Sum rules and physical limitations for passive metamaterials

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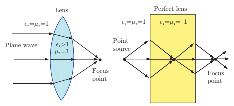
Outline

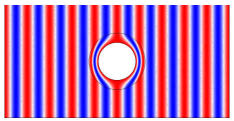
Trade-off: bandwidth versus performance for metamaterials

- Perfect lenses: bandwidth and resolution?
- Cloaking bandwidth and (extinction) cross section?
- High impedance surfaces: bandwidth, impedance, and thickness?
- Extra ordinary transmission: bandwidth, transmittance, and aperture fraction?
- Antennas: bandwidth, gain, and size?

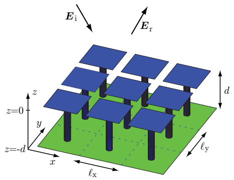
Here, we construct integral identities (sum rules) and physical bounds using properties such as causality, linearity, passivity, and time translational invariance to analyze these questions.



Perfect lens.



Free space cloaking.

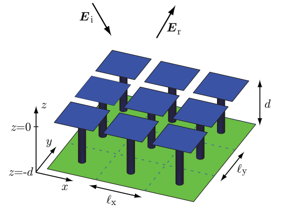


High-impedance surface.

Outline

High-impedance (artificial magnetic) surfaces

- PEC surfaces have low impedance, *i.e.*, short circuit currents give $Z = 0$. They also have reflection coefficients $\Gamma = (Z - Z_0)/(Z + Z_0) = -1$.
- PMC surfaces have high impedance and $\Gamma = 1$ (no phase shift).
- Useful for low-profile antennas ,*i.e.*, planar antenna elements can be placed above a PMC.
- Also useful to stop surface waves, *cf.*, hard and soft surfaces.

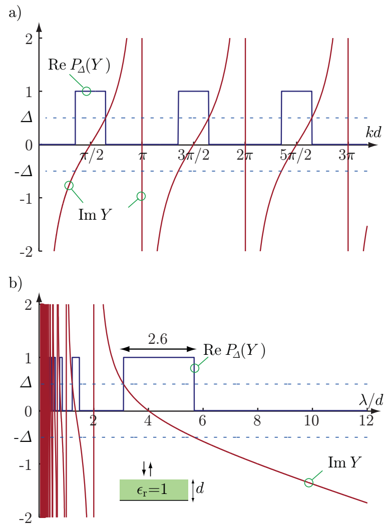


High-impedance surfaces are often composed by periodic structures above a PEC ground plane, here a mushroom structure.

For what bandwidth can a periodic structure above a PEC plane have 'high' impedance (reflection coefficient $\Gamma \approx 1$)?

'Simple' high-impedance surface

- A PEC ground plane at the distance $d = \lambda/4$ (quarter wavelength) gives a high impedance.
- Here, we use the (normalized) admittance $Y = Z_0/Z$ to quantify the bandwidth where $|Y| < \Delta$.
- Note that $\mathrm{Re}\, Y = 0$ for lossless structures.
- Construct $P_\Delta(Y)$ such that $P_\Delta(Y) = 1$ if $|Y| < \Delta$.
- We show that the area under the blue curve (lower figure, $\Delta = 1/2$) is π (a sum rule).



Sum rules and physical bounds

1. Find a (time domain) passive system (passive imply causal). Represent with either:
   - Impedance $Z(s)$, where $\mathrm{Re}\, Z(s) \geq 0$ for $\mathrm{Re}\, s > 0$ and $Z(s)$ analytic for $\mathrm{Re}\, s > 0$.
   - Reflection coefficient $\Gamma(s)$, where $|\Gamma(s)| \leq 1$ for $\mathrm{Re}\, s > 0$.

   where $s = \sigma + j\omega$ (*cf.*, Laplace transform).
2. Determine the low- and high-frequency asymptotic for $Z(s)$.
3. Have sum rules (integral identities), in particular with $Z(s) \sim a_1 s$ as $s \hat{\to} 0$ and $Z(s) \sim b_1 s$ as $s \hat{\to} \infty$

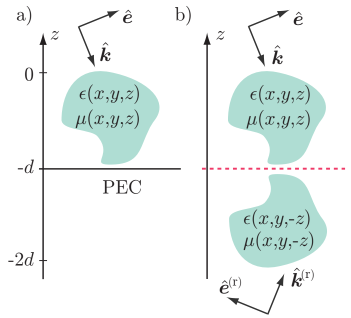
$$\frac{2}{\pi} \int_0^\infty \frac{\mathrm{Re}\, Z(j\omega)}{\omega^2} \, d\omega = a_1 - b_1 \leq a_1$$

Do not need to know the high-frequency limit for a bound.

Sum rules and constraints on passive systems J. Phys. A: Math. Theor. 44 145205, 2011

Low-frequency scattering ($\kappa \hat{\to} 0$)

Replace the ground plane with an incident wave and a mirror object.



Use $\kappa = \sigma + jk$, where $k = \omega/c_0$ is the wavenumber and $\sigma > 0$.

- Reflection coefficient

$$\Gamma(\kappa) \sim -1 + \kappa\big(2d + \gamma/A\big),$$

where γ/A is the magnetic polarizability per unit cell.
- Normalized impedance has

$$Z(\kappa) = \frac{1 + \Gamma(\kappa)}{1 - \Gamma(\kappa)} \sim \kappa(d + \gamma/(2A))$$

- Bound

$$d + \gamma/(2A) \leq \mu_s^{\max} d$$

where $\mu_s^{\max}$ is the maximal (static) permeability in the structure.

Sum rule for $|Y| < \Delta$

Interested in the bandwidth where

$$|Y| = 1/|Z| < \Delta$$

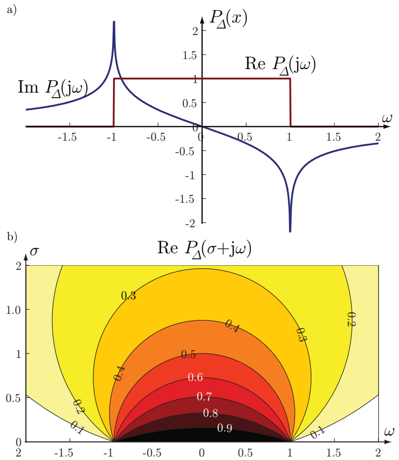
Solution:

- map $|Y| < \Delta$ and $\operatorname{Re} Y = 0$ to 1.
- compose with a positive real (PR) function, P_Δ, that has

$$\operatorname{Re} P_\Delta(\mathrm{j}k) = \begin{cases} 1 & -\Delta < k < \Delta \\ 0 & |k| > \Delta \end{cases}$$

That is

$$P_\Delta(\kappa) = \frac{1}{\pi}\int_{-\Delta}^{\Delta} \frac{1}{\mathrm{j}\xi + \kappa}\,\mathrm{d}\xi = \frac{1}{\mathrm{j}\pi}\ln\frac{\mathrm{j}\kappa - \Delta}{\mathrm{j}\kappa + \Delta}$$



Pulse PR function $P_\Delta(\sigma + \mathrm{j}\omega)$ with $\Delta = 1$.

Sum rule for $|Y| < \Delta$

Asymptotic

$$P_\Delta(\kappa) \sim \begin{cases} 1, & \text{as } \kappa \to 0 \\ \frac{2\Delta}{\pi\kappa}, & \text{as } \kappa \to \infty \end{cases}$$

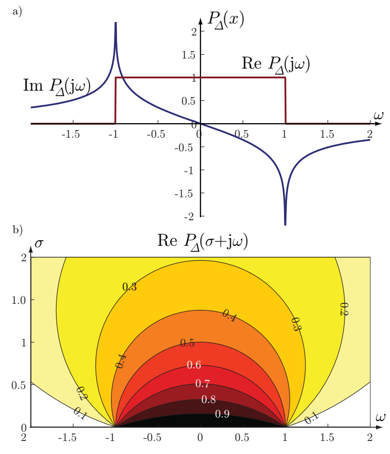
The composition $P_{\Delta 1}(\kappa) = P_\Delta(Y(\kappa))$

$$P_\Delta(Y(\kappa)) \sim \begin{cases} \frac{2}{\pi}\kappa(d + \gamma/(2A))\Delta & \kappa \hat{\to} 0 \\ o(\kappa) & \kappa \hat{\to} \infty. \end{cases}$$

Sum rule ($n = 1$ identity)

$$\int_0^\infty \frac{\operatorname{Re} P_\Delta(Y(\mathrm{j}k))}{k^2}\,\mathrm{d}k = \left(d + \frac{\gamma}{2A}\right)\Delta.$$

i.e., the area under $\operatorname{Re} P_\Delta(Y(\mathrm{j}k))/k^2$ is known.



The PR pulse function P_Δ with $\Delta = 1$.

Sum rule and bound

The integral identities for PR function give the $n = 1$ sum rule

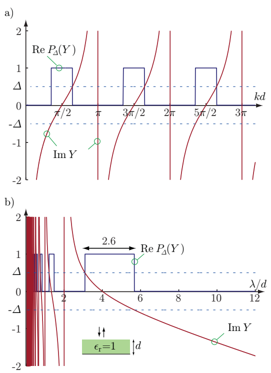
$$\int_0^\infty \frac{\operatorname{Re} P_\Delta(Y(\mathrm{j}k))}{k^2}\,\mathrm{d}k = \left(d + \frac{\gamma}{2A}\right)\Delta.$$

It is convenient to rewrite it into

$$\int_0^\infty \operatorname{Re} P_\Delta(Y(\lambda))\,\mathrm{d}\lambda = \left(d + \frac{\gamma}{2A}\right)2\pi\Delta,$$

where $\lambda = 2\pi/k$ denotes the wavelength.
Bound

$$\frac{B\lambda_0}{d} \leq 4\pi\mu_{\mathrm{s}}^{\max}\max_{\lambda\in\mathcal{B}}|Y(\lambda)|\begin{cases} 1 & \text{lossy case} \\ 1/2 & \text{lossless case.} \end{cases}$$



$\gamma = 0$ and $\Delta = 1/2$ gives π.

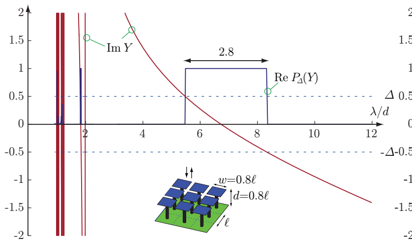
Note: the low loss case is close to the lossless case.

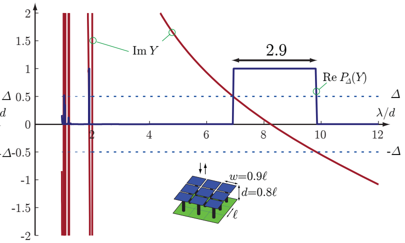
High-impedance surfaces

Physical bound

$$\frac{B\lambda_0}{d} \leq 4\pi\mu_{\mathrm{s}}^{\max}\max_{\lambda\in\mathcal{B}}|Y(\lambda)|\begin{cases} 1 & \text{lossy} \\ 1/2 & \text{lossless,} \end{cases}$$

Non-magnetic and $\max|Y| \leq 1/2$ gives the normalized bandwidth $B\lambda_0/d \leq \pi$.



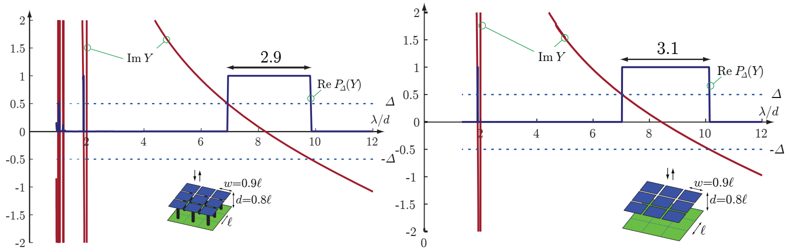


Can we approach the bound, π?

High-impedance surfaces

The via (PEC cylinder) connecting the patch with the ground has a negative magnetic polarizability $\gamma_{\mathrm{m}}/(2Ad) \approx -0.08$. Remove the via to get a patch structure.

$\max|Y| \leq 1/2$ gives the normalized bandwidth $B\lambda_0/d \leq \pi - 0.08 \approx 3.06$.

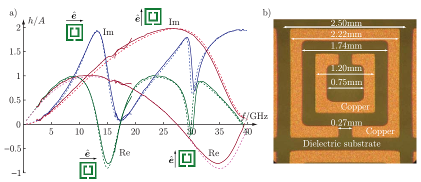


Result approach π as the distance between the patches decreases, *e.g.*, $w = 0.99\ell$ gives 3.12.

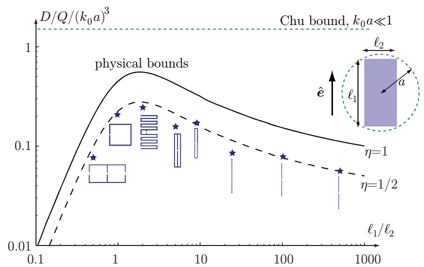
Outline

Sum rules and bounds on metamaterials

- Transmission blockage: low transmission for low-pass structures.
- Extinction cross section: scattered and absorbed power for low-pass structures.
- Extraordinary transmission for thin structures.
- Superluminal transmission ($n < 1$).
- Perfect lens ($\epsilon_{\mathrm{r}} = \mu_{\mathrm{r}} = -1$).
- Absorbers: absorption over a bandwidth.
- Antennas: bandwidth for given size.



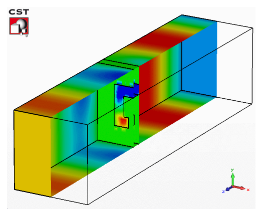
Extinction cross section.

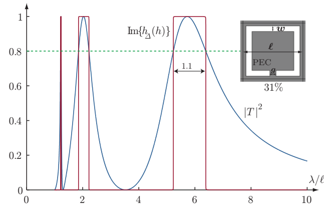


Antenna bounds (planar geometry).

Extraordinary transmission through PEC sheets

Over what bandwidth can at least 80% of the power be transmitted?



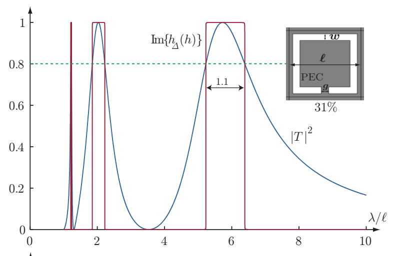


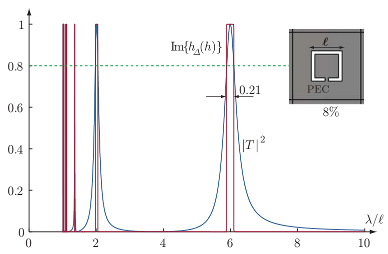
- Construct a sum rule for $|T|^2 \geq 0.8$, *i.e.*, $\Delta = 0.5$ below

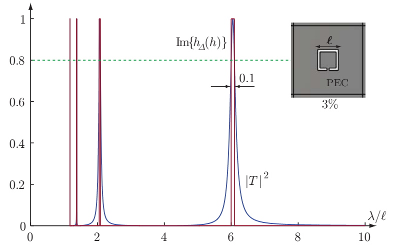
$$\int_0^{\infty} \mathrm{Im}\{h_\Delta(h(\lambda))\}\,\mathrm{d}\lambda = \frac{\gamma\Delta\pi}{A}$$

- Example with an aperture array of SRR in a PEC sheet.
- The area under $\mathrm{Im}\{h_\Delta(h(\lambda/\ell))\}$ is known: 1.56.
- Bandwidth with $|T|^2 \geq 0.8$ is ≈ 1.1 (bound 1.56).

Extraordinary transmission: arrays of SRR

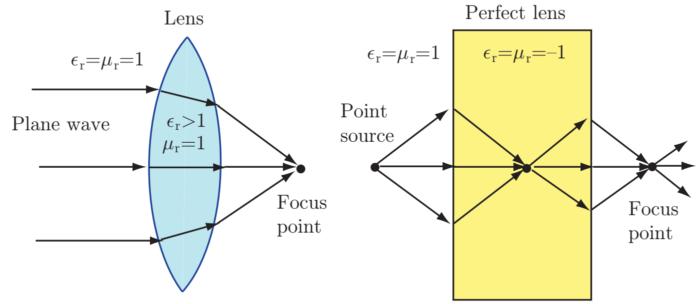






- Focusing of power $|T|^2 A/\mathcal{A}$.
- Increase the unit cell size, $A = \ell^2$, with $\ell = 2.5, 5, 7.5$.
- Aperture areas $31\%, 8\%, 3\%$.

Perfect lens



- Metamaterials realized as periodic structures.
- For what bandwidth is it possible to design a periodic structure that has the properties of a perfect lens?
- Bounds on the temporal dispersion of $\epsilon_r(\omega)$ and $\mu_r(\omega)$.
- Bounds on all periodic realizations with $\Gamma \approx 0$ and $T \approx e^{jkd}$.

Passive constitutive relations

The linear, causal, time translational invariant, continuous, non-magnetic, and isotropic constitutive relations are

$$\boldsymbol{D}(t) = \epsilon_0\epsilon_\infty \boldsymbol{E}(t) + \epsilon_0 \int_{\mathbb{R}} \chi_{ee}(t-t')\boldsymbol{E}(t')\,dt'$$

where $\chi_{ee}(t) = 0$ for $t < 0$, the dependence of the spatial coordinates is suppressed, and $\epsilon_\infty > 0$ is the instantaneous response. Passive if

$$0 \leq \int_{-\infty}^{T} \boldsymbol{E}(t)\cdot\frac{\partial \boldsymbol{D}(t)}{\partial t}\,dt = \epsilon_0 \int_{-\infty}^{T}\int_{\mathbb{R}} \boldsymbol{E}(t)\cdot\frac{\partial}{\partial t}\left(\epsilon_\infty\delta(t-t') + \chi_{ee}(t-t')\right)\boldsymbol{E}(t')\,dt'\,dt$$

for all times T and fields $\boldsymbol{E}$.

- Similarly for the magnetic fields.
- The presented results are also valid for the diagonal elements of general bi-anisotropic constitutive relations.
- Time-domain model, *e.g.*, used in FDTD.
- Fourier transform to get the frequency-domain model $\boldsymbol{D}(\omega) = \epsilon_0\epsilon(\omega)\boldsymbol{E}(\omega)$. Passivity imply that $h(\omega) = \omega\epsilon(\omega)$ is a Herglotz function, *i.e.*, $h(z)$ is analytic and $\operatorname{Im}\{h(z)\} \geq 0$ for $\operatorname{Im} z > 0$.

Constraints on the temporal dispersion of metamaterials

Answered Question

What is the minimum temporal dispersion of passive materials over bandwidths $\mathcal{B} = [\omega_1, \omega_2]$?
Want *e.g.*, permittivity $\epsilon(\omega) \approx \epsilon_m$ (similarly for $\mu(\omega)$ and $n(\omega)$)

Solution

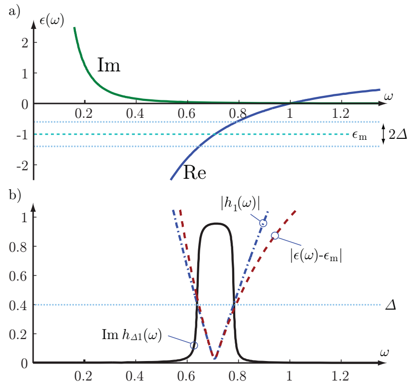
- no limitation for $\epsilon_\infty \leq \epsilon_m \leq \epsilon_s$ (static value).
- limitations for $\epsilon_m \leq \epsilon_\infty = \epsilon(\infty)$ (instantaneous value).

$$\max_{\omega\in\mathcal{B}} |\epsilon(\omega) - \epsilon_m| \geq \frac{B}{1+B/2}(\epsilon_\infty - \epsilon_m)\begin{cases}1/2 & \text{lossy case}\\ 1 & \text{lossless case,}\end{cases}$$

where $B = (\omega_2 - \omega_1)/\omega_0$ and $\omega_0 = (\omega_1 + \omega_2)/2$.

- limitations for $\epsilon_m \geq \epsilon_s = \epsilon(0)$ (static value).

Example: Drude model



$\mathrm{Im}\, h_{\Delta 1}(\omega)$ with $\Delta = 0.4$.
Sum rule

$$\int_0^\infty \mathrm{Im}\, h_{\Delta 1}(\omega)\,\mathrm{d}\omega = \frac{\omega_0 \Delta}{\epsilon_\infty - \epsilon_m}$$

The Drude model

$$\epsilon(\omega) = 1 + \frac{1}{-\mathrm{i}\omega(0.01 - \mathrm{i}\omega)},$$

has $\epsilon(\omega) \approx -1 = \epsilon_m$ for $\omega \approx 0.7$.

- The area under $\mathrm{Im}\, h_{\Delta 1}(\omega)$ is concentrated to the region where $|\epsilon(\omega) - \epsilon_m| \leq \Delta$.
- This area is known

$$\frac{\omega_0 \Delta}{\epsilon_\infty - \epsilon_m} \approx \frac{0.7 \cdot 0.4}{1-(-1)} = 0.14$$

- area $\approx$ height $\times$ width gives the bandwidth, *i.e.*, bandwidth ≈ 0.14.

Temporal dispersion: constraints

Interval $\mathcal{B} = [\omega_1, \omega_2]$ with fractional bandwidth $B = (\omega_2 - \omega_1)/\omega_0$, $\omega_0 = (\omega_1 + \omega_2)/2$
ϵ_s =static, ϵ_∞ =instantaneous, ϵ_m =target values.

1. $\epsilon_m < \epsilon_\infty$:

$$\max_{\omega\in\mathcal{B}} |\epsilon(\omega) - \epsilon_m| \geq \frac{B}{1+B/2}(\epsilon_\infty - \epsilon_m)\begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case,} \end{cases}$$

2. without static conductivity

$$\max_{\omega\in\mathcal{B}} \frac{|\epsilon(\omega) - \epsilon_m|}{|\epsilon(\omega) - \epsilon_\infty|} \geq \frac{B}{1+B/2}\,\frac{\epsilon_s - \epsilon_m}{\epsilon_s - \epsilon_\infty}\begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case,} \end{cases}$$

3. artificial magnetism $\mu_m > \mu_s$

$$\max_{\omega\in\mathcal{B}} \frac{|\mu(\omega) - \mu_m|}{|\mu(\omega) - \mu_\infty|} \geq \frac{B}{1+B/2}\,\frac{\mu_m - \mu_s}{\mu_s - \mu_\infty}\begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case,} \end{cases}$$

Sum rules and physical bounds on passive metamaterials, New Journal of Physics, Vol. 12, pp. 043046-, 2010.

Outline

Conclusions

- Sum rules for high-impedance surfaces, extraordinary transmission, and temporal dispersion.
- Often polarizability of the structure.
- Physical bounds on the bandwidth.
- They are all extreme cases, *e.g.*, $T = 0$ or $T = -1$ for the low-pass FSS ($T(f=0) = 1$), $T = 1$ for the bandpass FSS ($T(f=0) = 0$), and $T = \mathrm{e}^{\mathrm{j}kd}$ for the negative refractive index (wrong direction for the phase).

Why physical bounds?

- Realistic expectations. Possible/impossible.
- Possible design improvements. Is it worth it?
- Figure of merit for a design.

Static and instantaneous parameter values

The static, *e.g.*, $\epsilon_{\mathrm{s}} = \epsilon(0)$, and instantaneous (or high frequency), *e.g.*, $\epsilon_\infty = \epsilon(\infty)$, values are used in the bounds. Some properties of the static properties are known:

- The static permittivity and permeability are well defined.
- Can be determined by homogenization techniques.
- The effective material parameters are bounded by the parameters of the included materials.

Instantaneous (or high frequency) properties:

- Hard to define for heterogeneous materials.
- Necessary in time-domain constitutive relations, *cf.*, well-posedness of the PDE and FDTD simulations (time step).
- Often suggested that $\epsilon_\infty = 1$ and $\mu_\infty = 1$ (contradictions with diamagnetism, $\mu_\infty < 1$).
- Often suggested that $\sqrt{\epsilon_\infty \mu_\infty} = n_\infty \geq 1$ (wavefront speed less than the speed of light in vacuum).