Lecture on multidimensional passivity with applications to electrodynamics

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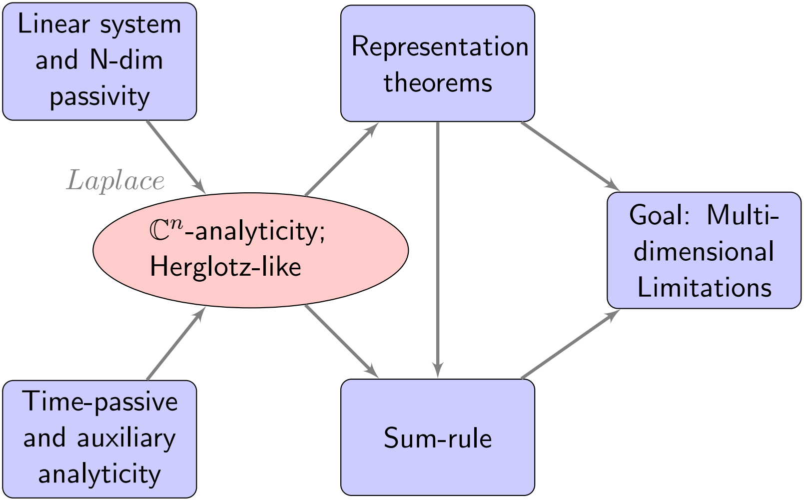
Plan of the lecture

Key ingredients

- First part is to show Analyticity and positivity properties.
  - Passive multi-dimensional systems have analytic and positive type kernels.
- Second part we examine types of representation theorems
  - Cauchy-type representation theorems (boundedness, dispersion rel.)
  - Herglotz-type/Schwartz kernel.
- Parallel to the above investigation we look on a few applications

Most of the material in these lectures are based on the books
Vladimirov, Methods of the theory of Generalized Functions, 2002
Reed & Simon Methods of Modern Mathematical Physics Part II, Fourier Analysis, Self-Adjointness 2003,
King, Hilbert Transforms, 2009

Goal: Limitations of measurable quantities



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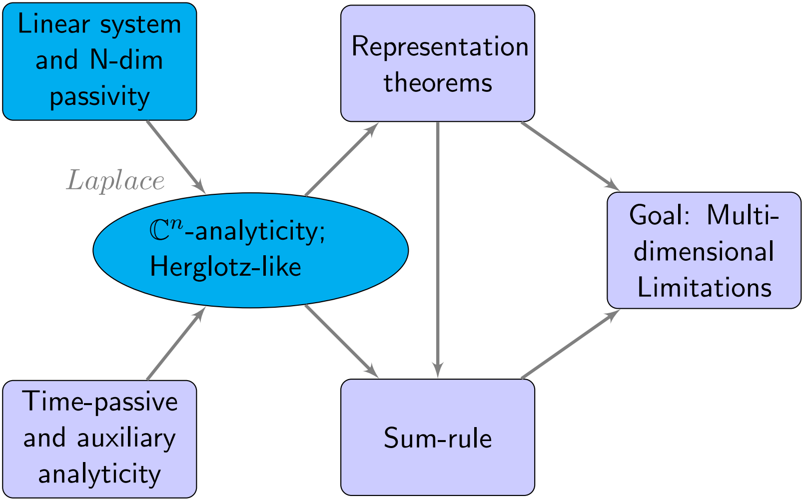

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Definition of linear system [Vladimirov 2002]

Input: $u(x) = (u_1(x), \dots, u_N(x))$. **Output:** $f(x) = (f_1, \dots, f_N)$.

- Linearity. If u_a generates f_a, and u_b generates f_b then $\alpha u_a + \beta u_b$ generates $\alpha f_a + \beta f_b$.
- Reality: If u is real, then f is real-valued.
- Continuity: If $u_j \to 0$ for all $j \in [1, N]$ in $\mathcal{E}'$ then $f_k \to 0$ in $\mathcal{D}'$ for all k.
- Translational invariance: If $f(x)$ is associated with $u(x)$ then for any translation $h \in \mathbb{R}^n$ to the original perturbed $u(x+h)$ there corresponds a response perturbation $f(x+h)$

There exists a unique $N \times N$ matrix $Z(x)$, with $Z_{jk} \in \mathcal{D}'(\mathbb{R}^n)$ such that $f = Z * u$.

$\mathcal{D}$ is smooth functions of compact support. $\mathcal{D}'$ is the space of generalized functions. $\mathcal{E}'$ is the space of generalized functions with compact support.

Admittance-passivity

A function Z is *admittance passive* relative to the cone Γ if for any $\phi(x) \in \mathcal{D}^{\times N}$ then $\langle Z, \phi * \phi^* \rangle \geq 0$, or (if $Z \in L^1_{loc}$ then

$$\mathrm{Re} \int_{-\Gamma} (Z * \phi) \cdot \bar{\phi} \, \mathrm{d}x = \mathrm{Re} \int_{-\Gamma} \int (Z(x-y)\phi(y)) \cdot \bar{\phi}(x) \, \mathrm{d}y \, \mathrm{d}x \geq 0$$

Examples:

- 1-dim, linear 1-port circuit theory RLC-nets with zero initial conditions.
- n-dim, linear n-port circuit theory with RLC-components, with zero initial conditions.
- All passive Cauchy systems with constant matrices have Z_j real symmetric of the form $\sum_j Z_j \partial_j + Z_0$, with $\sum_j q_j Z_j \geq 0, \forall q \in \mathrm{int}\, C^*$ and $\mathrm{Re}\, Z_0 \geq 0$. (Maxwell, Linear acoustics, light cone) [Vladimirov 20.6 Thm 1]

Physical system and cones

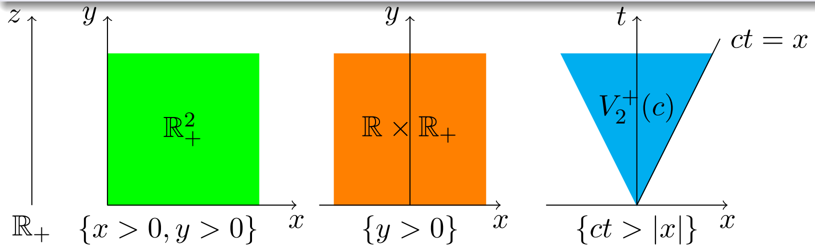
Cones are important for passivity

- The generalization of the half-line support that gives Herglotz functions is a cone.
- Passivity for a N-dimensional linear systems can be defined with respect to some cone.
- Functions with support in a cone (causality) have nice properties when the are Fourier/Laplace transformed.
- Next we examine some basic properties of cones.

Cone

- A cone $\Gamma \subset \mathbb{R}^n$, with vertex 0 is a set such that if $x \in \Gamma$, then $\lambda x \in \Gamma$ for all $\lambda > 0$.
- The conjugate Γ^* to the cone $\Gamma \subset \mathbb{R}^n$ is the set

$$\Gamma^* = \{\xi \in \mathbb{R}^n : \xi \cdot x \geq 0, \text{ for all } x \in \Gamma \subset \mathbb{R}^n\}$$



Cones in $\mathbb{R}$ and in $\mathbb{R}^2$.

Other cones: positive hermitian matrices and the functions of positive type, $f : \mathbb{R}^n \mapsto \mathbb{C}$ such that $\{f(\lambda_i - \lambda_j)\}_{ij}$ positive matrix in $\mathbb{C}^N$ for all N.

Conjugate cones – Quiz

$$\Gamma^* = \{\xi \in \mathbb{R}^n : \xi \cdot x \geq 0, \text{ for all } x \in \Gamma \subset \mathbb{R}^n\}$$

Example 1: Determine Γ^* for $\Gamma = \mathbb{R}_+$

We seek all $\xi \in \mathbb{R}$ such that $\xi x \geq 0$, for $x > 0$. Clearly $\Gamma^*_{R_+} = \{\xi \in \mathbb{R} : \xi \geq 0\} = \bar{\Gamma}_{\mathbb{R}_+}$.

Quiz

Determine Γ^* for the cone $\Gamma = \mathbb{R}^2_+$.

Conjugate cones – Quiz

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Quiz

Determine Γ^* for the cone $\Gamma = \mathbb{R}^2_+$.

Alternatives

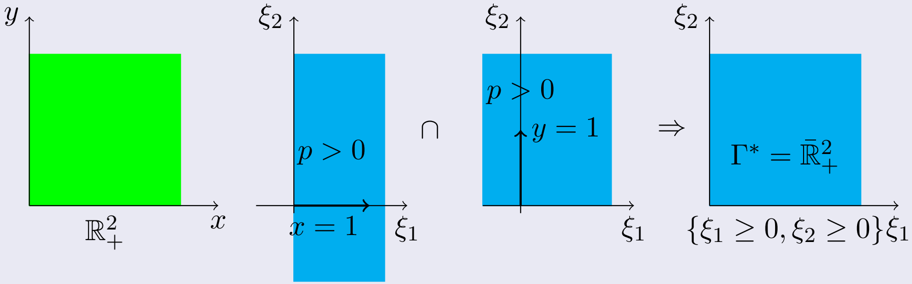
1. $\{(\xi_1, \xi_2) \in \mathbb{R}^2\}$ i.e. all ξ_1 and ξ_2 in $\mathbb{R}$
2. $\{(\xi_1, \xi_2) \in \mathbb{R}^2$, such that $\xi_1 > 0$, $\xi_2 > 0\}$
3. $\{(\xi_1, \xi_2) \in \mathbb{R}^2$, such that $\xi_1 \geq 0$ and $\xi_2 \in \mathbb{R}\}$
4. $\{(\xi_1, \xi_2) \in \mathbb{R}^2$, such that $\xi_1 \geq 0$ and $\xi_2 \geq 0\}$

Answer

$$\Gamma^* = \{\xi \in \mathbb{R}^n : \xi \cdot x \geq 0, \text{ for all } x \in \Gamma \subset \mathbb{R}^n\}$$

Conjugate cone for Γ^* for $\Gamma = \mathbb{R}^2_+$

We seek all $\xi \in \mathbb{R}^2$ such that $p = \xi \cdot x \geq 0$, for all $x \in \Gamma_{\mathbb{R}^2_+}$.



Interior vectors in Γ, i.e. $\boldsymbol{u}_1 = (1, \varepsilon)$, $\boldsymbol{u}_2 = (\varepsilon, 1)$.

Acute cones

Example: The cone $\mathbb{R} \times \mathbb{R}_+$ have $\operatorname{int} \Gamma^* = \varnothing$

A cone is **acute** if it does not contain an integral straight line.
Acute cones have $\operatorname{int} \Gamma^* \neq 0$.

Exercise L.1:

Show that a) that $(V_2^+(c))^* = \bar{V}_2^+(1/c)$. b) Determine $(\mathbb{R} \times \mathbb{R}_+)^*$.

Examples of acute cones, Vladimirov 4.4

$$\mathbb{R}_+^1, \quad \mathbb{R}_+^n = \{x : x_1 > 0, x_2 > 0, \ldots x_n > 0\}, \ (\mathbb{R}_+^n)^* = \overline{\mathbb{R}_+^n},$$

$$C = \{x \in \mathbb{R}^n : \hat{\boldsymbol{e}}_i \cdot x > 0 \ \forall i\}, \ C^* = \{\xi : \xi = \sum_k \lambda_k \hat{\boldsymbol{e}}_k, \lambda_k \geq 0\},$$

$$\text{if } \{\hat{\boldsymbol{e}}_i\} \text{ is a } \ \mathbb{R}^n - \text{basis}$$

$$V_4^+ = \{x = (x_0, \boldsymbol{x}) : x_0 > |\boldsymbol{x}|\} \subset \mathbb{R}^4, \ (V_4^+)^* = \bar{V}_4^+$$

$$P_n \subset \mathbb{R}^{n^2}, \text{ positive hermitian matrices}, \ P_n^* = P_n$$

Strong passivity

$$\mathrm{Re}\int_{-\Gamma}\langle Z*\phi,\phi\rangle\,\mathrm{d}x\geq 0\ \forall\ \phi\in\mathcal{D}^{\times N}\qquad(1)$$

Implies

$$\int_{-\Gamma+x_*}\langle Z*\phi,\phi\rangle\,\mathrm{d}x\geq 0,\ \forall\phi\in\mathcal{D}^{\times N},\forall x_*\in\mathbb{R}^n\qquad(2)$$

Proof: $\phi_{x_*}(x)=\phi(x+x_*)\in\mathcal{D}^{\times N}$. Time translational invariance gives:

$$0\leq\mathrm{Re}\int_{-\Gamma}\langle Z*\phi_{x_*},\phi_{x_*}\rangle\,\mathrm{d}x=\mathrm{Re}\int_{-\Gamma}\langle (Z*\phi)(x+x_*),\phi(x+x_*)\rangle\,\mathrm{d}x$$
$$=\int_{-\Gamma+x_*}\langle Z*\phi,\phi\rangle\,\mathrm{d}x.\ \square\qquad(3)$$

[Bochner-Schwartz Thm] $f\in\mathcal{D}'$, $f\gg 0\Leftrightarrow f=\mathcal{F}[\mu]$, and $\mu,f\in\mathcal{S}'$, $\mu\geq 0$ measure.

Passivity yields causality; [Youla, Castriota, Carlin, 1959]

A linear passive system as defined above implies causality, i.e. that

$$\operatorname{supp} Z(x) \subset \Gamma \tag{4}$$

Proof [Vladimirov]: Given $\phi, \psi \in \mathcal{D}^{\times N}$, $\lambda \in \mathbb{R}$. Let $\phi \to \phi + \lambda\psi$. Thus

$$0 \leq \int_{-\Gamma} \langle \phi, \phi \rangle \, \mathrm{d}x + \lambda \int_{-\Gamma} \langle Z * \phi, \psi \rangle + \langle Z * \psi, \phi \rangle \, \mathrm{d}x + \lambda^2 \int_{-\Gamma} \langle \psi, \psi \rangle \, \mathrm{d}x \tag{5}$$

We have a quadratic equation in λ that is non-negative this implies:

$$\Big[\int_{-\Gamma} \langle Z*\phi, \psi \rangle \, \mathrm{d}x + \int_{-\Gamma} \langle Z*\psi, \phi \rangle \, \mathrm{d}x \Big]^2 \leq 4 \int_{-\Gamma} \langle Z*\phi, \phi \rangle \, \mathrm{d}x \int_{-\Gamma} \langle Z*\psi, \psi \rangle \, \mathrm{d}x$$

If $\operatorname{supp} \phi \subset \mathbb{R}^n \backslash (-\Gamma)$. Then $\int_{-\Gamma} \langle Z * \phi, \psi \rangle \, \mathrm{d}x = 0$
$\forall \psi \Rightarrow Z * \phi = 0, x \in \Gamma$. Let $x = 0$ then $(Z(-x'), \phi(x')) = 0$
$\forall \phi \in \mathcal{D}(\mathbb{R}^n \backslash (-\Gamma)) \Rightarrow Z(-x) = 0$, $\forall x \in \mathbb{R}^n \backslash (-\Gamma)$.$\square$

Remarks on passivity

- Another passivity concepts. 1D (circuit) n-Port passivity: $E(x) = \sup_{T>0;x} \int_0^T -\langle v(t;x), i(t;x)\rangle \, \mathrm{d}t$. A system is passive if $E(x) < \infty$ for all allowed states x. [Wyatt etal: Energy concepts in state-space 1981]
- Multidimensional (nonlinear) state space passivity: Given a system N, with power $p(\boldsymbol{t})$ supplied to the system, where $\boldsymbol{t} \in \mathbb{R}^n$. N is multidimensionally passive system if there exists a *stored energy vector* $\boldsymbol{W}_s$ such that

$$p(t) \geq \sum_k \partial_{t_k} \hat{\boldsymbol{W}}_k(\boldsymbol{t}), \text{ where } \hat{\boldsymbol{W}}_s(\boldsymbol{t}) = \boldsymbol{W}_s(\boldsymbol{q}(\boldsymbol{t}), \boldsymbol{t}). \tag{6}$$

  for admissible states $\boldsymbol{q}$. [A. Fettweis, S. Basu, Multidimensional causality and passivity of linear and nonlinear system arising from physics, 2011]
- Linear passive systems without translational invariance are also studied in Drozhzhinov 1981.

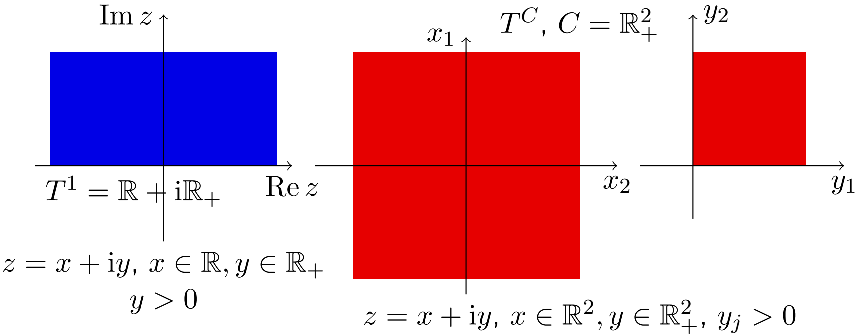
Tubular neighbourhood of a cone

Definition: Tubular neighbourhood

The tubular neighbourhood of a convex cone $\Gamma \subset \mathbb{R}^n$ is the set

$$T^C = \mathbb{R}^n + \mathrm{j}C, \; C = \operatorname{int}\Gamma^* \tag{7}$$

Thus $z = x + \mathrm{j}y \in T^C \Rightarrow x \in \mathbb{R}^n,\; y \in C$.



Analyticity and growth condition of $\mathcal{L}[Z]$

Recall: Fourier transforms, functions - $\mathcal{F}u(\xi) = \int_{\mathbb{R}^n} u(x)\mathrm{e}^{\mathrm{j}\xi\cdot x}\,\mathrm{d}x$, $\xi \in \mathbb{R}^n$.
Distributions $\mathcal{F}Z(\phi) := Z(\mathcal{F}\phi).(= \int (Z, \mathrm{e}^{\mathrm{j}\xi\cdot x})\hat{\phi}(\xi)\,\mathrm{d}x$ Z compact support).
(Bilateral) Laplace transform:
$(\mathcal{L}u)(z) := \mathcal{F}[u\mathrm{e}^{-\eta\cdot x}] = \int \mathrm{e}^{-\eta\cdot x}\mathrm{e}^{\mathrm{j}\xi\cdot x}u(x)\,\mathrm{d}x$, $z = \xi + \mathrm{j}\eta$
$\eta \in \Gamma^* = \{\eta : \eta \cdot x \geq 0, x \in \Gamma\}$.

Theorem of analyticity [Reed & Simon II.IX.16]

Let $Z \in \mathcal{S}'(\Gamma)$, then $\mathcal{L}(Z)$ is analytic in T^C, $C = \operatorname{int}\Gamma^*$ (Note interior region). Furthermore

$$|\mathcal{F}[Z](\xi + \mathrm{j}\eta)| \leq |P(\xi + \mathrm{j}\eta)|(1 + [\mathrm{dist}(\eta, \partial\Gamma^*)]^{-N}) \quad (8)$$

for a suitable polynomial P and positive integer N.

Note: This is a generalization of the Paley-Wiener theorem, by Schwartz 1934, Gårding and Bros-Epstein-Glaser 1967 and others.
Note: A kind of converse of the theorem exists in R&S.

Passive linear system in Laplace domain – first goal

Properties

Let $Z \in \mathcal{S}'$, and $Z(z) := \mathcal{F}(Z)$, where Z is a passive linear system.

- $Z(z)$ is analytic in T^C. Furthermore if $Z \in \mathcal{D}$ and it is linear passive then $Z \in \mathcal{S}$.
- The condition of reality: $Z(z) = \bar{Z}(-\bar{z})$, $z \in T^C$.
- Positivity $\operatorname{Re} Z(z) \geq 0$, $z \in T^C$.
- $Z(z)$ corresponds to a passive operator if Z is a positive real matrix function in T^C.
- (Bros-Epstein-Glaser) If Γ is a open convex cone and $Z \in \mathcal{S}'$ with support in $\bar{\Gamma}$, Then there exists a polynomially bounded function G with support in $\bar{\Gamma}$ and a partial differential operator $P(D)$ such that $Z = P(D)G$.

Quiz

What role does the conjugate cone play in the Laplace-transform of $Z(\boldsymbol{x})$?

What role does the conjugate cone play in the Laplace-transform of $Z(\boldsymbol{x})$?

Alternatives

- Support of $\mathcal{L}Z(\boldsymbol{k})$
- Region of analyticity of $\mathcal{L}Z$
- Region of passivity of $Z(\boldsymbol{x})$

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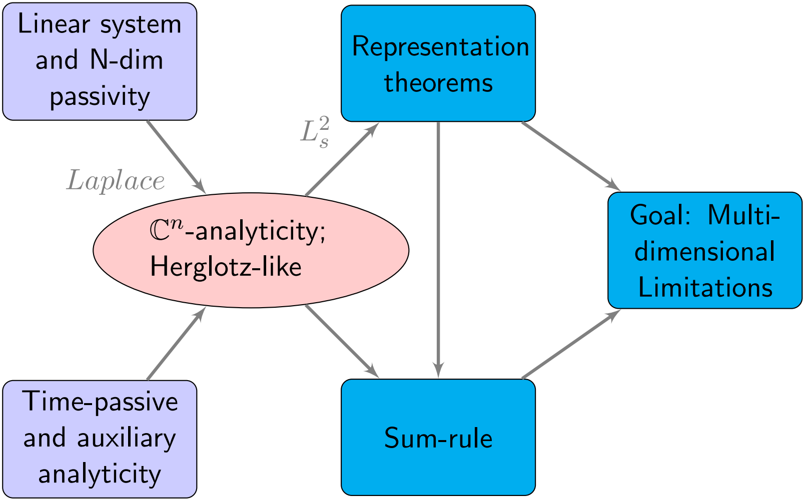

Linear system and N-dim passivity
Laplace
Time-passive and auxiliary analyticity
$\mathbb{C}^n$-analyticity; Herglotz-like
L^2_s
Representation theorems
Sum-rule
Goal: Multi-dimensional Limitations

Efforts toward definitions

Definition: Representation theorem [Wikipedia]

"A representation theorem is a theorem that states that every structure with a certain property is isomorphic to another (abstract or concrete) structure."

Isomorphorphic (iso = equal morphe=shape).
Roughly: We search for identifications between two ways (here) to describe system responses, seen as a method to describe a class/set of functions.

Definition Sum-rule [e.g. Bernland 2012]

A sum or integral [of a family of functions] that relates to a 'fixed value'

Note 1: Dispersion relations are an example of representation theorems, they can be made into 'sum-rules' by studying it at origin, or apply transformations.
Note 2: King in Hilbert transform also call some representation theorems for sum-rules, but mostly lean towards the latter description. See examples and discussion in Chapt. 19.

Classes of representation theorems

Classes of representation theorems for systems

- Spectral decomposition. Certain 'nice' (e.g. self-adjoint) operators are fully described through their spectral measure.
- Cauchy-kernel representations (Hilbert-transform pairs; Cauchy-Bochner transform (relations to (Cauchy-)Szegö and Poisson-kernel representations)
- Reproducing kernel Hilbert spaces; Riez representation theorem. E.g. band-limited functions. Centered around the boundedness of the evaluation operator.
- Herglotz representation (1D) of certain holomorphic functions. All Herglotz-functions can be characterized through two constants and a class of positive Borel-measures. (Schwartz-representations).

These categories are not independent. [wikipedia] The spectral representation of Positive symmetric L^2-integral operators is also a reproducing kernel hilbert space.

Titchmarsh thm [e.g. Bernland's PhD thesis, Lund '12]

1-dimensional case. Let $g(\omega) = \mathcal{F}(f)(\omega)$ belong to L^2 for $\omega \in \mathbb{R}$, if $g = \mathcal{F}f$ satisfy one of the below properties then it satisfy all of them and $\mathcal{F}f(\omega)$ is a causal transform

- $f(t) = 0$ for $t < 0$ (causality)
- 1:st Plemelj formula (Hilbert-transform pair)

$$\mathrm{Re}\, g(\omega) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\omega - \xi| > \varepsilon} \frac{\mathrm{Im}\, g(\xi)}{\omega - \xi} \,\mathrm{d}\xi$$

- 2:nd Plemelj formula

$$\mathrm{Im}\, g(\omega) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\omega - \xi| > \varepsilon} \frac{\mathrm{Re}\, g(\xi)}{\omega - \xi} \,\mathrm{d}\xi$$

- $g(\omega + \mathrm{i}\sigma)$ is holomorphic in $\mathbb{C}^+ = \{(\omega, \sigma) \subset \mathbb{R}^2 : \sigma > 0\}$ and $\int_{\mathbb{R}} f(x + \mathrm{i}y) \,\mathrm{d}x < \infty$ and $y > 0$

The 1 & 2:nd Plemelj-formula with symmetry $g(-\omega) = \bar{g}(\omega)$, $\omega \in \mathbb{R}$ yield the **Kramers-Kronig relation** or a **dispersion relation** when applied to material coefficients. Cone is $\Gamma = \mathbb{R}_+$; Cauchy kernel: i/z

Observations

Note 1: Passive linear system + L^2

Note that passivity for (Vladimirov-)linear system $g(t)$ yields causality and thus analyticity of $g(s)$, and hence if $g(\omega) \in \mathrm{L}^2$ we can apply Titchmarsh theorem

Note 2: Dispersion relation as representation thm

Titchmars theorem is a 1 dimension a representation theorem of $\operatorname{Re} g$ in terms of $\operatorname{Im} g$, for causal and L^2-bounded functions.
Causality or analyticity can be used as assumptions to provide the dispersion relation. The L^2-bound can be the limiting.

Note 3: $g(\omega)$ can be seen as the boundary value of a holomorphic $g(s)$, and it is here that the L^2 assumptions are placed.
Note 4: The signs in the Hilbert-transform pair depend on the definition of the Fourier-transform.

Examples

Refractive index; non-conducting, non-magnetic material [e.g. King]

The complex refractive index $N = n + \mathrm{i}\kappa$ and Titchmarsh theorem yields:

$$\kappa(\omega) = \frac{1}{\pi} P \int_{\mathbb{R}} \frac{n(\omega') - 1}{\omega - \omega'} \,\mathrm{d}\omega' \tag{9}$$

$$n(\omega) - 1 = \frac{-1}{\pi} P \int_{\mathbb{R}} \frac{\kappa(\omega')}{\omega - \omega'} \,\mathrm{d}\omega' \tag{10}$$

Using symmetry we can rewrite them into the standard Kramers-Kronig relation for n, κ.

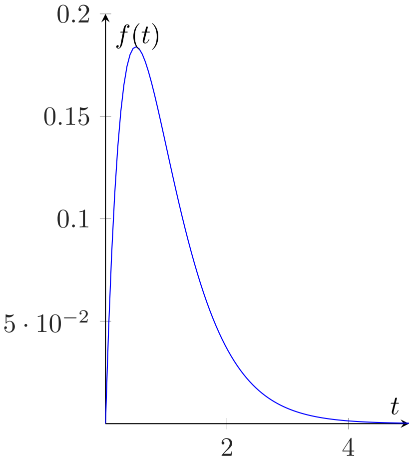
Example: Dielectric constant

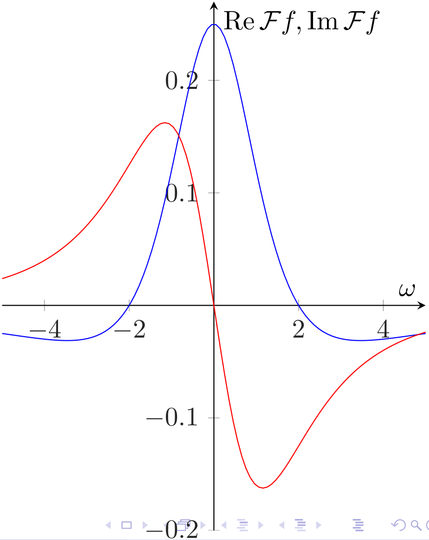
Apropriate assumptions on $\varepsilon(\omega)$ (bounded, continuous, asymptotic etc.) we have the dispersion relation [Landau etal; King; Bernland]

$$\operatorname{Re} \varepsilon(\omega) = \varepsilon_\infty + \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|\xi - \omega| > \varepsilon} \frac{\operatorname{Im}(\varepsilon(\xi))}{\xi - \omega} \,\mathrm{d}\xi, \ \omega \in \mathbb{R}$$

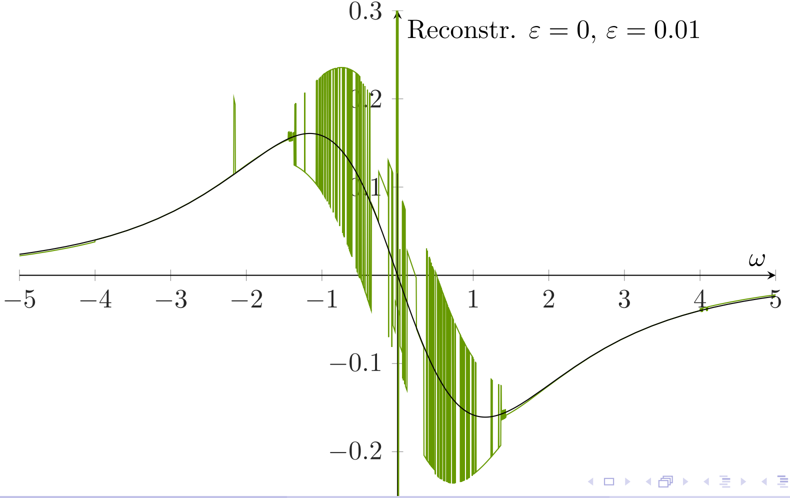
Numeric example

Causal, L^2, function: $f(t) = t\mathrm{e}^{-2t}/2$, $t > 0$, zero elsewere.





Mathmatica: $\frac{-1}{\pi}$`NIntegral[` $\frac{1}{z+I\varepsilon-\omega}\,\mathrm{Re}(\mathcal{F}(f))(\omega),\{\omega,-\infty,z,\infty\}$`]` $=\mathcal{H}_\varepsilon[\mathcal{F}(f)]$



Exercise L.2 – verify a 1d-dispersion relation

Choose a L^2, causal function $f(t)$. We are interested in the Hilbert-transform pair for of $f(\omega)$. Examine the numerical convergence of the Hilbert-transform pair for your choice of function f, similarly to the example above. I.e. regularize the integral kernel of the Hilbert transform with a small imaginary epsilon $\mathcal{H} \to \mathcal{H}_\varepsilon$. How well (as a function of $\varepsilon \to 0$) does your numerical algorithm of $\mathcal{H}_\varepsilon[\mathrm{Im}\,\mathcal{F}f]$ converge to the real part? How does this compare if you take the Cauchy-principal value integral with some absence δ around the singularity; does it converge better numerically?

Hints – signs in the Hilbert transform and the sign-convention in the Fourier-transform are closely connected.

Three function spaces

Titchmarch theorem is for L^2-functions $g(\omega)$. We need a suitable generalization for n-dimensional functions

Spaces

- L^2_s: $g(\xi) \in \mathrm{L}^2_s$ if
  $\|g\|_{(s)} = \int (1+|\xi|^2)^s |g(\xi)|^2 \, \mathrm{d}\xi = \|g(\xi)(1+|\xi|^2)^{s/2}\| < \infty$.
- $H_s = \{f \in \mathcal{D}' : f = \mathcal{F}(g), g \in \mathrm{L}^2_s\}$, $\|f\|_s = \|g\|_{(s)}$.
- $H^{(s)} = \{f \in$ Holomorphic in T^C:
  $\|f\|^{(s)} = \sup_{y \in C} \|f(x+\mathrm{i}y)\|_s < \infty\}$, weighted Hardy space

Properties

- $H_s \in \bar{C}_0^\ell$, ℓ integer, $\ell < s - n/2$.
- H_s is the set of functions $f \in H_{s-1}$, where $\partial_j f \in H_{s-1}$.

Cauchy(-Szegö) Kernels $\mathcal{K}_C$ [Vladimirov 10.2]

- The Cauchy kernel for a connected open cone in $\mathbb{R}^n$ with vertex 0 is:

$$\mathcal{K}_C(z) = \int_{C^*} \mathrm{e}^{\mathrm{i}z\cdot\xi}\,\mathrm{d}\xi = F[\theta_{C^*}\mathrm{e}^{-y\cdot\xi}],\ z = x + \mathrm{i}y$$

  Here θ_{C^*} is the characteristic-function of C^*, the conjugate cone.
- $\mathcal{K}_{\mathbb{R}^n_+}(z) = \frac{\mathrm{i}^n}{z_1\cdots z_n} \Rightarrow \mathcal{K}_1(x) = \frac{\mathrm{i}}{x+\mathrm{i}0} = \pi\delta(x) + \mathrm{i}P\frac{1}{x}$.
- $\mathcal{K}_{V^+}(z) = 2^n\pi^{(n-1)/2}\Gamma(\frac{n+1}{2})(-z^2)^{-\frac{n+1}{2}}$, $z \in T^{V^+}$, $z^2 = z_0^2 - z_1^2 - \cdots - z_n^2$.
- $$\mathcal{K}_{P^n}(Z) = \pi^{n(n-1)/2}\mathrm{j}^{n^2}\frac{1!\ldots(n-1)!}{(\det Z)^n},\ Z \in T^{P_n},$$

Properties: $\mathcal{K}_{-C}(x) = (-1)^n\mathcal{K}_C(x)$, $x \in C \cup (-C)$;
$\mathrm{Im}\,\mathcal{K}_C(x) = \frac{1}{2\mathrm{i}}\mathcal{F}(\theta_{C^*} - \theta_{-C^*})$. $\mathcal{K}_C$ holomorphic in T^C

Cauchy-Kernel in higher dimension

Calculate explicitly the Cauchy kernel $\mathcal{K}_{\mathbb{R}^n_+}$ for $\mathbb{R}^n_+$ for $n = 1, 2, 3$ and determine its distribution on the boundary when $\Gamma \ni \boldsymbol{y} \to 0$. A partial solution is given on previous (and later) slides. Be explicit and do the missing steps.

Transform of a boundary function to T^C

Cauchy-Bochner tranform [V10.3]

Let $g \in \mathrm{L}^2_s$, define $f(x) = \mathcal{F}[g] \in H_s$, $x \in \mathbb{R}^n$, the **Cauchy-Bochner-transform** is, $z \in T^C \cup T^{-C}$:

$$f(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{K}_C(z - x') f(x') \, \mathrm{d}x' = \frac{1}{(2\pi)^n} (f(x'), \mathcal{K}(z - x')),$$

Note $f(z)$ holomorphic in $T^C \cup T^{-C}$.

Example: The time-cone, $\mathbb{R}_+$

The conjugate cone $C^* = \bar{\mathbb{R}}_+$. Clearly $\mathcal{K}_{\mathbb{R}+}(z) = \int_0^\infty \mathrm{e}^{\mathrm{i}zt} \, \mathrm{d}t = \frac{\mathrm{i}}{z}$. The Cauchy-Bochner transform becomes:

$$f(z) = \frac{1}{2\pi\mathrm{i}} P \int_{\mathbb{R}} \frac{f(x')}{x' - z} \, \mathrm{d}x', \;\; z \notin \mathbb{R} \tag{11}$$

which looks like a Cauchy-integral over the line.

Generalized Titchmarsh's theorem (n-dimensional)

Theorem II (V10.6) Generalized Hilbert-transform relation

Let $f_+ = \mathcal{F}g \in H_s$, *i.e.*, $g \in \mathrm{L}_s^2$ the following things are equivalent:

- $\operatorname{supp} g = \operatorname{supp} F^{-1}(f_+) \subset C^*$. [$g$ is causal]
- (Hilbert-transform pair)

$$\operatorname{Re} f_+(x) = \frac{-2}{(2\pi)^n} \int_{\mathbb{R}^n} (\operatorname{Im} f_+)(x')(\operatorname{Im} \mathcal{K}_C)_+(x - x')\, \mathrm{d}x',$$

$$\operatorname{Im} f_+(x) = \frac{2}{(2\pi)^n} \int_{\mathbb{R}^n} (\operatorname{Re} f_+)(x')(\operatorname{Im} \mathcal{K}_C)_+(x - x')\, \mathrm{d}x',$$

- f_+ is a boundary value of some $f \in H^{(s)}(T^C)$. (Holomorphic in T^C)

Note: $\operatorname{Re} f_+$ and $\operatorname{Im} f_+$ form a Hilbert-transform pair. Here

$$(\operatorname{Im} \mathcal{K}_C)_+(x) = \operatorname{Im} \left(\mathrm{i}^n \Gamma(n) \int_{S^{n-1} \cap C^*} \frac{\mathrm{d}\sigma}{[x \cdot \sigma + \mathrm{i}0]^n} \right)$$

Quiz

What type of object is $(\mathrm{Im}\,\mathcal{K}_C)_+(x - x')$ and why?

- Its an analytic function
- Its a locally integrable function
- Its a distribution

1-dim case

We have $\mathcal{K}_{\mathbb{R}_+}(z) = \frac{\mathrm{i}}{z}$. Note that

$$\lim_{y\to 0} \mathcal{K}_{\mathbb{R}_+}(x+\mathrm{i}y) = \lim_{y\to 0} \frac{\mathrm{i}}{x+\mathrm{i}y} = \mathrm{i}P\frac{1}{x} + \pi\delta(x) \qquad (12)$$

Thus $\operatorname{Im}\mathcal{K}_{\mathbb{R}_+}(x) = P\frac{1}{x}$. The theorem II hence become the Hilbert-transform pair, with the first as:

$$\operatorname{Re} f_+(x) = \frac{-1}{\pi} P \int_{\mathbb{R}} \frac{\operatorname{Im} f_-(x')}{x-x'}\,\mathrm{d}x'. \qquad (13)$$

Applications were shown above for $f(t) = t\mathrm{e}^{-2t}$.
As a representation theorem, we note that the real part of all $f \in H_s$ yields the imaginary part. We have a representative structure.

2-dim case

We have $\mathcal{K}_{\mathbb{R}^2_+}(z) = \frac{-1}{z_1 z_2}$. Note that in distributional sense

$$\lim_{y\to 0} \mathcal{K}_{\mathbb{R}^2_+}(\boldsymbol{x} + \mathrm{i}\boldsymbol{y}) = -(P\frac{1}{x_1} - \mathrm{i}\pi\delta(x_1))(P\frac{1}{x_2} - \mathrm{i}\pi\delta(x_2)) \qquad (14)$$

Thus $\operatorname{Im}\mathcal{K}_{\mathbb{R}^2_+}(x) = \pi P\frac{1}{x_1}\delta(x_2) + \pi P\frac{1}{x_2}\delta(x_1)$. The theorem II 'Hilbert-transform' pair becomes:

$$\operatorname{Re} f_+(x) = \frac{-1}{2\pi} P \int_{\mathbb{R}} \frac{\operatorname{Im} f_+(x', x_2)}{x_1 - x'} + \frac{\operatorname{Im} f_+(x_1, x')}{x_2 - x'} \,\mathrm{d}x' \qquad (15)$$

Consider $g(x_1, x_2) = \theta(x_1)\theta(x_2)\mathrm{e}^{-ax_1 - ax_2}$. Fourier-transform: $f = \mathcal{F}g = \frac{1}{(a+\mathrm{i}\omega_1)(b+\mathrm{i}\omega_2)}$. Thus

$$\operatorname{Im} f = -\frac{b\omega_1 + a\omega_2}{(a^2+\omega_1^2)(b^2+\omega_2^2)}, \ \operatorname{Re} f = \frac{ab - \omega_1\omega_2}{(a^2+\omega_1^2)(b^2+\omega_2^2)} \qquad (16)$$

Note that $\mathcal{H}\frac{1}{x^2+a^2} = \frac{y}{a(a^2+y^2)}$, $\mathcal{H}\frac{x}{x^2+a^2} = \frac{-a}{(a^2+y^2)}$, thus

$$\mathcal{H}_{s\to\omega_1} \operatorname{Im} f(s, \omega_2) = \frac{\omega_1\omega_2 - ab}{(\omega_1^2 + a^2)(\omega_2^2 + b^2)} = \mathcal{H}_{s\to\omega_2} \operatorname{Im} f(\omega_1, s)$$
$$= -\frac{1}{2}\operatorname{Re} f \qquad (17)$$

We have hence showed that $\operatorname{Re} f$ and $\operatorname{Im} f$ indeed are a 'Hilbert-transform' pair under the Cauchy-kernel.

Exercise L.4

Verify either numerically or analytically a dispersion relation in 2-dimensions. Choose a 2d-function H_s, causal function, or do a step-by-step verification of the above case.

Observation: Real and imaginary part of the kernel $\mathcal{L}Z$ for a passive system are connected with through the **Cauchy**-kernel $\mathcal{K}_C$, which depends on domain, (cone) Γ of the variables $x \in \Gamma$.

Case 1: Light cone $\Gamma = V_n^+$

- Dispersion-relations for solutions to Cauchy-problem in homogeneous space, $(t, x) \in V_n^+$. [Vladimirov 2002].
- Spatial dispersion properties V_4^+.

Case 2: Cone $\Gamma = \mathbb{R}_+^N$ – independent variables

Examples:

- Nonlinear susceptibility, variables $\{\omega_k\} \in \mathbb{R}_+^n$.
- Certain elements of nonlinear circuit theory

Spatial dispersion

$$\partial_t \boldsymbol{D}(\boldsymbol{r},t) = \partial_t \int_{-\infty}^{t} \mathrm{d}t' \int_{\mathbb{R}^3} \varepsilon(\boldsymbol{r}-\boldsymbol{r}', t-t')\boldsymbol{E}(\boldsymbol{r}',t') \qquad (18)$$

Applications:

- Optical activity requires spatial dispersion. Magneto-optic media include weak spatial dispersion $\sim$(molecule diam)/λ.
- Plasma physics and EM-propagation in metals at low temperatures at radio-frequency. The effects can be strong.
- Graphene sheets (2D) at low THz-frequencies $\sigma = \sigma_0 + a_j \partial_{x_j} + b_{jk}\partial^2_{x_j x_k}$ (sum-notation), σ_0, a_j and b_{jk} are matrices. [Gomez-Diaz etal 2013]
- Periodic structures (homogenisation) $D_i = \varepsilon_{ij}E_j + \alpha_{jkr}\partial_{x_r}E_j + \beta_{ijrs}\partial^2_{x_r x_s}E_j$ [Ciattoni etal 2015].
- High-impedance surface $\varepsilon = (\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}})\varepsilon_t + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\varepsilon_{zz}$, $\varepsilon_{zz} = (1 - \frac{k_p}{\varepsilon_t k_0^2 - q_z^2})\varepsilon_t$, where k_p is the plasma wave number. [Luukkonen etal 2008]

Multiple approaches give the same EM-field response

Equivalent representations

Different spatial dispersion representations have equivalent EM-field-response. There are at-least three different models, perhaps the most efficient is $\mu = 1$ and

$$\varepsilon_{ik}(\omega, \boldsymbol{k}) = (\delta_{ij} - \frac{k_i k_j}{k^2})\varepsilon_t(\omega, \boldsymbol{k}) + \frac{k_i k_j}{k^2}\varepsilon_{zz}(\omega, \boldsymbol{k}). \quad (19)$$

There are a 1-1 relation between the σ and the $(\varepsilon,\ \mu = 1)$ representations, and between scalar (ε, μ)-representations. These equivalences appear through different models for $\rho, \boldsymbol{J}$ as induced sources.

Boundary conditions: continuous tangential $\boldsymbol{E}$, and B_n. However both tangential B and H may have discontinuity at boundary.
Symmetries: $\varepsilon(\omega, \boldsymbol{r}, \boldsymbol{r}') = \varepsilon^*(-\omega, \boldsymbol{r}, \boldsymbol{r}')$, $\varepsilon(\omega, \boldsymbol{k}) = \varepsilon^*(-\omega, -\boldsymbol{k})$.
Key references: [A. A. Rukhadze and V.P. Silin, *Electrodynamics of media with spatial dispersion* 1961]

What cone Γ is appropriate

Passive operator with no spatial dispersion:

$$\int_{-\infty}^{T} \boldsymbol{E}(t) \cdot \frac{\partial \boldsymbol{D}(t)}{\mathrm{d}t}\,\mathrm{d}t \geq 0, \tag{20}$$

How to generalize this to include spatial dispersion? – we need $\mathrm{Re}\int_{-\Gamma} \langle \partial_t(\varepsilon * \boldsymbol{E}), \boldsymbol{E}\rangle \,\mathrm{d}x > 0$, here $x = (x_0, \boldsymbol{r})$. That is

$$\int_{-\infty}^{0} \mathrm{d}t \int_{|\boldsymbol{r}|\leq -ct} \mathrm{d}V\, \boldsymbol{E}(\boldsymbol{r},t)\cdot\partial_t \int_{-\infty}^{t} \mathrm{d}t' \int_{\mathbb{R}^3} \mathrm{d}V' (\varepsilon(\boldsymbol{r}-\boldsymbol{r}', t-t')\boldsymbol{E}(\boldsymbol{r}',t')) \geq 0$$

$\boldsymbol{E} \in \mathcal{D}^{\times N}$.
Vacuum light cone, $V^+(x) \to V^+(1)$ [change of variables].
Note: dimension of $\varepsilon(\boldsymbol{r}, t)$ and $\varepsilon(t)$ is different.

Spatial dispersion

Assume that the operator $\varepsilon(\boldsymbol{r}, t)$ is passive and have support in $V^+(1)$. Hence $\varepsilon(\boldsymbol{k}, \omega)$ is analytic in T^C. If, in addition, $\varepsilon(\boldsymbol{r}, \omega) \in H_s$, then by Thm II

$$\operatorname{Re} \varepsilon(\omega, \boldsymbol{k}) = \frac{-2}{(2\pi)^n} (\operatorname{Im} \mathcal{K}_{V^+})_+ * \operatorname{Im} \varepsilon =$$

$$\frac{-2}{(2\pi)^n} \int_{\mathbb{R}} \int_{\mathbb{R}^3} (\operatorname{Im} \mathcal{K}_{V^+})_+ (\omega - \omega', \boldsymbol{k} - \boldsymbol{k}') \operatorname{Im} \varepsilon(\omega', \boldsymbol{k}') \, \mathrm{d}\omega' \, \mathrm{d}V_{k'}$$

Note:

1. An explicit form of $(\operatorname{Im} \mathcal{K}_{V^+})_+$, can be expressed in terms of the generalized functions $P^{(k)} \frac{1}{\sigma \cdot x}$ and $\delta^{(k)}(\sigma \cdot x)$. I.e. we have let $C \ni y \to 0$.
2. Application to periodic structures.

Case 2: n-dimensional Hilbert transform on cone $\mathbb{R}_+^n$

$$(H_n f)(x) = \frac{1}{\pi^n} P \int_{\mathbb{R}^n} f(s) \Pi_{k=1}^n \frac{1}{x_k - s_k} \,\mathrm{d}s$$

Furthermore we have that $(H_n^2 f)(x) = (-1)^n f(x)$

Examples: King 2009

$$H_n[\sin(a \cdot s)](x) = \begin{cases} (-1)^{(n-1)/2} \cos(a \cdot x) \Pi_k \operatorname{sgn} a_k & n \text{ odd} \\ (-1)^{n/2} \sin(a \cdot x) \Pi_k \operatorname{sgn} a_k & n \text{ even} \end{cases}$$

$$H_n[\cos(a \cdot s)](x) = \begin{cases} (-1)^{(n-1)/2} \sin(a \cdot x) \Pi_k \operatorname{sgn} a_k & n \text{ odd} \\ (-1)^{n/2} \cos(a \cdot x) \Pi_k \operatorname{sgn} a_k & n \text{ even} \end{cases}$$

$$H_n[\mathrm{e}^{\mathrm{j} a \cdot s}](x) = (-1)^n \mathrm{e}^{\mathrm{j} a \cdot x} \Pi_k \operatorname{sgn} a_k$$

$$H_n[\mathrm{e}^{-as^2}](x) = (-\mathrm{j})^n \mathrm{e}^{-ax^2} \Pi_k \operatorname{erf}(\mathrm{j} x_k \sqrt{a})$$

H_n is a special case of a Calderón-Zygmund singular operator.

Nonlinear electric susceptibilities:

$$\boldsymbol{P}(t) = \sum_n \boldsymbol{P}^{(n)}(t),$$

where

$$P_k^{(n)}(\omega) = \varepsilon_0 \int_{\mathbb{R}} \mathrm{d}\omega_1 E_{\ell_1}(\omega_1) \cdots \int_{\mathbb{R}} \mathrm{d}\omega_n E_{\ell_n}(\omega_n) \cdot \chi^{(n)}_{k\ell_1\ell_2\cdots\ell_n}(\omega_1, \ldots, \omega_n)\delta(\omega - \omega_1 - \omega_2 - \cdots \omega_n)$$

n-dimensional dispersion relation

Ref: Peiponen 1988 (see also King: Hilbert transforms Chapt 22.9)

$$\mathrm{Re}\,\chi^{(n)}(\omega_1, \ldots, \omega_n) = \frac{\mathrm{j}^{n+1}}{\pi^n} P \int_{\mathbb{R}} \cdots P \int_{\mathbb{R}} \frac{\mathrm{Im}\,\chi^{(n)}(\omega_1', \ldots, \omega_n')\,\mathrm{d}\omega_1' \cdots \mathrm{d}\omega_n'}{(\omega_1 - \omega_1') \cdots (\omega_n - \omega_n')}$$

Using that $\chi^{(n)}(\omega_1,\dots,\omega_k,\dots,\omega_n) = \mathrm{O}(\omega_k^{-1-\delta})$ as $\omega \to \infty$, the result:

Sum-rule nonlinear susceptibility [Peiponen 1988]:

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} (\omega_1 \cdots \omega_n)^{s-1} [\chi^{(n)}(\omega_1,\dots,\omega_n)]^t \, \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_n = 0$$

where $s = 1, 2, \dots$, $t = 1, 2, \dots$, $s \leq t$.

Observations: There exists N-dim sum-rules, useful in nonlinear optics Analyticity of $\chi^{(n)}$ supplies additional relations, see e.g. King 2009 **Note:** These the claimed dispersion-relations differ from the passive system approach outlined above if n even. They are discussed in King.

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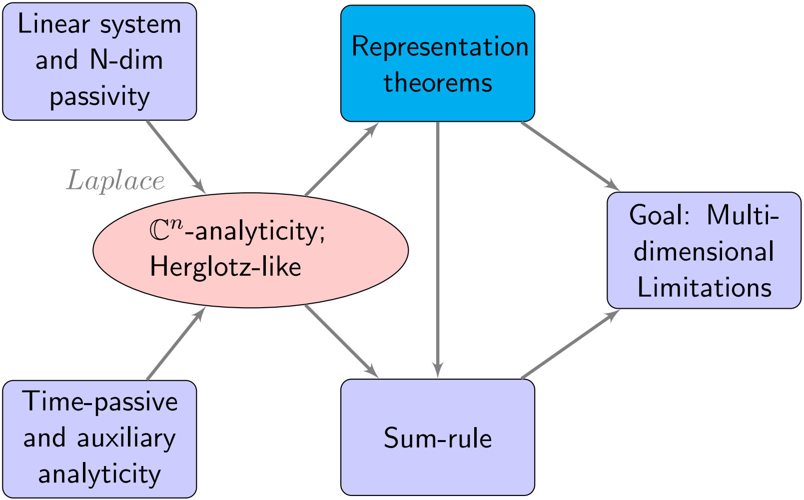
Goal: Limitations of measurable quantities



Poisson Kernel

- $\mathcal{P}_C(x,y) = \frac{\mathcal{K}_C(x+\mathrm{i}y)}{\pi^n \mathcal{K}_C(\mathrm{i}y)}, \quad (x,y) \in T^C$
- $\mathcal{P}_{\mathbb{R}^n_+}(x,y) = \frac{y_1 \cdots y_n}{\pi^n |z_1|^2 \cdots |z_n|^2}$
- $\mathcal{P}_{V+}(x,y) = \frac{2^n \Gamma(\frac{n+1}{2})}{\pi^{\frac{n+3}{2}}} \frac{(y^2)^{\frac{n+1}{2}}}{|(x+\mathrm{i}y)^2|^{n+1}}$

Schwartz kernel

- $\mathcal{S}_C(z,z^0) = \frac{2\mathcal{K}_C(z)\mathcal{K}_C(-\overline{z^0})}{(2\pi)^n \mathcal{K}_C(z-\overline{z^0})} - \mathcal{P}_C(x_0, y_0)$
-
$$\mathcal{S}_{\mathbb{R}^n_+} = \frac{2\mathrm{i}^n}{(2\pi)^n} \left(\frac{1}{z_1} - \frac{1}{\overline{z_1^0}} \right) \cdots \left(\frac{1}{z_n} - \frac{1}{\overline{z_n^0}} \right) - \mathcal{P}_{\mathbb{R}^n_+}(x^0, y^0)$$
- $\mathcal{S}_{V+}$ is also known explicitly.

The generalized Schwartz representation theorem

If C is an acute regular* cone, then any function f holomorphic and

$$|f(z)| \leq M(1+|z|^2)^{\alpha/2}(1+[\delta(y)]^{-\beta}),\ \ z \in T^C \tag{21}$$

for some M, α, β. Then there exists a boundary value f_+ such that

$$f(z) = \mathrm{i}(\operatorname{Im} f_+(x'), \mathcal{S}_C(z-x', z^0-x')) + \operatorname{Re} f(z_0),\ \ z^0, z \in T^C, \tag{22}$$

* A cone is regular if $\mathcal{K}_C$ is a divisor of the algebra $H(C)$. $\mathbb{R}^n_+$ and $V^+ \subset \mathbb{R}^4$ are regular. $n \leq 3$ every convex acute solid cone is regular.

Representation Theorem [Vladimirov 17.2]

Properties of Holomorphic functions with non-negative imaginary part

Let $u \in \mathcal{P}_+(T^C)$ then $0 \leq u(x,y) = \operatorname{Im} f \in H_+(T^C)$ and $\mu = u(x,+0)$ is a non-negative tempered measure, and u have the representation:

$$u(x,y) = \int_{\mathbb{R}^n} \mathcal{P}_C(x-x',y)\mu(\mathrm{d}x') + v_C(y),\ (x,y) \in T^C$$

where $v_C > 0$ continuous, $v_C \to 0$, as $C \ni y \to 0$. If the cone also is regular, then it also have a Schwartz representation. For $\mathbb{R}^n_+$ it is

$$f(z) = \mathrm{i}\int_{\mathbb{R}^n} \mathcal{S}_{\mathbb{R}^n_+}(z-x';z^0-z')\mu(\mathrm{d}x') + (a,z) + b(z^0),\ z,z^0 \in T^C,$$

where $\mu = \operatorname{Im} f_+$, $b(z^0) = \operatorname{Re}(f(z^0)) - (a,x^0)$, $a_j = \lim_{y_j\to 0} \frac{\operatorname{Im} f(\mathrm{i}y)}{y_j}$, $j = 1\ldots,n$, $y \in \bar{\mathbb{R}}^n$

Note: 1) $u \in \mathcal{P}_+$ is pluriharmonic and positive functions, i.e. $\partial_{z_j}\partial_{\overline{z_k}}u = 0$, $u \geq 0$. H_+ functions are holomorphic with non-negative imaginary part.

Note 2) [V18.2 Thm I] For $n = 1$ this is Nevanlinna representation. With $z^0 = \mathrm{i}$, we have

$$f(z) = b + az + \mathrm{i}\int \mathcal{S}_{\mathbb{R}^1}(z - x'; \mathrm{i} - z')\mu(\mathrm{d}x') =$$

$$b + az + \frac{\mathrm{i}}{\pi}\int \mathrm{i}(\frac{1}{z - x'} - \frac{1}{-\mathrm{i} - x'}) - \frac{1}{1 + (x')^2}\mu(\mathrm{d}x') =$$

$$b + az + \frac{1}{\pi}\int \frac{1}{x' - z} - \frac{x'}{1 + (x')^2}\mu(\mathrm{d}x')$$

which we recognize as the Herglotz-representation.

Note 3) For $n > 1$ there are some questions about the terms a and b raised by Annemarie last time, about their interpretation.

Conclusions

Conclusions

- Linear passive system in a cone have positivity and analyticity in T^C.
- Dispersion-relation follows from the Cauchy-Szegö representation, under H_s-constraints. (generalized Titchmarch theorems)
- Representations similar to Herglotz follow from the Schwartz-kernel.
- First applications are found. Limitations are still missing

Outlook/question

- Given $f(u,t)$, passive in t, what properties of u enables representations/sum-rules.