Sum Rules and Physical Bounds in Electromagnetics

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Outline

1 Sum rules in EM
   Finite objects

2 Conclusions

3 References
Sum rules and physical bounds on passive systems

General simple approach

1. Identify a linear and passive system.
2. Construct a Herglotz (or similarly a positive real) function \( h(z) \) that models the parameter of interest.
3. Investigate the asymptotic expansions of \( h(z) \) as \( z \to 0 \) and \( z \to \infty \).
4. Use integral identities for Herglotz functions to relate the dynamic properties to the asymptotic expansions.
5. Bound the integral.

Examples: Matching networks [2, 3], Radar absorbers [11], Antennas [6, 7, 4], Scattering [12, 1], High-impedance surfaces [9], Metamaterials [5], Extraordinary transmission [8], Periodic structures [10]
Integral identities for Herglotz functions

Herglotz functions with the symmetry \( h(z) = -h^*(-z^*) \) (real-valued in the time domain) have asymptotic expansions (\( N_0 \geq 0 \) and \( N_\infty \geq 0 \))

\[
\begin{align*}
\text{as } z \to 0 : & \quad h(z) = \sum_{n=0}^{N_0} a_{2n-1} z^{2n-1} + o(z^{2N_0-1}) \\
\text{as } z \to \infty : & \quad h(z) = \sum_{n=0}^{N_\infty} b_{1-2n} z^{1-2n} + o(z^{1-2N_\infty})
\end{align*}
\]

where \( \to \) denotes limits in the Stoltz domain \( 0 < \theta \leq \arg(z) \leq \pi - \theta \). They satisfy the identities (\( 1 - N_\infty \leq n \leq N_0 \))

\[
\begin{align*}
\lim_{\epsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\epsilon}^{1/\epsilon} \frac{\text{Im} \, h(x + iy)}{x^{2n}} \, dx = a_{2n-1} - b_{2n-1} = & \begin{cases} 
-b_{2n-1} & \text{if } n < 0 \\
 a_{-n} - b_{-n} & \text{if } n = 0 \\
 a_{-n} - b_{-n} & \text{if } n = 1 \\
 a_{2n-1} & \text{if } n > 1 
\end{cases}
\end{align*}
\]
Integral identities for Herglotz functions

Common cases

**Known low-frequency expansion** \((a_1 \geq 0)\):

\[
h(z) \sim \begin{cases} 
a_1 z & \text{as } z \to 0 

b_1 z & \text{as } z \to \infty
\end{cases}
\]

that gives the \(n = 1\) identity (we drop the limits for simplicity)

\[
\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{1/\varepsilon} \frac{\text{Im} \ h(x + iy)}{x^2} \, dx \overset{\text{def}}{=} \frac{2}{\pi} \int_{0}^{\infty} \frac{\text{Im} \ h(x)}{x^2} \, dx = a_1 - b_1 \leq a_1
\]

**Known high-frequency expansion** (short times) \((b_{-1} \leq 0)\):

\[
h(z) \sim \begin{cases} 
a_{-1} / z & \text{as } z \to 0 

b_{-1} / z & \text{as } z \to \infty
\end{cases}
\]

that gives the \(n = 0\) identity

\[
\frac{2}{\pi} \int_{0}^{\infty} \text{Im} \ h(x) \, dx = a_{-1} - b_{-1} \leq -b_{-1}.
\]
Forward scattering sum rule

\[ \hat{\epsilon} \hat{k} \quad \epsilon(r) \mu(r) \quad \hat{\epsilon} \hat{k} \quad \epsilon_0 \quad \mu_0 \]

Assumptions:

- Finite scattering object composed of a linear, passive, and time translational invariant medium.
- Incident linearly polarized plane wave.

From physics:

- The propagation (wavefront) speed is limited by the speed of light.
- Optical theorem (energy conservation).
- Induced dipole moment in the static limit.
- Shadow scattering.
Forward scattering sum rule

\[
\epsilon_0 \quad \mu_0
\]

Use the \( n = 1 \) identity with
\[
a_1 = \gamma = \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e}) \text{ and } b_1 = 0, \text{ i.e.,}
\]
\[
\frac{2}{\pi} \int_0^\infty \frac{\sigma_{\text{ext}}(k)}{k^2} \, dk = \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e})
\]

or written in the free-space wavelength \( \lambda = 2\pi/k \)
\[
\frac{1}{\pi^2} \int_0^\infty \sigma_{\text{ext}}(\lambda) \, d\lambda = \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e})
\]
- Causality in the sense that the scattered field cannot precede the incident field in the forward direction.
- Causal impulse response $h_t(t)$.
- Analytic transfer function $h(k)$, (Fourier, Laplace transform of $h_t(t)$, where $k = 2\pi f/c_0$ denotes the free-space wavenumber.).
Causality in the sense that the scattered field cannot precede the incident field in the forward direction.

Causal impulse response $h_t(t)$.

Analytic transfer function $h(k)$, (Fourier, Laplace transform of $h_t(t)$, where $k = 2\pi f/c_0$ denotes the free-space wavenumber.).
Energy conservation (passivity)

\[ W_{\text{ext},\tau} = - \int_{-\infty}^{T} \int_{\partial V} \left( E_i(t, r) \times H_s(t, r) + E_s(t, r) \times H_i(t, r) \right) \cdot \hat{n}(r) \, dS \, dt. \]

simplify to

\[ W_{\text{ext}} = \int_{-\infty}^{T} \int_{\mathbb{R}} E(t) h(t-t) E(\tau) \, dt \, d\tau \geq 0 \]

for all \( E \) implying

\[ \text{Im } h(k) = \sigma_{\text{ext}}(k) \geq 0 \quad \text{for } \text{Im } k > 0 \]

cf., the optical theorem.

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Low-frequency asymptotic expansion

\[ h(k) = \hat{e} \cdot \gamma_e \hat{e} k + O(k^2) \text{ as } k \to 0 \) (Kleinman & Senior 1986).

- Polarizability dyadic \( \gamma_e \).
- Induced dipole moment \( p = \epsilon_0 \gamma_e E_0 \hat{e} \).
- Variational principles \( \gamma_e \leq \gamma_\infty \) (Jones 1985, Sjöberg 2009).
- High contrast polarizability dyadic \( \gamma_\infty \).
High-frequency asymptote

▶ Shadow scattering (Peierls 1979, Gustafsson et al. 2008).
▶ $\mathrm{Im} \ h(k) = \sigma_{\text{ext}}(k) \leq 2A$ on average as $k \to \infty$. $h(k) \to 0$, i.e., for $0 < \delta < \arg k < \pi - \delta$.
▶ the extinction paradox.
Forward scattering sum rule

\[ \epsilon_0 \mu_0 \]

\[ \hat{e} \quad \hat{k} \]

\[ \epsilon(r) \mu(r) \]

\[ \hat{e} \quad \hat{k} \]

Use the \( n = 1 \) identity with

\[ a_1 = \gamma = \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e}) \]

and \( b_1 = 0 \), i.e.,

\[ \frac{2}{\pi} \int_0^\infty \frac{\sigma_{\text{ext}}(k)}{k^2} \, dk = \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e}) \]

or written in the free-space wavelength \( \lambda = 2\pi/k \)

\[ \frac{1}{\pi^2} \int_0^\infty \sigma_{\text{ext}}(\lambda) \, d\lambda = \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e}) \]
Polarizability dyadic and induced dipole moment

The induced dipole moment can be written

\[ p = \epsilon_0 \gamma_e \cdot E \]

where \( \gamma_e \) is the polarizability dyadic.

Example (Dielectric sphere)

A dielectric sphere with radius \( a \) and relative permittivity \( \epsilon_r \) has the polarizability dyadic

\[ \gamma_e = 4\pi a^3 \frac{\epsilon_r - 1}{\epsilon_r + 2} \mathbf{I} \rightarrow \gamma_\infty = 4\pi a^3 \mathbf{I} \]

as \( \epsilon_r \to \infty \).

Analytic expressions for spheroids, elliptic discs, half spheres, hollow half spheres, touching spheres,...
Extinction cross sections for $a = 50 \text{ nm}$ spheres

$$\sigma_{\text{ext}} = \sigma_a + \sigma_s = \frac{P_a + P_s}{|E_i|^2 / 2\eta_0}$$

Sum of the scattered and absorbed powers divided by the incident power flux. Integrate over the free-space wavelength $\lambda = \frac{2\pi}{k}$

$$\frac{1}{\pi^2} \int_0^\infty \sigma_{\text{ext}}(\lambda) \, d\lambda = \hat{e} \cdot \gamma_e \cdot \hat{e} = 4\pi a^3$$

The areas under the curves are identical.

*Time-domain approach to the forward scattering sum rule, Proc.R.Soc.A, 2010*
Forward scattering measurement of a dipole antenna.

- Loaded, short, and open circuit.
- Length 15 cm and 0.5 GHz to 6 GHz.

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<th>in cm³</th>
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Physical bounds on antennas

Given a geometry, e.g., sphere, rectangle, spheroid, or cylinder. How does $D/Q$ (directivity bandwidth product) depend on the geometry for optimal antennas?

$$\frac{D}{Q} \leq \frac{\eta k_0^3}{2\pi} \left( \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e}) \right)$$

based on

- Passive materials
- Antenna forward scattering
- Identities for Herglotz function

Absorption Efficiency and Physical Bounds on Antennas, Int. J. of Antennas and Propagat., 946746, 2010
http://www.mathworks.com/matlabcentral/fileexchange/26806-antennaq

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Circumscribing rectangles

\[ \frac{D}{Q}/(k_0 a)^3 \]

Chu bound, \( k_0 a \ll 1 \)

physical bounds

\( \hat{\epsilon} \)

\( \ell_1 \)

\( \ell_2 \)

\( \eta = 1 \)

\( \eta = 1/2 \)

\( \ell_1/\ell_2 \)
Chu bound, $k_0 a \ll 1$

physical bounds

$D/Q/(k_0 a)^3$
How can we measure the polarizability?

- Change of capacitance in a parallel plate capacitor.
- The polarizability in a parallel plate waveguide.
- The periodic polarizability for symmetric objects.

Objects with increasing distance between the coins. Large separation of charge give a large polarizability.

High-contrast polarizability dyadics: $\gamma_\infty$

$\gamma_\infty$ is determined from the induced normalized surface charge density, $\rho$, as

$$\hat{e} \cdot \gamma_\infty \cdot \hat{e} = \frac{1}{E_0} \int_{\partial V} \hat{e} \cdot r \rho(r) \, dS$$

where $\rho$ satisfies the integral equation

$$\int_{\partial V} \frac{\rho(r')}{4\pi|r - r'|} \, dS' = E_0 r \cdot \hat{e} - V_n$$

with the constraints of zero total charge

$$\int_{\partial V_n} \rho(r) \, dS = 0$$

Can also use FEM (Laplace equation).
Removal of metal from a square plate

\[ \frac{\gamma_\infty}{\ell^3} \]

\[ \frac{A}{\ell^2} \]

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Removal of metal from a square plate and circular disk

\[
\frac{16}{3} a^3 (\hat{x} \hat{x} + \hat{y} \hat{y})
\]

\[
a^3 (4.3 \hat{x} \hat{x} + 4.5 \hat{y} \hat{y})
\]

\[
a^3 (4.0 \hat{x} \hat{x} + 4.8 \hat{y} \hat{y})
\]

\[
1.04 \ell^3 (\hat{x} \hat{x} + \hat{y} \hat{y})
\]

\[
\ell^3 (0.94 \hat{x} \hat{x} + 0.96 \hat{y} \hat{y})
\]

\[
\ell^3 (0.51 \hat{x} \hat{x} + 0.93 \hat{y} \hat{y})
\]
Sum rules for passive systems

- Spherical modes
- Extra ordinary transmission
- Forward scattering
- Scattering
- Antennas
- Absorbers
- High-impedance surfaces
- Cross section
- Blockage
- Extra ordinary transmission
- Periodic structures
- Dispersion
- Artificial $\mu$
- $\epsilon$ near 0
- Negative refraction
- Matching

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Sum rules for passive systems

- Spherical modes
  - $|\Gamma| \ll 1$
  - $\Gamma(0) = 1$
- Extra ordinary transmission
- Forward scattering
  - $\text{Re} Y_{\text{big}}$
  - $\text{Y}(0) = 0$
- Artificial $\mu$
  - $|Y| \approx 0$
  - $\text{Y}(0) = \infty$
- Dispersion
  - $\epsilon$ near 0
  - $|Y| \approx 0$
  - $\text{Y}(\infty) = \infty$
- Negative refraction
  - $|Y| \approx 0$
  - $\text{Y}(\infty) = \infty$
- Antennas
  - $|\Gamma| \ll 1$
  - $\Gamma(0) = -1$
- Absorbers
  - $|\Gamma| \ll 1$
  - $\Gamma(0) = -1$
- High-impedance surfaces
  - $|Y| \approx 0$
  - $\text{Y}(0) = \infty$
- Cross section
  - $\text{Re} Y_{\text{big}}$
  - $\text{Y}(0) = 0$
- Blockage
  - $|T| \ll 1$
  - $T(0) = 1$
- Extra ordinary transmission
  - $|\text{Y}| \approx 0$
  - $\text{Y}(0) = \infty$
- Matching
  - $|\Gamma| \ll 1$
  - $\Gamma(0) = 1$

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Basically four (or two) cases

**admittance** want
large $\text{Re} \ Y(\omega_0)$ with $Y(0) = 0$.
forward scattering (cross section)
Use identity

**$S$-parameter** want
$|S'(\omega_0)| \leq \delta$ with $|S'(0)| = 1$.
absorber, matching, blockage, modes, ...
Use log+identity

**admittance** want
small $|Y(\omega_0)| \leq \delta$ with $Y(0) = \infty$.
high impedance surface, temporal dispersion.
Use pulse+identity

**$S$-parameter** want
$S'(\omega_0) \approx 1$ with $S'(0) = -1$.
high impedance surface, extra ordinary transmission
Use Cayley+pulse+identity
1 Sum rules in EM
   Finite objects

2 Conclusions

3 References
Conclusions

- Sum rules derived from integral identities for Herglotz functions.
- Physical bounds.
- Extinction cross section, antennas, extraordinary transmission, transmission blockage, high-impedance surfaces, radar absorbers, temporal dispersion, perfect lenses, artificial magnetism.

Why physical bounds for passive systems?
- Realistic expectations. Possible/impossible.
- Possible design improvements. Is it worth it?
- Figure of merit for a design.
- Use of active and/or non-linear systems.
Passive systems and Herglotz functions

Antennas

Forward scattering

Temporal dispersion in metamaterials

Periodic structures

Extraordinary transmission
Outline

1. Sum rules in EM
   Finite objects

2. Conclusions

3. References


