Sum rules and physical limitations for passive metamaterials

Mats Gustafsson and Daniel Sjöberg
Department of Electrical and Information Technology
Lund University, Sweden

Trade-off: bandwidth versus performance for metamaterials

- Perfect lenses: bandwidth and resolution?
- Cloaking bandwidth and (extinction) cross section?
- High impedance surfaces: bandwidth, impedance, and thickness?
- Extraordinary transmission: bandwidth, transmittance, and aperture fraction?
- Antennas: bandwidth, gain, and size?

Here, we construct integral identities (sum rules) and physical bounds using properties such as causality, linearity, passivity, and time translational invariance to analyze these questions.
High-impedance (artificial magnetic) surfaces

- PEC surfaces have low impedance, i.e., short circuit currents give $Z = 0$. They also have reflection coefficients $\Gamma = (Z - Z_0) / (Z + Z_0) = -1$.
- PMC surfaces have high impedance and $\Gamma = 1$ (no phase shift).
- Useful for low-profile antennas, i.e., planar antenna elements can be placed above a PMC.
- Also useful to stop surface waves, cf., hard and soft surfaces.

For what bandwidth can a periodic structure above a PEC plane have ‘high’ impedance (reflection coefficient $\Gamma \approx 1$)?

Sum rules and physical bounds

1. Find a (time domain) passive system (passive imply causal). Represent with either:
   - Impedance $Z(s)$, where $\text{Re} Z(s) \geq 0$ for $\text{Re} s > 0$ and $Z(s)$ analytic for $\text{Re} s > 0$.
   - Reflection coefficient $\Gamma(s)$, where $|\Gamma(s)| \leq 1$ for $\text{Re} s > 0$, where $s = \sigma + j\omega$ (cf., Laplace transform).
2. Determine the low- and high-frequency asymptotic for $Z(s)$.
3. Have sum rules (integral identities), in particular with $Z(s) \sim a_1 s$ as $s \to 0$ and $Z(s) \sim b_1 s$ as $s \to \infty$

   \[ \frac{2}{\pi} \int_{0}^{\infty} \frac{\text{Re} Z(j\omega)}{\omega^2} \, d\omega = a_1 - b_1 \leq a_1 \]

   Do not need to know the high-frequency limit for a bound.

'Simple' high-impedance surface

- A PEC ground plane at the distance $d = \lambda/4$ (quarter wavelength) gives a high impedance.
- Here, we use the (normalized) admittance $Y = Z_0 / Z$ to quantify the bandwidth where $|Y| < \Delta$.
- Note that $\text{Re} Y = 0$ for lossless structures.
- Construct $P_\Delta(Y)$ such that $P_\Delta(Y) = 1$ if $|Y| < \Delta$.
- We show that the area under the blue curve (lower figure, $\Delta = 1/2$) is $\pi$ (a sum rule).

Low-frequency scattering ($\kappa \to 0$)

Replace the ground plane with an incident wave and a mirror object.

- Reflection coefficient
  \[ \Gamma(\kappa) \sim -1 + \kappa(2d + \gamma/A), \]
  where $\gamma/A$ is the magnetic polarizability per unit cell.
- Normalized impedance has
  \[ Z(\kappa) = \frac{1 + \Gamma(\kappa)}{1 - \Gamma(\kappa)} \sim \kappa(d + \gamma/(2A)) \]
- Bound
  \[ d + \gamma/(2A) \leq \mu_s^{\text{max}} d \]
  where $\mu_s^{\text{max}}$ is the maximal (static) permeability in the structure.

High-impedance surfaces are often composed by periodic structures above a PEC ground plane, here a mushroom structure.

Also useful to stop surface waves, cf., hard and soft surfaces.

Use $\kappa = \sigma + jk$, where $k = \omega / c_0$ is the wavenumber and $\sigma > 0$. 

Sum rule for $|Y| < \Delta$

Interested in the bandwidth where

$$|Y| = 1/|Z| < \Delta$$

Solution:

- map $|Y| < \Delta$ and $\text{Re} Y = 0$ to 1.
- compose with a positive real (PR) function, $P_\Delta$, that has

$$\text{Re} P_\Delta(jk) = \begin{cases} 1 & \quad -\Delta < k < \Delta \\ 0 & \quad |k| > \Delta \end{cases}$$

That is

$$P_\Delta(\kappa) = \frac{1}{\pi} \int_{-\Delta}^{\Delta} \frac{1}{j \pi + \kappa} \, dk = \frac{1}{j \pi} \ln \frac{j \kappa - \Delta}{j \kappa + \Delta}$$

Pulse PR function $P_\Delta(\sigma + jw)$ with $\Delta = 1$.

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Sum rule and bound

The integral identities for PR function give the $n = 1$ sum rule

$$\int_0^{\infty} \text{Re} \frac{P_\Delta(\lambda)}{k^2} \, dk = \left( d + \frac{\gamma}{2A} \right) \Delta.$$  

It is convenient to rewrite it into

$$\int_0^{\infty} \text{Re} P_\Delta(\lambda) \, d\lambda = \left( d + \frac{\gamma}{2A} \right) 2\pi \Delta,$$

where $\lambda = 2\pi/k$ denotes the wavelength.

Bound

$$B\lambda_0/d \leq 4\pi \mu_s \max_{\lambda \in B} |Y(\lambda)| \begin{cases} 1 & \text{lossy case} \\ 1/2 & \text{lossless case} \end{cases}$$

Note: the low loss case is close to the lossless case.

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High-impedance surfaces

**Physical bound**

$$B\lambda_0/d \leq 4\pi \mu_s \max_{\lambda \in B} |Y(\lambda)| \begin{cases} 1 & \text{lossy,} \\ 1/2 & \text{lossless} \end{cases}$$

Non-magnetic and $\max |Y| \leq 1/2$ gives the normalized bandwidth $B\lambda_0/d \leq \pi$.

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Can we approach the bound, $\pi$?
High-impedance surfaces

The via (PEC cylinder) connecting the patch with the ground has a negative magnetic polarizability $\gamma_m/(2Ad) \approx -0.08$. Remove the via to get a patch structure.

\[
\max |Y| \leq 1/2 \text{ gives the normalized bandwidth } B_{\lambda_0}/d \leq \pi - 0.08 \approx 3.06.
\]

Result approach $\pi$ as the distance between the patches decreases, e.g., $w = 0.99\ell$ gives 3.12.

Sum rules and bounds on metamaterials

- Transmission blockage: low transmission for low-pass structures.
- Extinction cross section: scattered and absorbed power for low-pass structures.
- Extraordinary transmission for thin structures.
- Superluminal transmission ($n < 1$).
- Perfect lens ($\epsilon_r = \mu_r = -1$).
- Absorbers: absorption over a bandwidth.
- Antennas: bandwidth for given size.

Extraordinary transmission through PEC sheets

Over what bandwidth can at least 80% of the power be transmitted?

\[
\int_0^\infty \text{Im}\{h_{\Delta}(h(\lambda))\} \, d\lambda = \frac{\gamma \Delta \pi}{A}
\]

- Example with an aperture array of SRR in a PEC sheet.
- The area under $\text{Im}\{h_{\Delta}(h(\lambda))\}$ is known: 1.56.
- Bandwidth with $|T|^2 \geq 0.8$ is $\approx 1.1$ (bound 1.56).
Extraordinary transmission: arrays of SRR

- Focusing of power $|T|^2 A/A$.
- Increase the unit cell size, $A = \ell^2$, with $\ell = 2.5, 5, 7.5$.
- Aperture areas $31\%, 8\%, 3\%$.

Mats Gustafsson, Department of Electrical and Information Technology, Lund University, Sweden

Perfect lens

- Metamaterials realized as periodic structures.
- For what bandwidth is it possible to design a periodic structure that has the properties of a perfect lens?
- Bounds on the temporal dispersion of $\epsilon_\tau(\omega)$ and $\mu_\tau(\omega)$.
- Bounds on all periodic realizations with $\Gamma \approx 0$ and $T \approx e^{jkd}$.

Mats Gustafsson, Department of Electrical and Information Technology, Lund University, Sweden

Passive constitutive relations

The linear, causal, time translation invariant, continuous, non-magnetic, and isotropic constitutive relations are

$$D(t) = \epsilon_0\epsilon\omega E(t) + \epsilon_0 \int_{\mathbb{R}} \chi_{\omega}(t-t') E(t') \, dt'$$

where $\chi_{\omega}(t) = 0$ for $t < 0$, the dependence of the spatial coordinates is suppressed, and $\epsilon_\infty > 0$ is the instantaneous response. Passive if

$$0 \leq \int_{-\infty}^{T} E(t) \frac{\partial D(t)}{\partial t} \, dt = \epsilon_0 \int_{-\infty}^{T} E(t) \frac{\partial}{\partial t} (\epsilon_\infty \delta(t-t') + \chi_{\omega}(t-t')) E(t') \, dt' \, dt$$

for all times $T$ and fields $E$.

- Similarly for the magnetic fields.
- The presented results are also valid for the diagonal elements of general bi-anisotropic constitutive relations.
- Time-domain model, e.g., used in FDTD.
- Fourier transform to get the frequency-domain model $D(\omega) = \epsilon_0\epsilon(\omega) E(\omega)$. Passivity imply that $h(\omega) = \omega\epsilon(\omega)$ is a Herglotz function, i.e., $h(z)$ is analytic and $\text{Im}\{h(z)\} \geq 0$ for $\text{Im} z > 0$.

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Constraints on the temporal dispersion of metamaterials

Answered Question

What is the minimum temporal dispersion of passive materials over bandwidths $B = [\omega_1, \omega_2]$?

Want e.g., permittivity $\epsilon(\omega) \approx \epsilon_m$ (similarly for $\mu(\omega)$ and $n(\omega)$)

Solution

- no limitation for $\epsilon_\infty \leq \epsilon_m \leq \epsilon_s$ (static value).
- limitations for $\epsilon_m \leq \epsilon_\infty = \epsilon(\infty)$ (instantaneous value).

$$\max_{\omega \in B} |\epsilon(\omega) - \epsilon_m| \geq \frac{B}{1 + B/2}(\epsilon_\infty - \epsilon_m) \left\{ \begin{array}{ll} 1/2 & \text{lossy case} \vspace{0.5em} \\ 1 & \text{lossless case,} \end{array} \right.$$

where $B = (\omega_2 - \omega_1)/\omega_0$ and $\omega_0 = (\omega_1 + \omega_2)/2$.

- limitations for $\epsilon_m \geq \epsilon_s = \epsilon(0)$ (static value).
Example: Drude model

The Drude model

$$\varepsilon(\omega) = 1 + \frac{1}{-i\omega(0.01 - i\omega)},$$

has $$\varepsilon(\omega) \approx -1 = \varepsilon_m$$ for $$\omega \approx 0.7$$.

- The area under $$\text{Im} h_{\Delta 1}(\omega)$$ is concentrated to the region where $$|\varepsilon(\omega) - \varepsilon_m| \leq \Delta$$.
- This area is known

$$\frac{\omega_0 \Delta}{\varepsilon_\infty - \varepsilon_m} \approx 0.7 \cdot 0.4 \approx 0.14$$

- area $$\approx$$ height $$\times$$ width gives the bandwidth, i.e., bandwidth $$\approx 0.14$$.

Temporal dispersion: constraints

Interval $$B = [\omega_1, \omega_2]$$ with fractional bandwidth $$B = (\omega_2 - \omega_1)/\omega_0$$, 

$$\varepsilon_s = \text{static}, \quad \varepsilon_\infty = \text{instantaneous}, \quad \varepsilon_m = \text{target values}.$$

1. $$\varepsilon_m < \varepsilon_\infty$$:

$$\max_{\omega \in B} |\varepsilon(\omega) - \varepsilon_m| \geq \frac{B}{1 + B/2} (\varepsilon_\infty - \varepsilon_m) \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{cases}$$

2. without static conductivity

$$\max_{\omega \in B} |\varepsilon(\omega) - \varepsilon_\infty| \geq \frac{B}{1 + B/2} \varepsilon_s - \varepsilon_\infty \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{cases}$$

3. artificial magnetism $$\mu_m > \mu_s$$

$$\max_{\omega \in B} |\mu(\omega) - \mu_m| \geq \frac{B}{1 + B/2} \mu_s - \mu_\infty \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{cases}$$


Outline

1. Bandwidth versus performance for metamaterials
2. High-impedance surfaces
3. Sum rules and bounds on metamaterials
   - Extraordinary transmission
   - Perfect lens
   - Temporal dispersion
4. Conclusions

Conclusions

- Sum rules for high-impedance surfaces, extraordinary transmission, and temporal dispersion.
- Often polarizability of the structure.
- Physical bounds on the bandwidth.
- They are all extreme cases, e.g., $$T = 0$$ or $$T = -1$$ for the low-pass FSS ($$T(f = 0) = 1$$), $$T = 1$$ for the bandpass FSS ($$T(f = 0) = 0$$), and $$T = e^{jkd}$$ for the negative refractive index (wrong direction for the phase).

Why physical bounds?

- Realistic expectations. Possible/impossible.
- Possible design improvements. Is it worth it?
- Figure of merit for a design.
Static and instantaneous parameter values

The static, e.g., $\epsilon_s = \epsilon(0)$, and instantaneous (or high frequency), e.g., $\epsilon_\infty = \epsilon(\infty)$, values are used in the bounds. Some properties of the static properties are known:

- The static permittivity and permeability are well defined.
- Can be determined by homogenization techniques.
- The effective material parameters are bounded by the parameters of the included materials.

Instantaneous (or high frequency) properties:

- Hard to define for heterogeneous materials.
- Necessary in time-domain constitutive relations, cf., well-posedness of the PDE and FDTD simulations (time step).
- Often suggested that $\epsilon_\infty = 1$ and $\mu_\infty = 1$ (contradictions with diamagnetism, $\mu_\infty < 1$).
- Often suggested that $\sqrt{\epsilon_\infty \mu_\infty} = n_\infty \geq 1$ (wavefront speed less than the speed of light in vacuum).