Passive approximation and optimization

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Abstract

An approximation problem is formulated where it is required to approximate a general "well-behaved" function on an interval of the real axis based on the set of Herglotz functions.

1 Introduction

An approximation problem is formulated where it is required to approximate a general "well-behaved" function on an interval of the real axis based on the set of Herglotz functions. Applications are *e.g.*, with the optimal realizations of passive structures presented in [12].

2 Basic properties of Herglotz functions

2.1 Integral representation

A Herglotz function $h(\omega)$ is an analytic function with the property $\operatorname{Im} h(\omega) \geq 0$ for $\omega \in \mathbb{C}_+ = \{\omega \in \mathbb{C} | \operatorname{Im} \omega > 0\}$. It can be shown that $h(\omega)$ is a Herglotz function if and only if it can be represented as

$$h(\omega) = b_1\omega + \alpha + \int_{-\infty}^{\infty} \frac{1+\xi\omega}{\xi-\omega} \, \mathrm{d}\sigma(\xi), \tag{1}$$

where $b_1 \geq 0$, $\alpha \in \mathbb{R}$ is a real valued constant and $\sigma(\xi)$ a nondecreasing function of bounded variation, or equivalently, that $\mathrm{d}\sigma(\xi)$ is a finite positive Borel measure, see *e.g.*, [1, 9, 13]. It is also useful to introduce the positive Borel measure $\mathrm{d}\beta(\xi) = (1+\xi^2)\,\mathrm{d}\sigma(\xi)$ so that

$$h(\omega) = b_1\omega + \alpha + \int_{-\infty}^{\infty} \left(\frac{1}{\xi-\omega} - \frac{\xi}{1+\xi^2} \right) \mathrm{d}\beta(\xi), \tag{2}$$

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where $\int_{\mathbb{R}} \mathrm{d}\beta(\xi)/(1+\xi^2) < \infty$. The spectral function $\beta(\xi)$, or equivalently, the positive measure $\mathrm{d}\beta(\xi)$ is uniquely defined by the Herglotz function $h(\omega)$ from the following properties

$$\begin{cases} \beta(\xi_2) - \beta(\xi_1) = \lim\limits_{y\to 0+} \dfrac{1}{\pi} \displaystyle\int_{\xi_1}^{\xi_2} \operatorname{Im} h(\xi + \mathrm{i}y)\,\mathrm{d}\xi, \\ \beta(\xi) = \dfrac{\beta(\xi+0) + \beta(\xi-0)}{2}, \\ \beta(0) = 0, \end{cases} \tag{3}$$

and which motivates the simplified notation $\mathrm{d}\beta(\xi) = \frac{1}{\pi} \operatorname{Im} h(\xi + \mathrm{i}0)\,\mathrm{d}\xi$, see *e.g.*, [9, 13]. The following notation will also be used here where $\mathrm{d}\sigma(\xi) = \sigma'(\xi)\,\mathrm{d}\xi$ and $\mathrm{d}\beta(\xi) = \beta'(\xi)\,\mathrm{d}\xi$ and where the measures consist of measurable functions multiplied by the Lebesgue measure plus point measures and hence that $\sigma'(\xi)$ and $\beta'(\xi)$ can be understood in the sense of distributional derivatives of the spectral functions $\sigma(\xi)$ and $\beta(\xi)$, respectively. It follows immediately that $\beta'(\xi) = (1+\xi^2)\sigma'(\xi)$.

Let $h(\xi)$ denote the distributional (or weak) limit of the Herglotz function $h(\omega)$ as $\operatorname{Im}\omega \to 0+$ and $\xi \in \mathbb{R}$. It can then be shown that

$$\operatorname{Im} h(\xi) = \pi\beta'(\xi), \tag{4}$$

see *e.g.*, [2]. Formally, the relationship (4) is readily obtained from the representation (1) or (2) by applying the distributional limit

$$\lim_{\operatorname{Im}\omega\to 0+} \frac{1}{\xi-\omega} = \frac{1}{\xi-\omega'} + \mathrm{i}\pi\delta(\xi-\omega'), \tag{5}$$

where (the distribution) $\frac{1}{\xi-\omega'}$ denotes the Cauchy principal value integral (*cf.*, the Hilbert transform) and $\omega' = \operatorname{Re}\omega$, see *e.g.*, [10, 13].

In the sequel, the notation $\operatorname{Im} h(\xi)$ with $\xi \in \mathbb{R}$ will be used synonymously to denote the measure $\pi\,\mathrm{d}\beta(\xi)$ as well as the distributional limit $\operatorname{Im} h(\xi + \mathrm{i}0)$.

2.1.1 Symmetric Herglotz functions

Symmetric Herglotz functions satisfy the symmetry requirement

$$h(\omega) = -h^*(-\omega^*), \tag{6}$$

where $\omega \in \mathbb{C}_+$ and $\mathrm{d}\beta(\xi)$ is an even measure with $\mathrm{d}\beta(-\xi) = \mathrm{d}\beta(\xi)$. The even symmetry of the measure can also be expressed as

$$\operatorname{Im} h(\xi) = \operatorname{Im} h(-\xi), \tag{7}$$

where $\xi \in \mathbb{R}$. For symmetric Herglotz functions, $\alpha = 0$ in (1) and (2) and the representation formula (2) can furthermore be simplified as

$$h(\omega) = b_1\omega + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\xi-\omega} \operatorname{Im} h(\xi)\,\mathrm{d}\xi, \tag{8}$$

where $\omega \in \mathbb{C}_+$. Formally, the distributional limit (5) immediately yields the following representation of the real part of $h(\omega)$ in terms of its imaginary part

$$\operatorname{Re} h(\omega) = b_1\omega + \frac{1}{\pi}\fint_{-\infty}^{\infty} \frac{1}{\xi-\omega} \operatorname{Im} h(\xi)\,\mathrm{d}\xi, \tag{9}$$

where $\omega \in \mathbb{R}$, $b_1 \geq 0$ and where the notation $⨍$ indicates a distributional convolution which can be interpreted in the sense of the Cauchy principal value integral for all ω where $\operatorname{Re} h(\omega)$ is point-wise defined (*e.g.*, on intervals in $\mathbb{R}$ where $h(\omega)$ is regular).

It should be noted that the Hilbert transform relationship (9), which is derived from the properties of the Herglotz function and its representation formulas displayed in (1), (2) and (8), is only a one-way Hilbert transform relationship taking $\operatorname{Im} h(\xi)$ to $\operatorname{Re} h(\xi)$. This is in contrast to the general L^p cases ($1 < p < \infty$) where the Hilbert transform operator H satisfies $H^2 = -I$ where I is the identity operator on L^p [10]. Further, the Hilbert transform Hf acting on a function $f \in L^1$ is not necessarily in L^1, and the operator H is unbounded on L^∞ [10]. It is therefore interesting to note that the Hilbert transform relationship (9), which is derived directly from the properties of passivity, is not directly related to any L^p spaces which is the common practice in most literature on causalily, *cf.*, the Plemelj formulas or the Kramers-Kronig relations [10, 13].

Note that the typical feature of a Herglotz function representing a passive system, is that the measure is positive with $\operatorname{Im} h(\xi) \geq 0$. It is finally noted that the real part of $h(\xi)$ is odd symmetric with $\operatorname{Re} h(-\xi) = -\operatorname{Re} h(\xi)$.

2.2 Asymptotic properties and sum rules

Let $o(\cdot)$ denote the order symbol as defined in [14], and let $\omega \hat{\to} 0$ mean that $|\omega| \to 0$ in the cone $\varphi \leq \arg \omega \leq \pi - \varphi$ for any $\varphi \in (0, \pi/2]$, and similarly for $\omega \hat{\to} \infty$. Symmetric Herglotz functions have the general asymptotic behavior $h(\omega) = a_{-1}\omega^{-1} + o(\omega^{-1})$ as $\omega \hat{\to} 0$ and where $a_{-1} \leq 0$, and $h(\omega) = b_1\omega + o(\omega)$ as $\omega \hat{\to} \infty$ and where $b_1 \geq 0$.

Suppose now that the small and large argument asymptotic behavior of a symmetric Herglotz function $h(\omega)$ is given by a partial expansion based on odd order coefficents as follows

$$h(\omega) = \begin{cases} a_{-1}\omega^{-1} + a_1\omega + \ldots + a_{2N_0-1}\omega^{2N_0-1} + o(\omega^{2N_0-1}) & \text{as } \omega \hat{\to} 0, \\ b_1\omega + b_{-1}\omega^{-1} + \ldots + b_{1-2N_\infty}\omega^{1-2N_\infty} + o(\omega^{1-2N_\infty}) & \text{as } \omega \hat{\to} \infty, \end{cases} \tag{10}$$

where N_0 and N_∞ are non-negative integers (or possibly infinity) and where $1 - N_\infty \leq N_0$. It is noted that all the odd order coefficients $\{a_{2n-1}\}_{n=0}^{N_0}$ and $\{b_{1-2n}\}_{n=0}^{N_\infty}$ above are real valued due to the symmetry assumption. In this case, it is possible to show that the following integral identities (sum rules) hold

$$\frac{2}{\pi}\int_{0+}^{\infty} \frac{\operatorname{Im} h(\xi)}{\xi^{2n}}\,\mathrm{d}\xi \overset{\text{def}}{=} \lim_{\varepsilon \to 0+} \lim_{y \to 0+} \frac{2}{\pi}\int_{\varepsilon}^{1/\varepsilon} \frac{\operatorname{Im} h(\xi + \mathrm{i}y)}{\xi^{2n}}\,\mathrm{d}\xi = a_{2n-1} - b_{2n-1}, \tag{11}$$

for $n = 1 - N_\infty, \ldots, N_0$, see *e.g.*, [1, 3]. It is noted that point masses should be included in the sum rule (11) above, except for the point mass at 0. In particular, consider the case with $\mathrm{d}\beta(\xi) = \delta(\xi)\,\mathrm{d}\xi$ and $h(\omega) = -\omega^{-1}$ and where $a_{-1} = b_{-1} = -1$. The $n = 0$ sum rule in this case is $2\int_{0+}^{\infty} \delta(\xi)\,\mathrm{d}\xi = a_{-1} - b_{-1} = 0$. It is finally observed that a combination such as $N_0 = N_\infty = 0$ is not valid in (10) and (11).

2.3 The class Carathéodory

The class Carathéodory [1] consists of all analytic functions defined on the unit disc and which have positive real part. The representation of this class is given by the formula

$$h_{\mathrm{d}}(z) = -\mathrm{i}\alpha + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z}\,\mathrm{d}\sigma_{\mathrm{d}}(\theta), \tag{12}$$

where $\alpha = \mathrm{Re}\{\mathrm{i}h_{\mathrm{d}}(0)\}$, $\sigma_{\mathrm{d}}(\theta)$ is a finite positive Borel measure ($\int_{-\pi}^{\pi}\mathrm{d}\sigma_{\mathrm{d}}(\theta) < \infty$) and $|z| < 1$. Here, the following notation is used $\mathrm{d}\sigma_{\mathrm{d}}(\theta) = \sigma_{\mathrm{d}}'(\theta)\,\mathrm{d}\theta$ where the measure consists of a measurable function multiplied by the Lebesgue measure plus a point measure and hence that $\sigma_{\mathrm{d}}'(\theta)$ can be understood in the sense of a distributional derivative of the spectral function $\sigma_{\mathrm{d}}(\theta)$. Symmetric Carathéodory functions have the properties $h_{\mathrm{d}}(z) = h_{\mathrm{d}}^*(z^*)$ and $\mathrm{d}\sigma_{\mathrm{d}}(\theta) = \mathrm{d}\sigma_{\mathrm{d}}(-\theta)$ and hence that $\alpha = 0$.

Let $h_{\mathrm{d}}(\mathrm{e}^{\mathrm{i}\theta})$ denote the distributional (or weak) limit of the Carathéodory function $h_{\mathrm{d}}(r\mathrm{e}^{\mathrm{i}\theta})$ as $r \to 1-$ and $\theta \in [-\pi, \pi]$. It can then be shown that

$$\mathrm{Re}\, h_{\mathrm{d}}(\mathrm{e}^{\mathrm{i}\theta}) = \sigma_{\mathrm{d}}'(\theta), \tag{13}$$

see *e.g.*, [2].

The relation between the representations (1) and (12) is readily obtained by identifying $h(\omega) = \mathrm{i}h_{\mathrm{d}}(z)$ via the Cayley transform

$$\begin{cases} z = \dfrac{\omega - \mathrm{i}}{\omega + \mathrm{i}} \\ \omega = \mathrm{i}\dfrac{1+z}{1-z} \end{cases} \qquad \begin{cases} \mathrm{e}^{\mathrm{i}\theta} = \dfrac{\xi - \mathrm{i}}{\xi + \mathrm{i}} \\ \xi = \mathrm{i}\dfrac{1+\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}} = -\cot\dfrac{\theta}{2}, \end{cases} \tag{14}$$

which leads to the following relationships between the measures

$$\mathrm{d}\sigma(\xi) = \frac{1}{2\pi}\,\mathrm{d}\sigma_{\mathrm{d}}(-2\cot^{-1}\xi), \tag{15}$$

or

$$\beta'(\xi) = \frac{1}{\pi}\sigma_{\mathrm{d}}'(-2\cot^{-1}\xi), \tag{16}$$

and where $\beta'(\xi) = (1+\xi^2)\sigma'(\xi)$, $\theta = -2\cot^{-1}\xi$ and $\mathrm{d}\theta = \frac{2}{1+\xi^2}\,\mathrm{d}\xi$ have been used. Obviously, it is also seen that $\mathrm{Im}\, h(\xi) = \mathrm{Re}\, h_{\mathrm{d}}(\mathrm{e}^{\mathrm{i}\theta})$ where $\theta = -2\cot^{-1}\xi$.

A formal derivation of (13) can be obtained from the following distributional limit

$$\lim_{r\to 1-}\frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} = -\mathrm{i}\cot(\frac{\theta - \theta'}{2}) + 2\pi\delta(\theta - \theta'), \tag{17}$$

where $z = r\mathrm{e}^{\mathrm{i}\theta'}$ and $\cot(\frac{\theta-\theta'}{2})$ denotes the corresponding Cauchy principal value integral. In the sequel and unless otherwise is stated, the notation $\delta(\theta - \theta')$ is to be understood in the sense of a 2π-periodic delta-distribution if θ is an angle representing the unit circle. The result (17) can be derived directly from (5) by using (14) together with $\mathrm{e}^{\mathrm{i}\theta'} = (\omega' - \mathrm{i})/(\omega' + \mathrm{i})$, $\omega' = -\cot\frac{\theta'}{2}$, as well as the distributional identity

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x - x_0), \tag{18}$$

which is valid for sufficiently regular functions $f(x)$ with $f'(x_0) \neq 0$.

By employing (14), (16) and (18), the following canonical relationships regarding point masses can be obtained

$$\begin{cases} \sigma_{\mathrm{d}}'(\theta) & \Leftrightarrow \quad \sigma'(\xi) & \Leftrightarrow \quad \mathrm{i}h_{\mathrm{d}}(z) = h(\omega), \\ \sigma_{\mathrm{d}}'(\theta) = 2\pi\delta(\theta - \theta_0) & \Leftrightarrow \quad \sigma'(\xi) = \delta(\xi - \xi_0) & \Leftrightarrow \quad \mathrm{i}\dfrac{\mathrm{e}^{\mathrm{i}\theta_0} + z}{\mathrm{e}^{\mathrm{i}\theta_0} - z} = \dfrac{1 + \xi_0\omega}{\xi_0 - \omega}, \\ \sigma_{\mathrm{d}}'(\theta) = 2\pi\delta(\theta) & \Leftrightarrow \quad \sigma'(\xi) = \delta(\xi - \xi_0)|_{\xi_0 = \pm\infty} & \Leftrightarrow \quad \mathrm{i}\dfrac{1+z}{1-z} = \omega, \\ \sigma_{\mathrm{d}}'(\theta) = 2\pi\delta(\theta - \pi) & \Leftrightarrow \quad \sigma'(\xi) = \delta(\xi) & \Leftrightarrow \quad \mathrm{i}\dfrac{1-z}{1+z} = -\dfrac{1}{\omega}, \end{cases} \tag{19}$$

where $\xi = -\cot(\theta/2)$ and $\xi_0 = -\cot(\theta_0/2)$.

3 Examples of Herglotz functions

3.1 Canonical examples

Compositions of two Herglotz functions $h_1(\omega)$ and $h_2(\omega)$ can be used to generate a new Herglotz function $h(\omega) = h_2(h_1(\omega))$. Another useful composition is based on the Carathéodory functions $h_{\mathrm{d}}(z)$ with the property that $\operatorname{Re} h_{\mathrm{d}}(z) \geq 0$ when $|z| < 1$. Let $h_1(\omega)$ denote a Herglotz function satisfying the strict inequality $\operatorname{Im} h_1(\omega) > 0$ when $\operatorname{Im} \omega > 0$. Then $|\mathrm{e}^{\mathrm{i}h_1(\omega)}| < 1$ for $\operatorname{Im} \omega > 0$, and the composition $h(\omega) = \mathrm{i}h_{\mathrm{d}}(\mathrm{e}^{\mathrm{i}h_1(\omega)})$ generates a new Herglotz function.

3.1.1 The Herglotz function $\tan(\omega)$

Let the Carathéodory function $h_{\mathrm{d}}(z)$ be defined by

$$h_{\mathrm{d}}(z) = \frac{1-z}{1+z}, \tag{20}$$

where the spectral measure is $\operatorname{Re} h_{\mathrm{d}}(\mathrm{e}^{\mathrm{i}\theta}) = \sigma'_{\mathrm{d}}(\theta) = 2\pi\delta(\theta - \pi)$. Then it is found that the Herglotz function

$$h(\omega) = \mathrm{i}h_{\mathrm{d}}(\mathrm{e}^{\mathrm{i}2\omega}) = \mathrm{i}\frac{1-\mathrm{e}^{\mathrm{i}2\omega}}{1+\mathrm{e}^{\mathrm{i}2\omega}} = \tan(\omega), \tag{21}$$

has the spectral measure

$$\operatorname{Im} h(\xi) = \operatorname{Re} h_{\mathrm{d}}(\mathrm{e}^{\mathrm{i}2\xi}) = 2\pi \sum_{l=-\infty}^{\infty} \delta(2\xi - \pi + l2\pi), \tag{22}$$

or

$$\operatorname{Im} h(\xi) = \pi \sum_{l=-\infty}^{\infty} \delta(\xi - \frac{\pi}{2} + l\pi). \tag{23}$$

It is noted that the amplitudes of the delta-distributions in (23) are the negative residues at the poles of the meromorphic function $\tan(\omega)$.

The asymptotic expansions of the Herglotz function in (21) are given by

$$\tan(\omega) = \begin{cases} \omega + o(\omega) & \text{as } \omega \hat{\to} 0, \\ \mathrm{i} + o(1) = o(\omega) & \text{as } \omega \hat{\to} \infty, \end{cases} \tag{24}$$

and the sum rule (11) for $n = 1$ yields

$$\frac{2}{\pi}\int_{0+}^{\infty} \frac{\operatorname{Im} h(\xi)}{\xi^2}\,\mathrm{d}\xi = \frac{1}{\pi^2}\sum_{l=-\infty}^{\infty} \frac{1}{(\frac{1}{2} - l)^2} = 1. \tag{25}$$

3.1.2 The Herglotz function $\tan(-1/\omega)$

In the same way as above with $h_{\mathrm{d}}(z)$ defined as in (20), the Herglotz function

$$h(\omega) = \mathrm{i}h_{\mathrm{d}}(\mathrm{e}^{\frac{\mathrm{i}2}{-\omega}}) = \mathrm{i}\frac{1-\mathrm{e}^{\frac{\mathrm{i}2}{-\omega}}}{1+\mathrm{e}^{\frac{\mathrm{i}2}{-\omega}}} = \tan(-1/\omega), \tag{26}$$

is found to have the spectral measure

$$\operatorname{Im} h(\xi) = \operatorname{Re} h_{\mathrm{d}}(\mathrm{e}^{\frac{\mathrm{i}2}{-\xi}}) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\frac{2}{-\xi} - \pi + l2\pi), \tag{27}$$

or

$$\operatorname{Im} h(\xi) = \pi \sum_{l=-\infty}^{\infty} \frac{1}{(l\pi - \frac{\pi}{2})^2} \delta(\xi - \frac{1}{l\pi - \frac{\pi}{2}}), \tag{28}$$

where (18) has been used. It is noted that the amplitudes of the delta-distributions in (28) are the negative residues at the poles of the meromorphic function $\tan(-1/\omega)$ where the domain of meromorphicity is $\mathbb{C} \setminus \{0\}$. It also is noted that the point $\omega = 0$ is not a singular point of $\tan(-1/\omega)$, but it is a limit point of the sequence of poles.

The asymptotic expansions of the Herglotz function in (26) are given by

$$\tan(-1/\omega) = \begin{cases} \mathrm{i} + o(1) = o(\omega^{-1}) & \text{as } \omega \hat{\to} 0, \\ -\dfrac{1}{\omega} + o(\omega^{-1}) & \text{as } \omega \hat{\to} \infty, \end{cases} \tag{29}$$

and the sum rule (11) for $n = 0$ yields

$$\frac{2}{\pi} \int_{0+}^{\infty} \operatorname{Im} h(\xi) \, \mathrm{d}\xi = \frac{1}{\pi^2} \sum_{l=-\infty}^{\infty} \frac{1}{(l - \frac{1}{2})^2} = 1, \tag{30}$$

which gives the same series as in (25).

3.2 Passive material models

In electromagnetics, the response of a linear, time translation invariant and isotropic electric material to an applied electric field intensity $\boldsymbol{E}(t)$ is given in terms of a time domain constitutive relation $\boldsymbol{D}(t) = \epsilon_0 \epsilon_{\mathrm{r}}(t) * \boldsymbol{E}(t)$ where $\boldsymbol{D}(t)$ is the electric flux density, $\epsilon_{\mathrm{r}}(t)$ the real valued relative permittivity convolution kernel and ϵ_0 the permittivity of vacuum, see *e.g.*, [8]. Based on Poyntings theorem [8] and according to the definitions made in [15], the material (or more precisely, the convolution operator $\epsilon_{\mathrm{r}}(t)$) is said to be passive if

$$\int_{-\infty}^{T} \boldsymbol{E}(t) \cdot \frac{\partial \boldsymbol{D}}{\partial t} \, \mathrm{d}t \geq 0, \tag{31}$$

for all T, and for all electric fields that are testing functions with compact support. If the passive convolution operator is of slow growth (a Schwartz distribution), then this energy expression is valid also for the testing functions of rapid descent [15]. It can be shown that this passivity condition is equivalent to the condition that

$$h(\omega) = \omega \epsilon(\omega), \tag{32}$$

is a Herglotz function where $\epsilon(\omega)$ is the analytic Fourier transform of the convolution kernel $\epsilon_{\mathrm{r}}(t)$

$$\epsilon(\omega) = \int_{0}^{\infty} \epsilon_{\mathrm{r}}(t) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}t, \tag{33}$$

and where $\omega \in \mathbb{C}_+$, see *e.g.*, [7, 15].

3.2.1 Derivation of the Herglotz condition for a passive system

To derive the passivity condition (32) we proceed as follows, see also [15]. Consider a linear and time translation invariant (LTI) system based on a time domain derivative and a real valued convolution kernel $\epsilon_{\mathrm{r}}(t)$ of slow growth, and where $x(t)$ and $y(t)$ are any valid and possibly complex valued input and output signals, respectively. In general, we write the input-output relationship as

$$x(t) \mapsto y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\{\epsilon_{\mathrm{r}}(t) * x(t)\}, \tag{34}$$

where $*$ denotes the time domain convolution. For complex valued signals it is seen that

$$\begin{cases} \operatorname{Re} x(t) \mapsto \operatorname{Re} y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\{\epsilon_{\mathrm{r}}(t) * \operatorname{Re} x(t)\} \\ \operatorname{Im} x(t) \mapsto \operatorname{Im} y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\{\epsilon_{\mathrm{r}}(t) * \operatorname{Im} x(t)\}. \end{cases} \tag{35}$$

A characterization of passivity for all possible input signals (testing functions of rapid descent) is now given by the equivalent statements

$$\operatorname{Re} \int_{-\infty}^{T} x^*(t) y(t)\,\mathrm{d}t \geq 0 \Leftrightarrow \begin{cases} \int_{-\infty}^{T} \operatorname{Re} x(t) \operatorname{Re} y(t)\,\mathrm{d}t \geq 0 \\ \int_{-\infty}^{T} \operatorname{Im} x(t) \operatorname{Im} y(t)\,\mathrm{d}t \geq 0. \end{cases} \tag{36}$$

To derive the Herglotz condition, let $x(t) = \mathrm{e}^{-\mathrm{i}\omega t}$ for $-\infty < t < a$, where $T < a < \infty$ and $\operatorname{Im} \omega > 0$. Then

$$\operatorname{Re}\left\{\int_{-\infty}^{T} x^*(t) y(t)\,\mathrm{d}t\right\} = \operatorname{Re}\left\{\int_{-\infty}^{T} \mathrm{e}^{\mathrm{i}\omega^* t} \frac{\mathrm{d}}{\mathrm{d}t} \int_{0}^{\infty} \epsilon_{\mathrm{r}}(\tau) \mathrm{e}^{-\mathrm{i}\omega(t-\tau)}\,\mathrm{d}\tau\,\mathrm{d}t\right\}$$
$$= \operatorname{Re}\left\{\int_{-\infty}^{T} \mathrm{e}^{-\mathrm{i}(\omega-\omega^*)t}\,\mathrm{d}t(-\mathrm{i}\omega) \int_{0}^{\infty} \epsilon_{\mathrm{r}}(\tau) \mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau\right\} = \frac{\mathrm{e}^{2 \operatorname{Im} \omega T}}{2 \operatorname{Im} \omega} \operatorname{Im}\{\omega\epsilon(\omega)\}.$$

Hence, for a passive material (system) it follows that $\operatorname{Im}\{\omega\epsilon(\omega)\} \geq 0$. It can also be shown that any Herglotz function can represent a realization of a passive LTI system [15].

In the following, the well known and frequently used conductivity, Debye, Lorentz' and Drude material models will be described. As will be seen, the corresponding Herglotz functions have a very high degree of regularity since they can all be continued to analytic functions that are regular in a neighbourhood of the real line $\mathbb{R}$. However, there is no guarantee that this is a property that will hold for a physical system in general. In particular, it can be shown that there exist Herglotz functions that can not be continued outside $\mathbb{C}_+$.

3.2.2 Conductivity model

The conductivity model is used to model electrical conductivity and is given by

$$\epsilon(\omega) = \epsilon_\infty + \mathrm{i}\frac{\sigma}{\omega\epsilon_0}, \tag{37}$$

where $\epsilon_\infty > 0$ is the optical response and $\sigma \geq 0$ the conductivity. The corresponding time domain response is given by

$$\epsilon_{\mathrm{r}}(t) = \epsilon_\infty \delta(t) + \frac{\sigma}{\epsilon_0} H(t), \tag{38}$$

where $H(t)$ is the Heaviside unit step function. It is noted that the Fourier transform of $\epsilon_{\mathrm{r}}(t)$ is a distribution $\epsilon(\xi) = \epsilon_\infty + \mathrm{i}\frac{\sigma}{\epsilon_0}\frac{1}{\xi} + \frac{\sigma}{\epsilon_0}\pi\delta(\xi)$ and where $\mathrm{i}\frac{1}{\omega} \mapsto \mathrm{i}\frac{1}{\xi} + \pi\delta(\xi)$ as $\operatorname{Im}\omega \to 0+$. The corresponding Herglotz function and its asymptotics are given by

$$h(\omega) = \omega\epsilon_\infty + \mathrm{i}\frac{\sigma}{\epsilon_0} = \begin{cases} o(\omega^{-1}) & \text{as } \omega \hat{\to} 0, \\ \omega\epsilon_\infty + o(\omega) & \text{as } \omega \hat{\to} \infty. \end{cases} \tag{39}$$

The Herglotz function $h(\omega)$ can be continued to an analytic function regular in $\mathbb{C}$. The spectral measure is given by

$$\operatorname{Im} h(\xi) = \frac{\sigma}{\epsilon_0}, \tag{40}$$

for all $\xi \in \mathbb{R}$. There are no sum rules (11) that can be applied to a Herglotz function $h(\omega)$ with asymptotics as in (39).

3.2.3 Debye model

The Debye model is used to model the response of dielectric media and is given by

$$\epsilon(\omega) = \epsilon_\infty + \frac{\epsilon_\mathrm{s} - \epsilon_\infty}{1 - \mathrm{i}\omega\tau}, \tag{41}$$

where $\epsilon_\infty > 0$ is the optical response, $\epsilon_\mathrm{s} > 0$ the static response and $\tau > 0$ the relaxation time. The corresponding time domain response is given by

$$\epsilon_\mathrm{r}(t) = \epsilon_\infty \delta(t) + \frac{\epsilon_\mathrm{s} - \epsilon_\infty}{\tau} \mathrm{e}^{-t/\tau} H(t). \tag{42}$$

The corresponding Herglotz function and its asymptotics are given by

$$h(\omega) = \omega\epsilon_\infty + \omega \frac{\epsilon_\mathrm{s} - \epsilon_\infty}{1 - \mathrm{i}\omega\tau} = \begin{cases} \omega\epsilon_\mathrm{s} + o(\omega) & \text{as } \omega \hat{\to} 0, \\ \omega\epsilon_\infty + o(\omega) & \text{as } \omega \hat{\to} \infty. \end{cases} \tag{43}$$

The Herglotz function $h(\omega)$ can be continued to a function meromorphic in $\mathbb{C} \setminus \{-\mathrm{i}\frac{1}{\tau}\}$ with a single pole at $\omega = -\mathrm{i}\frac{1}{\tau}$. The spectral measure is given by

$$\operatorname{Im} h(\xi) = \frac{\xi^2 \tau (\epsilon_\mathrm{s} - \epsilon_\infty)}{1 + \xi^2 \tau^2}, \tag{44}$$

where $\xi \in \mathbb{R}$. There is a sum rule (11) for $n = 1$ yielding

$$\frac{2}{\pi} \int_{0+}^{\infty} \frac{\operatorname{Im} h(\xi)}{\xi^2} \, \mathrm{d}\xi = \frac{2}{\pi} \int_{0+}^{\infty} \frac{\operatorname{Im} \epsilon(\xi)}{\xi} \, \mathrm{d}\xi = \epsilon_\mathrm{s} - \epsilon_\infty. \tag{45}$$

It is noted that the sum rule (45) reveals a general feature for all passive materials having the same asymptotic expansion as in (43), namely that $\epsilon_\mathrm{s} \geq \epsilon_\infty$, and if $\epsilon_\mathrm{s} = \epsilon_\infty$ then $\operatorname{Im} h(\xi) = 0$ for all $\xi \in \mathbb{R}$ and the material is loss-less.

3.2.4 Lorentz' model

The Lorentz' model is used to model the response of a plasma and is given by

$$\epsilon(\omega) = \epsilon_\infty - \frac{\omega_\mathrm{p}^2}{\omega^2 + \mathrm{i}\omega\nu - \omega_0^2} \tag{46}$$

where $\epsilon_\infty > 0$ is the optical response, $\omega_\mathrm{p} > 0$ the plasma frequency, $\omega_0 > 0$ the resonance frequency and $\nu > 0$ the collision frequency. The corresponding time domain response is given by

$$\epsilon_\mathrm{r}(t) = \epsilon_\infty \delta(t) + \frac{\omega_\mathrm{p}^2}{\nu_0} \mathrm{e}^{-\nu t/2} \sin(\nu_0 t) H(t), \tag{47}$$

where $\nu_0 = \sqrt{\omega_0^2 - \nu^2/4}$ and $\omega_0 \geq \nu/2$. The corresponding Herglotz function and its asymptotics are given by

$$h(\omega) = \omega\epsilon_\infty - \frac{\omega\omega_{\mathrm{p}}^2}{\omega^2 + \mathrm{i}\omega\nu - \omega_0^2} = \begin{cases} \omega\epsilon_{\mathrm{s}} + o(\omega) & \text{as } \omega\hat{\to}0, \\ \omega\epsilon_\infty - \omega^{-1}\omega_{\mathrm{p}}^2 + o(\omega^{-1}) & \text{as } \omega\hat{\to}\infty, \end{cases} \tag{48}$$

where $\epsilon_{\mathrm{s}} = \epsilon_\infty + \omega_{\mathrm{p}}^2/\omega_0^2$ and $\omega_0 \neq 0$. The Herglotz function $h(\omega)$ can be continued to a function meromorphic in $\mathbb{C} \setminus \{\omega_1, \omega_2\}$ with two poles at $\omega_{1,2} = -\mathrm{i}\nu/2 \pm \sqrt{-\nu^2/4 + \omega_0^2}$. The spectral measure is given by

$$\operatorname{Im} h(\xi) = \frac{\xi^2\omega_{\mathrm{p}}^2\nu}{(\xi^2 - \omega_0^2)^2 + \xi^2\nu^2}, \tag{49}$$

where $\xi \in \mathbb{R}$. When $\omega_0 \neq 0$ there is a sum rule (11) for $n = 1$ just as in (45). There is also a sum rule (11) for $n = 0$ yielding

$$\frac{2}{\pi}\int_{0+}^{\infty} \operatorname{Im} h(\xi)\,\mathrm{d}\xi = \omega_{\mathrm{p}}^2, \tag{50}$$

and which is valid also when $\omega_0 = 0$ (see the Drude model below).

3.2.5 Drude model

The Drude model is used to model the conduction of charges in metals and is a special case of the Lorent'z model with $\omega_0 = 0$. Hence

$$\epsilon(\omega) = \epsilon_\infty - \frac{\omega_{\mathrm{p}}^2}{\omega(\omega + \mathrm{i}\nu)} = \epsilon_\infty + \mathrm{i}\frac{\sigma_0}{\omega\epsilon_0}\frac{1}{1 - \mathrm{i}\omega/\nu}, \tag{51}$$

where $\epsilon_\infty > 0$ is the optical response, $\omega_{\mathrm{p}} > 0$ the plasma frequency, $\nu > 0$ the collision frequency and $\sigma_0 = \frac{\omega_{\mathrm{p}}^2\epsilon_0}{\nu}$ the static conductivity. The corresponding time domain response is given by

$$\epsilon_{\mathrm{r}}(t) = \epsilon_\infty\delta(t) + \frac{\sigma_0}{\epsilon_0}(1 - \mathrm{e}^{-\nu t})H(t). \tag{52}$$

The corresponding Herglotz function and its asymptotics are given by

$$h(\omega) = \omega\epsilon_\infty + \mathrm{i}\frac{\sigma_0}{\epsilon_0}\frac{1}{1 - \mathrm{i}\omega/\nu} = \begin{cases} \mathrm{i}\sigma_0/\epsilon_0 + o(1) = o(\omega^{-1}) & \text{as } \omega\hat{\to}0, \\ \omega\epsilon_\infty - \omega^{-1}\omega_{\mathrm{p}}^2 + o(\omega^{-1}) & \text{as } \omega\hat{\to}\infty, \end{cases} \tag{53}$$

where $\omega_{\mathrm{p}}^2 = \sigma_0\nu/\epsilon_0$. The Herglotz function $h(\omega)$ can be continued to a function meromorphic in $\mathbb{C} \setminus \{-\mathrm{i}\nu\}$ with a single pole at $\omega = -\mathrm{i}\nu$. The spectral measure is given by

$$\operatorname{Im} h(\xi) = \frac{\sigma_0}{\epsilon_0}\frac{1}{1 + \xi^2/\nu^2}, \tag{54}$$

where $\xi \in \mathbb{R}$. There is no sum rule (11) for $n = 1$ as in (45). However, the sum rule (11) for $n = 0$, and hence (50) is valid also when $\omega_0 = 0$. Note that (50) can also be written

$$\frac{2}{\pi}\int_0^\infty \xi \operatorname{Im} \epsilon(\xi) = \omega_{\mathrm{p}}^2. \tag{55}$$

Hence, a measurement (or a priori knowledge) of the static conductivity σ_0 as well as the mean value (first moment) of $\operatorname{Im} \epsilon(\xi)$ will completely determine the Drude model through (55) and finally that the collision frequency is given by $\nu = \frac{\omega_{\mathrm{p}}^2\epsilon_0}{\sigma_0}$.

3.3 Pulse functions

In [10] can be found a large number of useful examples of various functions and their Hilbert transforms. Below is given two important examples, the rectangular and the triangular pulse functions. It is left as an exercise for the reader to verify the integrals given below.

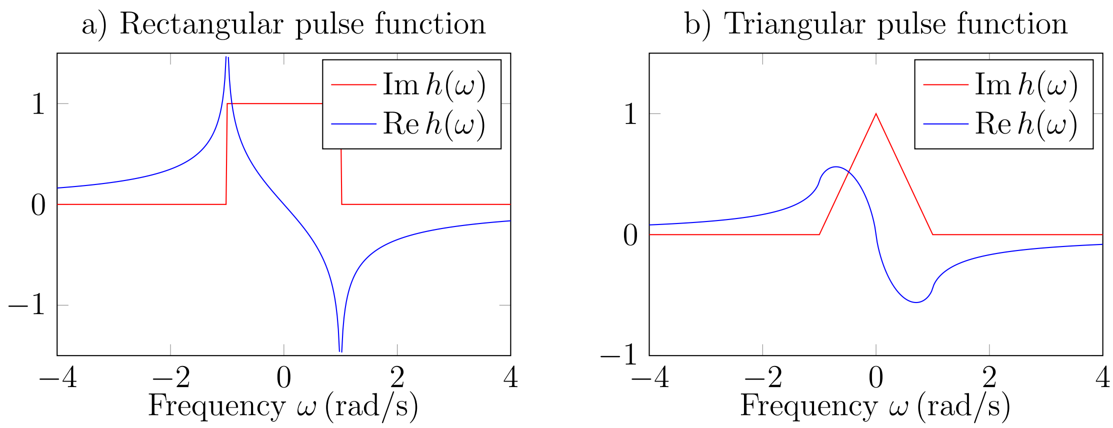

Figure 1: Pulse functions $\mathrm{Im}\, h(\omega) = p_a(\omega)$ and their (negative) Hilbert transforms $\mathrm{Re}\, h(\omega) = \frac{1}{\pi} \fint \frac{1}{\xi - \omega} p_a(\xi)\, \mathrm{d}\xi$. a) Rectangular pulse function (parameter $a = 1$) and its (negative) Hilbert transform. Note that the rectangular pulse is discontinuous and its Hilbert transform has logarithmic singularities at $\pm a$. b) Triangular pulse function (parameter $a = 1$) and its (negative) Hilbert transform.

3.3.1 Rectangular pulse

A symmetric and positive measure is given by the rectangular pulse function $p_a(\xi)$

$$\mathrm{Im}\, h(\xi) = p_a(\xi) = \begin{cases} 1 & |\xi| \leq a, \\ 0 & |\xi| > a, \end{cases} \tag{56}$$

where $\xi \in \mathbb{R}$ and $a > 0$. The corresponding Herglotz function $h(\omega)$ is obtained by evaluating the standard (Lebesgue) integral

$$h(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - \omega} p_a(\xi)\, \mathrm{d}\xi = \frac{1}{\pi} (\ln(\omega - a) - \ln(\omega + a)), \tag{57}$$

where $\mathrm{Im}\, \omega > 0$. The distributional limit on the real axis is obtained by evaluating the corresponding Cauchy principal value integral

$$h(\omega) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{1}{\xi - \omega} p_a(\xi)\, \mathrm{d}\xi + \mathrm{i} p_a(\omega) = \frac{1}{\pi} (\ln |\omega - a| - \ln |\omega + a|) + \mathrm{i} p_a(\omega), \tag{58}$$

where $\omega \in \mathbb{R}$.

3.3.2 Triangular pulse

A symmetric and positive measure is given by the triangular pulse function $p_a(\xi)$

$$\mathrm{Im}\, h(\xi) = p_a(\xi) = \begin{cases} 1 - \dfrac{|\xi|}{a} & |\xi| \leq a \\ 0 & |\xi| > a, \end{cases} \tag{59}$$

where $\xi \in \mathbb{R}$ and $a > 0$. The corresponding Herglotz function $h(\omega)$ is obtained by evaluating the standard (Lebesgue) integral

$$h(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - \omega} p_a(\xi) \, \mathrm{d}\xi = \frac{1}{\pi a} (2\omega \ln \omega - (\omega - a) \ln(\omega - a) - (\omega + a) \ln(\omega + a)), \quad (60)$$

where $\operatorname{Im} \omega > 0$. The distributional limit on the real axis is obtained by evaluating the corresponding Cauchy principal value integral

$$\begin{aligned} h(\omega) &= \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{1}{\xi - \omega} p_a(\xi) \, \mathrm{d}\xi + \mathrm{i} p_a(\omega) \\ &= \frac{1}{\pi a} (2\omega \ln |\omega| - (\omega - a) \ln |\omega - a| - (\omega + a) \ln |\omega + a|) + \mathrm{i} p_a(\omega), \quad (61) \end{aligned}$$

where $\omega \in \mathbb{R}$.

4 A brief summary on convex optimization

Below is given a brief summary on some of the central concepts in convex optimization. Most of the properties stated below are rather simple and can be proven directly by the reader by employing the definitions of convex sets and convex functions. References have been given for the more advanced properties. For a more complete treatise on convex optimization and related algorithms, see *e.g.*, [4, 5, 11].

- **Convex sets:** A set $S \subset \mathbb{R}^n$ is *convex* if

$$\mathbf{x}_1, \mathbf{x}_2 \in S \Rightarrow (1 - t)\mathbf{x}_1 + t\mathbf{x}_2 \in S, \quad (62)$$

for all $0 < t < 1$. Geometrically this means that the straight line between any two points in a convex set remains in the set, see Figure 2 for an illustration of the two-dimensional case.

Figure 2: Illustration of a) A convex set in $\mathbb{R}^2$. b) A non-convex set in $\mathbb{R}^2$.

- **Intersection of convex sets:** The intersection $\cap_\alpha S_\alpha$ of convex sets S_α is a convex set.

- **Convex combinations:** Let $S \subset \mathbb{R}^n$ be an arbitrary set. A *convex combination* of the elements $\mathbf{x}_i \in S$ is a positive linear combination

$$\mathbf{z} = \sum_{i=1}^{m} t_i \mathbf{x}_i \quad \text{where} \quad t_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{m} t_i = 1.$$

If S is a convex set, it can be shown (by induction) that any convex combination of elements of S belongs to S.

Proof: Suppose that $\mathbf{x}_1$, $\mathbf{x_2}$ and $\mathbf{x_3}$ belong to the convex set S. Let

$$\mathbf{z} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + t_3\mathbf{x}_3,$$

where $t_1 + t_2 + t_3 = 1$ and $t_3 \neq 1$ (otherwise $\mathbf{z} = \mathbf{x}_3 \in S$). Then

$$\mathbf{z} = (1-t_3) \underbrace{\left(\frac{t_1}{1-t_3}\mathbf{x}_1 + \frac{t_2}{1-t_3}\mathbf{x}_2\right)}_{\in S \text{ since } \frac{t_1}{1-t_3} + \frac{t_2}{1-t_3} = 1} + t_3\mathbf{x}_3 \in S.$$

- **Convex hull:** Let $S \subset \mathbb{R}^n$ be an arbitrary set. The *convex hull* $\mathrm{conv}(S)$ of S is the set of all convex combinations of elements in S. It can be shown that $\mathrm{conv}(S)$ is the smallest convex set containing S, *i.e.*,

$$\mathrm{conv}(S) = \cap_\alpha T_\alpha$$

where the intersection is taken over all convex sets T_α containing S, *i.e.*, $S \subset T_\alpha$.

- **Convex functions:** Let f be a function defined on the convex set $S \subset \mathbb{R}^n$. The function f is said to be a *convex function* if

$$\mathbf{x}_1, \mathbf{x}_2 \in S \Rightarrow f((1-t)\mathbf{x}_1 + t\mathbf{x}_2) \leq (1-t)f(\mathbf{x}_1) + tf(\mathbf{x}_2), \tag{63}$$

for all $0 < t < 1$. Geometrically this means that a convex function f is always less then or equal to a corresponding linear (or affine) function passing through the values $f(\mathbf{x}_1)$ and $f(\mathbf{x}_2)$ where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two points in the convex set S, see Figure 3 for an illustration of the one-dimensional case.

If the function f satisfies a strict inequality in (63), it is said to be a *strictly convex function*. A function f is said to be *concave* if $-f$ is convex.

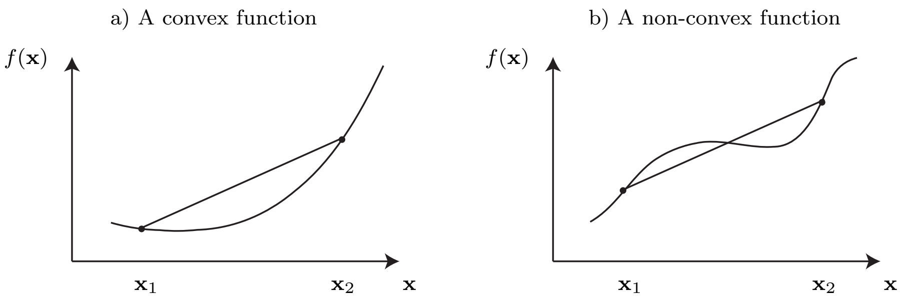

Figure 3: Illustration of a) A convex function on $S \subset \mathbb{R}$. b) A non-convex function on $S \subset \mathbb{R}$.

- **Linear functions:** A linear (or affine) function f is both convex and concave (but it is not strictly convex). In particular, if f is linear (or affine), both f and $-f$ are convex.

- **Positive linear combinations of convex functions:** Let f_1 and f_2 be convex functions defined on a convex set S. Then the positive linear combination

$$f = \alpha_1 f_1 + \alpha_2 f_2$$

(with $\alpha_1 > 0$ and $\alpha_2 > 0$) is a convex function. If any one of f_1 and f_2 is strictly convex, then the function f is strictly convex.

- **Differentiable convex functions:** Let f be a two times differentiable continuous function defined on the open convex set $S \subset \mathbb{R}^n$ ($f \in C^2(S)$). It can then be shown that f is a convex function if and only if the Hessian $\mathbf{H}_{ij}(\mathbf{x}) = \partial_{x_i}\partial_{x_j} f(\mathbf{x})$ is a positively semidefinite matrix for all $\mathbf{x} \in S$, *i.e.*,

$$\mathbf{H}(\mathbf{x}) \geq 0 \ \forall \mathbf{x} \in S \Leftrightarrow f \text{ convex.}$$

  If the Hessian $\mathbf{H}(\mathbf{x})$ is positively definite for all $\mathbf{x} \in S$, then f is strictly convex, *i.e.*,

$$\mathbf{H}(\mathbf{x}) > 0 \ \forall \mathbf{x} \in S \Rightarrow f \text{ strictly convex,}$$

  but the converse is not true (take *e.g.*, $f(x) = x^4$), see *e.g.*, [4].

- **Quadratic forms:** Let $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} + c$ be a quadratic form where $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ vectors, c a constant and $(\cdot)^{\mathrm{T}}$ denotes the transpose. The function f is convex if and only if the Hessian matrix $\mathbf{A}$ is positively semidefinite, *i.e.*,

$$\mathbf{A} \geq 0 \Leftrightarrow f \text{ convex.}$$

  The function f is strictly convex if and only if the Hessian matrix $\mathbf{A}$ is positively definite, *i.e.*,

$$\mathbf{A} > 0 \Leftrightarrow f \text{ strictly convex.}$$

- **Level set:** Let g be a convex function defined on the convex set $S \subset \mathbb{R}^n$ and α an arbitrary real number. Then the level set $S_\alpha = \{\mathbf{x} \in S | g(\mathbf{x}) \leq \alpha\}$ is a convex set.

- **Continuity:** Let f be a convex function defined on the convex set $S \subset \mathbb{R}^n$. Then f is a continuous function on the interior of S [4].

- **Convex optimization:** A convex optimization problem in general is a problem of the form

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in S, \end{array} \tag{64}$$

  where f is a convex function defined on the convex set S. It is common that the convex set S is given in the form $S = \{\mathbf{x} \in \mathbb{R}^n | g_i(\mathbf{x}) \leq 0\}$ where $\{g_i(\mathbf{x})\}_{i=1}^M$ is a set of convex functions representing the *convex constraints*.

- **A local minimum is also a global minimum:** One of the most important properties of a convex optimization problem is that any local minimum is also a global minimum[1]. Below are given some definitions and a proof.

  Consider the convex optimization problem (64) where f is a convex function defined on the convex set S.

  - If $\mathbf{x} \in S$ it is called a *feasible point*.
  - The function f has a *global minimum* at $\mathbf{x}$ if

$$f(\mathbf{x}) \leq f(\boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in S$$

[1] Here the term "minimum" refer to the minimizing point $\mathbf{x}$ and the term "minimum value" to the corresponding minimum value $f(\mathbf{x})$.

– The function f has a *local minimum* at $\mathbf{x}$ if

$$f(\mathbf{x}) \leq f(\boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in S_r = \{\boldsymbol{\xi} \in S | \|\boldsymbol{\xi} - \mathbf{x}\| < r\}$$

for *some* r-neighbourhood S_r with radius $r > 0$, and where $\|\cdot\|$ is a suitable norm.

Proof: Let $\mathbf{x} \in S$ be a local minimum and S_r a corresponding r-neighbourhood. Suppose now that $\mathbf{x}$ is not a global minimum, then there is a feasible $\mathbf{y} \in S$ with minimum value $f(\mathbf{y}) < f(\mathbf{x})$. Since $\mathbf{x}$ is a local minimum it follows that $\mathbf{y} \notin S_r$ and hence $\|\mathbf{y} - \mathbf{x}\| \geq r$. Define the point

$$\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y} \quad \text{where} \quad t = \frac{r}{2\|\mathbf{y} - \mathbf{x}\|} \leq \frac{1}{2},$$

so that

$$\|\mathbf{z} - \mathbf{x}\| = t\|\mathbf{y} - \mathbf{x}\| = \frac{r}{2},$$

and hence $\mathbf{z} \in S_r$. On the other hand, it follows by the convexity of f that

$$f(\mathbf{z}) = f((1-t)\mathbf{x} + t\mathbf{y}) \leq (1-t)f(\mathbf{x}) + tf(\mathbf{y}) < f(\mathbf{x}),$$

which contradicts the assumption that $\mathbf{x}$ is a local minimum.

- **If f is strictly convex then a minimum is also a unique minimum:**
Proof: Let $\mathbf{x} \in S$ be a minimum so that $f(\mathbf{x}) \leq f(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in S$. Suppose that $\mathbf{y} \in S$ where $\mathbf{y} \neq \mathbf{x}$ is also a minimum so that $f(\mathbf{y}) = f(\mathbf{x})$. Let $\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y}$ where $0 < t < 1$ and hence $\mathbf{z} \in S$. Then by the strict convexity

$$f(\mathbf{z}) = f((1-t)\mathbf{x} + t\mathbf{y}) < (1-t)f(\mathbf{x}) + tf(\mathbf{y}) = f(\mathbf{x}),$$

which contradicts the assumption that $\mathbf{x}$ is a minimum.

- **The norm:** Any norm $f(\mathbf{x}) = \|\mathbf{x}\|$ is a convex function on $\mathbb{R}^n$ since by the triangle inequality

$$f((1-t)\mathbf{x}_1 + t\mathbf{x}_2) = \|(1-t)\mathbf{x}_1 + t\mathbf{x}_2\| \leq (1-t)\|\mathbf{x}_1\| + t\|\mathbf{x}_2\| = (1-t)f(\mathbf{x}_1) + tf(\mathbf{x}_2),$$

where $0 < t < 1$.

Comments: It follows from the properties of the norm that the minimum of $f(\mathbf{x}) = \|\mathbf{x}\|$ is the unique vector $\mathbf{x} = \mathbf{0}$. Note however that $f(\mathbf{x}) = \|\mathbf{x}\|$ is not a strictly convex function. To see this, choose *e.g.*, $\mathbf{x}_2 = \alpha\mathbf{x}_1$ where $\alpha > 1$ and verify that $f((1-t)\mathbf{x}_1 + t\mathbf{x}_2) = (1-t)f(\mathbf{x}_1) + tf(\mathbf{x}_2)$ for all $0 < t < 1$. Note however that the function $f(\mathbf{x}) = \|\mathbf{x}\|_2^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$ is strictly convex.

- **Norm of a linear (or affine) form:** The norm of a linear (or affine) form is a convex function on $\mathbb{R}^n$. Consider *e.g.*, the function $f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|$ where $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{x}$ an $n \times 1$ vector and $\mathbf{b}$ an $m \times 1$ vector. The function $f(\mathbf{x})$ is convex since by the triangle inequality

$$\begin{aligned} f((1-t)\mathbf{x}_1 + t\mathbf{x}_2) &= \|\mathbf{A}((1-t)\mathbf{x}_1 + t\mathbf{x}_2) - \mathbf{b}\| \\ &= \|(1-t)(\mathbf{A}\mathbf{x}_1 - b) + t(\mathbf{A}\mathbf{x}_2 - \mathbf{b})\| \leq (1-t)\|\mathbf{A}\mathbf{x}_1 - \mathbf{b}\| + t\|\mathbf{A}\mathbf{x}_2 - \mathbf{b}\| \\ &= (1-t)f(\mathbf{x}_1) + tf(\mathbf{x}_2). \end{aligned}$$

Comments: Note that if the Euclidean norm $\|\cdot\|_2$ is used, if $m > n$ and if the rank of the matrix $\mathbf{A}$ is $\mathrm{rank}\{\mathbf{A}\} = n$, then the minimum of f (the least squares solution) is unique (even though the function f is not strictly convex).

It is evident that the well-posedness of the minimization of $f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|$ depends on the properties of the matrix (or operator) $\mathbf{A}$. If the matrix $\mathbf{A}$ is not full rank and has a non-trivial null space, then the optimization problem can be regularized by adding a strictly convex penalty term as *e.g.*, $f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 + \alpha\|\mathbf{x}\|_2^2$ where $\alpha > 0$ (*cf.*, Tikhonov regularization). Now the new function f is strictly convex, and if α is sufficiently small the optimal solution will coincide with the pseudo-inverse.

- **Disciplined convex programming in Matlab:** As an example, consider the following convex optimization problem related to the normed error

$$\begin{aligned} &\text{minimize} && \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_p \\ &\text{subject to} && \mathbf{B}\mathbf{x} \le \mathbf{c} \\ & && \mathbf{C}\mathbf{x} = \mathbf{d}, \end{aligned} \tag{65}$$

where the $n \times 1$ vector $\mathbf{x}$ is the unknown optimization variable and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given matrices and $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are given vectors of suitable dimensions. Here, $\|\cdot\|_p$ denotes that a p-norm is used where $p \ge 1$. The max-norm is denoted $\|\cdot\|_\infty$.

Once formulated on a standard form the problem (65) can be solved efficiently by using the CVX Matlab software for disciplined convex programming [6] which is available via the link http://cvxr.com/cvx/. Here, the following lines in Matlab may be used

```
cvx_begin
    variable x(n)
    minimize( norm(A*x-b,p) )
    subject to
    B*x<=c
    C*x==d
cvx_end
```

where `n`, `p` (`p` ≥ 1 or `p` $=$ `Inf`), `A`, `B`, `C`, `b`, `c` and `d` have been defined in the preamble.

5 Approximation of Herglotz functions

Consider the following general approximation problem

$$\begin{aligned} &\text{minimize} && \|h(\xi) - f(\xi)\|_\Omega \\ &\text{subject to} && f(\xi) \text{ continuous on } \Omega \\ & && h(\xi) \overset{\text{def}}{=} h(\xi + \mathrm{i}0) \exists \text{ continuous on } \Omega \end{aligned} \tag{66}$$

where the approximation domain Ω is a closed and bounded subset of $\mathbb{R}$. Here, the objective is to find an approximating symmetric Herglotz function $h(\omega)$ which is regular in a neighbourhood of $\mathbb{C}^+ \cup \Omega$ and which approximates a given (generally complex valued) function $f(\xi)$ which is continuous on Ω. The norm $\|\cdot\|_\Omega$ can be any norm defined on the interval Ω, such as with L^p-norms ($p \ge 1$), the L^∞-norm, and the corresponding weighted variants, etc. Since $h(\xi) = (-\mathcal{H} + \mathrm{i}\mathcal{I})\,\mathrm{Im}\,h(\xi)$ (where $\mathcal{H}$ denotes the Hilbert transform operator and $\mathcal{I}$ the identity operator) is a linear form operating on $\mathrm{Im}\,h(\xi)$, and any norm of a linear form yields a convex function, the problem (66) is a convex optimization problem. In particular, if $\operatorname{Im} h(\xi) \geq 0$ is treated as a linear constraint on the optimization variable $\operatorname{Im} h(\xi)$, it is a convex constraint. Additional convex constraints can also be added to (66), such as additional sum rule constraints, etc.

It can be readily shown that (66) can have non-trivial solutions. Suppose for example that

$$f(\omega) = -h_0(\omega), \tag{67}$$

is the negative of a Herglotz function $h_0(\omega)$ which can be continued analytically to a neighbourhood of $\mathbb{C}^+ \cup \Omega$, and which has the high-frequency asymptotic expansion

$$h_0(\omega) = b_1^0 \omega + o(\omega), \tag{68}$$

as $\omega \hat{\to} \infty$. It can then be shown that the following non-trivial bound holds

$$\sup_{\xi\in\Omega} |h(\xi) - f(\xi)| \geq b_1^0 \frac{1}{2} |\Omega|, \tag{69}$$

where $|\Omega|$ is the length of the interval Ω, see also [7].

5.1 Derivation of a non-trivial bound

To prove (69) we follow the technique outlined in [7] which is based on a composition using the auxiliary Herglotz function $h_\Delta(z)$, defined by the square pulse $p_\Delta(\xi)$ as a generating measure (as in (56) above), and its associated sum rules. The *auxiliary Herglotz function* and its asymptotics are given by

$$h_\Delta(z) = \frac{1}{\pi}\int_{-\Delta}^{\Delta} \frac{1}{\xi - z} \,\mathrm{d}\xi = \frac{1}{\pi} \ln \frac{z-\Delta}{z+\Delta} = \begin{cases} \mathrm{i} + o(1) & \text{as } z\hat{\to}0, \\ \dfrac{-2\Delta}{\pi z} + o(z^{-1}) & \text{as } z\hat{\to}\infty, \end{cases} \tag{70}$$

where $\operatorname{Im} h_\Delta(z) \geq \frac{1}{2}$ for $|z| \leq \Delta$ and $\operatorname{Im} z \geq 0$.

Next, the *composite Herglotz function* $\widetilde{h}(\omega)$ is defined by

$$\widetilde{h}(\omega) = h_\Delta(h(\omega) + h_0(\omega)), \tag{71}$$

where $h(\omega) + h_0(\omega) = (b_1 + b_1^0)\omega + o(\omega)$ as $\omega\hat{\to}\infty$, yielding the asymptotics of $\widetilde{h}(\omega)$

$$\widetilde{h}(\omega) = \begin{cases} o(\omega^{-1}) & \text{as } \omega\hat{\to}0, \\ \dfrac{-2\Delta}{\pi(b_1 + b_1^0)}\omega^{-1} + o(\omega^{-1}) & \text{as } \omega\hat{\to}\infty. \end{cases} \tag{72}$$

The *sum rule* (11) for $n = 0$ is given by

$$\frac{2}{\pi}\int_0^\infty \operatorname{Im} \widetilde{h}(\xi)\,\mathrm{d}\xi = a_{-1} - b_{-1} = \frac{2\Delta}{\pi(b_1 + b_1^0)}. \tag{73}$$

Let $\Delta = \sup_{\xi\in\Omega} |h(\xi) + h_0(\xi)|$, then the following *integral inequalities* follow

$$\frac{1}{\pi}|\Omega| \leq \frac{2}{\pi}\int_\Omega \underbrace{\operatorname{Im} \widetilde{h}(\xi)}_{\geq\frac{1}{2}} \mathrm{d}\xi \leq \frac{2}{\pi}\int_0^\infty \operatorname{Im} \widetilde{h}(\xi)\,\mathrm{d}\xi = \frac{2}{\pi(b_1 + b_1^0)} \sup_{\xi\in\Omega} |h(\xi) + h_0(\xi)|, \tag{74}$$

or

$$\sup_{\xi\in\Omega} |h(\xi) + h_0(\xi)| \geq (b_1 + b_1^0)\frac{1}{2}|\Omega|, \tag{75}$$

where $|\Omega| = \int_\Omega \mathrm{d}\xi$. The non-trivial bound (69) is obtained by chosing $b_1 = 0$.

5.1.1 Permittivity functions and metamaterials

A passive dielectric material where the real part of the permittivity function is negative over a frequency interval is sometimes regarded a *metamaterial*, *cf.*, [7]. Let $h(\omega) = \omega\epsilon(\omega)$ with optical response $b_1 = \epsilon_\infty$, and let $f(\omega) = \omega\epsilon_{\rm t}$ with target permittivity $\epsilon_{\rm t} < 0$. Then $h_0(\omega) = -f(\omega) = -\omega\epsilon_{\rm t}$ with asymptotic coefficient $b_1^0 = -\epsilon_{\rm t}$. It follows from (75) that the following lower bound must hold when approximating a metamaterial over a bandwidth

$$\sup_{\xi\in\Omega} |\epsilon(\xi) - \epsilon_{\rm t}| \geq \frac{(\epsilon_\infty - \epsilon_{\rm t})\frac{1}{2}B}{1+\frac{B}{2}}, \tag{76}$$

where $\Omega = \omega_0[1-\frac{B}{2}, 1+\frac{B}{2}]$, and where ω_0 is the center frequency and B the relative bandwidth with $0 < B < 2$, see also [7].

5.2 Discretization and optimization

In the following, we will let ω denote the frequency variable on the real line, $\omega \in \mathbb{R}$, and $h(\omega)$ the analytic continuation of the approximating Herglotz function which is assumed to be regular in a neighbourhood of $\mathbb{C}_+ \cup \Omega$.

5.2.1 Discretization

Consider the general approximation problem (66) where the positive and symmetric measure $\operatorname{Im} h(\omega)$ is supported on $\Omega_1 \cup -\Omega_1$ where Ω_1 is the right-sided interval $\Omega_1 = [\omega_1, \omega_4]$ and $-\Omega_1 = [-\omega_4, -\omega_1]$. The approximation domain (where the norm $\|\cdot\|_\Omega$ is defined) is $\Omega = [\omega_2, \omega_3] \subset \Omega_1$, and where $0 \leq \omega_1 < \omega_2 < \omega_3 < \omega_4$. In the limit, the upper boundary ω_4 may approach infinity.

A simple discretization of the intervals Ω and Ω_1 is defined based on M and N $(M < N)$ uniformly sampled frequency points, respectively, in such a way that the sampling interval is $\Delta\omega = \frac{\omega_4-\omega_1}{N-1}$, and $\omega_1 = n_1\Delta\omega$, $\omega_2 = n_2\Delta\omega$, $\omega_3 = n_3\Delta\omega$ and $\omega_4 = n_4\Delta\omega$ and where $0 \leq n_1 < n_2 < n_3 < n_4$ are integers.

A discretization of the positive and symmetric measure $\operatorname{Im} h(\omega)$ is defined by the following piece-wise linear (roof-top) approximation

$$\operatorname{Im} h(\omega) = \sum_{n=0}^{N-1} x_n \frac{(2-\delta_{n+n_1,0})}{2}\left[p(\omega - (n+n_1)\Delta\omega) + p(\omega + (n+n_1)\Delta\omega)\right], \tag{77}$$

where $x_n \geq 0$ are non-negative optimization variables and $p(\omega)$ the triangular pulse function given by

$$p(\omega) = \begin{cases} 1 - \dfrac{|\omega|}{\Delta\omega} & |\omega| \leq \Delta\omega, \\ 0 & |\omega| > \Delta\omega, \end{cases} \tag{78}$$

see also (59).

The real part $\operatorname{Re} h(\omega)$ is given by (9) and yields the following linear form

$$\operatorname{Re} h(\omega) = b_1\omega + \sum_{n=0}^{N-1} x_n \frac{(2-\delta_{n+n_1,0})}{2}\left[\hat{p}(\omega - (n+n_1)\Delta\omega) + \hat{p}(\omega + (n+n_1)\Delta\omega)\right], \tag{79}$$

where $b_1 \geq 0$ and $x_n \geq 0$ are the non-negative optimization variables and $\hat{p}(\omega)$ the (negative) Hilbert transform of the triangular pulse function given by

$$\begin{aligned}\hat{p}(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\xi-\omega} p(\xi)\,\mathrm{d}\xi \\ &= \left(2\omega\ln|\omega| - (\omega-\Delta\omega)\ln|\omega-\Delta\omega| - (\omega+\Delta\omega)\ln|\omega+\Delta\omega|\right)/(\pi\Delta\omega), \quad (80)\end{aligned}$$

with $\hat{p}(\pm\Delta\omega) = \mp(2\ln 2)/\pi$ and $\hat{p}(0) = 0$.

Based on the triangular (roof-top) approximation (77), the sum rule in (11) can similarly be represented by a linear form. In particular, the following case will be useful here

$$\frac{2}{\pi}\int_0^{\infty} \frac{\operatorname{Im} h(\xi)}{\xi^2}\,\mathrm{d}\xi = \frac{2}{\pi\Delta\omega}\sum_{n=0}^{N-1} x_n \ln\frac{(n+n_1)^2}{(n+n_1-1)(n+n_1+1)}, \quad (81)$$

assuming that $n_1 > 1$. Note that convergence of the integral in (81) requires that $\operatorname{Im} h(\xi) = o(\xi)$ when $\xi \to 0$, and hence that $\omega_1 > 0$ when a piece-wise linear approximation is employed.

5.2.2 Disciplined convex programming in Matlab

Based on the discretization of the domains Ω and Ω_1, as well as the discretization of the function $h(\omega)$ given by (77) and (79), the "Herglotz function"[2]

$$h(\omega) = \operatorname{Re} h(\omega) + \mathrm{i}\operatorname{Im} h(\omega), \quad (82)$$

can now be represented in matrix form as

$$\mathbf{h} = (\mathbf{H} + \mathrm{i}\mathbf{I})\mathbf{x} \quad (83)$$

where $\mathbf{h}$ is an $M \times 1$ vector defined on the approximation domain Ω, $\mathbf{x}$ an $N \times 1$ vector of elements x_n representing the unknown measure defined on Ω_1, $\mathbf{H}$ an $M \times N$ matrix representing the (negative) Hilbert transform as in (79) and $\mathbf{I}$ an identity matrix of suitable dimension giving the restriction of the measure $\mathbf{x}$ on Ω. Note that the rows of the matrices $\mathbf{H}$ and $\mathbf{I}$ correspond to the approximation domain Ω and the columns to the measure domain Ω_1.

In (83), the matrices can easily be defined to include the variable b_1 as well as the variables x_n defined in (79). For numerical reasons it may be important to scale the factor $b_1\omega = \tilde{b}_1\frac{\omega}{\omega_0}$ when b_1 is an unknown, so that the scaled variable $\tilde{b}_1 = b_1\omega_0$ has the same physical dimension as the x_n variables, in particular when the center frequency ω_0 is large.

The general approximation problem (66) can now be written in the standard form

$$\begin{array}{ll}\text{minimize} & \|\mathbf{A}\mathbf{x} - \mathbf{f}\|_{\Omega} \\ \text{subject to} & \mathbf{x} \geq \mathbf{0},\end{array} \quad (84)$$

where $\mathbf{A} = \mathbf{H} + \mathrm{i}\mathbf{I}$ is described above and $\mathbf{f}$ is an $M \times 1$ vector representing the function $f(\xi)|_{\Omega}$ to be approximated. Once formulated on standard form, the convex optimization problem (84) can be solved efficiently by using the CVX Matlab software [6] for disciplined convex programming which is available via the link http://cvxr.com/cvx/. Here, the following lines in Matlab may be used

[2]A more correct phrasing would be: "the analytic continuation of the Herglotz function evaluated on $\Omega \subset \mathbb{R}$".

$$\begin{array}{l} \texttt{cvx\_begin} \\ \qquad \texttt{variable x(N)} \\ \qquad \texttt{minimize(norm(A * x - f, Inf))} \\ \qquad \texttt{subject to} \\ \qquad \texttt{x >= 0} \\ \texttt{cvx\_end} \end{array} \tag{85}$$

where N, $\texttt{A} = \texttt{H} + \texttt{1i} * \texttt{I}$ and f (corresponding to N, $\mathbf{A} = \mathbf{H} + \mathrm{i}\mathbf{I}$ and $\mathbf{f}$) have been defined in the preamble, and where the L^∞ (supremum) norm has been chosen.

5.2.3 Approximation of metamaterials

To illustrate some possible generalizations of the convex optimization formulation given above, we consider the problem

$$\begin{array}{ll} \text{minimize} & \sup\limits_{\xi\in\Omega} \dfrac{1}{\xi}|h(\xi) - f(\xi)| \\ \text{subject to} & \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\operatorname{Im} h(\xi)}{\xi^2}\,\mathrm{d}\xi \leq \epsilon_{\mathrm{s}}^{\max} - \epsilon_\infty, \end{array} \tag{86}$$

where the objective is to approximate a given metamaterial with target permittivity $\epsilon_{\mathrm{t}} < 0$ as represented by the function $f(\omega) = \omega\epsilon_{\mathrm{t}}$, and where the approximation is based on a passive dielectric material with $h(\omega) = \omega\epsilon(\omega)$. The additional constraint in (86) corresponds to the sum rule given in (11) and where the maximum static permittivity $\epsilon_{\mathrm{s}}^{\max}$ and the optical response ϵ_∞ have been prescribed. Note also that the weighted norm $\sup \frac{1}{\xi}|h(\xi) - f(\xi)| = \sup|\epsilon(\xi) - \epsilon_{\mathrm{t}}|$ is used in (86), as well as in (76). The optimization formulation (86) (with or without the sum-rule constraint) can be used to study the realizability of a metamaterial and the ultimate bound given by (76).

The approximation problem (86) can now be written in the standard form

$$\begin{array}{ll} \text{minimize} & \|\epsilon_\infty\boldsymbol{\omega} + \mathbf{A}\mathbf{x} - \mathbf{f}\|_\Omega \\ \text{subject to} & \mathbf{x} \geq \mathbf{0}, \\ & \mathbf{B}\mathbf{x} \leq b, \end{array} \tag{87}$$

where $\mathbf{A} = \mathbf{H} + \mathrm{i}\mathbf{I}$ and $\mathbf{f}$ have been defined as described in section 5.2.2 above, $\boldsymbol{\omega}$ is an $M \times 1$ vector of frequency samples corresponding to the approximation domain Ω, $\mathbf{B}$ an $1 \times N$ vector representing the discretized sum-rule (81) and $b = \epsilon_{\mathrm{s}}^{\max} - \epsilon_\infty$.

A disciplined convex programming [6] to solve (87) can now be formulated based on the following lines in Matlab

$$\begin{array}{l} \texttt{cvx\_begin} \\ \qquad \texttt{variable x(N)} \\ \qquad \texttt{minimize(norm((epsinf * Omega + A * x - f)./Omega, Inf))} \\ \qquad \texttt{subject to} \\ \qquad \texttt{x >= 0} \\ \qquad \texttt{B * x <= b} \\ \texttt{cvx\_end} \end{array} \tag{88}$$

where N, A, f, epsinf, Omega, B and b (corresponding to N, $\mathbf{A}$, $\mathbf{f}$, ϵ_∞, $\boldsymbol{\omega}$, $\mathbf{B}$ and b) have been defined in the preamble.

6 Exercises

6.1 Approximation of $\tan(\omega)$

A detailed study of the Herglotz function $h(\omega) = \tan(\omega)$ is given in section 3.1.1 Note in particular that the distributional limit of $h(\omega) = \tan(\omega)$ as $\operatorname{Im}\omega \to 0+$ is given by

$$h(\xi) \overset{\text{def}}{=} h(\xi + \mathrm{i}0) = \tan(\xi) + \mathrm{i}\pi \sum_{l=-\infty}^{\infty} \delta(\xi - \frac{\pi}{2} + l\pi). \tag{89}$$

Develop a Matlab code that specifies and solves the optimization problem (66) discretized and reformulated as in (84) and solved numerically based on disciplined convex programming as in (85), and where the function to be approximated is given by

$$f(\xi) = \tan(\xi), \tag{90}$$

comprising N data points for $\xi \in \Omega$. Let the approximating Herglotz function be represented as in (77) and (79) without high-frequency coefficient, *i.e.*, $b_1 = 0$.

This problem formulation can be interpreted as a study of the properties of a passive analytic continuation based on the finite input data defined by (90).

In sections 6.4.1 and 6.4.2 are given Matlab codes as an example implementation of this problem generating the results illustrated in Figure 4 below.

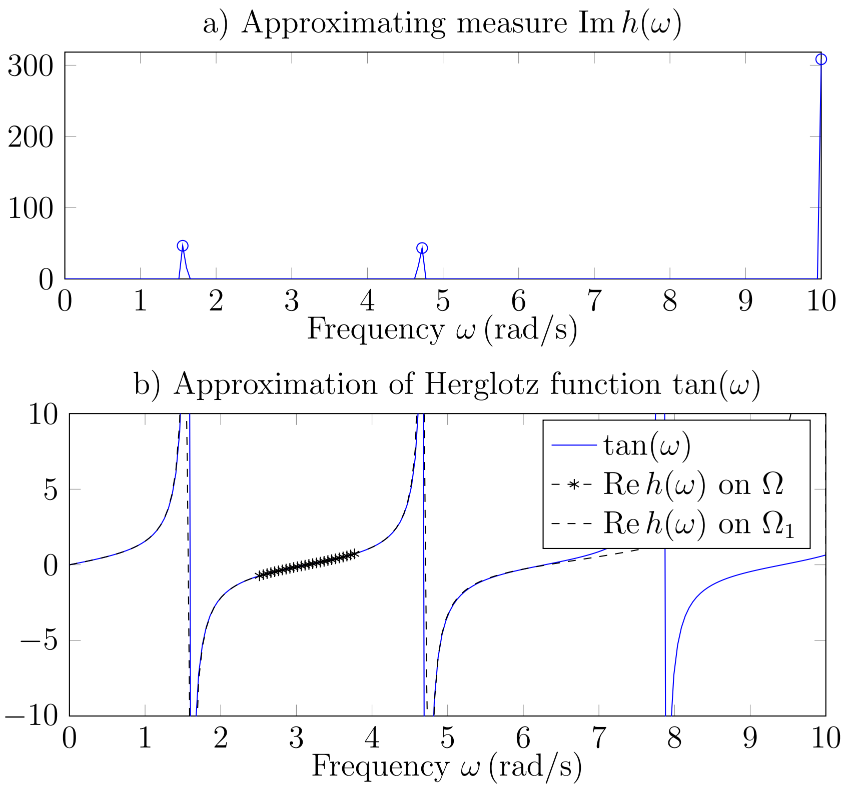

Figure 4: Approximation of the Herglotz function $\tan(\omega)$. a) The approximating measure $\operatorname{Im} h(\omega)$ based on $N = 200$ variables. b) The real part $\operatorname{Re} h(\omega)$ on $\Omega = [\omega_2, \omega_3] = \omega_0[1 - \frac{B}{2}, 1 + \frac{B}{2}]$ and $\Omega_1 = [\omega_1, \omega_4]$. Here, $\omega_0 = \pi$, $B = 0.4$, $\omega_1 = 0$ and $\omega_4 = 10$.

6.2 Approximation of $\tan(-1/\omega)$

A detailed study of the Herglotz function $h(\omega) = \tan(-1/\omega)$ is given in section 3.1.2 Note in particular that the distributional limit of $h(\omega) = \tan(-1/\omega)$ as $\mathrm{Im}\,\omega \to 0+$ is given by

$$h(\xi) \stackrel{\text{def}}{=} h(\xi + \mathrm{i}0) = \tan(-1/\xi) + \mathrm{i}\pi \sum_{l=-\infty}^{\infty} \frac{1}{(l\pi - \frac{\pi}{2})^2} \delta(\xi - \frac{1}{l\pi - \frac{\pi}{2}}). \tag{91}$$

Develop a Matlab code that specifies and solves the optimization problem (66) discretized and reformulated as in (84) and solved numerically based on disciplined convex programming as in (85), and where the function to be approximated is given by

$$f(\xi) = \tan(-1/\xi), \tag{92}$$

comprising N data points for $\xi \in \Omega$. Let the approximating Herglotz function be represented as in (77) and (79) without high-frequency coefficient, *i.e.*, $b_1 = 0$.

This problem formulation can be interpreted as a study of the properties of a passive analytic continuation based on the finite input data defined by (92).

Some input data and results of an example implementation of this problem is illustrated in Figure 5 below.

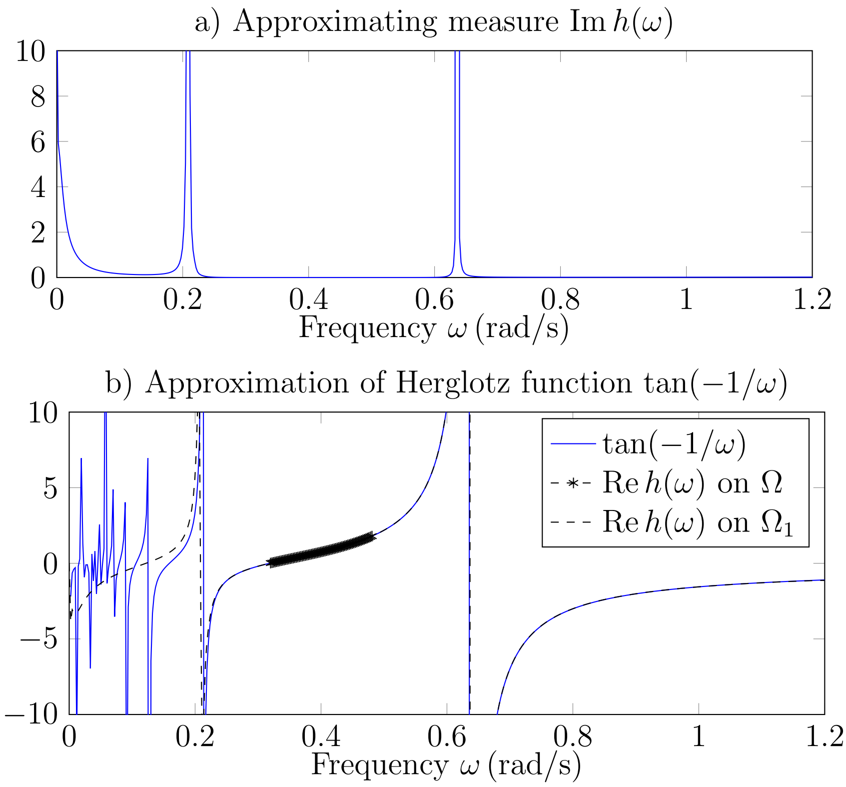

Figure 5: Approximation of the Herglotz function $\tan(-1/\omega)$. a) The approximating measure $\mathrm{Im}\,h(\omega)$ based on $N = 500$ variables. b) The real part $\mathrm{Re}\,h(\omega)$ on $\Omega = [\omega_2, \omega_3] = \omega_0[1-\frac{B}{2}, 1+\frac{B}{2}]$ and $\Omega_1 = [\omega_1, \omega_4]$. Here, $\omega_0 = 0.4$, $B = 0.4$, $\omega_1 = 0$ and $\omega_4 = 1.2$. The "random" behavior of the plotted $\tan(-1/\omega)$ in $[0, 0.1]$ is only due to undersampling.

6.3 Approximation of metamaterials

Consider the approximation of metamaterials as described in section 5.2.3. Develop a Matlab code that specifies and solves the optimization problem (86) discretized and reformulated as in (87) and solved numerically based on disciplined convex programming as in (88). The function to be approximated is given by

$$f(\xi) = \xi\epsilon_{\mathrm{t}}, \tag{93}$$

comprising N data points for $\xi \in \Omega$ and where the target permittivity is $\epsilon_{\mathrm{t}} = -1$. Let the approximating Herglotz function be represented as in (77) and (79) with high-frequency coefficient $b_1 = \epsilon_\infty = 1$.

- Solve the problem (86) without the sum-rule constraint, and compare with the theoretical lower bound given by (76). Some input data and results of an example implementation of this problem are illustrated in Figures 6, 7 and 8.

  Comment: for a Herglotz function with a point-mass at $\xi = 0$, it is noted that the distributional limit for $\omega \in \mathbb{R}$ is given by

  $$\omega\epsilon(\omega) = -\frac{1}{\pi}\frac{1}{\omega} + \mathrm{i}\delta(\omega) = \omega\left(-\frac{1}{\pi}\frac{1}{\omega^2} - \mathrm{i}\delta'(\omega)\right),$$

  where the permittivity function $\epsilon(\omega)$ is identified in the last line.

- Solve the problem (86) including the sum-rule constraint based on $\epsilon_{\mathrm{s}}^{\max}$, *e.g.*, with $\epsilon_{\mathrm{s}}^{\max} = 5$. Some input data and results of an example implementation of this problem are illustrated in Figures 9 and 10.

  Comment: for a Herglotz function with point-masses at $\xi = \pm\omega_0$, it is noted that the distributional limit for $\omega \in \mathbb{R}$ is given by

  $$\begin{aligned}\omega\epsilon(\omega) &= -\frac{1}{\pi}\left(\frac{1}{\omega-\omega_0} + \frac{1}{\omega+\omega_0}\right) + \mathrm{i}\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right) \\ &= \omega\left(-\frac{2}{\pi}\frac{1}{(\omega-\omega_0)(\omega+\omega_0)} + \mathrm{i}\left(\frac{1}{\omega_0}\delta(\omega-\omega_0) - \frac{1}{\omega_0}\delta(\omega+\omega_0)\right)\right),\end{aligned}$$

  where the permittivity function $\epsilon(\omega)$ is identified in the last line.

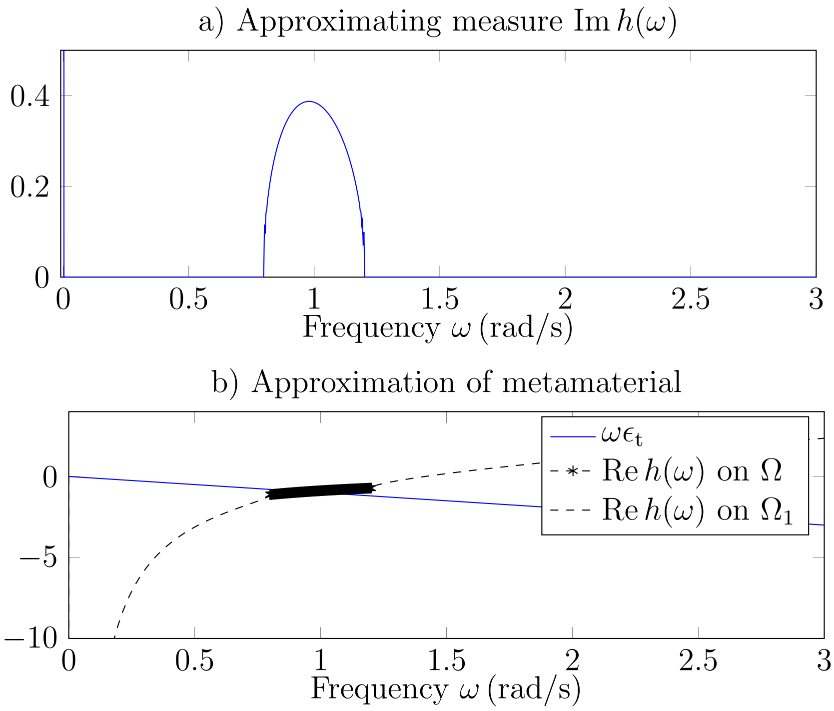

Figure 6: Approximation of metamaterial with target permittivity $\epsilon_{\mathrm{t}} = -1$ and optical response $\epsilon_\infty = 1$. a) The approximating measure $\operatorname{Im} h(\omega)$ based on $N = 1000$ variables. b) The real part $\operatorname{Re} h(\omega)$ on $\Omega = [\omega_2, \omega_3] = \omega_0[1 - \frac{B}{2}, 1 + \frac{B}{2}]$ and $\Omega_1 = [\omega_1, \omega_4]$. Here, $\omega_0 = 1$, $B = 0.4$, $\omega_1 = 0$ and $\omega_4 = 3$.

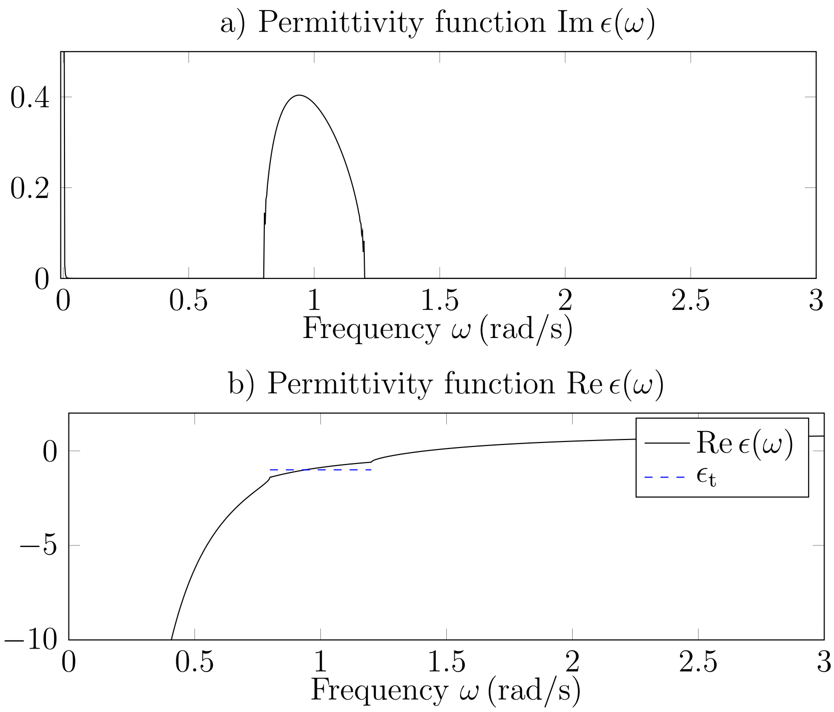

Figure 7: Approximation of metamaterial with target permittivity $\epsilon_{\mathrm{t}} = -1$ and optical response $\epsilon_\infty = 1$. a) The imaginary part $\operatorname{Im} \epsilon(\omega)$ based on $N = 1000$ variables. b) The real part $\operatorname{Re} \epsilon(\omega)$ on $\Omega = [\omega_2, \omega_3] = \omega_0[1 - \frac{B}{2}, 1 + \frac{B}{2}]$ and $\Omega_1 = [\omega_1, \omega_4]$. Here, $\omega_0 = 1$, $B = 0.4$, $\omega_1 = 0$ and $\omega_4 = 3$. The approximation error is $\max_\Omega |\epsilon(\omega) - \epsilon_{\mathrm{t}}| = 0.40$.

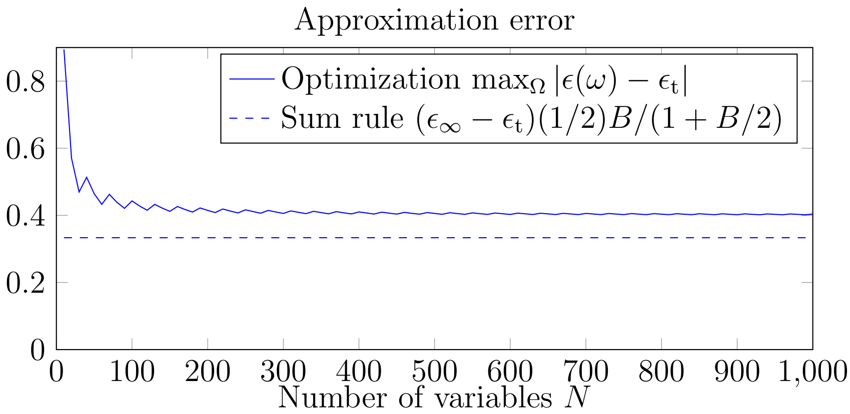

Figure 8: Approximation error $\max_\Omega |\epsilon(\omega) - \epsilon_t|$ as a function of the number of variables N in the optimization problem to approximate a metamaterial with target permittivity ϵ_t over an interval Ω. Here, $\omega_0 = 1$, $B = 0.4$, $\omega_1 = 0$ and $\omega_4 = 3$.

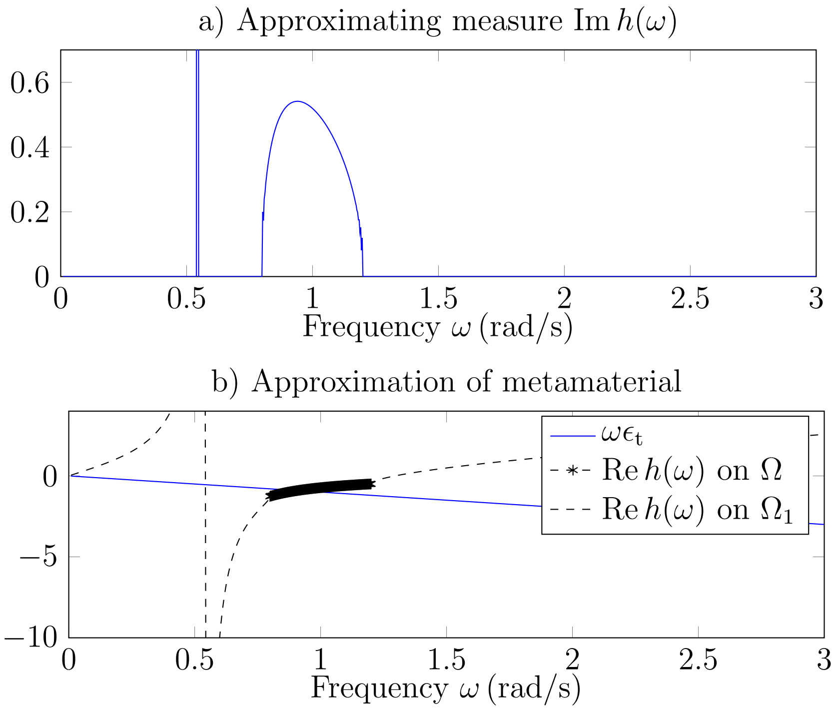

Figure 9: Approximation of metamaterial with target permittivity $\epsilon_t = -1$, optical response $\epsilon_\infty = 1$ and static permittivity $\epsilon_s = 5$. a) The approximating measure $\mathrm{Im}\, h(\omega)$ based on $N = 1000$ variables. b) The real part $\mathrm{Re}\, h(\omega)$ on $\Omega = [\omega_2, \omega_3] = \omega_0[1 - \frac{B}{2}, 1 + \frac{B}{2}]$ and $\Omega_1 = [\omega_1, \omega_4]$. Here, $\omega_0 = 1$, $B = 0.4$, $\omega_1 = 0.01$ and $\omega_4 = 3$.

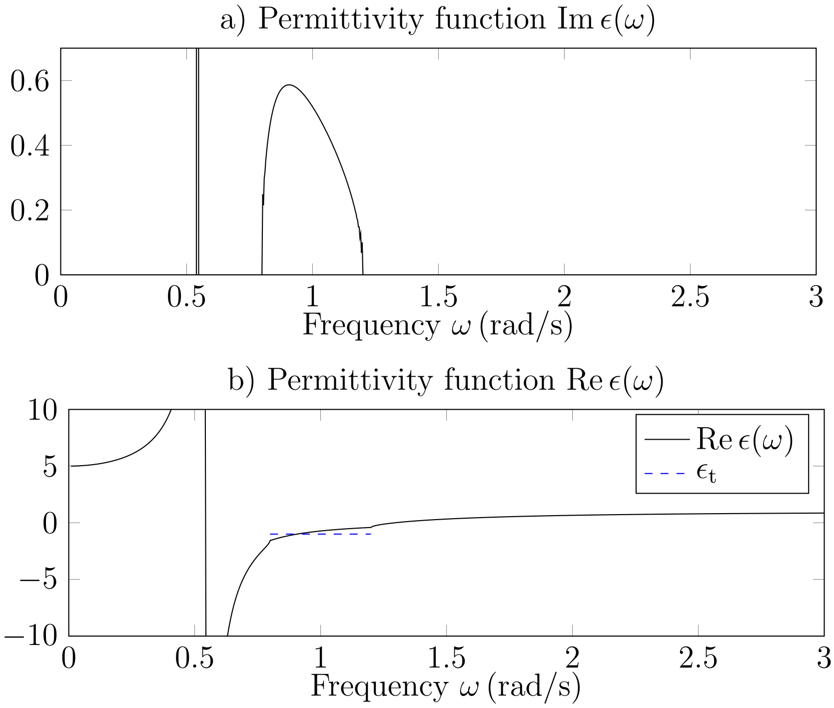

Figure 10: Approximation of metamaterial with target permittivity $\epsilon_{\mathrm{t}} = -1$, optical response $\epsilon_\infty = 1$ and static permittivity $\epsilon_{\mathrm{s}} = 5$. a) The imaginary part $\operatorname{Im}\epsilon(\omega)$ based on $N = 1000$ variables. b) The real part $\operatorname{Re}\epsilon(\omega)$ on $\Omega = [\omega_2, \omega_3] = \omega_0[1 - \frac{B}{2}, 1 + \frac{B}{2}]$ and $\Omega_1 = [\omega_1, \omega_4]$. Here, $\omega_0 = 1$, $B = 0.4$, $\omega_1 = 0.01$ and $\omega_4 = 3$. The approximation error is $\max_\Omega |\epsilon(\omega) - \epsilon_{\mathrm{t}}| = 0.59$.

6.4 Matlab codes

The CVX Matlab software for disciplined convex programming [6] can be downloaded by following this link: http://cvxr.com/cvx/.

6.4.1 A Matlab subroutine to compute $\hat{p}(\omega)$

Below is given a Matlab subroutine to compute the (negative) Hilbert transform $\hat{p}(\omega)$ of the triangular pulse function $p(\omega)$ as defined in (78) and (80).

```
function [ph]=phat(omega,a)

index0=find(omega==0);
indexa=find(omega==a);
indexma=find(omega==-a);

if ~isempty(index0)
omega(index0)=2*a;
end
if ~isempty(indexa)
omega(indexa)=2*a;
end
if ~isempty(indexma)
omega(indexma)=2*a;
end

ph=2*omega.*log(abs(omega))-(omega-a).*log(abs(omega-a))...
-(omega+a).*log(abs(omega+a));
ph=ph/(pi*a);

if ~isempty(index0)
ph(index0)=0;
end
if ~isempty(indexa)
ph(indexa)=-2*log(2)/pi;
end
if ~isempty(indexma)
ph(indexma)=2*log(2)/pi;
end
```

6.4.2 A Matlab code to approximate the function $\tan(\omega)$

Below is given a Matlab code to approximate the function $\tan(\omega)$ based on partial data according to the optimization problem formulated in section 6.1.

```
% Approximation of tan(omega)
clear

% Problem parameters
```

```
omega0=pi; % center frequency
B=0.4; % Relative bandwidth, B<2

% Frequency domain for measure, Omega1
N=200; % Number of optimization variables (measure)
omega1=0; % Lower frequency limit
omega4=10; % Upper frequency limit
a=(omega4-omega1)/(N-1); % Frequency step
n1=round(omega1/a);
n4=round(omega4/a);
Omega1=(n1:n4)'*a; % Frequency domain for measure

% Frequency domain for approximation, Omega
omega2=omega0*(1-B/2); % Lower frequency limit
omega3=omega0*(1+B/2); % Upper frequency limit
n2=round(omega2/a);
n3=round(omega3/a);
Omega=(n2:n3)'*a; % Frequency domain for approximation
M=length(Omega);

figure(1),clf
plot(Omega1,tan(Omega1))
hold on
plot(Omega,tan(Omega),'r*')
axis([0 omega4 -10 10])
hold off

% Build constraint matrices for Omega (Hilbert transform H)
H=zeros(M,N);
for n=0:N-1
    if n+n1==0
    H(:,n+1)=phat(Omega,a);
    else
    H(:,n+1)=phat(Omega-(n+n1)*a,a)+phat(Omega+(n+n1)*a,a);
    end
end

A=H+1i*[zeros(M,n2-n1),eye(M),zeros(M,n4-n3)];

f=tan(Omega);

% Solve convex optimization problem using CVX
cvx_begin
    variable x(N)
    minimize( norm(A*x-f,Inf) )
    subject to
    x>=0
cvx_end
```

```
h=A*x;

% Build complete matrices for Omega1 (Hilbert transform H1)
H1=zeros(N,N);
for n=0:N-1
    if n+n1==0
    H1(:,n+1)=phat(Omega1,a);
    else
    H1(:,n+1)=phat(Omega1-(n+n1)*a,a)+phat(Omega1+(n+n1)*a,a);
    end
end

A1=H1+1i*eye(N);

h1=A1*x;

figure(2),clf
plot(Omega1,tan(Omega1))
hold on
plot(Omega,real(h),'k*--')
plot(Omega1,real(h1),'k--')
axis([0 omega4 -10 10])
hold off
xlabel('Frequency (rad/s)')
ylabel('Herglotz function tan(omega)')
title(['Approximation of Herglotz function tan(omega), N=' ...
 num2str(N)])
legend('tan(omega)','Re h','Re h1')

[sortx,sortindx]=sort(x,'descend');
figure(3)
plot(Omega1,x)
hold on
plot(Omega1(sortindx(1:3)),x(sortindx(1:3)),'o')
hold off
axis([0 omega4 0 max(x)+10])
xlabel('Frequency (rad/s)')
ylabel('Measure Im h')
title(['Approximating measure Im h, N=' num2str(N)])
```

Acknowledgment

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