# Sum Rules and Physical Bounds in Electromagnetics

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# Outline

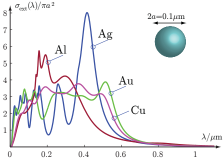
1. **Sum rules in EM**
   Finite objects
2. **Conclusions**
3. **References**

# Sum rules and physical bounds on passive systems

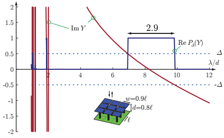
## General simple approach

1. Identify a linear and passive system.
2. Construct a Herglotz (or similarly a positive real) function $h(z)$ that models the parameter of interest.
3. Investigate the asymptotic expansions of $h(z)$ as $z\hat{\to}0$ and $z\hat{\to}\infty$.
4. Use integral identities for Herglotz functions to relate the dynamic properties to the asymptotic expansions.
5. Bound the integral.

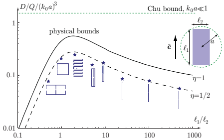
**Examples:** Matching networks [2, 3], Radar absorbers [11], Antennas [6, 7, 4], Scattering [12, 1], High-impedance surfaces [9], Metamaterials [5],Extraordinary transmission [8],Periodic structures [10]




Cross sections of nano spheres.




High-impedance surface.

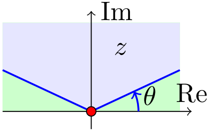



Antenna $D/Q$.

# Integral identities for Herglotz functions

Herglotz functions with the symmetry $h(z) = -h^*(-z^*)$ (real-valued in the time domain) have asymptotic expansions ($N_0 \geq 0$ and $N_\infty \geq 0$)

$$\begin{cases} h(z) = \sum_{n=0}^{N_0} a_{2n-1} z^{2n-1} + o(z^{2N_0-1}) & \text{as } z \hat{\to} 0 \\ h(z) = \sum_{n=0}^{N_\infty} b_{1-2n} z^{1-2n} + o(z^{1-2N_\infty}) & \text{as } z \hat{\to} \infty \end{cases}$$




where $\hat{\to}$ denotes limits in the Stoltz domain $0 < \theta \leq \arg(z) \leq \pi - \theta$ ??. They satisfy the identities ($1 - N_\infty \leq n \leq N_0$)

$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{\operatorname{Im} h(x + \mathrm{i}y)}{x^{2n}} \, \mathrm{d}x = a_{2n-1} - b_{2n-1} = \begin{cases} -b_{2n-1} & n < 0 \\ a_{-1} - b_{-1} & n = 0 \\ a_1 - b_1 & n = 1 \\ a_{2n-1} & n > 1 \end{cases}$$

Bernland, Luger, Gustafsson, *Sum rules and constraints on passive systems*, J. Phys. A: Math. Theor., 2011.

# Integral identities for Herglotz functions

## Common cases

**Known low-frequency expansion** $(a_1 \geq 0)$:

$$h(z) \sim \begin{cases} a_1 z & \text{as } z \hat{\to} 0 \\ b_1 z & \text{as } z \hat{\to} \infty \end{cases}$$

that gives the $n = 1$ identity (we drop the limits for simplicity)

$$\lim_{\varepsilon\to 0^+} \lim_{y\to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{1/\varepsilon} \frac{\operatorname{Im} h(x+\mathrm{i}y)}{x^2}\,\mathrm{d}x \overset{\text{def}}{=} \frac{2}{\pi} \int_0^{\infty} \frac{\operatorname{Im} h(x)}{x^2}\,\mathrm{d}x = a_1 - b_1 \leq a_1$$

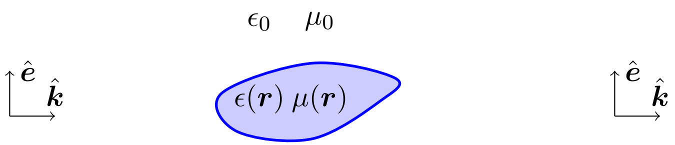
**Known high-frequency expansion (short times)** $(b_{-1} \leq 0)$:

$$h(z) \sim \begin{cases} a_{-1}/z & \text{as } z \hat{\to} 0 \\ b_{-1}/z & \text{as } z \hat{\to} \infty \end{cases}$$

that gives the $n = 0$ identity

$$\frac{2}{\pi} \int_0^{\infty} \operatorname{Im} h(x)\,\mathrm{d}x = a_{-1} - b_{-1} \leq -b_{-1}.$$

# Forward scattering sum rule



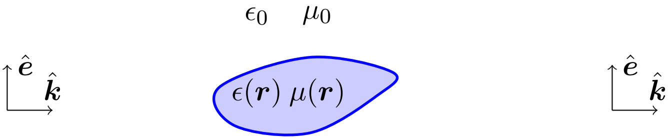

**Assumptions:**

- Finite scattering object composed of a linear, passive, and time translational invariant medium.
- Incident linearly polarized plane wave.

**From physics:**

- The propagation (wavefront) speed is limited by the speed of light.
- Optical theorem (energy conservation).
- Induced dipole moment in the static limit.
- Shadow scattering

# Forward scattering sum rule



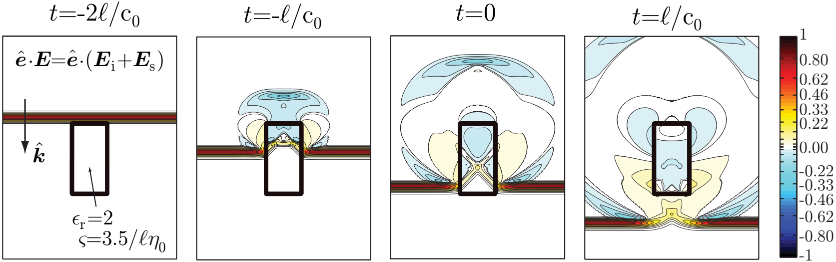

Use the $n = 1$ identity with
$a_1 = \gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$ and $b_1 = 0$, *i.e.*,

$$\frac{2}{\pi} \int_0^\infty \frac{\sigma_{\mathrm{ext}}(k)}{k^2} \,\mathrm{d}k = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$$

or written in the free-space wavelength $\lambda = 2\pi/k$

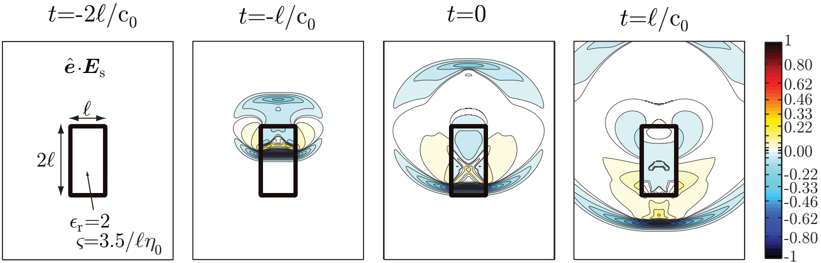
$$\frac{1}{\pi^2} \int_0^\infty \sigma_{\mathrm{ext}}(\lambda) \,\mathrm{d}\lambda = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$$

# Propagation speed limited by the speed of light




- Causality in the sense that the scattered field cannot precede the incident field in the forward direction.
- Causal impulse response $h_t(t)$.
- Analytic transfer function $h(k)$, (Fourier, Laplace transform of $h_t(t)$, where $k = 2\pi f/c_0$ denotes the free-space wavenumber.).

# Propagation speed limited by the speed of light




- Causality in the sense that the scattered field cannot precede the incident field in the forward direction.
- Causal impulse response $h_t(t)$.
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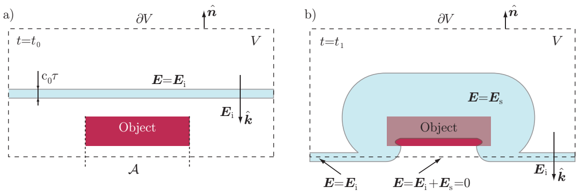
# Energy conservation (passivity)




$$W_{\text{ext},\tau} = -\int_{-\infty}^{T}\int_{\partial V}\big(\boldsymbol{E}_{\rm i}(t,\boldsymbol{r})\times\boldsymbol{H}_{\rm s}(t,\boldsymbol{r})+\boldsymbol{E}_{\rm s}(t,\boldsymbol{r})\times\boldsymbol{H}_{\rm i}(t,\boldsymbol{r})\big)\cdot\hat{\boldsymbol{n}}(\boldsymbol{r})\,\mathrm{dS}\,\mathrm{d}t$$

simplify to

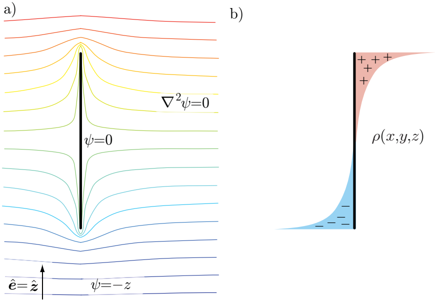
$$W_{\text{ext}} = \int_{-\infty}^{T}\int_{\mathbb{R}} E(t)h_{\rm t}(\tau-t)E(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \geq 0$$

for all $E$ implying

$$\operatorname{Im} h(k) = \sigma_{\text{ext}}(k) \geq 0 \quad \text{for } \operatorname{Im} k > 0$$

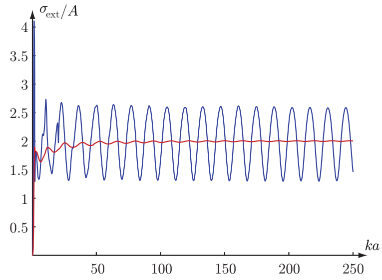
*cf.*, the optical theorem.

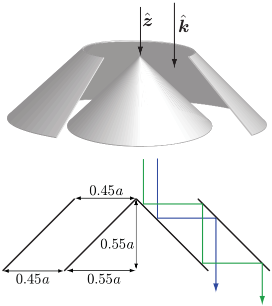
# Low-frequency asymptotic expansion




- $h(k) = \hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\hat{\boldsymbol{e}}k + \mathcal{O}(k^2)$ as $k\to 0$ (Kleinman&Senior 1986).
- Polarizability dyadic $\boldsymbol{\gamma}_{\mathrm{e}}$.
- Induced dipole moment $\boldsymbol{p} = \epsilon_0\boldsymbol{\gamma}_{\mathrm{e}}E_0\hat{\boldsymbol{e}}$.
- Variational principles $\boldsymbol{\gamma}_{\mathrm{e}} \leq \boldsymbol{\gamma}_{\infty}$ (Jones 1985, Sjöberg 2009).
- High contrast polarizability dyadic $\boldsymbol{\gamma}_{\infty}$.

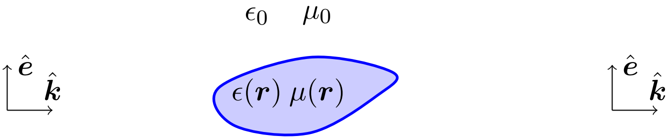
# High-frequency asymptote




- Shadow scattering (Peierls 1979, Gustafsson *etal* 2008).
- $\operatorname{Im} h(k) = \sigma_{\rm ext}(k) \leq 2A$ on average as $k \to \infty$. $h(k) \hat{\to} 0$, *i.e.*, for $0 < \delta < \arg k < \pi - \delta$.
- the extinction paradox.

# Forward scattering sum rule




Use the $n = 1$ identity with
$a_1 = \gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$ and $b_1 = 0$, *i.e.*,

$$\frac{2}{\pi} \int_0^\infty \frac{\sigma_{\mathrm{ext}}(k)}{k^2} \, \mathrm{d}k = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$$

or written in the free-space wavelength $\lambda = 2\pi/k$

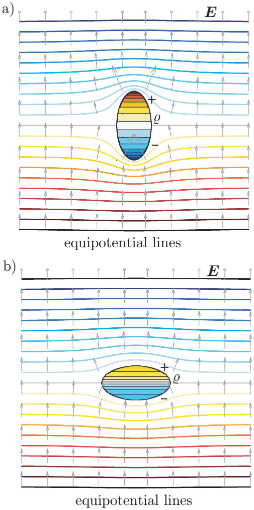
$$\frac{1}{\pi^2} \int_0^\infty \sigma_{\mathrm{ext}}(\lambda) \, \mathrm{d}\lambda = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$$

# Polarizability dyadic and induced dipole moment

The induced dipole moment can be written

$$\boldsymbol{p} = \epsilon_0 \boldsymbol{\gamma}_{\mathrm{e}} \cdot \boldsymbol{E}$$

where $\boldsymbol{\gamma}_{\mathrm{e}}$ is the polarizability dyadic.




## Example (Dielectric sphere)

A dielectric sphere with radius $a$ and relative permittivity $\epsilon_{\mathrm{r}}$ has the polarizability dyadic

$$\boldsymbol{\gamma}_{\mathrm{e}} = 4\pi a^3 \frac{\epsilon_{\mathrm{r}} - 1}{\epsilon_{\mathrm{r}} + 2} \mathbf{I} \to \boldsymbol{\gamma}_{\infty} = 4\pi a^3 \mathbf{I}$$

as $\epsilon_{\mathrm{r}} \to \infty$.

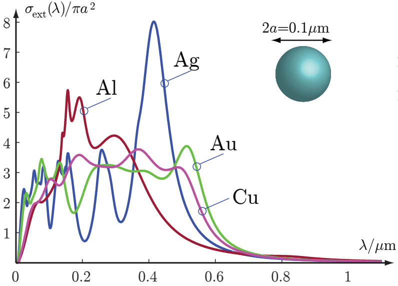
Analytic expressions for spheroids, elliptic discs, half spheres, hollow half spheres, touching spheres,...

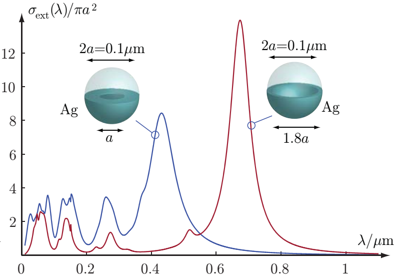
# Extinction cross sections for $a = 50\,\text{nm}$ spheres

$$\sigma_{\text{ext}} = \sigma_{\text{a}} + \sigma_{\text{s}} = \frac{P_{\text{a}} + P_{\text{s}}}{|\boldsymbol{E}_{\text{i}}|^2 / 2\eta_0}$$

Sum of the scattered and absorbed powers divided by the incident power flux. Integrate over the free-space wavelength $\lambda = 2\pi/k$

$$\frac{1}{\pi^2} \int_0^\infty \sigma_{\text{ext}}(\lambda)\, \mathrm{d}\lambda = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\text{e}} \cdot \hat{\boldsymbol{e}} = 4\pi a^3$$

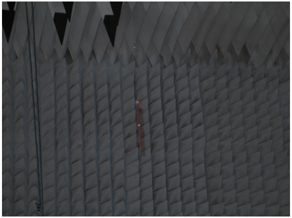


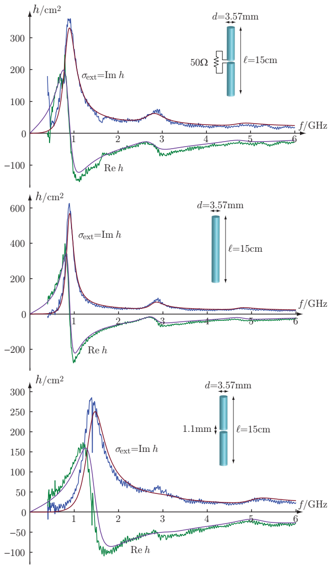




The areas under the curves are identical.

*Time-domain approach to the forward scattering sum rule*, Proc.R.Soc.A, 2010

# Forward scattering of antennas






- Forward scattering measurement of a dipole antenna.
- Loaded, short, and open circuit.
- Length $15\,\mathrm{cm}$ and $0.5\,\mathrm{GHz}$ to $6\ \mathrm{GHz}$.

| | in $\mathrm{cm}^3$ | loaded | short | open |
|---|---|---|---|---|
| sim: | $\gamma$ | 661 | 661 | 291 |
| sim: | $\frac{2}{\pi}\int_0^{k_2}\frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k$ | 644 | 644 | 265 |
| meas: | $\frac{2}{\pi}\int_{k_1}^{k_2}\frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k$ | 605 | 670 | 322 |

*Forward scattering of loaded and unloaded antennas*, IEEE-TAP, 2012.

# Physical bounds on antennas

Given a geometry, *e.g.*, sphere, rectangle, spheroid, or cylinder. How does $D/Q$ (directivity bandwidth product) depend on the geometry for optimal antennas?

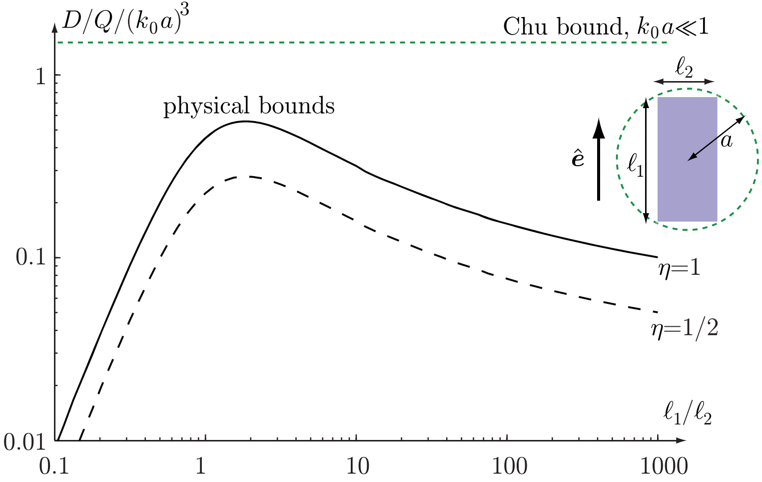
$$\frac{D}{Q} \leq \frac{\eta k_0^3}{2\pi}\big(\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}}+(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}})\cdot\boldsymbol{\gamma}_{\mathrm{m}}\cdot(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}})\big)$$

based on

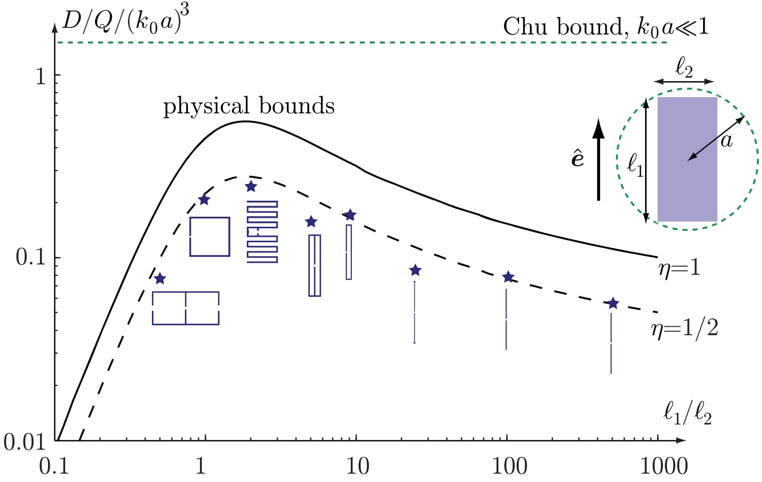
- Passive materials
- Antenna forward scattering
- Identities for Herglotz function

Physical limitations on antennas of arbitrary shape Proc R. Soc. A, 463. 2589-2607, 2007.
Illustrations of new physical bounds on linearly polarized antennas, IEEE Trans. Antennas Propagat., 2009.
Absorption Efficiency and Physical Bounds on Antennas, Int. J. of Antennas and Propagat., 946746, 2010
http://www.mathworks.com/matlabcentral/fileexchange/26806-antennaq

# Circumscribing rectangles

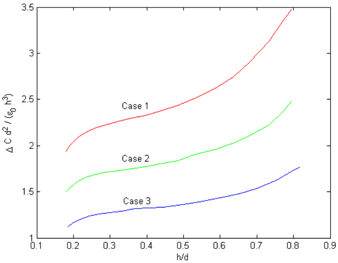


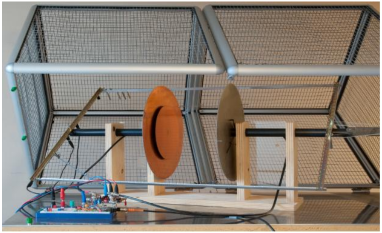
# Circumscribing rectangles

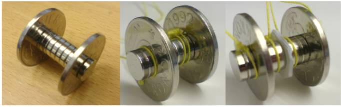


# How can we measure the polarizability?

- Change of capacitance in a parallel plate capacitor.
- The polarizability in a parallel plate waveguide.
- The periodic polarizability for symmetric objects.





**Case 1** **Case 2** **Case 3**

Objects with increasing distance between the coins.
Large separation of charge give a large polarizability.

D. Lovrić, *Theoretical and experimental studies of polarizability dyadics* , 2011. Kristensson, *The polarizability and the capacitance change of a bounded object in a parallel plate capacitor*, Physica Scripta, 2012.

# High-contrast polarizability dyadics: $\gamma_\infty$

$\gamma_\infty$ is determined from the induced normalized surface charge density, $\rho$, as

$$\hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_\infty \cdot \hat{\boldsymbol{e}} = \frac{1}{E_0} \int_{\partial V} \hat{\boldsymbol{e}} \cdot \boldsymbol{r} \rho(\boldsymbol{r}) \, \mathrm{dS}$$

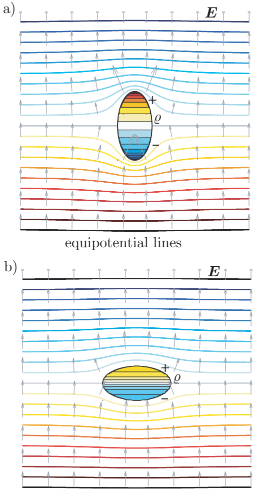
where $\rho$ satisfies the integral equation

$$\int_{\partial V} \frac{\rho(\boldsymbol{r}')}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{dS}' = E_0 \boldsymbol{r} \cdot \hat{\boldsymbol{e}} - V_n$$

with the constraints of zero total charge

$$\int_{\partial V_n} \rho(\boldsymbol{r}) \, \mathrm{dS} = 0$$

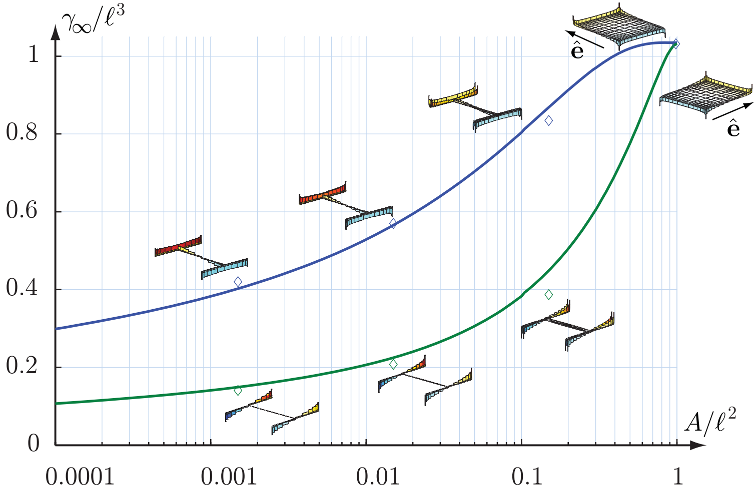
Can also use FEM (Laplace equation).



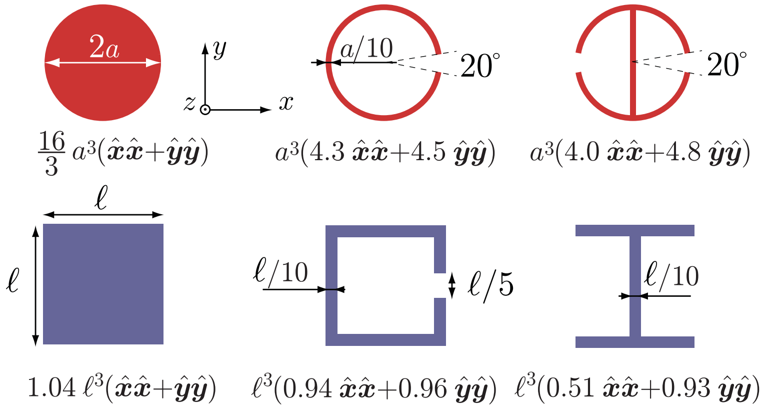
equipotential lines

equipotential lines

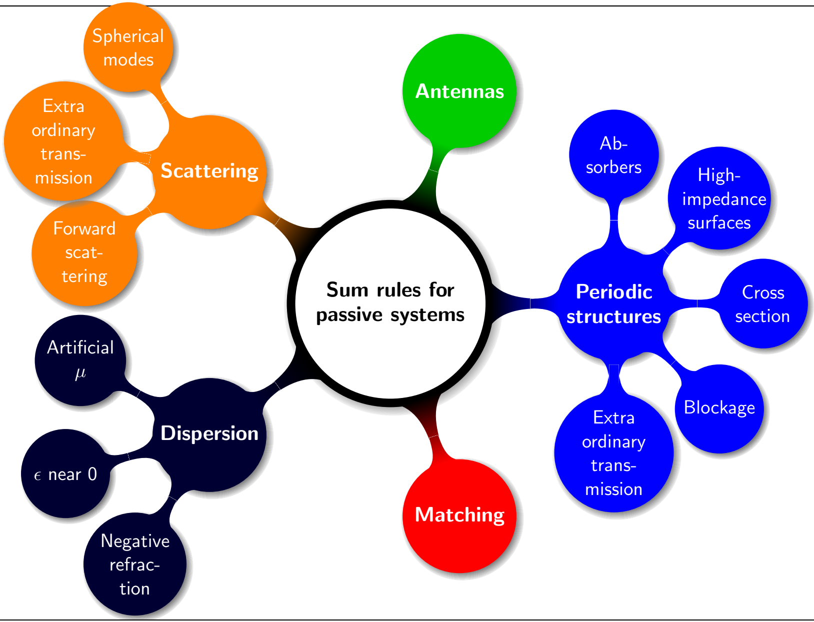
# Removal of metal from a square plate



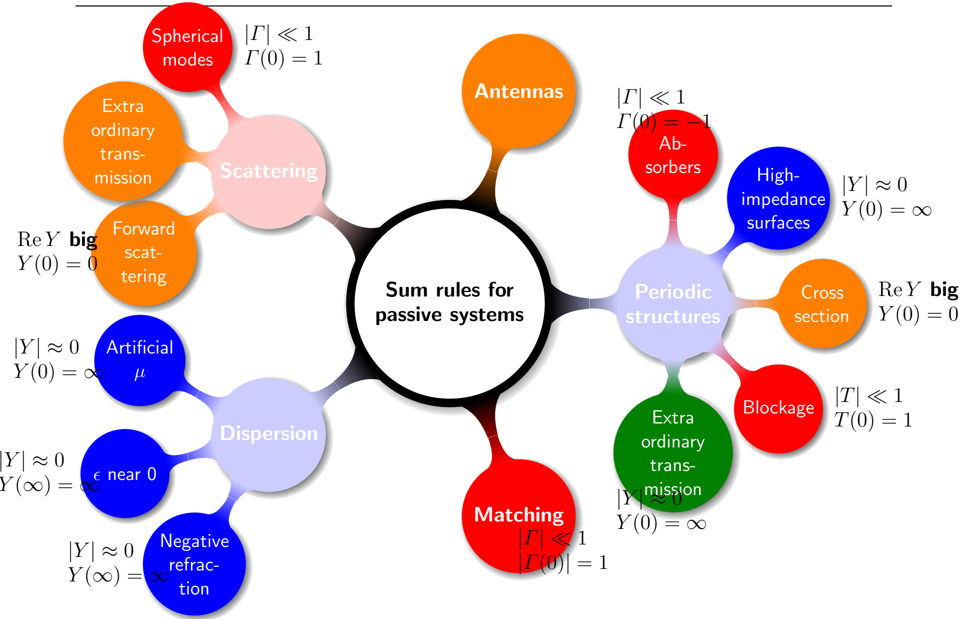
# Removal of metal from a square plate and circular disk



# Sum rules for passive systems



# Sum rules for passive systems



# Basically four (or two) cases

**admittance** want
large $\mathrm{Re}\, Y(\omega_0)$ with $Y(0) = 0$.
forward scattering (cross section)
Use identity

**S-parameter** want
$|S(\omega_0)| \leq \delta$ with $|S(0)| = 1$.
absorber, matching, blockage, modes, ...
Use log+identity

**admittance** want
small $|Y(\omega_0)| \leq \delta$ with $Y(0) = \infty$.
high impedance surface, temporal dispersion.
Use pulse+identity

**S-parameter** want
$S(\omega_0) \approx 1$ with $S(0) = -1$.
high impedance surface, extra ordinary transmission
Use Cayley+pulse+identity

# Outline

1. **Sum rules in EM**
   Finite objects

2. **Conclusions**

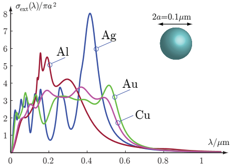
3. **References**

# Conclusions

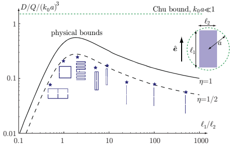
- Sum rules derived from integral identities for Herglotz functions.
- Physical bounds.
- Extinction cross section, antennas, extra ordinary transmission, transmission blockage, high-impedance surfaces, radar absorbers, temporal dispersion, perfect lenses, artificial magnetism.

Why physical bounds for passive systems?

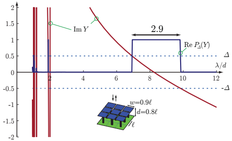
- Realistic expectations. Possible/impossible.
- Possible design improvements. Is it worth it?
- Figure of merit for a design.
- Use of active and/or non-linear systems.



Cross sections of nano spheres.



Antenna $D/Q$.



High-impedance surface.

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