Topics in Analysis

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Content:

Our lectures are divided into four sections:

1. Review of complex analysis in one variable
   and introduction to complex analysis in several variales.
2. Introduction to the theory of measure and integration.
   Lebesgue measure and integral on the real line.
   Complex measures and the Riesz representation formula.
   Weak convergence of measures.
3. Application of measure theory in complex analysis,
   representation formulas and Herglotz functions.
4. A few concepts from distribution theory.

1. Review of complex analysis in one variable and introduction to several variables

Contents:

1.1 Differentiable maps

Let $X \subset \mathbb{R}^N$ be open, $a \in X$, and $f : X \to \mathbb{R}^M$ be a map.
Definition: The function f is said to be differentiable at a if there exists a linear map $L : \mathbb{R}^N \to \mathbb{R}^M$ such that

$$\frac{\|f(a+V) - f(a) - L(V)\|}{\|V\|} \to 0 \qquad V \to 0.$$

The linear map L is unique. It is called the differential of f at a and is denoted by

$$d_a f \quad \text{or} \quad Df(a).$$

The differential

Denote the variables by $s = (s_1, \ldots, s_N)$, $V = (V_1, \ldots, V_N)$ and write $f = (f_1, \ldots, f_M)$.

The map f is differentiable if and only if each of the real valued functions f_j is differentiable.

Then all the partial derivatives $\partial_k f_j(a)$ exist and the differential $d_a f_j(V)$ acting on the vector V is given as

$$d_a f_j(V) = \sum_{k=1}^{N} \partial_k f_j(a) V_k = \sum_{k=1}^{N} \frac{\partial f_j}{\partial s_k}(a) V_k.$$

Observe that the value of the differential $d_a f(V)$ of the map f is a vector in $\mathbb{R}^M$. We can view it as a function of the point a and the vector V. For fixed a it is linear in V.

Differentials of the coordinate functions

The coordinate functions $s \mapsto s_k$ have the differential $d_a s_k(V) = V_k$ at very point a in $\mathbb{R}^N$.
Since they are independent of a we denote them by ds_k.

Hence the coordinates of the linear map $d_a f$ are

$$d_a f_j = \sum_{k=1}^{N} \frac{\partial f_j}{\partial s_k}(a) ds_k.$$

We write

$$d_a f = (d_a f_1, \ldots, d_a f_M).$$

Definition The map f is said to be continuously differentiable if it is differentiable and the partial derivatives $\partial f_j / \partial s_k$ are continuous functions on X.
We let $C^1(X, \mathbb{R}^M)$ denote the set of all continuously differentiable maps on X with values in $\mathbb{R}^M$.
We write $C^1(X)$ in the special case $M = 2$ and view the elements as the complex valued functions.

Differentiable functions of two variables

Now we identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$.

Let $X \subset \mathbb{R}^2 = \mathbb{C}$, $a \in X$, denote the real coordinates by (x, y) and introduce the complex coordinates $z = x + iy$, $\bar{z} = x - iy$.

We view a function $f : X \to \mathbb{C}$ as a map $f = u + iv : X \to \mathbb{R}^2$.

$$\begin{aligned} d_a f &= d_a u + i d_a v \\ &= \left(\frac{\partial u}{\partial x}(a)dx + \frac{\partial u}{\partial y}(a)dy\right) + i\left(\frac{\partial v}{\partial x}(a)dx + \frac{\partial v}{\partial y}(a)dy\right) \\ &= \left(\frac{\partial u}{\partial x}(a) + i\frac{\partial v}{\partial x}(a)\right)dx + \left(\frac{\partial u}{\partial y}(a) + i\frac{\partial v}{\partial y}(a)\right)dy \\ &= \frac{\partial f}{\partial x}(a)dx + \frac{\partial f}{\partial y}(a)dy. \end{aligned}$$

We have $dz = dx + idy$, $d\bar{z} = dx - idy$, which implies that

$$dx = \tfrac{1}{2}\big(dz + d\bar{z}\big) \qquad \text{and} \qquad dy = \tfrac{1}{2i}\big(dz - d\bar{z}\big)$$

We continue our calculation

$$\begin{aligned} d_a f &= \frac{\partial f}{\partial x}(a)dx + \frac{\partial f}{\partial y}(a)dy \\ &= \frac{\partial f}{\partial x}(a)\frac{1}{2}\big(dz + d\bar{z}\big) + \frac{\partial f}{\partial y}(a)\frac{1}{2i}\big(dz - d\bar{z}\big) \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x}(a) + \frac{1}{i}\frac{\partial f}{\partial y}(a)\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x}(a) - \frac{1}{i}\frac{\partial f}{\partial y}(a)\right)d\bar{z} \end{aligned}$$

Wirtinger derivatives

The *Wirtinger derivatives* of the function f are defined as

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

The *Wirtinger operators* are the corresponding partial differential operators.

We conclude that

$$d_a f = \frac{\partial f}{\partial x}(a)dx + \frac{\partial f}{\partial y}(a)dy = \frac{\partial f}{\partial z}(a)dz + \frac{\partial f}{\partial \bar{z}}(a)d\bar{z}.$$

Generalization to higher dimensions

Assume now that X is an open subset of $\mathbb{C}^n$. Every function of n complex variables $z = (z_1, \ldots, z_n)$, where $z_j = x_j + iy_j$, can be considered as a function of $N = 2n$ variables $s = (x_1, y_1, \ldots, x_n, y_n)$ and the differential is

$$\begin{aligned} d_a f &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a) dx_j + \frac{\partial f}{\partial y_j}(a) dy_j \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(a) dz_j + \frac{\partial f}{\partial \bar{z}_j}(a) d\bar{z}_j \\ &= \partial_a f + \bar{\partial}_a f, \end{aligned}$$

$$\partial_a f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(a) dz_j \qquad \text{and} \qquad \bar{\partial}_a f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}(a) d\bar{z}_j.$$

1.2 Analytic (holomorphic) functions

$\mathbb{C}$-differentiable functions:

Let X be a subset of $\mathbb{C}$ and $f = u + iv$ be a function.

Definition: The function f is said to be $\mathbb{C}$ differentiable at the point $a \in X$ if the limit

$$\lim_{\mathbb{C} \ni h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The limit is denoted by $f'(a)$ and is called the $\mathbb{C}$ *derivative of the function f at the point a.*

Observe that

$$\lim_{\mathbb{C} \ni h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0.$$

This shows that if f is $\mathbb{C}$ differentiable at a then f is differentiable and $d_a f(h) = f'(a)h$.

Conversely, if f is differentiable at a and $d_a f(h) = Ah$ for some complex number A, then f is $\mathbb{C}$ differentiable and $f'(a) = A$.

Cauchy-Riemann equation

If f is $\mathbb{C}$ differentiable at a, then

$$f'(a) = \lim_{\mathbb{R}\ni h\to 0} \frac{f(a+h)-f(a)}{h} = \frac{\partial f}{\partial x}(a)$$

and

$$f'(a) = \lim_{\mathbb{R}\ni h\to 0} \frac{f(a+ih)-f(a)}{ih} = \frac{1}{i}\frac{\partial f}{\partial y}(a).$$

Hence we conclude that f satisfies the *Cauchy-Riemann equation*

$$\frac{\partial f}{\partial \bar{z}}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(a) - \frac{1}{i}\frac{\partial f}{\partial y}(a)\right) = 0.$$

Conversely, if f is differentiable at a and $\frac{\partial f}{\partial \bar{z}}(a) = 0$, then for $h \in \mathbb{C}$

$$d_a f(h) = \frac{\partial f}{\partial z}(a)h,$$

which implies

$$\frac{f(a+h) - f(a)}{h} - \frac{\partial f}{\partial z}(a) = \frac{f(a+h) - f(a) - d_a f(h)}{h} \to 0$$

as $h \to 0$ and we conclude that f is $\mathbb{C}$ differentiable at a with

$$f'(a) = \frac{\partial f}{\partial z}(a).$$

Cauchy-Riemann equations

If we write $f(z) = u(x, y) + iv(x, y)$, then

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}}(a) &= \frac{1}{2}\left(\frac{\partial f}{\partial x}(a) - \frac{1}{i}\frac{\partial f}{\partial y}(a)\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x}(a) - \frac{\partial v}{\partial y}(a)\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y}(a) + \frac{\partial v}{\partial x}(a)\right)\end{aligned}$$

Hence the Cauchy-Riemann equation is equivalent to the system of equations

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a).$$

They are called *Cauchy-Riemann equations*.

Analytic (holomorphic) functions

Let X be an open subset of $\mathbb{C}^n$ (where of course $\mathbb{C}^1 = \mathbb{C}$).

Definition:

A function $f \in C^1(X)$ is said to be is *analytic* if the Cauchy-Riemann-equations are satisfied:

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \qquad j = 1, \ldots, n.$$

The set of all analytic functions on X is denoted by $\mathcal{O}(X)$.

Hartogs theorem

It is actually possible to give a weaker definition:

Theorem (Hartogs 1906): Let X be an open subset of $\mathbb{C}^n$ and $f : X \to \mathbb{C}$ be a function, and assume that for every point $a \in X$ and every j the function

$$\zeta \mapsto f(a_1, \ldots, a_{j-1}, \zeta, a_{j+1}, \ldots, a_n)$$

is analytic in a neighbourhood of a_j, then $f \in \mathcal{O}(X)$

Observe: There are no a priori regularity conditions on the function f, like differentiability or continuity.

1.3 The Cauchy formula

Path integrals

Let X be a domain in $\mathbb{C}$ and $f \in C^1(X)$ and $\gamma : [a, b] \to X$, $\gamma(t) = \alpha(t) + i\beta(t)$ be a piecewise smooth path in X.
We define four types of path integrals

$$\int_\gamma f\,dx = \int_a^b f(\gamma(t))\alpha'(t)\,dt,$$

$$\int_\gamma f\,dy = \int_a^b f(\gamma(t))\beta'(t)\,dt,$$

$$\int_\gamma f\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_\gamma f\,dx + i\int_\gamma f\,dy = \int_\gamma f\,(dx + idy),$$

$$\int_\gamma f\,d\bar{z} = \int_a^b f(\gamma(t))\overline{\gamma'(t)}\,dt = \int_\gamma f\,dx - i\int_\gamma f\,dy = \int_\gamma f\,(dx - idy).$$

Boundary integrals

Let X be a domain in $\mathbb{C}$ and $f, g \in C^1(X)$.
Let $\Omega \subset X$ be a bounded domain with boundary $\partial\Omega \subset X$ which can be parametrized by a number of simple piecewise smooth curves $\gamma_j = \alpha_j + i\beta_j : [a_j, b_j] \to X$, $j = 1, \dots, N$ with positive orientation.

Then we define the boundary integral

$$\int_{\partial\Omega} f\,dx + g\,dy = \sum_{j=1}^{N} \int_{\gamma_j} f\,dx + g\,dy$$

We allow the functions to be complex valued, so in general the values of the integrals are complex numbers.

The Green theorem

Theorem (Green):

$$\int_{\partial\Omega} f\, dx + g\, dy = \iint_{\Omega} (\partial_x g - \partial_y f)\, dxdy.$$

It is usually proved for real valued functions, but if we apply it for real and imaginary parts separately, then we conclude that it even holds for complex valued functions.

The Cauchy theorem follows from the Green theorem

The Green theorem gives

$$\begin{aligned}\int_{\partial\Omega} f\,dz &= \int_{\partial\Omega} f\,(dx + i dy)\\ &= \int_{\partial\Omega} f\,dx + if\,dy\\ &= \iint_{\Omega} \left(i\partial_x f - \partial_y f\right) dxdy\\ &= i\iint_{\Omega} \left(\partial_x f + i\partial_y f\right) dxdy\\ &= 2i\iint_{\Omega} \partial_{\bar z} f\,dxdy.\end{aligned}$$

Theorem (Cauchy):

For every $f \in \mathcal{O}(X)$ we have

$$\int_{\partial\Omega} f\,dz = 0.$$

Theorem (Cauchy-Pompeiu formula):

For every $f \in C^1(X)$ and every $z \in \Omega$ that

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi} \iint_{\Omega} \frac{\partial_{\bar{\zeta}} f(\zeta)}{\zeta - z}\, d\xi d\eta,$$

where the complex variable in the second integral is $\zeta = \xi + i\eta$.

Theorem(Cauchy formula):

If $f \in \mathcal{O}(X)$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

The Cauchy formula for derivatives

Assume now that $\partial\Omega$ is parametrized by the paths $\gamma_j : [a_j, b_j] \to \mathbb{C}$, $j = 1, \dots, N$. If we apply the Cauchy formula and parametrize the integrals, we get

$$f(x+iy) = \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{a_j}^{b_j}\frac{f(\gamma_j(t))}{\gamma_j(t)-x-iy}\gamma_j'(t)\,dt, \qquad f \in \mathcal{O}(X).$$

We may differentiate f by taking derivatives under the integral sign,

$$f'(z) = \partial_x f(z) = \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{a_j}^{b_j}\frac{f(\gamma_j(t))}{(\gamma_j(t)-x-iy)^2}\gamma_j'(t)\,dt.$$

We see that f' is an analytic function in X and with the same argument we may differentiate under the integral sign again and get

$$f''(z) = \partial_x^2 f(z) = \frac{2}{2\pi i} \sum_{j=1}^{N} \int_{a_j}^{b_j} \frac{f(\gamma_j(t))}{(\gamma_j(t) - x - iy)^3} \gamma_j'(t)\, dt.$$

Continuing in this way we see that f is infinitely differentiable and each derivative is analytic.

By induction we define higher $\mathbb{C}$ derivatives $f^{(n)}$ of f by

$$f^{(0)} = f, \qquad f^{(n)} = \big(f^{(n-1)}\big)', \quad n \geq 1.$$

and arrive at:

Theorem (Cauchy formula for derivatives):

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Power series expansions

Let $f \in \mathcal{O}(X)$, $r > 0$ and assume that the closed disc $\overline{D}(a,r) \subset X$. Then the Cauchy formula gives

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad z \in D(a,r).$$

Observe that

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{\zeta - a} \cdot \frac{1}{1 - (z-a)/(\zeta - a)} \\ &= \frac{1}{\zeta - a} \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^n} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}} \end{aligned}$$

We put the series into the Cauchy formula and interchange the order of the sum and the integral

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta \right) (z-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

1.4 The Cauchy-formula in a polydisc

Multi-index notation

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers including 0. A *multi-index* is an element

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n.$$

For each multi-index α we define the length of α by

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

the factorial of α by,

$$\alpha! = \alpha_1! \cdots \alpha_n!,$$

for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ the *monomial*

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$

and the differential operator ∂^α by

Polydics

A set of the form

$$D = D(a, r) = D(a_1, r_1) \times \cdots \times D(a_n, r_n)$$

where $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $r = (r_1, \ldots, r_n) \in (\mathbb{R}_+^*)^n$ is called a *polydisc* with *center* a and *multi-radius* r.
The distinguished boundary of D is the product of the boundaries of the discs

$$\partial_0 D = \partial D(a_1, r_1) \times \cdots \times \partial D(a_n, r_n).$$

A Cauchy formula in a polydisc

Let $f \in \mathcal{O}(X)$ be an analytic function on an open subset X of $\mathbb{C}^n$, for $n \geq 2$, and assume that $\overline{D} \subset X$. By the Cauchy formula in the first variable gives

$$f(z) = \frac{1}{(2\pi i)} \int_{\partial D(a_1, r_1)} \frac{f(\zeta_1, z_2, \dots, z_n)}{(\zeta_1 - z_1)} \, d\zeta_1, \qquad z = (z_1, \dots, z_n) \in D.$$

We iterate this formula and get the Cauchy formula

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n, \qquad z \in D.$$

A Cauchy formula for partial derivatives in a polydisc

From the Cauchy formula we see that f is infinitely differentiable each partial derivative is analytic, and by differentiating with respect to the variables z_j we get:

Theorem (Cauchy formula for partial derivatives):

$$\partial^\alpha f(z) = \frac{\alpha!}{(2\pi i)^n} \int_{\partial_0 D} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1)^{\alpha_1+1} \cdots (\zeta_n - z_n)^{\alpha_n+1}} \, d\zeta_1 \cdots d\zeta_n.$$

Power series expansion:

In the Cauchy formula each of the factors has a power series expansion

$$\frac{1}{\zeta_j - z_j} = \sum_{\alpha_j \in \mathbb{N}} \frac{(z_j - a_j)^{\alpha_j}}{(\zeta_j - a_j)^{\alpha_j + 1}}$$

We can expand this in a power series

$$\frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} = \sum_{\alpha \in \mathbb{N}^n} \frac{(z - a)^{\alpha}}{(\zeta - a)^{\alpha + 1}}$$

where $\alpha + 1 = (\alpha_1 + 1, \ldots, \alpha_n + 1)$.
In the same way as above we arrive at the power series expansion

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} \frac{\partial^{\alpha} f(a)}{\alpha!} (z - a)^{\alpha}, \qquad z \in D.$$

1.5 Domains of holomorphy

Holomorphic maps:

Let X be open in $\mathbb{C}^n$. A map $F = (F_1, \dots, F_m) : X \to \mathbb{C}^m$ is said to be holomorphic if $F_j \in \mathcal{O}(X)$ for all j.

If $m = n$ and F is bijective onto its image $Y = F(X)$, then we say that F is biholomorphic.

We say that two domains X and Y are biholomorphically equivalent if there exists a biholomorphic map on X with $Y = F(X)$.

The Riemann Mapping Theorem:

If X is a simply connected domain in $\mathbb{C}$, $\varnothing \neq X$ and $X \neq \mathbb{C}$, then X is biholomorphically equivalent to $\mathbb{D}$ the unit disc in $\mathbb{C}$.

This does not generalize to higher dimensions:

Theorem: If $n > 1$, then the unit ball

$$\mathbb{B}_n = \{z \in \mathbb{C}^n ;\ |z_1|^2 + \cdots + |z_n|^n < 1\}$$

in $\mathbb{C}^n$ is not biholomorphically equivalent to a polydisc

Domains of holomorphy

With the aid of the Weierstrass product theorem it is possible to show that for every open X in $\mathbb{C}$ there exits $f \in \mathcal{O}(X)$, which *can not* be extended to a holomorphic function in a neighbourhood of a boundary point of X.

Theorem (Hartogs 1906): Let X be an open subset of $\mathbb{C}^n$, $n > 1$, K be a compact subset of X such that $X \setminus K$ is connected. Then every function $f \in \mathcal{O}(X \setminus K)$ extends uniquely to a function $F \in \mathcal{O}(X)$.

We have a little bit technical definition:

Definition: An open set X is said to be a *domain of holomorphy* if there *do not exist* non-empty open sets X_1 and X_2 with X_2 is connected, $X_2 \not\subset X$, and $X_1 \subset X \cap X_2$, such that for every $f \in \mathcal{O}(X)$ there exists $F \in \mathcal{O}(X_2)$ such that $f = F$ on X_1.

There exist many equivalent characterizations of domains of holomorphy: (Oka, Cartan, Bremermann, and Norguet, c.a. 1937-1954).

1.6 Integral representation formulas in several variables

Cauchy-Fantappiè-Leray formula

Let Ω be a domain in X with smoooth boundary $\partial\Omega$ in X and assume that Ω is defined by the function ϱ in the sense that

$$\Omega = \{z \in X\,;\ \varrho(z) < 0\}$$

and the gradient

$$\varrho' = (\partial\varrho/\partial z_1, \dots, \partial\varrho/\partial z_n)$$

is non-zero at every boundary point.

Then for every $f \in \mathcal{O}(X)$ we have

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial\Omega} \frac{f(\zeta)\, \partial\varrho \wedge (\bar\partial\partial\varrho)^{n-1}}{\langle \varrho'(\zeta), \zeta - z\rangle^n}, \qquad z \in \Omega.$$

1.7 Complex analysis and potential theory

Let X be and open subset of $\mathbb{R}^n$.

The Laplace operator:

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

Definition:

A function $u : X \to \mathbb{R}$ is said to be *harmonic* if it is in $C^2(X)$ and satisfies the *Laplace equation*,

$$\Delta u = 0.$$

We let $\mathcal{H}(X)$ denote the set of all harmonic funtions on X.

Properties of harmonic functions

(i) The set $\mathcal{H}(X)$ is a real vector space.

(ii) Every function $u \in \mathcal{H}(X)$ satiesfies the mean value property,

$$u(a) = \frac{1}{c_n r^{n-1}} \int_{\partial B(a,r)} u(y)\, dS(y),$$

where c_n is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

(iii) In two variables the Laplace operator is

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

This implies that $\mathrm{Re} f, \mathrm{Im} f \in \mathcal{H}(X)$ for every $f \in \mathcal{O}(X)$.

Conversely, *if* X is simply connected then every $u \in H(X)$ is of the form $u = \mathrm{Re} f$ for some $f \in \mathcal{O}(X)$.

Subharmonic functions

Definition

A function $u : X \to \mathbb{R} \cup \{0\}$, where X is an open subset of $\mathbb{R}^n$ is said to be *subharmonic* if

(i) u is upper-semicontinuous, which means that it can be written as a limit of a decreasing sequence of continuous functions, and

(ii) u satisfies the *sub-mean value property*, which means that for every $a \in X$ such that $\overline{B}(a, r) \subset X$ we have

$$u(a) \leq \frac{1}{c_n r^{n-1}} \int_{\partial B(a,r)} u(y)\, dS(y).$$

We denote the set of all subharmonic functions on X by $\mathcal{SH}(X)$.

Properties of subharmonic functions:

Elementary properties:

- Every convex function $u : X \to \mathbb{R}$ is subharmonic.
- If $u \in \mathcal{SH}(X)$ and $\varphi : I \to \mathbb{R}$ is convex and increasing on an interval I containing the image of u, then $\varphi \circ u \in \mathcal{SH}(X)$.

Theorem (The maximum principle)

An upper semi-continuous function u on X is subharmonic if and only if it has the property:
For every compact $K \subset X$ and $h \in C(K) \cap \mathcal{H}(\text{int}K)$, with $u \leq h$ on ∂K it follows $u \leq h$ on K.

Theorem

A function $u \in C^2(X)$ is subharmonic if and only if

$$\Delta u \geq 0.$$

Subharmonic functions in the plane

Assume now that $X \subset \mathbb{R}^2 = \mathbb{C}$.

Examples:

- Convex functions on convex domains X.
- $\varphi = \log|f|$, $f \in \mathcal{O}(X)$.
- $\varphi = \log(|f_1|^{p_1} + \cdots + |f_m|^{p_m})$, $f_j \in \mathcal{O}(X)$, $p_j \geq 0$.

The sub-mean value property:

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta, \qquad a \in X, 0 < r < d(a, \partial X)$$

Plurisubharmonic functions

Let X be an open subset of $\mathbb{C}^n$.

Definition:

A function $u : X \to \mathbb{R} \cup \{-\infty\}$, where X is said to be *plurisubharmonic* if

(i) u is upper semicontinuous, and

(ii) for every point $a \in X$ and every $w \in \mathbb{C}^n$ the function

$$\Omega_{a,w} \ni \tau \mapsto u(a + \tau w)$$

is subharmonic in the open subset

$$\Omega_{a,w} = \{\tau \in \mathbb{C}\,;\, a + \tau w \in X\}.$$

We denote the set of all plurisubharmonic functions on X by $\mathcal{PSH}(X)$.

Remark: On can say that an upper semicontinuous function is said to plurisubharmonic if it is subharmonic along every complex line.

Theorem

1.8 Existence theory for the Cauchy Riemann system

The inhomogeneous Cauchy-Riemann equations

Assume that we have a function $g \in C^{\infty}(X)$ and that we want to modify g, so that it becomes holomorphic, i.e., we want to find $u \in C^{\infty}(X)$ such that

$$F = g - u \in \mathcal{O}(X).$$

Then u has to satisfy the inhomogeneous Cauchy-Riemann equations

$$\frac{\partial u}{\partial \bar{z}_j} = f_j, \qquad \text{with} \qquad f_j = \frac{\partial g}{\partial \bar{z}_j}$$

Hörmander's existence theorem

Let X be a domain of holomorphy, $f_j \in C^1(X)$ be functions on X satisfying

$$\frac{\partial f_j}{\partial \bar{z}_k} = \frac{\partial f_k}{\partial \bar{z}_j}.$$

and let $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ be a *plurisubharmonic function*. Then there exists a function u on X satisfying the Cauchy-Riemann equations

$$\frac{\partial u}{\partial \bar{z}_j} = f_j,$$

with the L^2-estimate of u in terms of $f = (f_1, \ldots, f_n)$,

$$\int_X |u|^2 (1 + |z|^2)^{-2} e^{-\varphi} \, d\lambda \leq \int_X |f|^2 e^{-\varphi} \, d\lambda$$

References

Robert E. Greene and Steven G. Krantz *Function theory of one complex variable.* 3rd ed., Graduate Studies in Mathematics 40. AMS, Providence, RI, 2006.

Lars Hörmander, *An introduction to complex analysis in several variables*, 3rd ed. North-Holland Mathematical Library, 7. North-Holland Publishing Co., Amsterdam, 1990.

Steven G. Krantz *Function theory of several complex variables.* 2nd ed., AMS Chelsea Publishing, Providence, RI, 2001.

2. Measure and integration

Summary:

We are going to give a meaning to expressions like

$$\int_X f \, d\mu$$

which we read as the integral of the function f on the set X with respect to the measure μ.

The steps in this process are the following:

(i) On the set X we have a σ-algebra $\mathcal{X}$ of subsets.

(ii) We look at functions $f : X \to \mathbb{R}$ such that for every $a, b \in \mathbb{R}$

$$\{x \in X \, ; \; f(x) \in]a, b[\} \in \mathcal{X}$$

and call them $\mathcal{X}$-measurable functions.

(iii) We look at characteristic functions χ_A of subsets A of X,

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in X \setminus A. \end{cases}$$

and observe that they are measurable if and only if $A \in \mathcal{X}$.

(iv) On the σ-algebra $\mathcal{X}$ we define the measure μ.

(v) We define the integral of the $\mathcal{X}$-measurable functions in such a way that

$$f \mapsto \int_X f \, d\mu$$

is a linear operation over $\mathbb{R}$ and

$$\int_X \chi_A \, d\mu = \mu(A).$$

(vi) We extend the integral to complex valued functions so that the operation of integrating functions becomes $\mathbb{C}$-linear.

Motivation for the integrals:

The Riemann integral is incomplete and too strict

It is hard to tell when we may shift the operations like:

- $\displaystyle\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx.$
- $\displaystyle\sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx = \int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx.$
- $\displaystyle\lim_{t\to t_0}\int_a^b f(x,t)\,dx = \int_a^b \lim_{t\to t_0} f(x,t)\,dx.$
- $\displaystyle\frac{\partial}{\partial t}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx.$

and this is even worse for intergrals over infinite intervals.

A larger class of integrable functions is needed and a more flexible definition of integral!

Motivation for measures

- We need a uniform approach to deal with concepts like, length, area, and volume and treat them as functions which associate to sets A positive real value which interpreted as length, area or volume.
- We need a uniform approach to the concept of probability.

The solution to the first problem is to introduce the Lebesgue measure on $\mathbb{R}^n$. (Lebesgue, 1904.)

The solution to the second problem is the introduction of a probability space $(\Omega, \mathcal{F}, P)$ where

- Ω is a set and its elements ar called *outcomes*.
- $\mathcal{F}$ is a collection of subsets of Ω and its elements are called *events*.
- P is a function which associates to every event E a number in $P(E) \in [0, 1]$, the probability of the event E.

(Kolmogorov, 1932.)

2.1 Sets, algebras and σ-algebras

Let X be any set and let $\mathcal{P}(X)$ be the power set of X consisting of all subsets of X. For every $A \in \mathcal{P}(X)$ we let A^c and $X \setminus A$ denote the complement of A which is defined as the set of all elements in X that are not in A,

$$A^c = \{x \in X \, ; \, x \notin A\}.$$

If A and B are subsets of X, then we define their union by

$$A \cup B = \{x \in X \, ; \, x \in A \text{ or } x \in B\}$$

and their intersection by

$$A \cap B = \{x \in X \, ; \, x \in A \text{ and } x \in B\}.$$

… more generally

if I is a set and $A_\alpha \in \mathcal{P}(X)$ for all $\alpha \in I$, then we define the union of $(A_\alpha)_{\alpha \in I}$ by

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X\,;\, x \in A_\alpha \text{ for some } \alpha \in I\}$$

and the intersection of $(A_\alpha)_{\alpha \in I}$ as

$$\bigcap_{\alpha \in I} A_\alpha = \{x \in X\,;\, x \in A_\alpha \text{ for all } \alpha \in I\}.$$

In particular, for $I = \varnothing$ we have

$$\bigcup_{\alpha \in \varnothing} A_\alpha = \varnothing \qquad \text{and} \qquad \bigcap_{\alpha \in \varnothing} A_\alpha = X.$$

De Morgan's rules

for the complements of unions and intersections are

$$\Big(\bigcup_{\alpha\in I} A_\alpha\Big)^c = \bigcap_{\alpha\in I} A_\alpha^c \qquad \text{and} \qquad \Big(\bigcap_{\alpha\in I} A_\alpha\Big)^c = \bigcup_{\alpha\in I} A_\alpha^c$$

for any set I.

Algebras of sets

Definition: Let X be a set and let $\mathcal{X} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. We say that $\mathcal{X}$ is an *algebra of subsets of X* if it satisfies

(i) $\varnothing \in \mathcal{X}$.

(ii) If $A \in \mathcal{X}$, then $A^c \in \mathcal{X}$.

(iii) If $A, B \in \mathcal{X}$, then $A \cup B \in \mathcal{X}$.

An immediate consequence of (i) and (ii) is that $X \in \mathcal{X}$. From (iii) it follows by induction that if $A_1, \ldots, A_n \in \mathcal{X}$ then $\bigcup_{j=1}^n A_j \in \mathcal{X}$ and de Morgan's rules give that

$$\bigcap_{j=1}^{n} A_j = \Big(\bigcup_{j=1}^{n} A_j^c\Big)^c \in \mathcal{X}.$$

σ-algebras

Defintion Let X be a set and $\mathcal{X} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. We say that $\mathcal{X}$ is a *σ-algebra on X* if it is an algebra of subsets of X and satisfies

(iii)′ If $A_j \in \mathcal{X}$, for $j = 1, 2, 3, \dots$, then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{X}$.

A pair $(X, \mathcal{X})$ where X is a set and $\mathcal{X}$ is a σ- algebra on X is called a *measurable space*.
The sets in $\mathcal{X}$ are called *measurable sets*.

Observe: (iii)′ implies (iii), for if $A, B \in \mathcal{X}$ and we set $A_1 = A, A_2 = B$, and $A_j = \emptyset$ for $j \geq 3$, then $A \cup B = \bigcup_{j=1}^{\infty} A_j \in \mathcal{X}$. We also have for every $A_j \in \mathcal{X}, j = 1, 2, \dots$

$$\bigcap_{j=1}^{\infty} A_j = \Big(\bigcup_{j=1}^{\infty} A_j^c\Big)^c \in \mathcal{X}.$$

Natural σ-algebras

(i) On every set X we have the smallest σ-algebra, $\{\varnothing, X\}$, and the largest σ- algebra, $\mathcal{P}(X)$.

(ii) If $\{\mathcal{X}_\alpha\}_{\alpha \in I}$ is a collection of σ-algebras on X, then $\bigcap_{\alpha \in I} \mathcal{X}_\alpha$ is a σ-algebra on X. A similar result holds for algebras of subsets of X.

(iii) If $\mathcal{A} \subseteq \mathcal{P}(X)$ is any collection of subsets of X, then the intersection of all σ-algebras containing $\mathcal{A}$, is a σ-algebra and it is the smallest σ-algebra containing $\mathcal{A}$. We call it the *σ-algebra generated by $\mathcal{A}$* and denote it by $\mathcal{A}^\sigma$.

(iv) The *Borel algebra* on $\mathbb{R}^n$ is the smallest σ-algebra containing all the open subsets.
We can write it as $\mathcal{B}_{\mathbb{R}^n} = \mathcal{A}^\sigma$, where
$\mathcal{A} = \{B(a, r)\,;\, a \in \mathbb{R}^n, r > 0\}$.
The class $\mathcal{A}$ can be chosen in various other ways, for example as the set of all closed n-rectangles.

2.2 Measures

A collection $\{A_\alpha\}_{\alpha \in I}$ of sets is said to be *disjoint* if $A_\alpha \cap A_\beta = \varnothing$ for $\alpha \neq \beta$.

Definition Let X be a set and let $\mathcal{X}$ be an algebra of subsets of X. A *measure* on $\mathcal{X}$ is a function $\mu : \mathcal{X} \to [0, +\infty] = \overline{\mathbb{R}}_+$ satisfying

(i) $\mu(\varnothing) = 0$ and

(ii) if $(A_j)_{j=1}^\infty$ is a disjoint collection in $\mathcal{X}$ and $\bigcup_{j=1}^\infty A_j \in \mathcal{X}$, then

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j).$$

If $\mathcal{X}$ is a σ-algebra on X and μ is a measure on $\mathcal{X}$, then the triple $(X, \mathcal{X}, \mu)$ is called a *measure space*.

The measure μ is said to be finite if $\mu(X) < +\infty$ and it is said to be *σ-finite* if there exist $X_j \in \mathcal{X}$ for $j = 1, 2, 3, \ldots$ such that $X = \bigcup_{j=1}^{\infty} X_j$ and $\mu(X_j) < +\infty$.
The measure μ is called a *probability measure* if $\mathcal{X}$ is a σ-algebra and $\mu(X) = 1$.
A set $E \in \mathcal{X}$ is called a *null set* or a *μ-null set* if $\mu(E) = 0$.
If $\mathcal{X}$ is a σ-algebra then the measure space $(X, \mathcal{X}, \mu)$ is said to be *complete* if every subset of a null set is measurable, i.e. if $E \in \mathcal{X}$, $\mu(E) = 0$, and $F \subseteq E$, then $F \in \mathcal{X}$.

Remarks: (i) If $\mathcal{X}$ is a σ-algebra, the condition $\bigcup_{j=1}^{\infty} A_j \in \mathcal{X}$ is superfluous.

(ii) If $(A_j)_{j=1}^{n}$ is a finite collection of sets in $\mathcal{X}$ which are disjoint, then we set $A_j = \varnothing$ for $j > n$ and get $\bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{\infty} A_j \in \mathcal{X}$ and

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

Examples of measures

Our main example will be the Lebesgue measure defined on the Lebesgue algebra on the real line. It takes quite a lot of work until we get there.

Examples: (i) The *counting measure* on a set X is

$$\tau_X : \mathcal{P}(X) \to [0, +\infty], \qquad \tau_X(A) = \#A = \text{number of elements in } A,$$

(ii) The *Dirac measure at the point a* in a set $X \neq \varnothing$ is

$$\delta_a : \mathcal{P}(X) \to [0, +\infty], \qquad \delta_a(A) = \chi_A(a) = \begin{cases} 1, & a \in A, \\ 0, & a \notin A, \end{cases} \qquad A \in \mathcal{P}(X).$$

(iii) Let X be any set, $f : X \to [0, +\infty]$ be function, and set $\mathcal{X} = \mathcal{P}(X)$. Then

$$\mu(A) = \sum_{x \in A} f(x) = \sup \left\{ \sum_{x \in B} f(x) \, ; \, B \subseteq A, B \text{ finite} \right\}, \qquad A \in \mathcal{P}(X),$$

is a measure on $\mathcal{X}$. Here the sum over the empty set has to be taken as 0. If f is the constant function 1, then μ is the counting measure on X. If $f = \chi_{\{a\}}$, i.e. $f(x) = 1$ for $x = a$ and $f(x) = 0$

Some properties of measures

Theorem: Let $(X, \mathcal{X}, \mu)$ be a measure space.

(i) If $A, B \in \mathcal{X}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$ and if $\mu(A) < +\infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(ii) If $(A_j)_{j=1}^{\infty}$ is an increasing sequence in $\mathcal{X}$ then

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} \mu(A_j).$$

(iii) If $(A_j)_{j=1}^{\infty}$ is a decreasing sequence in $\mathcal{X}$ and $\mu(A_1) < +\infty$, then

$$\mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} \mu(A_j).$$

(iv) If $(A_j)_{j=1}^{\infty}$ is a sequence in $\mathcal{X}$, then

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j).$$

2.3 Measurable maps and measurable functions

Image and preimage

Let X and Y be sets and $f : X \to Y$ be a map. Then f generates two maps, the *image map*

$$f : \mathcal{P}(X) \to \mathcal{P}(Y), \qquad f(A) = \{y \in Y\,;\, y = f(x),\, x \in A\},$$

and the *preimage map*

$$f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X), \qquad f^{-1}(B) = \{x \in X\,;\, f(x) \in B\}.$$

If $\mathcal{A} \subseteq \mathcal{P}(X)$, then we define the *image* $f(\mathcal{A}) \subseteq \mathcal{P}(Y)$ of $\mathcal{A}$ by

$$f(\mathcal{A}) = \{f(A)\,;\, A \in \mathcal{A}\}$$

and if $\mathcal{B} \subseteq \mathcal{P}(Y)$, then we define the *preimage* $f^{-1}(\mathcal{B}) \subseteq \mathcal{P}(X)$ of $\mathcal{B}$ by

$$f^{-1}(\mathcal{B}) = \{f^{-1}(B)\,;\, B \in \mathcal{B}\}.$$

If $A \in \mathcal{P}(Y)$, I is a set, and $(A_\alpha)_{\alpha \in I}$ is a collection in $\mathcal{P}(Y)$, then

$$f^{-1}(A^c) = f^{-1}(A)^c, \quad f^{-1}\Big(\bigcup_{\alpha \in I} A_\alpha\Big) = \bigcup_{\alpha \in I} f^{-1}(A_\alpha),$$

$$\text{and} \quad f^{-1}\Big(\bigcap A_\alpha\Big) = \bigcap f^{-1}(A_\alpha).$$

Properties of the preimage map

Proposition Let X and Y be sets and $f : X \to Y$ be a map.

(i) If $\mathcal{Y}$ is a σ-algebra on Y, then $f^{-1}(\mathcal{Y})$ is a σ-algebra on X.

(ii) If $\mathcal{X}$ is a σ-algebra on $\mathcal{X}$, then $\mathcal{Y} = \{B \in \mathcal{P}(Y)\,;\, f^{-1}(B) \in \mathcal{X}\}$ is a σ-algebra on Y.

(iii) If $\mathcal{A} \subseteq \mathcal{P}(Y)$, then $f^{-1}(\mathcal{A})^{\sigma} = f^{-1}(\mathcal{A}^{\sigma})$.

Measurable functions

Theorem and definition: Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be measurable spaces and $f : X \to Y$ be a map. Then the following are equivalent:

(i) $f^{-1}(\mathcal{Y}) \subseteq \mathcal{X}$.

(ii) $f^{-1}(\mathcal{A}) \subseteq \mathcal{X}$ for every $\mathcal{A} \subseteq \mathcal{P}(Y)$ with $\mathcal{A}^\sigma = \mathcal{Y}$.

(iii) $f^{-1}(\mathcal{A}) \subseteq \mathcal{X}$ for some $\mathcal{A} \subseteq \mathcal{P}(Y)$ with $\mathcal{A}^\sigma = \mathcal{Y}$.

We say that f is *measurable* or more precisely $(\mathcal{X}, \mathcal{Y})$-*measurable* if these conditions hold. In particular, we say that a function $f : X \to \mathbb{R}$ is *measurable* or $\mathcal{X}$-*measurable* if it is measurable with $\mathcal{B}_\mathbb{R}$ in the role of $\mathcal{Y}$.

Descriptions of $\mathcal{B}_{\mathbb{R}}$

Proposition:

We have $\mathcal{B}_{\mathbb{R}} = \mathcal{A}^{\sigma}$ if $\mathcal{A}$ is one of the classes

$$
\begin{array}{rlrl}
\text{(i)} & \mathcal{A} = \{]\alpha, +\infty[\,;\ \alpha \in \mathbb{R}\}, & \text{(v)} & \mathcal{A} = \{]\alpha, \beta[\,;\ \alpha, \beta \in \mathbb{R}, \alpha \leq \beta \\
\text{(ii)} & \mathcal{A} = \{[\alpha, +\infty[\,;\ \alpha \in \mathbb{R}\}, & \text{(vi)} & \mathcal{A} = \{[\alpha, \beta]\,;\ \alpha, \beta \in \mathbb{R}, \alpha \leq \beta \\
\text{(iii)} & \mathcal{A} = \{]-\infty, \alpha[\,;\ \alpha \in \mathbb{R}\}, & \text{(vii)} & \mathcal{A} = \{]\alpha, \beta]\,;\ \alpha, \beta \in \mathbb{R}, \alpha \leq \beta \\
\text{(iv)} & \mathcal{A} = \{]-\infty, \alpha]\,;\ \alpha \in \mathbb{R}\}, & \text{(viii)} & \mathcal{A} = \{[\alpha, \beta[\,;\ \alpha, \beta \in \mathbb{R}, \alpha \leq \beta
\end{array}
$$

In all these cases we can replace $\mathbb{R}$ by $\mathbb{Q}$.

A small part of the proof: Denote the sets in (i)-(viii) by $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_8$. It is clear that $\mathcal{A}_5^{\sigma} = \mathcal{B}_{\mathbb{R}}$. Observe that

$$\mathcal{A}_6 \ni [\alpha, \beta] = \bigcap_{j=1}^{\infty}]\alpha_j, \beta_j[\in \mathcal{A}_5^{\sigma}, \qquad \text{where } \alpha_j \nearrow \alpha \text{ and } \beta_j \searrow \beta.$$

This implies $\mathcal{A}_6^{\sigma} \subseteq \mathcal{A}_5^{\sigma}$. We also have

$$\mathcal{A}_5 \ni]\alpha, \beta[= \bigcup^{\infty} [\alpha_j, \beta_j] \in \mathcal{A}_6^{\sigma}, \qquad \text{where } \alpha_j \searrow \alpha \text{ and } \beta_j \nearrow \beta.$$

Characteristic functions

Definition: For every A is a subset of we define the *characteristic function χ_A of A in X* by

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in X \setminus A. \end{cases}$$

Proposition: Let $(X, \mathcal{X})$ be a measurable space and $A \subseteq X$. Then the characteristic function χ_A of A is measurable if and only if A is measurable.

Proof: Let $\alpha \in \mathbb{R}$. Then

$$\chi_A^{-1}(]\alpha, +\infty[) = \{x \in X \,;\, \chi_A(x) > \alpha\} = \begin{cases} \emptyset, & \alpha \geq 1, \\ A, & 0 \leq \alpha < 1, \\ X, & \alpha < 0. \end{cases}$$

The statement follows from Theorem and Proposition (i) above.

The extended real line

If we let $+\infty$ and $-\infty$ be two symbols which are not real numbers and add them to the real line, then we get the *extended real line*

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

We define a topology on $\overline{\mathbb{R}}$ by saying that $U \subseteq \overline{\mathbb{R}}$ is open if it is a union of intervals of the type

$$[-\infty, \alpha[, \quad]\beta, \gamma[, \quad]\delta, +\infty], \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}.$$

This topology generates the Borel algebra $\mathcal{B}_{\overline{\mathbb{R}}}$ on $\overline{\mathbb{R}}$.

Defintion: Let $(X, \mathcal{X})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$ be a function. We say that f is *measurable* or $\mathcal{X}$*-measurable* if it is $(\mathcal{X}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. We denote by $M_{\mathbb{R}}(X, \mathcal{X})$ the set of all measurable functions $f : X \to \overline{\mathbb{R}}$ and by $M^+_{\mathbb{R}}(X, \mathcal{X})$ the set of all measurable functions $f : X \to [0, +\infty]$.

Extension of the operations on $\overline{\mathbb{R}}$

$$
\begin{aligned}
a + (+\infty) &= +\infty, \quad a \in \mathbb{R} \\
a + (-\infty) &= -\infty, \quad a \in \mathbb{R} \\
(+\infty) + (+\infty) &= +\infty \\
(-\infty) + (-\infty) &= -\infty \\
a \cdot +\infty &= \begin{cases} +\infty & a > 0 \\ 0 & a = 0 \\ -\infty & a < 0 \end{cases} \\
a \cdot (-\infty) &= \begin{cases} -\infty & a > 0 \\ 0 & a = 0 \\ +\infty & a < 0 \end{cases} \\
(-\infty)(+\infty) &= -\infty \\
(+\infty)(+\infty) &= +\infty \\
(-\infty)(-\infty) &= +\infty.
\end{aligned}
$$

The Borel algebra $\mathcal{B}_{\overline{\mathbb{R}}}$

We have a similar description for the Borel algebra $\mathcal{B}_{\overline{\mathbb{R}}}$ as for $\mathcal{B}_{\mathbb{R}}$.
Proposition We have $\mathcal{B}_{\overline{\mathbb{R}}} = \mathcal{A}^{\sigma}$ where $\mathcal{A}$ is one of the sets

$$
\begin{array}{rlrl}
\text{(i)} & \mathcal{A} = \{]\alpha, +\infty]\,;\, \alpha \in \mathbb{R}\}, & \text{(v)} & \mathcal{A} = \{[\alpha, \beta]\,;\, \alpha, \beta \in \overline{\mathbb{R}}, \alpha \leq \beta\}, \\
\text{(ii)} & \mathcal{A} = \{[\alpha, +\infty]\,;\, \alpha \in \mathbb{R}\}, & \text{(vi)} & \mathcal{A} = \{]\alpha, \beta]\,;\, \alpha, \beta \in \overline{\mathbb{R}}, \alpha \leq \beta\}. \\
\text{(iii)} & \mathcal{A} = \{[-\infty, \alpha[\,;\, \alpha \in \mathbb{R}\}, & \text{(vii)} & \mathcal{A} = \{[\alpha, \beta[\,;\, \alpha, \beta \in \overline{\mathbb{R}}, \alpha \leq \beta\}, \\
\text{(iv)} & \mathcal{A} = \{[-\infty, \alpha]\,;\, \alpha \in \mathbb{R}\}. & &
\end{array}
$$

In all these cases $\mathbb{R}$ may be replaced by $\mathbb{Q}$.

Comparison of functions with values in $\overline{\mathbb{R}}$ and in $\mathbb{R}$

Proposition: Let $(X, \mathcal{X})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$ be a function. Then f is measurable if and only if the function $g : X \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} f(x), & f(x) \in \mathbb{R} \\ 0, & f(x) \in \{-\infty, +\infty\} \end{cases}$$

is measurable and the sets $f^{-1}(\{-\infty\})$ and $f^{-1}(\{+\infty\})$ are measurable.

Theorem: If $(f_n)_{n \in N}$ is a sequence in $M_{\mathbb{R}}(X, \mathcal{X})$, then

$$\inf_{n \in \mathbb{N}} f_n, \quad \sup_{n \in \mathbb{N}} f_n, \quad \liminf_{n \to \infty} f_n \quad \text{and} \quad \limsup_{n \to \infty} f_n$$

are measurable. If f_n is convergent at every point $x \in X$, then $\lim_{n \to \infty} f_n$ is measurable.

Theorem: Let $(X, \mathcal{X})$ be a measurable space, $M_{\mathbb{R}}(X, \mathcal{X})$ denote the set of all measurable functions with values in $\overline{\mathbb{R}}$, and add the values $+\infty$ and $-\infty$ according to the rule
$(+\infty) + (-\infty) = (-\infty) + (+\infty) = 0$.

(i) If $f, g \in M_{\mathbb{R}}(X, \mathcal{X})$, then $f + g, fg \in M_{\mathbb{R}}(X, \mathcal{X})$.

(ii) If $\varphi :]\alpha, \beta[\to \mathbb{R}$ is a continuous function which extends continuously to a function $[\alpha, \beta] \to \overline{\mathbb{R}}$, $f \in M_{\mathbb{R}}(X, \mathcal{X})$, and $f(X) \subseteq [\alpha, \beta]$, then $\varphi \circ f \in M_{\mathbb{R}}(X, \mathcal{X})$.

(iii) If $f, g, h \in M_{\mathbb{R}}(X, \mathcal{X})$ then the intermediate function midfunction$\{f, g, h\}$ is in $M_{\mathbb{R}}(X, \mathcal{X})$.

(iv) If $f \in M_{\mathbb{R}}(X, \mathcal{X})$, then the functions f_+, f_-, $|f|^p$ for $p \in \mathbb{R}$, and $\log |f|$ are in $M_{\mathbb{R}}(X, \mathcal{X})$, where we take $|0|^p = +\infty$ for $p < 0$, $|\pm\infty| = +\infty$, $\log 0 = -\infty$ and $\log +\infty = +\infty$.

Simple functions

Let X be a set and $f : X \to \overline{\mathbb{R}}$ be a function. We say that f is *simple* if it only takes finitely many values. If we let $a_1, \ldots, a_n$ be the different values of f and set $E_j = f^{-1}(\{a_j\})$, then the sets E_j are disjoint, their union is X, and

$$f = \sum_{j=1}^{n} a_j \chi_{E_j}.$$

This is called the *standard representation* of the simple function f.

Theorem: Let $(X, \mathcal{X})$ be a measurable space. Every measurable function $f : X \to \overline{\mathbb{R}}_+ = [0, +\infty]$ is the limit of an increasing sequence of real valued simple measurable functions on X. If f is bounded, then the convergence is uniform.

2.4 Integrals

Let $(X, \mathcal{X}, \mu)$ be a measure space.

(i) For every $E \in \mathcal{X}$ we define the *integral* of the characteristic function χ_E of E *with respect to the measure* μ by

$$\int_X \chi_E \, d\mu = \mu(E).$$

(ii) For every simple measurable function $\varphi : X \to \overline{\mathbb{R}}_+$ with standard representation

$$\varphi = \sum_{j=1}^{n} a_j \chi_{E_j}$$

we define the *integral of* φ *with respect to the measure* μ by

$$\int_X \varphi \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j).$$

(iii) For every measurable function $f : X \to \overline{\mathbb{R}}_+$,
i.e. $f \in M^+_{\mathbb{R}}(X, \mathcal{X})$, we define the *integral of f with respect to the measure μ* by

$$\int_X f \, d\mu = \sup\{ \int_X \varphi \, d\mu \, ; \, \varphi \in M^+_{\mathbb{R}}(X, \mathcal{X}), \varphi \text{ is simple}, \varphi \leq f \}.$$

We say that f is *integrable with respect to μ* if $\int_X f \, d\mu < +\infty$

(iv) If $f : X \to \overline{\mathbb{R}}$ is measurable, i.e. $f \in M_{\mathbb{R}}(X, \mathcal{X})$, then we say that f is *integrable with respect to μ* if both f_+ and f_- are integrable and we define the *integral of f with respect to μ* by

$$\int_X f \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu.$$

(v) We say that a measurable function $f : X \to \mathbb{C}$ is *integrable with respect to* μ if both $\text{Re}f$ and $\text{Im}f$ are integrable with respect to μ and we define the *integral of* f *with respect to* μ by

$$\int_X f \, d\mu = \int_X \text{Re}f \, d\mu + i \int_X \text{Im}f \, d\mu.$$

(vi) If E is a measurable set and $f : X \to \overline{\mathbb{R}}$ or $f : X \to \mathbb{C}$ is a measurable function, then we say that f is *integrable on* E *with respect to* μ if $f\chi_E$ is an integrable function and then we define the *integral of* f *on* E by

$$\int_E f \, d\mu = \int_X f\chi_E \, d\mu.$$

Null sets

Let $(X, \mathcal{X}, \mu)$ be a measure space.

Definition:

(i) A μ-null set is a measurable set E such that $\mu(E) = 0$.

(ii) We say that two functions f and g on X with values in $\overline{\mathbb{R}}$ or $\mathbb{C}$ are *equal μ-almost everywhere* or *almost everywhere with respect to μ*, if $\{x \in X\,;\, f(x) \neq g(x)\}$ is contained in a μ-null set.

(iii) Let $Q(x)$ be a statement depending on $x \in X$. We say that $Q(x)$ holds *μ-almost everywhere* if $\{x \in X\,;\, Q(x) \text{ is not true}\}$ is contained in a μ-null set.

We abbreviate the term *μ-almost everywhere* as μ-a.e.. When there is no ambiguity about the measure we are referring to we drop the prefix μ-.

Monotone convergence and Fatou lemma

Monotone convergence theorem:

For every increasing sequence (f_n) of functions in $M^+_{\mathbb{R}}(X, \mathcal{X})$ we have

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu.$$

Corollary:

If (f_n) is a sequence in $M^+_{\mathbb{R}}(X, \mathcal{X})$, then

$$\sum_{n=1}^{\infty} \int_X f_n \, d\mu = \int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu.$$

The Fatou Lemma:

If (f_n) is a sequence in $M^+_{\mathbb{R}}(X, \mathcal{X})$, then

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

The Lebesgue dominated convergence theorem

Dominated convergence theorem:

Let $(X, \mathcal{X}, \mu)$ be a measure space, (f_n) be a sequence of integrable functions, and f a measurable function such that $f_n \to f$ a.e. and let g be an integrable function such that $|f_n| \le g$ a.e. for all n. Then f is integrable and

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Integrals depending continuously on a parameter

Theorem:
Let $(X, \mathcal{X}, \mu)$ be a measure space, $T \subseteq \mathbb{R}^d$.
If $f : X \times T \to \overline{\mathbb{R}}$ or $f : X \times T \to \mathbb{C}$ is a function such that

(i) for every $t \in T$ the function $X \ni x \mapsto f(x,t)$ is integrable,

(ii) there exists a neighbourhood V of t_0 and an integrable function $g : X \to \overline{\mathbb{R}}_+$ such that $|f(x,t)| \leq g(x)$ for all $t \in V$ and almost all $x \in X$ (outside a null set $N(t)$ which may depend on t), and

(iii) $t_0 \in T$, and $T \ni t \mapsto f(x,t)$ is continuous at t_0 for almost all $x \in X$.

Then

$$\lim_{T \ni t \to t_0} \int_X f(x,t)\, d\mu(x) = \int_X f(x,t_0)\, d\mu(x).$$

Differentiation under the integral sign

Theorem:
Let $(X, \mathcal{X}, \mu)$ be a measure space, $T \subseteq \mathbb{R}$ be an interval, $t_0 \in T$, and $f : X \times T \to \overline{\mathbb{R}}$ or $f : X \times T \to \mathbb{C}$ be a function such that

(i) for every $t \in T$ the function $X \ni x \mapsto f(x,t)$ is integrable,

(ii) for almost all $x \in X$ the function $T \ni t \mapsto f(x,t)$ has a continuous partial derivative $\partial f(x,t)/\partial t$, and

(iii) there exists a neighbourhood V of t_0 and an integrable function $g : X \to \mathbb{R}$ such that $\left|\partial f(x,t)/\partial t\right| \leq g(x)$ for almost all $x \in X$ (outside a null set $N(t)$ which may depend on t).

Then the function $F(t) = \int_X f(x,t)\, d\mu(x)$ is differentiable at the point t_0 and

$$F'(t_0) = \int_X \frac{\partial f}{\partial t}(x, t_0)\, d\mu(x).$$

Integrals depending analytically on a parameter

Theorem:

Let $(X, \mathcal{X}, \mu)$ be a measure space, $T \subseteq \mathbb{C}$ be an open set, and $f : X \times T \to \mathbb{C}$ be a function such that

(i) for every $z \in T$ the function $X \ni x \mapsto f(x,z)$ is integrable,

(ii) for almost all $x \in X$ the function $T \ni z \mapsto f(x,z)$ is analytic, and

(iii) every point $z_0 \in T$ has a neighbourhood V and an integrable function $g : X \to \mathbb{R}$ such that $\big|f(x,z)\big| \leq g(x)$ for almost all $x \in X$ (outside a null set $N(z)$ which may depend on z) and all $z \in V$.

Then the function $F(z) = \int_X f(x,z)\, d\mu(x)$ analytic in T and

$$F'(z) = \int_X \frac{\partial f}{\partial z}(x,z)\, d\mu(x).$$

2.5 The Lebesgue measure and integral on the real line

How do we measure length of $A \subset \mathbb{R}$?

It is clear at least if A is a finite interval, which can be open, half open or closed,

$$]a, b[, \quad [a, b[, \quad]a, b], \quad \text{or} \quad [a, b].$$

In all these cases we set

$$\ell(A) = b - a.$$

If A is an infinite interval

$$[a, +\infty[, \ ]a, +\infty[, \ ]-\infty, b] \ \text{ or } \ ]-\infty, b[$$

we set

$$\ell(A) = +\infty.$$

An algebra of intervals

Definition:

The set $\mathcal{A}_{\mathbb{R}}$ of all *finite unions* of intervals of the form

$$]a,b], \quad]a,+\infty[, \quad]-\infty,b], \text{ and }]-\infty,+\infty[=\mathbb{R}$$

is called an *interval algebra*.

Extension of the length function ℓ to $\mathcal{A}_{\mathbb{R}}$

It is easy to see that every $A \in \mathcal{A}$ can be written uniquely as a union of finitely many *disjoint* intervals $I_1, \ldots, I_N$ in $\mathcal{A}_{\mathbb{R}}$. We define

$$\ell(A) = \sum_{j=1}^{N} \ell(I_j).$$

Proposition:

The function ℓ is a measure on $\mathcal{A}_{\mathbb{R}}$.

Extension of measures on set algebras

Let $\mathcal{A}$ be a algebra of subsets of X and μ a measure on $\mathcal{A}$.

Definition:

The outer measure μ^* of μ is a set function defined on $\mathcal{P}(X)$ by

$$\mu^*(B) = \inf \Big\{ \sum_{j=1}^{\infty} \mu(A_j) \, ; \; B \subset \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A} \Big\}.$$

Basic properties of the outer measure:

(i) $\mu^*(\varnothing) = 0$

(ii) $\mu^*(B) \geq 0$ for all $B \in \mathcal{P}(X)$.

(iii) If $A \subseteq B$ the $\mu^*(A) \leq \mu^*(B)$.

(iv) $A \in \mathcal{A}$, then $\mu^*(A) = \mu(A)$.

(v) For every sequence (B_j) in $\mathcal{P}(X)$, $\mu^*\left(\bigcup_{j=1}^{\infty} B_j \right) \leq \sum_{j=1}^{\infty} \mu^*(B_j)$.

Definition:

A $E \in \mathcal{P}(X)$ is said to be μ^*-measurable if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E), \qquad A \in \mathcal{P}(X).$$

The collection $\mathcal{A}^*$ of all μ^*-measurable sets is denoted by $\mathcal{A}^*$.

Carathéodory extension theorem:

$\mathcal{A}^*$ is a σ-algebra containing $\mathcal{A}$.

If (E_j) is a disjoint sequence in $\mathcal{A}^*$, then $\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu^*(E_j)$.

Hahn extension theorem:

If μ is σ-finite, then μ^* is a unique extension of μ to a measure on $\mathcal{A}^*$.

Lebesgue measure and lebesgue integral

Definition:

- The σ-algebra $\mathcal{A}_{\mathbb{R}}^{*}$ is called the *Lebesgue algebra* on the real line and we denote it by $\mathcal{M}$.
- The measure ℓ^{*} on $\mathcal{M}$ is called the *Lebesgue measure* on $\mathbb{R}$ and we denote it by λ

Theorem: (i) $\mathcal{A}_{\mathbb{R}} \subsetneq \mathcal{B}_{\mathbb{R}} = \mathcal{A}_{\mathbb{R}}^{\sigma} \subsetneq \mathcal{M} \subsetneq \mathcal{P}(\mathbb{R})$.

(ii) If $f : [a, b] \to \mathbb{C}$ is Riemann integrable and if we continue f as 0 outside $[a, b]$, then f is a Lebesgue intrgrable function and

$$\int_a^b f(x)\,dx = \int_{\mathbb{R}} f\,d\lambda.$$

(iii) The Lebesgue algebra is complete in the sense that every subset of a null set is measurable and consequently a null set.

2.6 Few concepts from functional analysis

Vector spaces

Recall that a vector space V is a set with two operations, addition and multiplicaton with scalars. We only look scalars is $\mathbb{R}$ or $\mathbb{C}$. The elemets in V are called vectors and one of them is the zero vector denoted by 0.

The following properties are assumed to hold:

(i) $x + y = y + x$

(ii) $x + (y + z) = (x + y) + z$

(iii) $x + 0 = x$

(iv) for every x there exists an y such that $x + y = 0$

(v) $c(x + y) = cx + cy$

(vi) $a(bx) = (ab)x$

(vii) $(a + b)x = ax + bx$

(viii) $1x = x$

Norm

Recall that a function $x \mapsto \|x\|$ on a vector space V over $\mathbb{R}$ or $\mathbb{C}$ taking values in the set of positive numbers $\mathbb{R}_+ = \{x \in \mathbb{R}\,;\, x \geq 0\}$ is a *norm* if it satisfies

(i) $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|ax\| = |a|\|x\|$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$

A vector space V with a norm $\|\cdot\|$ is called a *normed space*.
We often say that normed space is a pair $(V, \|\cdot\|)$ where V is a vector space and $\|\cdot\|$ is a norm on V.
A subspace W of a normed space V is automatically a normed space with the norm that is given on V.

Examples of normed spaces – $\mathbb{R}^n$ and $\mathbb{C}^n$ with various norms.

(i) The number fields $\mathbb{R}$ and $\mathbb{C}$ are normed spaces with the absolute value norm, $\|x\| = |x|$.

(ii) The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are normed spaces with the usual euclidean norm $\|x\| = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{\frac{1}{2}}$.

(iii) For $1 \leq p < +\infty$ the spaces $\ell_p^n(\mathbb{R})$ and $\ell_p^n(\mathbb{C})$ are defined as the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively, with the p-norm $\|x\|_p = \left(|x_1|^p + \cdots + |x_n|^p\right)^{\frac{1}{p}}$.

(iv) The spaces $\ell_\infty^n(\mathbb{R})$ and $\ell_p^n(\mathbb{C})$ are defined as the vectors spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively, with the ∞-norm, which is also called the supremum norm, $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$.

Examples of normed spaces – spaces of sequences.

(v) For $1 \leq p < +\infty$ the spaces $\ell_p(\mathbb{R})$ and $\ell_p(\mathbb{C})$ are defined as the vectors spaces of all sequences (x_j) with $x_j \in \mathbb{R}$ and $x_j \in \mathbb{C}$, respectively, such that $\sum_{j=1}^{\infty} |x_j|^p < +\infty$, with the p-norm $\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{\frac{1}{p}}$.

(vi) The spaces $\ell_\infty(\mathbb{R})$ and $\ell_\infty(\mathbb{C})$ are defined as the vectors spaces of all sequences (x_j) with $x_j \in \mathbb{R}$ and $x_j \in \mathbb{C}$, respectively, such that $\sup_j |x_j| < +\infty$, with the supremum norm $\|x\|_\infty = \sup_j |x_j|$.

(vii) We let $\mathbf{c}(\mathbb{R})$ and $\mathbf{c}(\mathbb{C})$ denote the subspaces of $\ell_\infty(\mathbb{R})$ and $\ell_p(\mathbb{C})$, respectively, consisting of all convergent sequences.

(viii) We let $\mathbf{c}_0(\mathbb{R})$ and $\mathbf{c}_0(\mathbb{C})$ denoting the supspaces of $\ell_\infty(\mathbb{R})$ and $\ell_p(\mathbb{C})$, respectively, consisting of all convergent sequences tending to zero.

Examples of normed spaces – function spaces.

(ix) $C([a, b])$ consisting of all continuous complex valued on the interval $[a, b]$ with the maximum norm

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|.$$

(x) More generally, for every compact (closed and bounded) subset K of we define $C(K)$ as the space of all continuous complex valued functions on K with maximum norm

$$\|f\|_\infty = \sup_{x \in K} |f(x)|.$$

Banach spaces

Definition Let $(V, \|\cdot\|)$ be a normed vector space.

(i) A sequence (x_n) in V is said to be *convergent* if there exists x in V such that $\|x_n - x\| \to 0$ as $n \to \infty$. The vector x is unique and called the *limit of the sequence* (x_n). This means that for every $\varepsilon > 0$ there exists N_ε such that $\|x_n - x\| < \varepsilon$ for every $n \geq N_\varepsilon$.

(ii) A sequence (x_n) in V is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there exists N_ε such that $\|x_n - x_m\| < \varepsilon$ for every $n, m \geq N_\varepsilon$.

(iii) The normed space $(V, \|\cdot\|)$ is said to be *complete* if every Cauchy sequence in V is convergent.

(iv) A complete normed space $(V, \|\cdot\|)$ is called a *Banach space*.

All the normed spaces in the examples above are Banach spaces.

L^p-spaces

Let $(X, \mathcal{X}, \mu)$ be measure space and $1 \leq p < +\infty$.
We let $\tilde{L}^p(\mu)$ denote the set of all measurable functions $f : X \to \mathbb{C}$ such that

$$\int_X |f|^p \, d\mu < +\infty.$$

and define $\| \cdot \|_p$ on $\tilde{L}^p(\mu)$ by

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}$$

We observe that $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere. If there are null sets $E \neq \varnothing$, then $\|f\|_p = 0$ does not necessarily imply that $f = 0$. Hence

We let $L^p(\mu)$ denote the set of all functions in $f \in \tilde{L}^p(\mu)$ where we identify two functions as equal if they are equal almost everywhere.

With this identification of functions every function which is 0 almost everywhere is considered as equal to the null element in $L^p(\mu)$.

Theorem:

$L^p(\mu)$ is a Banach space for every measure μ and every $1 \leq p < +\infty$.

Inner product spaces

An *inner product* on real vector space is a function $V \times V \to \mathbb{R}$, $(x, y) \mapsto \langle x, y \rangle$ satisfying

(i) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

(ii) $\langle x, y \rangle = \langle y, x \rangle$.

(iii) $\langle x, x \rangle \geq 0$

(iv) $\langle x, x \rangle = 0$ if and only if 0.

As a consequence of (i) and (ii) we get

$$\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle.$$

An *inner product* on complex vector space is a function $V \times V \to \mathbb{C}$, $(x, y) \mapsto \langle x, y \rangle$ satisfying the conditions above, except that (ii) is replcaed by

(ii)' $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

As a consequence of (i) and (ii)' we get

$$\langle x, ay + bz \rangle = \bar{a}\langle x, y \rangle + \bar{b}\langle x, z \rangle.$$

A vector space with an inner product is called an *inner product space*

Hilbert spaces

Every inner product induces a norm by the formula

$$\|x\| = \sqrt{\langle x, x\rangle}, \qquad x \in V.$$

and it is called a *Hilbert space* if it is a Banach space with the induced norm.

Examples:

(i) The space $\mathbb{R}^n$ with the usual inner product $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$ is a Hilbert space.

(ii) The space $\mathbb{C}^n$ with the usual complex inner product $\langle x, y\rangle = \sum_{k=1}^{n} x_k \bar{y}_k$ is a Hilbert space.

(iii) The space $C([a, b])$ with

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

is an inner product space, but it is not a Hilbert space, because it is easy to find a Cauchy sequence in this space, which is convergent in the induced norm, but the limit is not a continuous function.

The Hilbert space $L^2(\mu)$

On $L^2(\mu)$ we have a natural form $(f,g) \mapsto \langle f,g\rangle$,

$$\langle f,g\rangle = \int_X f\overline{g}\,d\mu.$$

The Cauchy-Schwarz inequality implies that the right hand side is a well defined complex numer for every f and g in $\tilde{L}^2(\mu)$.
The form satisfies (i) $\langle af+bg,h\rangle = a\langle f,h\rangle + b\langle g,h\rangle$
(ii)' $\langle f,g\rangle = \overline{\langle g,f\rangle}$, and
(iii) $\langle f,f\rangle \geq 0$,

In $L^2(\mu)$ we identify functions that are equal almost everywhere, so we also have

(iv) $\langle f,f\rangle$=0 if and only if $f=0$.

Theorem:
$L^2(\mu)$) is a Hilbert space.

Bounded linear operators

Now we look at two normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$. A map $T : V \to W$ is said to be linear if

$$T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2).$$

We let $\mathcal{L}(V, W)$ denote the set of all linear maps $T : V \to W$.

The linear map T is said to be *bounded* if

$$\sup_{\|v\|_V = 1} \|T(v)\|_W < +\infty.$$

We let $\mathcal{B}(V, W)$ denote the set of all bounded $T \in \mathcal{L}(V, W)$.

If W is a Banach space, then $\mathcal{B}(V, W)$ is a Banach space with the norm

$$\|T\|_{V,W} = \sup_{\|v\|_V = 1} \|T(v)\|_W.$$

This norm is called the *operator norm* on $\mathcal{B}(X, Y)$.

Bounded linear functionals – Dual spaces.

If V is a vector space over $\mathbb{R}$, then we set of $V^* = \mathcal{B}(V, \mathbb{R})$ and if V is a vector space over $\mathbb{C}$ then we set $V^* = \mathcal{B}(V, \mathbb{C})$.
In both cases we call V^* the dual space of V.
In functional analysis it is very important to give descriptions of the the dual space V^* of a given vector space.

Duality in Hilbert spaces

Theorem (Riesz-Fréchet, 1907) Let H be a Hilbert space. Then for every $y \in H$ the function f_y defined by

$$f_y(x) = \langle x, y \rangle, \qquad x \in H,$$

is an element of H^*. The operator

$$T : H \to H^*, \qquad y \mapsto f_y,$$

satisfies

(i) T is conjugate-linear, i.e., $f_{y+z} = f_y + f_z$ and $f_{ay} = \bar{a} f_y$,

(ii) T is bijective, i.e., for every $f \in H^*$ there is a unique $y \in H$ such that $f = f_y$

(iii) T is an isometry, i.e., $\|f_y\|_{H^*} = \|y\|_H$.

Convergence issues – functions

Let X be any set and $f_n : X \to \mathbb{C}$, $n = 0, 1, 2, \dots,$ and $f : X \to \mathbb{C}$ be complex valued functions on X.

There are several convergence concepts for sequences of functions:

(i) Pointwise convergence: For every $x \in X$, $f_n(x) \to f(x)$.

(ii) Uniform convergence: $\sup_{x \in X} |f_n(x) - f(x)| \to 0$

2.7 Real and complex measures and duality

Let $(X, \mathcal{X})$ be a measureable space.

Real measures

A set function $\lambda : \mathcal{X} \to \overline{\mathbb{R}}$ is called a *real measure* (or a *signed measure* or a *charge*) on $\mathcal{X}$ if it satisfies

(i) $\lambda(\emptyset) = 0$,

(ii) $\lambda(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \lambda(A_j)$, if $(A_j)_{j=1}^{\infty}$ is disjoint in $\mathcal{X}$, and

(iii) λ assumes at most one of the values $+\infty$ and $-\infty$.

λ is said to be *finite valued* if it only takes finite values.

Examples:

If μ is a measure on $\mathcal{X}$, $f \in M_{\mathbb{R}}(X, \mathcal{X})$ is a function such that one of the functions f^+ or f^- is integrable with respect to μ, then we have a real measure μ_f

$$\mu_f(A) = \int_A f \, d\mu = \int_A f^+ \, d\mu - \int_A f^- \, d\mu, \qquad A \in \mathcal{X}.$$

Decomposition of real measures

Definition:

Let λ be a real measure on $\mathcal{X}$. A set $E \in \mathcal{X}$ is said to be

(i) *positive* with respect to λ if $\lambda(A) \geq 0$ for all $A \in \mathcal{X}$, $A \subseteq E$,

(ii) *negative* with respect to λ if $\lambda(A) \leq 0$ for all $A \in \mathcal{X}$, $A \subseteq E$, and

(iii) *null set* with respect to λ or a λ-*null set* if $\lambda(A \cap P) = 0$ for all $A \in \mathcal{X}$, $A \subseteq E$.

Hahn decomposition theorem:

Then there exists $P \in \mathcal{X}$ such that P is positive with respect to λ and $N = P^c$ is negative with respect to λ.

The positive, negative and total variations of λ

With the aid of Hahn decomposition theorem we can associate to each real measure, three positive measures, λ^+, λ^-, and $|\lambda|$,

Definition:

(i) *positive variation:* $\lambda^+(A) = \lambda(A \cap P)$.

(ii) *negative variation:* $\lambda^-(A) = \lambda(A \cap N)$.

(iii) *total variation:* $|\lambda|(A) = \lambda^+(A) + \lambda^-(A)$.

Observe that $\lambda = \lambda^+ - \lambda^-$.

For every real measure λ we have:

(i) $|\lambda(E)| \leq |\lambda|(E)$.

(ii) If $\lambda = \mu - \nu$ where μ and ν are measures, then $\lambda^+ \leq \mu$ and $\lambda^- \leq \nu$.

(iii) If λ is finite valued, then $|\lambda|$ is a finite measure.

(iv) If λ and τ are real measures and $\lambda + \tau$ is a well defined real measure, then

$$|\lambda + \tau| \leq |\lambda| + |\tau|$$

The Banach space of finite real valued measures

Let $\mathcal{M}_{\mathbb{R}}(X, \mathcal{X})$ set of all finite valued real measures. It is a vector space with the usual addition and multiplication by scalars

$$(\lambda + \nu)(A) = \lambda(A) + \nu(A), \qquad (c\lambda)(A) = c\lambda(A).$$

The function $\|\cdot\|$ on $\mathcal{M}_{\mathbb{R}}(X, \mathcal{X})$ defined

$$\|\lambda\| = |\lambda|(X)$$

is a norm on $\mathcal{M}_{\mathbb{R}}(X, \mathcal{X})$.

Theorem

$(\mathcal{M}_{\mathbb{R}}(X, \mathcal{X}), \|\cdot\|)$ is a Banach space.

A few more properties of real measures

For every real measure λ we have

(i) $|\lambda(E)| \leq |\lambda|(E)$.

(ii) If $\lambda = \mu - \nu$ where μ and ν are measures, then $\lambda^+ \leq \mu$ and $\lambda^- \leq \nu$.

(iii) If λ is a finite valued, then $|\lambda|$ is a finite measure.

(iv) If λ and τ are real measures and $\lambda + \tau$ is a well defined real measure, then

$$|\lambda + \tau| \leq |\lambda| + |\tau|$$

Integrals with respect to real measures

If f is integrable with respect to both λ^+ and λ^-, then we define the integral of f with respect to λ by

$$\int_X f\, d\lambda = \int_X f\, d\lambda^+ - \int_X f\, d\lambda^-.$$

We have

$$\left|\int_X f\, d\lambda\right| \le \int_X |f|\, d|\lambda|.$$

For a fixed $\lambda \in \mathcal{M}_{\mathbb{R}}(X, \mathcal{X})$ we have a linear functional

$$L^1(\lambda^+) \cap L^1(\lambda^-) \ni f \mapsto \int_X f\, d\lambda.$$

and for a fixed $f \in L^1(\lambda^+) \cap L^1(\lambda^-)$ we have a linear functional

$$\mathcal{M}_{\mathbb{R}}(X, \mathcal{X}) \ni \lambda \mapsto \int_X f\, d\lambda.$$

Bounded linear functionals on $BC_{\mathbb{R}}(X)$

Let X be an open subset of $\mathbb{R}^n$ and let $BC_{\mathbb{R}}(X)$ denote the set of all bounded real valued continuous functions on X.

$BC_{\mathbb{R}}(X)$ is a Banach space over $\mathbb{R}$ with the $\|\cdot\|_\infty$-norm.

Every finite valued real measure $\lambda \in \mathcal{M}_{\mathbb{R}}(X, \mathcal{B}_X)$ defines a linear functional

$$\Lambda(f) = \int_X f\, d\lambda, \qquad f \in BC(X),$$

and we have

$$|\Lambda(f)| \leq \int_X |f|\, d|\lambda| \leq \|f\|_\infty \int_X d|\lambda| = \|f\|_\infty \|\lambda\|,$$

which shows that Λ is a bounded linear functional on $BC_{\mathbb{R}}(X)$.

Riesz representation theorem: Every bounded linear functional Λ on $BC(X)$ is of this form and we have $\|\Lambda\| = \|\lambda\|$.

Complex measures

Definition:

A set function $\lambda : \mathcal{X} \to \mathbb{C}$ is said to be a *complex measure* if it satisfies

(i) $\lambda(\varnothing) = 0$,

(ii) $\lambda\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \lambda(A_j)$ if $(A_j)_{j=1}^{\infty}$ is disjoint sequence in $\mathcal{X}$ and the series is absolutely convergent.

Every complex valued function har a real and imaginary part, so we get two finite valueed real measures $\mathrm{Re}\lambda$ and $\mathrm{Im}\lambda$.
The decompositon gives us a decompostion of λ into four (positive) measures

$$\lambda = ((\mathrm{Re}\lambda)^{+} - (\mathrm{Re}\lambda)^{-}) + i((\mathrm{Im}\lambda)^{+} - (\mathrm{Im}\lambda)^{-}).$$

For every measurable $f : X \to \mathbb{C}$ which is integrable with respect to all the four measures we define

$$\int_X f\, d\lambda = \int_X f\, d(\mathrm{Re}\lambda) + i \int_X f\, d(\mathrm{Im}\lambda)$$

Elementary properties

The complex measures form a vector space over the complex numbers with the usual addition and multiplication

$$(\kappa + \lambda)(A) = \kappa(A) + \lambda(A), \quad (c\lambda)(A) = c\lambda(A), \qquad A \in \mathcal{X}.$$

We denote this space by $\mathcal{M}_{\mathbb{C}}(X, \mathcal{X})$.

Total variation:
The set function

$$|\lambda|(A) = \sup\{\sum_{j=1}^{\infty} |\lambda(A_j)|\,;\,(A_j)_{j=1}^{\infty} \text{ is a partition of } A\}.$$

is a finite measure on $\mathcal{X}$.

Theorem $\mathcal{M}_{\mathbb{C}}(X, \mathbb{C}(X, \mathcal{B}_X)$ with the norm $\lambda \mapsto |\lambda|(X)$ is a Banach space.

Riesz representation theorem

Every complex measure $\lambda \in \mathcal{M}_{\mathbb{C}}(X, \mathcal{B}_X)$ defines a linear functional over $\mathbb{C}$

$$\Lambda(f) = \int_X f \, d\lambda, \qquad f \in BC(X),$$

and we have

$$|\Lambda(f)| \leq \int_X |f| \, d|\lambda| \leq \|f\|_\infty \int_X d|\lambda| = \|f\|_\infty \|\lambda\|,$$

which shows that Λ is a bounded linear functional on $BC(X)$.

Riesz representation theorem: Every bounded linear functional Λ on $BC(X)$ is of this form and we have $\|\Lambda\| = \|\lambda\|$.

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3. Integral representations and Herglotz functions

Definiton of a Herglotz function

Definition: An analytic function h that maps the complex upper half plane $\mathbb{C}^+$ into the closed upper half plane $\overline{\mathbb{C}^+}$ is called a *Herglotz function* (or a *Nevanlinna function* or *Pick function* or *R-function*).

Examples:

(i) $h(z) = a + bz$, $a \in \mathbb{R}$, $b \geq 0$.

(ii) $h(z) = i$, $z \in \mathbb{C}^+$.

(iii) $h(z) = \sqrt{z}$, where $z \mapsto \sqrt{z}$ denotes the principal branch of the square root function, $\sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i}{2}\mathrm{Arg}z}$ and $\mathrm{Arg}z \in [-\pi, \pi[$

(iv) $h(z) = -\dfrac{m}{z - \alpha}$, $\alpha \in \mathbb{R}$, $m \geq 0$.

(v) $h(z) = -\dfrac{\log(1+z)}{z}$, where log is the principal branch of the logarithm.

Elementary properties of Herglotz functions

(i) If h_1 and h_2 are Herglotz functions, then $h_1 + h_2$ and $h_1 \circ h_2$ are Herglotz functions.

(ii) If h is a Herglotz function then (i) gives that $z \mapsto -1/h(z)$, $z \mapsto h(\sqrt{z})$, and $z \mapsto h(-1/z)$ are Herglotz functions.

(iii) If (h_n) is a sequence of Herglotz functions which converges locally uniformly to h, then h is a Herglotz function.

(iv) If (h_n) is a sequence of Herglotz functions which is bounded at one point $z_0 \in \mathbb{C}^+$, i.e., $\sup_n |h_n(z_0)| < +\infty$, then there exists a subsequence (h_{n_j}), which converges locally uniformly in $\mathbb{C}^+$.

(v) If h is a Herglotz function and $h(z_0) = a \in \mathbb{R}$ for some $z_0 \in \mathbb{C}^+$, then h is the constant function $h(z) = a$ for all z. (This is a direct consequence of the maximum principle.)

Integral representation of Herglotz functions

Theorem: An analytic function $h \in \mathcal{O}(\mathbb{C}^+)$ is a Herglotz function if and only if it can be written in the form

$$h(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\mu(t),$$

where $a \in \mathbb{R}$, $b \geq 0$, and μ is a positive Borel measure such that

$$\int_{\mathbb{R}} \frac{1}{1 + t^2} \, d\mu(t) < +\infty.$$

Remark: The term $\dfrac{t}{1 + t^2}$ is needed to ensure the convergence of the integral.

Back to the examples

$$h(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t),$$

(i) For $h(z) = a + bz$.

(ii) For $h(z) = i$, we have $a = 0$, $b = 0$ and $\mu = \frac{1}{\pi}\lambda$, where λ is the Lebesgue measure.

(iii) For $h(z) = \sqrt{z}$, we have $d\mu(t) = \sqrt{-t}\,\chi_{\mathbb{R}_-}(t)\,dt$.

(iv) For $h(z) = -\dfrac{m}{z-\alpha}$, $\mu = m\delta_\alpha$, the point measure at the point α with mass m.

Symmetry issues

Note that the integral representation

$$h(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t),$$

defines h even for $z \in \mathbb{C}^-$. Then we have the symmetry relation

$$h(\bar{z}) = \overline{h(z)}, \qquad z \in \mathbb{C} \setminus \mathbb{R}$$

Example: $h(z) = \begin{cases} i, & z \in \mathbb{C}^+, \\ -i, & z \in \mathbb{C}^-. \end{cases}$

A possible analytic continuation of h might not coincide with the symmetric continuation.

Example: The function $h(z) = -\dfrac{1}{z+i}$ has an analytic continuation to a rational function given by the same formula on $\mathbb{C} \setminus \{-i\}$.

The symmetric continuation to $\mathbb{C} \setminus \mathbb{R}$ is given by

$$h(z) = \overline{h(\bar{z})} = -\frac{1}{z-i}, \qquad z \in \mathbb{C}^-,$$

and thus has a jump across the real line.

How to recover a, b and μ?

$$h(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t)$$

(i) $a = \operatorname{Re} h(i)$.

(ii) $b = \lim_{y \to +\infty} \dfrac{h(iy)}{iy}$.

(iii) The Stieltjes inversion formula is

$$\mu([\alpha,\beta]) + \tfrac{1}{2}\mu(\{\alpha\}) + \tfrac{1}{2}\mu(\{\beta\}) = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_{\gamma_\varepsilon} h(z)\, dz$$

where the integration is counter clockwise along the closed rectangular path γ_ε, with vertices $\alpha - i\varepsilon$, $\beta - i\varepsilon$, $\beta + i\varepsilon$, and $\alpha + i\varepsilon$.

Remark: By using the symmetry we can write the integral in (iii) as

$$\mu([\alpha,\beta]) + \tfrac{1}{2}\mu(\{\alpha\}) + \tfrac{1}{2}\mu(\{\beta\}) = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi} \int_\alpha^\beta \operatorname{Im} h(x+i\varepsilon)\, dx$$

The behaviour of h close to the real line reflects properties of the measure μ. More prcisely, μ is the weak limit as $\varepsilon \searrow 0$ of the family of measures

$$d\mu_\varepsilon(x) = \operatorname{Im} h(x+i\varepsilon)\, dx$$

which means that

$$\int_{\mathbb{R}} \varphi\, d\mu = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \varphi(x) \operatorname{Im} h(x+i\varepsilon)\, dx,$$

for every continuous function φ on $\mathbb{R}$ with compact support.

More on the behaviour "at" the real line

$$h(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t)$$

Then for $N \geq 0$ the following statements are equivalent

(i) Asymptotic expansion with real constants a_j,

$$h(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots + a_{2N-1} z^{2N-1} + o(z^{2N-1}) \qquad \text{as } z \hat{\to} 0.$$

(ii) Moments of the measure $\mu_0 = \mu - \mu(\{0\})\,\delta_0$ are finite, i.e.,

$$\int_{\mathbb{R}} \frac{1}{t^{2N}} \frac{d\mu_0(t)}{1+t^2} < +\infty.$$

In this case

$$a_0 = a + \int_{\mathbb{R}} \frac{1}{t} \frac{d\mu_0(t)}{1+t^2},$$

$$a_1 = b + \int_{\mathbb{R}} \frac{d\mu_0(t)}{t^2},$$

$$a_n = \int_{\mathbb{R}} \frac{d\mu_0(t)}{t^{n+1}}, \qquad n = 2, 3, \ldots, 2N-1.$$

Sum rules

$$\lim_{\varepsilon \searrow 0+} \lim_{y \searrow 0+} \frac{1}{\pi} \int_{\varepsilon < |x| < 1/\varepsilon} \frac{1}{x^p} \operatorname{Im} h(x + iy)\, dx = a_{p-1} - b_{p-1}$$

for $p = 2 - 2M, 2 - 2M, \dots, 2N$,

$$h(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + \dots + a_{2N-1} z^{2N-1} + o(z^{2N-1}) \qquad \text{as} \quad z \hat{\to} 0,$$

$$h(z) = b_1 z + b_0 + \frac{b_1}{z} + \dots + \frac{b_{-(2M-1)}}{z^{2M-1}} + o\left(\frac{1}{z^{2M-1}}\right) \qquad \text{as} \quad z \hat{\to} \infty.$$

Nevanlinna-Pick interpolation problem

Classical problem:

For given $z_1, \dots, z_N \in \mathbb{C}^+$ and $w_1, \dots, w_N \in \mathbb{C}^+$ find a (or all) Herglotz functions h with $h(z_j) = w_j$ for $j = 1, \dots, N$.

Theorem:

This problem has a solution if and only if the matrix $P = (P_{jk})_{j,k=1}^N$ with entries

$$P_{jk} = \frac{w_j - \bar{w}_k}{z_j - \bar{z}_k}$$

is non-negative (positive semi-definite).

Remark:

If there is a solution, then it is either unique or there exist infinitely many.

Real interpolation points

Even real interpolation points can be included. For each such point x_j a real number d_j is given and the condition becomes

$$h(x_j) = w_j, \qquad \lim_{z \hat{\to} x_j} \frac{\operatorname{Im} h(z)}{\operatorname{Im} z} \leq d_j.$$

The corresponding elements of the Pick matrix become

$$P_{jk} = \begin{cases} \dfrac{w_j - \bar{w}_k}{z_j - \bar{z}_k}, & z_j \neq \bar{z}_k, \\ d_j, & z_j = x_j \in \mathbb{R}. \end{cases}$$

Observe:

The limit $\lim\limits_{z \to x} \dfrac{\operatorname{Im} h(z)}{\operatorname{Im} z}$ is a kind of symmetric derivative.
At points x of analyticity of h is equal to $h'(x)$.

Example:

For $h(z) = -\dfrac{1}{z+i}$ we have $\lim\limits_{z \hat{\to} x} \dfrac{\operatorname{Im} h(z)}{\operatorname{Im} z} = +\infty$.

Other representations

Theorem

A $h : \mathbb{C}^+ \to \mathbb{C}^+$ is a Herglotz function if and only if there exists a Hilbert space $\mathcal{H}$, a self-adjoint operator $A = A^*$, and an element $v \in \mathcal{H}$ such that with some $z_0 \in \mathbb{C}^+$ it holds

$$h(z) = h(\bar{z}_0) + (z - \bar{z}_0)\big((I + (z - z_0)(A - zI)^{-1}v, v\big), \qquad z \in \mathbb{C} \setminus \mathbb{R}.$$

Hence there are many – unitarily equivalent – representations.
If A is the multiplication operator in $L^2(\mu)$, then the integral representation formula is recovered.

4. Elements of distribution theory

Content:

4.1

4.2

4.1 Motivation and basic definitions

Basic idea

Let X be an open subset of $\mathbb{R}^n$. Every $f \in L^1_{\text{loc}}(X)$ defines a linear functional u_f

$$C_0(X) \ni \varphi \mapsto \int_X f\varphi \, dx$$

The space $C_0(X)$ is the subspace of $C(X)$ consisting of all continuous functions on R^n which vanish outside a compact subset of X.

The idea of distribution theory is to view the functions f, or rather the functionals u_f, as elements of a larger space $\mathcal{D}'(X)$ of continuous linear functionals on the space $\mathcal{D}(X) = C_0^\infty(X)$ of infinitely differentiable functions in $C_0(X)$, and to extend the operations of analysis of functions to this larger space $\mathcal{D}'(X)$.

Defintion of a distribution

Definition: A distribution u on an open subset X of $\mathbb{R}^n$ is a functional operating on $\mathcal{D}(X)$ which is linear,

$$u(\varphi+\psi) = u(\varphi)+u(\psi), \quad u(c\varphi) = cu(\varphi), \qquad \gamma, \psi \in \mathcal{D}(X),\ c \in \mathbb{C},$$

and continuous in the sense that $u(\varphi_j) \to u(\varphi)$ as $j \to \infty$ if $\varphi_j \to \varphi$ in $\mathcal{D}(\mathcal{X})$.

The set of all distributions on X is denoted by $\mathcal{D}'(X)$.

Convergence in $\mathcal{D}(X)$:

$\varphi_j \to \varphi$ in $\mathcal{D}(\mathcal{X})$ means that there exists a compact $K \subset X$ such that $\operatorname{supp} \varphi_j \subset K$ for all j, $\varphi_j \to \varphi$ and all partial derivatives $\partial^\alpha \varphi_j \to \partial^\alpha \varphi$ unformly as $j \to \infty$.

Examples:

- u_f, $f \in L^1(X)$.
- Every complex measure μ defines a distribution

$$\mu(\varphi) = \int_X \varphi \, d\mu, \qquad \varphi \in \mathcal{D}(X).$$

  in particular a Dirac-measure of a point $a \in X$,

$$\delta_a(\varphi) = \varphi(a), \qquad \varphi \in \mathcal{D}(X).$$

- Let (a_j) be a sequence and assume that $\{a_j\,;\, a_j\}$ is a discrete subset of X. Take multi-indices α_j and complex numbers c_j. Then

$$u(\varphi) = \sum_j c_j \partial^{\alpha_j} \varphi(a_j)$$

  is a distribution on X.

The order of a distribution

Theorem: A linear functional u on $\mathcal{D}(X)$ is a distribution if and only if for every compact subset K of X there exist constants $C > 0$ and $k \in \mathbb{N}$, such that

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |\partial^\alpha \varphi(x)| \qquad \varphi \in C_0^\infty(K).$$

Definition: If it is possible to choose the same number k for every compact subset K of X, then we say that the distribution u is of *finite order* and we define *the order* of u as the minimal such number.

Examples: The distributions u_f defined by $f \in L^1_{\text{loc}}(X)$ and by complex measures are of order 0.

The principal value distribution

The function $f(x) = 1/x$ is infinitely differentiable in $\mathbb{R} \setminus \{0\}$ but not locally integrable near 0.

The function f extends to a distribution on $\mathbb{R}$. The extension is called the *principal value* of the function $x \mapsto 1/x$.

It is defined by the formula

$$\mathrm{PV}\left(\frac{1}{x}\right)(\varphi) = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx, \qquad \varphi \in C_0^\infty(\mathbb{R}).$$

This distribution is of order 1.

The vector space $\mathcal{D}'(X)$

The set of all distributions $\mathcal{D}'(X)$ on X is a vector space with addition of two distributions u and v defined by

$$(u + v)(\varphi) = u(\varphi) + v(\varphi),$$

and the product of $\alpha \in \mathbb{C}$ and u is defined by

$$(\alpha u)(\varphi) = \alpha u(\varphi).$$

The support of a distribution

If $f \in C(X)$ then the support $\operatorname{supp} f$ of f is defined as the closure of $\{x \in X\,;\, f(x) \neq 0\}$ in X.

The set $Y = X \setminus \operatorname{supp} f$ is open and

$$u_f(\varphi) = 0, \qquad \varphi \in C_0^\infty(Y).$$

We define the support of a distribution by describing its complement.

Definition: The support $\operatorname{supp} u$ of $u \in \mathcal{D}'(X)$ is defined as the complement of the union of all open subsets U of X such that

$$u(\varphi) = 0, \qquad \varphi \in C_0^\infty(U).$$

It is clear that $\operatorname{supp} u_f = \operatorname{supp} f$ for all $f \in C(X)$.

Sequences of distributions

Let (u_j) be a sequence in $\mathcal{D}'(X)$.

Definition: We say that (u_j) *converges to* $u \in \mathcal{D}'(X)$ and denote it by $u_j \to u$ and $\lim_{j\to+\infty} u_j = u$, if

$$\lim_{j\to+\infty} u_j(\varphi) = u(\varphi), \qquad \varphi \in C_0^\infty(X).$$

If all the distributions u_j are of the form u_{f_j}, where f_j are locally integrable in X, then we say that (f_j) converges to u, *weakly* or *in a weak sense* or *in the sense of distributions*.
This means that

$$\int_X f_j(x)\varphi(x)\,dx \to u(\varphi), \qquad \varphi \in C_0^\infty(X).$$

We write $f_j \to u$ and $\lim\limits_{j\to+\infty} f_j = u$.

Two results on weak convergence

Theorem: Let X be an open subset of $\mathbb{R}^n$ and let (u_j) be a sequence in $\mathcal{D}'(X)$. If the sequence $(u_j(\varphi))$ of complex numbers is convergent for every $\varphi \in \mathbb{C}_0^\infty(X)$, then the sequence (u_j) has a limit u in $\mathcal{D}'(X)$.

Proposition: Assume that $f \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} f(x)\,dx = 1$, and define $f_\varepsilon(x) = \varepsilon^{-n} f((x-a)/\varepsilon)$.

Then f_ε tends to δ_a in the sense of distributions as $\varepsilon \to 0$.

Examples:

- The probability density function for the standard normal distribution is
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$
and the density function for the normal distribution with mean μ and variance σ^2 is
$$f_{\mu,\sigma}(x) = \sigma^{-1} f((x-\mu)/\sigma).$$
By Proposition $f_{\mu,\sigma} \to \delta_\mu$ as $\sigma \to 0$.
- Poisson kernel for the upper half plane is
$$P_{\mathbb{C}^+}(x,y) = \frac{y}{\pi(x^2+y^2)}, \qquad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$
If we set $f(x) = 1/\pi(x^2+1)$, then f satisfies the conditions in Proposition and $f_y(x) = P_{\mathbb{C}^+}(x,y)$. Hence
$$P_{\mathbb{C}^+}(\cdot,y) \to \delta_0, \qquad y \to 0+.$$

Limits of analytic functions

Theorem:
Let I be an open interval, $\gamma > 0$ and set

$$Z = \{z \in \mathbb{C}\, ;\, \mathrm{Re} z \in I, 0 < \mathrm{Im} z < \gamma\}.$$

If $f \in \mathcal{O}(Z)$ and for some $N \in \mathbb{N}$ and $C > 0$ we have

$$|f(z)| \leq C|\mathrm{Im} z|^{-N},$$

Then $f(\cdot + iy)$ has a limit $f_0 \in \mathcal{D}'(I)$ of order $\leq N + 1$ as $y \to 0$, i.e.,

$$\lim_{y \to 0} \int f(x + iy)\varphi(x)\, dx f_0(\varphi), \qquad \varphi \in \mathcal{D}(I).$$

A similar result holds in higher dimensions.

4.2 Multiplication of distributions by functions

Take $f \in L^1_{\text{loc}}(X)$, $\psi \in C^\infty(X)$, and $\varphi \in C_0^\infty(X)$.
Then $\psi f \in L^1_{\text{loc}}(X)$, $\psi\varphi \in C_0^\infty(X)$, and we get

$$u_{\psi f}(\varphi) = \int_X (\psi(x)f(x))\varphi(x)\,dx = \int_X f(x)(\psi(x)\varphi(x))\,dx = u_f(\psi\varphi).$$

Definition: For $\psi \in C^\infty(X)$ and $u \in \mathcal{D}(X)$ we define the product of ψu of ψ and u by

$$(\psi u)(\varphi) = u(\psi\varphi).$$

If $(\varphi_j)\varphi$ in $\mathcal{D}(X)$ then the sequence $(\psi\varphi_j) \to \psi\varphi$ in $\mathcal{D}(X)$.
Hence ψu is a distribution on X.

Examples:

- $x\,\mathrm{PV}\left(\frac{1}{x}\right) = 1.$

  In fact, for every $\varphi \in C_0^\infty(\mathbb{R})$ we have

$$(x\,\mathrm{PV}\left(\frac{1}{x}\right))(\varphi) = (\mathrm{PV}\left(\frac{1}{x}\right))(x\varphi)$$

$$= \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{x\varphi(x)}{x}\,dx = \int_{\mathbb{R}} 1\,\varphi(x)\,dx = u_1(\varphi).$$

- $(x-a)\delta_a = 0.$

  In fact, for every $\varphi \in C_0^\infty(\mathbb{R})$ we have

$$\big((x-a)\delta_a\big)(\varphi) = ((x-a)\varphi)(a) = 0.$$

4.3 Differentiaton of distributions

Motivation:
Let $f \in C^1(\mathbb{R})$. Then

$$u_{f'}(\varphi) = \int_{-\infty}^{+\infty} f'(x)\varphi(x)\,dx = \Big[f(x)\varphi(x)\Big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(x)\varphi'(x)\,dx$$
$$= -\int_{-\infty}^{+\infty} f(x)\varphi'(x)\,dx = -u_f(\varphi').$$

It is clear that $\varphi \mapsto -u_f(\varphi')$ is a linear functional on $C_0^\infty(\mathbb{R})$ and that it defines a distribution. If $f \in C^k(\mathbb{R})$, then we get by induction

$$u_{f^{(k)}}(\varphi) = (-1)^k u_f(\varphi^{(k)}). \tag{1}$$

This formula is the basis for our definition of the derivative of a distribution:

Differentiaton of distributions - definition

Definition: Let $u \in \mathcal{D}'(X)$ be a distribution on X in $\mathbb{R}$. Then its *derivative* is defined as the distribution

$$u'(\varphi) = -u(\varphi'), \qquad \varphi \in C_0^\infty(X),$$

and for every natural number $k > 0$ we define the k-th derivative $u^{(k)}$ of u as the distribution

$$u^{(k)}(\varphi) = (-1)^k u(\varphi^{(k)}), \qquad \varphi \in C_0^\infty(X).$$

If $u = u_f$, where $f \in L^1_{\text{loc}}(X)$, then $(u_f)'$ is called the *weak derivative* of f or the *derivative of f in the sense of distributions* and we then write f' for $(u_f)'$, when it is obvious that we are referring to the weak derivative.

The weak k-th derivative of $f \in C^k(X)$ is simply the distribution that $f^{(k)}$ defines, i.e.

$$(u_f)^{(k)} = u_{f^{(k)}}.$$

Partial derivatives of distributions - motivation

Let X be an open subset of $\mathbb{R}^n$ and $f \in C^1(X)$. Then the distribution $u_{\partial_j f}$ acting on $\varphi \in C_0^\infty(\mathbb{R}^n)$ is calculated by performing a partial integration with respect to the variable x_j,

$$u_{\partial_j f}(\varphi) = \int_{\mathbb{R}^n} \partial_j f(x)\varphi(x)\,dx = -\int_{\mathbb{R}^n} f(x)\partial_j\varphi(x)\,dx = -u_f(\partial_j\varphi).$$

It is clear that the linear functional $\varphi \mapsto -u_f(\partial_j\varphi)$ is a distribution. If $f \in C^k(\mathbb{R}^n)$, then by induction on k we get

$$u_{\partial^\alpha f}(\varphi) = (-1)^{|\alpha|} u_f(\partial^\alpha\varphi),$$

Partial derivatives of distributions - definition

Definition: Let u be a distribution on an open subset X of $\mathbb{R}^n$. Then the partial derivative $\partial_j u$ is defined by

$$\partial_j u(\varphi) = -u(\partial_j \varphi), \qquad \varphi \in C_0^\infty(X),$$

and for every multi-index α we define the partial derivative $\partial^\alpha u$ of u as the distribution

$$\partial^\alpha u(\varphi) = (-1)^{|\alpha|} u(\partial^\alpha \varphi), \qquad \varphi \in C_0^\infty(X).$$

The Leibniz rule

Proposition: If X is open in $\mathbb{R}^n$, $u \in \mathcal{D}'(X)$, and $\psi \in C^\infty(X)$, then we have Leibniz' rule

$$\partial_j(\psi u) = (\partial_j \psi)u + \psi \partial_j u$$

Examples:

- Let $H = \chi_{\mathbb{R}_+}$ denote the Heaviside-function,

$$\begin{aligned}(u_H)'(\varphi) &= -\int_{-\infty}^{+\infty} H(x)\varphi'(x)\,dx \\ &= -\int_0^{+\infty} \varphi'(x)\,dx = -\Big[\varphi(x)\Big]_0^{+\infty} = \varphi(0) = \delta_0(\varphi)\end{aligned}$$

The conclusion is

$$H' = \delta_0.$$

- The natural logarithm $f(x) = \ln|x|$ is is locally integrable on $\mathbb{R}$ and infinitely differentiable on $\mathbb{R} \setminus \{0\}$ with derivative $f'(x) = 1/x$. Its weak derivative is

$$\begin{aligned}\langle (\ln|\cdot|)', \varphi \rangle &= -\int_{\mathbb{R}} \ln|x| \varphi'(x)\, dx \\ &= \lim_{\varepsilon \to 0} -\int_{\varepsilon}^{+\infty} \ln x(\varphi'(x) + \varphi'(-x))\, dx \\ &= \lim_{\varepsilon \to 0} \left(-\ln \varepsilon(\varphi(\varepsilon) - \varphi(-\varepsilon)) + \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx \right).\end{aligned}$$

By Taylor's formula $\varphi(\varepsilon) - \varphi(-\varepsilon) = 2\varphi'(0)\varepsilon + O(\varepsilon^2)$, so the first term in the right hand side tends to 0 as $\varepsilon \to 0$. Hence we have

$$\langle (\ln|\cdot|)', \varphi \rangle = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx = \mathrm{PV}\left(\frac{1}{x}\right)(\varphi)$$

which tells us that $(\ln|x|)' = \mathrm{PV}(1/x)$ in the sense of distributions.

Partial differential equations

Linear partial differential operator:

with coefficients $a_\alpha \in C^\infty(X)$ is of the from

$$P(\partial) = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$$

Partial differential equations for distributions:

For a given *fin*$\mathcal{D}(X)$ we look for distributions $f \in \mathcal{D}(X)$ satisfying

$$P(\partial)u = f.$$

A *distribution solution* or a *weak solution* is a distribution u in $\mathcal{D}'(X)$ satisfying the equation.

If all the coefficients are constant functions, then we say that the operator $P(\partial)$ has *constant coefficients*.

Fundamental solutions

Definition:

$u \in \mathcal{D}'(\mathbb{R}^n)$ is called a *fundamental solution* of $P(\partial)$ if

$$P(\partial)u = \delta_0.$$

Examples:

- The Cauchy kernel $E(z) = \dfrac{1}{\pi} = \dfrac{1}{\pi(x+iy)}$ is a fundamental solution of the Cauchy-Riemann operator

$$P(\partial_x, \partial_y) = \tfrac{1}{2}\left(\partial_x + i\partial_y\right)$$

- The function $E(z) = \dfrac{1}{2\pi}\log|z|$ is a fundamental solution of the Laplace operator

$$\Delta = \partial_x^2 + \partial_y^2 = 4\partial_z\partial_{\bar{z}}.$$

4.4 Convolution of distributions

Convolution of functions:

If $f, \varphi : \mathbb{R}^n \to \mathbb{C}$ are two functions and $y \mapsto f(x-y)\varphi(y)$ is an integrable function for every $x \in \mathbb{R}^n$, then the convolution $f * \varphi$ is well defined by

$$f*\varphi(x) = \int_{\mathbb{R}^n} f(x-y)\varphi(y)\,dy = \int_{\mathbb{R}^n} f(y)\varphi(x-y)\,dy, \qquad x \in \mathbb{R}^n.$$

If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$f * \varphi(x) = u_f(\varphi(x-\cdot)), \qquad x \in \mathbb{R}^n,$$

where $\varphi(x-\cdot)$ denotes the function $y \mapsto \varphi(x-y)$.

Convolution of a distribution and a function:

For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ or $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in C^\infty(\mathbb{R}^n)$,

$$u * \varphi(x) = u(\varphi(x-\cdot)), \qquad x \in \mathbb{R}^n.$$

Theorem:

We have $u * \varphi \in C^\infty(\mathbb{R}^n)$

$$\partial^\alpha(u * \varphi) = (\partial^\alpha u) * \varphi = u * (\partial^\alpha \varphi).$$

Furthermore,

$$\operatorname{supp} u * \varphi \subseteq \operatorname{supp} u + \operatorname{supp} \varphi.$$

Convolution of two distributions:

Let u and v be two distributions and assume that one of them has compact support. Then the convolution is defined in such a way that $u * v \in \mathcal{D}'(\mathbb{R}^n)$,

$$((u * v) * \varphi)(x) = (u * (v * \varphi))(x), \qquad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

One proves that there exists a unique such distribution.

Properties of convolutions

Let $u_1, u_2, u_3 \in \mathcal{D}'(\mathbb{R}^n)$ and assume that two have compact support.

$$(u_1 * u_2) * u_3 = u_1 * (u_2 * u_3)$$

$$u_1 * u_2 = u_2 * u_1$$

We observe that δ is a unity for convolution

$$\delta * u = u * \delta = u, \qquad u \in \mathcal{D}'(\mathbb{R}^n).$$

$$\partial^\alpha (u_1 * u_2) = (\partial^\alpha u_1) * u_2 = u_1 * (\partial^\alpha u_2).$$

For every partial differential operator $P = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$ with constant coefficients we have

$$P(u_1 * u_2) = (Pu_1) * u_2 = u_1 * (Pu_2).$$

The role of fundamental solutions

If $P(\partial)$ is a partial differential operator and $E \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution and $f \in \mathcal{E}'(\mathbb{R}^n)$, then $u = E * f$ satisfies

$$P(\partial)u = P(\partial)(E * f) = (P(\partial)E) * f) = (\delta * f) = f.$$

4.4 Fourier transform

For $f \in L^1(\mathbb{R}^n)$ we define

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-i\langle x,\xi\rangle} f(x)\, dx, \qquad \xi \in \mathbb{R}^n,$$

where $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$.

Observation

We look for a possible formula for a Fourier-transform of a distribution:

$$
\begin{aligned}
u_{\widehat{f}}(\varphi) &= \int_{\mathbb{R}^n} \widehat{f}(y)\varphi(y)\,dy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-i\langle x,y\rangle} f(x)\,dx \right) \varphi(y)\,dy \\
&= \int_{\mathbb{R}^n} f(x) \left(\int_{\mathbb{R}^n} e^{-i\langle x,y\rangle} \varphi(y)\,dy \right) dx = \int_{\mathbb{R}^n} f(x)\widehat{\varphi}(x)\,dx \\
&= u_f(\widehat{\varphi}).
\end{aligned}
$$

Now the problem is that if φ is in $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$ then $\widehat{\varphi}$ is not in $\mathcal{D}(\mathbb{R}^n)$.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$

If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then from the basic properties of the Fourier transform wee get

$$\mathcal{F}\{x^\beta \partial^\alpha \varphi\}(\xi) = i^{|\alpha|+|\beta|} \partial_\xi^\beta (\xi^\alpha \mathcal{F}\varphi(\xi))$$

and

$$\xi^\beta \partial^\alpha \mathcal{F}\{\varphi\}(\xi) = (-i)^{|\alpha|+|\beta|} \mathcal{F}\{\partial_x^\beta (x^\alpha \varphi(x))\}(\xi).$$

Hence for Laplace operator Δ we get $\mathcal{F}\{\Delta\varphi\}(\xi) = -|\xi|^2 \mathcal{F}\{\varphi\}(\xi)$ and consequently for every $k \in \mathbb{N}$

$$\mathcal{F}\{(1-\Delta)^k \varphi\}(\xi) = (1+|\xi|^2)^k \mathcal{F}\{\varphi\}(\xi).$$

This formula gives us the estimate

$$|\widehat{\varphi}(\xi)| \leq \frac{\|(1-\Delta)^k \varphi\|_1}{(1+|\xi|^2)^k},$$

where $\|\cdot\|_1$ is the $L^1(\mathbb{R}^n)$-norm.
If we apply it with $(-ix)^\alpha \varphi(x)$ in the role of φ, then we get

$$|\xi^\beta \partial_\xi^\alpha \widehat{\varphi}(\xi)| \leq \frac{|\xi^\beta| \|(1-\Delta)^k (x^\alpha \varphi(x))\|_1}{(1+|\xi|^2)^k}.$$

The Schwartz space of functions

Definition:

We let $\mathcal{S}(\mathbb{R}^n)$ denote the space of all $\varphi \in C^\infty(\mathbb{R}^n)$ such that for every multi-indices α and β we have

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha \varphi(x)| < +\infty.$$

Definition

A sequence (φ_j) in $\mathcal{S}(\mathbb{R}^n)$ is said to converge to φ in $\mathcal{S}(\mathbb{R}^n)$ if for every multi-indices α and β we have

$$\sup_{x\in\mathbb{R}^n} |x^\beta (\partial^\alpha \varphi_j(x) - \partial^\alpha \varphi(x))| \to 0 \qquad j \to \infty.$$

We have $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ and the inclusion is continuous, i.e. if a sequence (φ_j) converges to φ in $C_0^\infty(\mathbb{R}^n)$ then it converges to φ in $\mathcal{S}(\mathbb{R}^n)$. The inclusion $\mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ is also continuous.

Theorem

The Fourier transformation $\mathcal{F}$ is a bijective continuous map on $\mathcal{S}(\mathbb{R}^n)$ with continuous inverse

$$\mathcal{F}^{-1}\varphi(x) = \frac{1}{(2\pi)^n}\mathcal{F}\varphi(-x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\varphi(\xi)\,d\xi.$$

The Schwartz space of distributions

Definition

A linear functional $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ called a *Schwartz distribution* or a *temperate distribution* if it is continuous in the sense that $u(\varphi_j) \to u(\varphi)$ for every sequence $\varphi_j \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$.

We let $\mathcal{S}'(\mathbb{R}^n)$ denote the space of all Schwartz distributions. A sequence (u_j) of Schwartz distributions is said to converge to u in $\mathcal{S}'(\mathbb{R}^n)$ if $u_j(\varphi) \to u(\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Various spaces of distributions

We have

$$\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n).$$

and that the inclusions are continuous.
This implies that every restriction of a distribution in $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{D}(\mathbb{R}^n)$ is an element of $\mathcal{D}'(\mathbb{R}^n)$ and every restriction of a distribution in $\mathcal{E}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ is an element of $\mathcal{S}'(\mathbb{R}^n)$, i.e.,

$$\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n).$$

Fourier transformation on $\mathcal{S}'(\mathbb{R}^n)$

Definition:

We define the *Fourier transformation* $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \mathcal{F}u, \varphi \rangle = \langle u, \mathcal{F}\varphi \rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The distribution $\mathcal{F}u$ is called the *Fourier transform* of u and is also denoted $\widehat{u}$ and $\mathcal{F}\{u\}$.

The Fourier inversion formula gives that

$$\mathcal{F}\mathcal{F}\varphi = (2\pi)^n \check{\varphi}, \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n), \text{ where } \check{\varphi}(x) = \varphi(-x), \quad x \in \mathbb{R}^n$$

This implies that for every $u \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$\langle \mathcal{F}\mathcal{F}u, \varphi \rangle = \langle u, \mathcal{F}\mathcal{F}\varphi \rangle = (2\pi)^n \langle u, \check{\varphi} \rangle = (2\pi)^n \langle \check{u}, \varphi \rangle,$$

where $\check{u}$ is the pull-back of u by the map $\check{\iota} : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto -x$.

Fourier transforms of derivatives

From the rules $\mathcal{F}\{\partial_j \varphi\}(\xi) = i\xi_j \mathcal{F}\{\varphi\}(\xi)$ and $\mathcal{F}\{x_j \varphi\}(\xi) = i\partial_j \mathcal{F}\{\varphi\}(\xi)$ which hold for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we get

$$\langle \mathcal{F}\{\partial_j u\}, \varphi \rangle = \langle u, -\partial_j \mathcal{F}\varphi \rangle = \langle u, \mathcal{F}\{i\xi_j \varphi(\xi)\}\rangle = \langle i\xi_j \mathcal{F}u, \varphi(\xi)\rangle.$$

and

$$\langle \mathcal{F}\{x_j u\}, \varphi \rangle = \langle u, x_j \mathcal{F}\{\varphi\}\rangle = \langle u, \mathcal{F}\{-i\partial_j \varphi\}\rangle = \langle i\partial_j \mathcal{F}u, \varphi \rangle$$

These formulas are equivalent to

$$\mathcal{F}\{\partial_j u\} = i\xi_j \mathcal{F}u \qquad \text{and} \qquad \mathcal{F}\{x_j u\} = i\partial_j \mathcal{F}u$$

This generalizes by induction

$$\mathcal{F}\{\partial^\alpha u\} = (i\xi)^\alpha \mathcal{F}u \qquad \text{and} \qquad \mathcal{F}\{x^\beta u\} = (i\partial)^\beta \mathcal{F}u, \qquad u \in \mathcal{S}'(\mathbb{R}^n).$$

If $P(\partial) = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha$ is a partial differential operator with constant coefficients, then

$$\mathcal{F}\{P(\partial)u\} = P(i\xi)\mathcal{F}u, \qquad u \in \mathcal{S}'(\mathbb{R}^n),$$

So the partial differential equation $P(\partial)u = f$ with $f \in \mathcal{S}'(\mathbb{R}^n)$ is transformed into the equation $P(i\xi)\widehat{u} = \widehat{f}$. If it is possible to define the quotient $\widehat{f}/P(i\xi)$ as a distribution in $\mathcal{S}'(\mathbb{R}^n)$, then the solution u is its inverse Fourier transform.

Fourier transforms of convolutions

Theorem

If $\psi \in C^\infty(\mathbb{R}^n)$ satisfies a growth bound of the form

$$|\partial^\alpha \psi(x)| \leq C_\alpha (1 + |x|)^{N_\alpha}, \qquad x \in \mathbb{R}^n$$

and $u \in \mathcal{S}'(\mathbb{R}^n)$ then the product ψu is in $\mathcal{S}'(\mathbb{R}^n)$.

Theorem:

If $u_1 \in \mathcal{E}'(\mathbb{R}^n)$ and $u_2 \in \mathcal{S}'(\mathbb{R}^n)$, then $u_1 * u_2 \in \mathcal{S}'(\mathbb{R}^n)$ and

$$\widehat{u_1 * u_2} = \widehat{u}_1 \widehat{u}_2.$$

Fundamental solutions

Recall that a distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is said to be a fundamental solution of the partial differential operator $P(\partial) = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$ with constant coefficients if

$$P(\partial)E = \delta.$$

The existence of a fundamental solution for every operator $P(\partial)$ was first proved independently by Ehrenpreis and Maglgrange around 1954.

If $E \in \mathcal{S}'(\mathbb{R}^n)$ then we have seen that $P(i\xi)\widehat{E} = 1$, for $\widehat{\delta} = 1$.

Hence E is in some sense the inverse Fourier transform of $\mathbb{R}^n \ni \xi \mapsto 1/P(i\xi)$.

Lojasiewicz and Hörmander proved independently around 1958 that every partial differential operator with constant coefficients has a fundamental solution $E \in \mathcal{S}'(\mathbb{R}^n)$.

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