \)

Notation: \( d\beta(\xi) = \beta'(\xi) \, d\xi = \frac{1}{\pi} \text{Im } h(\xi + i0) \, d\xi \equiv \frac{1}{\pi} \text{Im } h(\xi) \, d\xi \)
Symmetric Herglotz functions

Conjugate symmetry and symmetric measure

\[ h(\omega) = -h^*(-\omega^*) \iff \begin{align*}
\text{Im } h(-\xi) &= \text{Im } h(\xi) \\
\text{Re } h(-\xi) &= -\text{Re } h(\xi)
\end{align*} \]

Integral representation for \( \omega \in \mathbb{C}_+ \)

\[ h(\omega) = b_1 \omega + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - \omega} \text{Im } h(\xi) \, d\xi \]

Distributional limit on the real line \( \omega \in \mathbb{R} \)

\[ \frac{1}{\xi - \omega} \rightarrow \frac{1}{\xi - \omega} + i\pi \delta(\xi - \omega) \quad \text{as} \quad \text{Im } \omega \rightarrow 0+ \]

\[ \text{Re } h(\omega) = b_1 \omega + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - \omega} \text{Im } h(\xi) \, d\xi \]

where the integral is a Cauchy principal value integral
Suppose that the following asymptotic properties are valid

\[
h(\omega) = \begin{cases} 
  a_{-1} \omega^{-1} + a_1 \omega + \ldots + a_{2N_0 - 1} \omega^{2N_0 - 1} + o(\omega^{2N_0 - 1}) & \text{as } \omega \to 0 \\
  b_1 \omega + b_{-1} \omega^{-1} + \ldots + b_{1 - 2N_\infty} \omega^{1 - 2N_\infty} + o(\omega^{1 - 2N_\infty}) & \text{as } \omega \to \infty 
\end{cases}
\]

then the following integral identities (sum rules) hold

\[
\frac{2}{\pi} \int_0^\infty \frac{\text{Im} \, h(\xi)}{\xi^{2n}} \, d\xi \overset{\text{def}}{=} \lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{1/\varepsilon} \frac{\text{Im} \, h(\xi + iy)}{\xi^{2n}} \, d\xi = a_{2n - 1} - b_{2n - 1}
\]

for \( n = 1 - N_\infty, \ldots, N_0 \)
A general approximation problem

minimize \[ \| h(\xi) - f(\xi) \|_\Omega \]
subject to \[ f(\xi) \text{ continuous on } \Omega \]
\[ h(\xi) \overset{\text{def}}{=} h(\xi + i0) \text{ continuous on } \Omega \]

Approximation domain \( \Omega \) is a closed and bounded subset of \( \mathbb{R} \)
\( h(\omega) \) is a symmetric Herglotz function regular in a neighbourhood of \( \mathbb{C}^+ \cup \Omega \)

A non-trivial example

\(-f(\xi)\) can be continued to a Herglotz function \( h_0(\omega) \) which is analytic in a neighbourhood of \( \mathbb{C}^+ \cup \Omega \)

Large argument asymptotics: \( h_0(\omega) = b_1^0 \omega + o(\omega) \) as \( \omega \to \infty \)

Then \( \sup_{\xi \in \Omega} |h(\xi) - f(\xi)| \geq b_1^0 \frac{1}{2} |\Omega| \)

Derivation of sum rule and non-trivial bound

Following [1] above, employ the auxiliary Herglotz function

\[ h_\Delta(z) = \frac{1}{\pi} \int_{-\Delta}^{\Delta} \frac{1}{\xi - z} \, d\xi = \frac{1}{\pi} \ln \frac{z - \Delta}{z + \Delta} = \begin{cases} i + o(1) & \text{as } z \to 0 \\ \frac{-2\Delta}{\pi z} + o(z^{-1}) & \text{as } z \to \infty \end{cases} \]

where \( \text{Im} \, h_\Delta(z) \geq \frac{1}{2} \) for \( |z| \leq \Delta \) and \( \text{Im} \, z \geq 0 \)

**Composite Herglotz function**

\[ \tilde{h}(\omega) = h_\Delta(h(\omega) + h_0(\omega)) \]

where \( h(\omega) + h_0(\omega) = (b_1 + b_1^0)\omega + o(\omega) \) as \( \omega \to \infty \)

\[ \tilde{h}(\omega) = \begin{cases} o(\omega^{-1}) & \text{as } \omega \to 0 \\ \frac{-2\Delta}{\pi (b_1 + b_1^0)} \omega^{-1} + o(\omega^{-1}) & \text{as } \omega \to \infty \end{cases} \]

**Sum rule for** \( n = 0 \)

\[ \frac{2}{\pi} \int_{0}^{\infty} \text{Im} \, \tilde{h}(\xi) \, d\xi = a_{-1} - b_{-1} = \frac{2\Delta}{\pi (b_1 + b_1^0)} \]
Derivation of sum rule and non-trivial bound

Sum rule for $n = 0$

$$\frac{2}{\pi} \int_0^\infty \text{Im} \tilde{h}(\xi) \, d\xi = a_{-1} - b_{-1} = \frac{2\Delta}{\pi(b_1 + b_0^0)}$$

Let $\Delta = \sup_{\xi \in \Omega} |h(\xi) + h_0(\xi)|$, the following integral inequalities hold

$$\frac{1}{\pi} |\Omega| \leq \frac{2}{\pi} \int_\Omega \text{Im} \tilde{h}(\xi) \, d\xi \leq \frac{2}{\pi} \int_0^\infty \text{Im} \tilde{h}(\xi) \, d\xi = \frac{2}{\pi(b_1 + b_0^0)} \sup_{\xi \in \Omega} |h(\xi) + h_0(\xi)|$$

or

$$\sup_{\xi \in \Omega} |h(\xi) + h_0(\xi)| \geq (b_1 + b_0^0) \frac{1}{2} |\Omega| \quad \text{where} \quad |\Omega| = \int_\Omega d\xi$$

**Permittivity functions:** If $h(\omega) = \omega \epsilon(\omega)$ and $h_0(\omega) = -f(\omega) = -\omega \epsilon_t$ with $\epsilon_t < 0$, then $b_1 = \epsilon_\infty$ and $b_0^1 = -\epsilon_t$ and

$$\sup_{\xi \in \Omega} |\epsilon(\xi) - \epsilon_t| \geq \frac{(\epsilon_\infty - \epsilon_t) \frac{1}{2} B}{1 + \frac{B}{2}}$$

where $\Omega = \omega_0[1 - \frac{B}{2}, 1 + \frac{B}{2}]$ and where $\omega_0$ is the center frequency and $0 < B < 2$, see also [1].
Discretization

Discretization of positive measure on \( \Omega_1 = [\omega_1, \omega_4] \), \( \Delta \omega = \frac{\omega_4 - \omega_1}{N-1} \)

\[
\text{Im } h(\omega) = \sum_{n=0}^{N-1} x_n [p(\omega - (n + n_1)\Delta \omega) + p(\omega + (n + n_1)\Delta \omega)], \quad x_n \geq 0
\]

Triangular basis functions

\[
p(\omega) = \begin{cases} 
1 - \frac{|\omega|}{\Delta \omega} & |\omega| \leq \Delta \omega \\
0 & |\omega| > \Delta \omega
\end{cases}
\]

Hilbert transform on approximation domain \( \Omega = [\omega_2, \omega_3] \subset \Omega_1 \)

\[
\text{Re } h(\omega) = b_1 \omega + \sum_{n=0}^{N-1} x_n [\hat{p}(\omega - (n + n_1)\Delta \omega) + \hat{p}(\omega + (n + n_1)\Delta \omega)]
\]

\[
\hat{p}(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - \omega} p(\xi) \, d\xi
\]

\[
= \frac{1}{(\pi \Delta \omega)} \left( 2\omega \ln |\omega| - (\omega - \Delta \omega) \ln |\omega - \Delta \omega| - (\omega + \Delta \omega) \ln |\omega + \Delta \omega| \right)
\]
Approximation problem

\[
\begin{align*}
\text{minimize} & \quad \| h(\xi) - f(\xi) \|_\Omega \\
\text{subject to} & \quad f(\xi) \text{ continuous on } \Omega \\
& \quad h(\xi) \overset{\text{def}}{=} h(\xi + i0) \exists \text{ continuous on } \Omega
\end{align*}
\]

Disciplined convex programming in MATLAB [2]

\[
A = H + 1i \ast I \\
cvx\_begin \\
\quad \text{variable } x(N) \\
\quad \text{minimize}(\text{norm}(A \ast x - f, \text{Inf})) \\
\quad \text{subject to} \\
\quad x \geq 0 \\
cvx\_end
\]

Approximation of $\tan(\omega)$

$$h(\xi) \overset{\text{def}}{=} h(\xi + i0) = \tan(\xi) + i\pi \sum_{l=-\infty}^{\infty} \delta(\xi - \frac{\pi}{2} + l\pi)$$

(a) Approximating measure $\text{Im } h(\omega)$

(b) Approximation of Herglotz function $\tan(\omega)$

$\Omega$ and $\Omega_1$: $\Omega = [\omega_2, \omega_3]$ and $\Omega_1 = [\omega_1, \omega_4]$. Here, $\omega_0 = \pi$, $B = 0$.4, $\omega_1 = 0$ and $\omega_4 = 10$.

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Approximation of \( \tan\left(-\frac{1}{\omega}\right) \)

\[
h(\xi) \overset{\text{def}}{=} h(\xi + i0) = \tan\left(-\frac{1}{\xi}\right) + i\pi \sum_{l=-\infty}^{\infty} \frac{1}{(l\pi - \frac{\pi}{2})^2} \delta\left(\xi - \frac{1}{l\pi - \frac{\pi}{2}}\right)
\]

**Figure 2:** Approximation of the Herglotz function \( \tan\left(-\frac{1}{\omega}\right) \).

a) The approximating measure \( \text{Im} h(\omega) \) based on \( N = 500 \) variables.

b) The real part \( \text{Re} h(\omega) \) on \( \omega \in [0.2, 0.3] \) and \( \omega \in [0.1, 0.4] \). Here, \( \omega_0 = 0.4 \), \( B = 0.4 \), \( \omega_1 = 0 \) and \( \omega_4 = 1.2 \).
Approximation of metamaterial

Herglotz function \( h(\omega) = \omega \epsilon(\omega) \) with \( \epsilon_\infty = 1 \) (and \( \epsilon_s \overset{\text{def}}{=} \epsilon(0) \))

Metamaterial \( f(\omega) = \omega \epsilon_t \) for \( \omega \in \Omega \) and where \( \epsilon_t = -1 \)

Convex optimization problem

\[
\begin{align*}
\text{minimize} & \quad \sup_{\xi \in \Omega} \frac{1}{\xi} |h(\xi) - f(\xi)| \\
\text{subject to} & \quad \frac{2}{\pi} \int_0^{\infty} \frac{\text{Im} h(\xi)}{\xi^2} \, d\xi \leq \epsilon_s^{\text{max}} - \epsilon_\infty
\end{align*}
\]

Discretization of sum rule

\[
\frac{2}{\pi} \int_0^{\infty} \frac{\text{Im} h(\xi)}{\xi^2} \, d\xi = \frac{2}{\pi \Delta \omega} \sum_{n=0}^{N-1} x_n \ln \left( \frac{(n + n_1)^2}{(n + n_1 - 1)(n + n_1 + 1)} \right)
\]

assuming that \( n_1 > 1 \) where \( \omega_1 = n_1 \Delta \omega \) (\( \text{Im} h(\xi) = o(\xi) \))
Approximation of metamaterial

**Disciplined convex programming in MATLAB [2]**

\[ \text{Omega;} \quad \% \text{ Approximation domain} \]
\[ A = H + 1i \ast I; \quad \% \text{ Representation of Herglotz function} \]
\[ B; \quad \% \text{ Representation of sum rule} \]
\[ b = \text{epss} - \text{epsinf}; \quad \% \text{ Static and optical responses} \]

```matlab
cvx_begin
    variable x(N)
    minimize(norm((epsinf \ast Omega + A \ast x - f)./Omega, Inf))
    subject to
    x >= 0
    B \ast x <= b

    cvx_end
```

Approximation of metamaterial

Herglotz function \( h(\omega) = \omega \epsilon(\omega) \) with \( \epsilon_\infty = 1 \)

Metamaterial \( f(\omega) = \omega \epsilon_t \) for \( \omega \in \Omega \) and where \( \epsilon_t = -1 \)

![Graph of approximating measure \( \text{Im} h(\omega) \)]

![Graph of approximation of metamaterial \( \omega \epsilon_t \) on \( \Omega \) and where \( \epsilon_t = -1 \)]
Approximation of metamaterial (permittivity)

Note that $\omega \epsilon(\omega) = -\frac{1}{\pi} \frac{1}{\omega} + i \delta(\omega) = \omega \left( -\frac{1}{\pi} \frac{1}{\omega^2} - i \delta'(\omega) \right)$
Convergence of approximation (metamaterial)

Figure 4: Approximation of metamaterial with target permittivity $t$.

Figure 5: Approximation error $\max_{\Omega} |\epsilon(\omega) - \epsilon_t|$ as a function of the number of variables $N$.

Here, the relative bandwidth is $B = 0.4$.
Approximation of metamaterial with static permittivity

Herglotz function \( h(\omega) = \omega \epsilon(\omega) \) with \( \epsilon_\infty = 1 \) and \( \epsilon_{s}^{\text{max}} = 5 \)

Metamaterial \( f(\omega) = \omega \epsilon_t \) for \( \omega \in \Omega \) and where \( \epsilon_t = -1 \)

**Figure 6:** Approximation of metamaterial with target permittivity \( \epsilon_t \) = 1, optical response \( \epsilon_1 = 1 \) and static permittivity \( \epsilon_s = 5 \).

**a)** Approximating measure \( \text{Im} \ h(\omega) \) based on \( N = 1000 \) variables.

**b)** The real part \( \text{Re} \ h(\omega) \) on \( \Omega = [\omega_2, \omega_3] \) and \( \Omega_1 = [\omega_1, \omega_4] \). Here, \( \omega_0 = 1 \), \( B = 0.4 \), \( \omega_1 = 0.01 \) and \( \omega_4 = 3 \).
Approximation of metamaterial (permittivity)

Note that $\omega\epsilon(\omega) = \omega \left( -\frac{2}{\pi} \frac{1}{(\omega-\omega_0)(\omega+\omega_0)} + \frac{i}{\omega_0} \delta(\omega - \omega_0) - \frac{i}{\omega_0} \delta(\omega + \omega_0) \right)$
Summary and future work

- A general approximation problem has been formulated based on the class of symmetric Herglotz functions.
- The approximation problem can be discretized and solved by using a disciplined convex programming in MATLAB which facilitates a straightforward addition of auxiliary convex constraints.
- Future work aims at
  - Rigorous mathematical formulation and proof of convergence
  - Further applications of dispersion analysis and optimization of passive materials and structures
  - Optimization of matrix valued Herglotz functions for complex materials

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