Physical bounds on antennas of arbitrary shape

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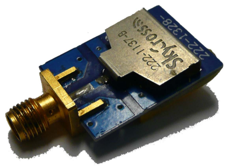
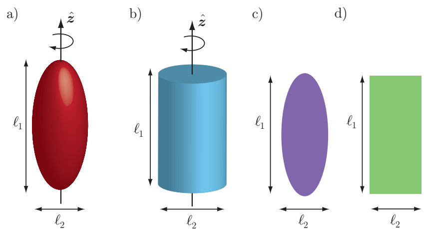
Outline

Physical bounds on antennas

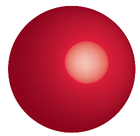


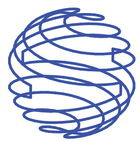


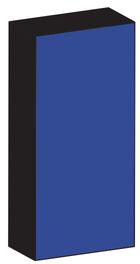
- Properties of the best antenna confined to a given (arbitrary) geometry, *e.g.*, spheroid, cylinder, elliptic disk, and rectangle.
- Tradeoff between performance and size.
- Performance in
  - Directivity bandwidth product: D/Q (half-power $B \approx 2/Q$).
  - Partial realized gain: $(1-|\Gamma|^2)G$ over a bandwidth.

Background

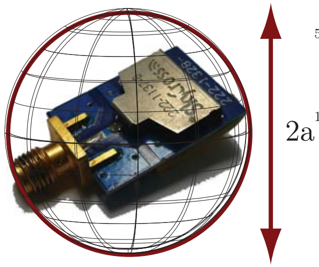
- 1947 Wheeler: *Bounds based on circuit models.*
- 1948 Chu: *Bounds on Q and D/Q for spheres.*
- 1964 Collin & Rothchild: *Closed form expressions of Q for arbitrary spherical modes, see also* Harrington, Collin, Fantes, Maclean, Gayi, Hansen, Hujanen, Sten, Thiele, Best, Yaghjian, ... *(most are based on Chu's approach using spherical modes.)*
- 1999 Foltz & McLean, 2001 Sten, Koivisto, and Hujanen: *Attempts for bounds in spheroidal volumes.*
- 2006 Thal: *Bounds on Q for small hollow spherical antennas.*
- 2007 Gustafsson, Sohl, Kristensson: *Bounds on D/Q for arbitrary geometries (and Q for small antennas).*
- 2010 Yaghjian & Stuart: *Bounds on Q for dipole antennas in the limit $ka \to 0$.*
- 2011 Vandenbosch: *Bounds on Q for small (non-magnetic) antennas in the limit $ka \to 0$.*
- 2011 Chalas, Sertel, and Volakis: *Bounds on Q using characteristic modes.*
- 2011 Gustafsson, Cismasu, Jonsson: *Optimal charge and current distributions on antennas.*

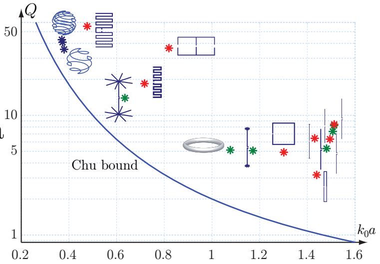






Background: Chu bound (sphere)





Calculation of the stored energy and radiated power outside a sphere with radius a gives the Chu-bounds (1948) for omni-directional antennas, *i.e.*,

$$Q \geq Q_{\mathrm{Chu}} = \frac{1}{(k_0a)^3} + \frac{1}{k_0a} \quad \text{and} \quad \frac{D}{Q} \leq \frac{3}{2Q_{\mathrm{Chu}}} \approx \frac{3}{2}(k_0a)^3$$

for $k_0a \ll 1$, where $k = k_0$ is the resonance wavenumber $k = 2\pi/\lambda = 2\pi f/c_0$.

Outline

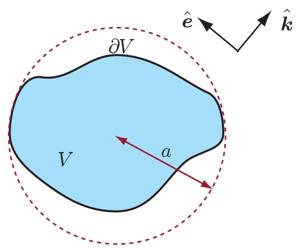
1. Motivation and background
2. **Antenna bounds based on forward scattering**
3. Antenna bounds and optimal currents based on stored energy
4. Conclusions

New physical bounds on antennas (2007)

Given a geometry, V, *e.g.*, sphere, rectangle, spheroid, or cylinder. Determine how D/Q (directivity bandwidth product) for optimal antennas depends on size and shape of the geometry.

Solution:

$$\frac{D}{Q} \leq \frac{\eta k_0^3}{2\pi}(\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}})\cdot\boldsymbol{\gamma}_{\mathrm{m}}\cdot(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}}))$$



is based on

- Antenna forward scattering
- Mathematical identities for Herglotz functions

M. Gustafsson, C. Sohl, G. Kristensson: Physical limitations on antennas of arbitrary shape Proceedings of the Royal Society A, 2007
M. Gustafsson, C. Sohl, G. Kristensson: Illustrations of new physical bounds on linearly polarized antennas IEEE Trans. Antennas Propagat. 2009

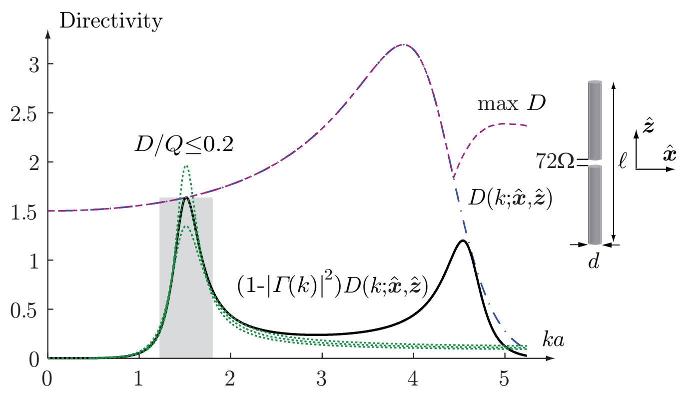
Antenna identity (sum rule)

Lossless linearly polarized antennas

$$\int_0^\infty \frac{(1-|\Gamma(k)|^2)D(k;\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{k^4}\,\mathrm{d}k = \frac{\eta}{2}(\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}})\cdot\boldsymbol{\gamma}_{\mathrm{m}}\cdot(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}}))$$

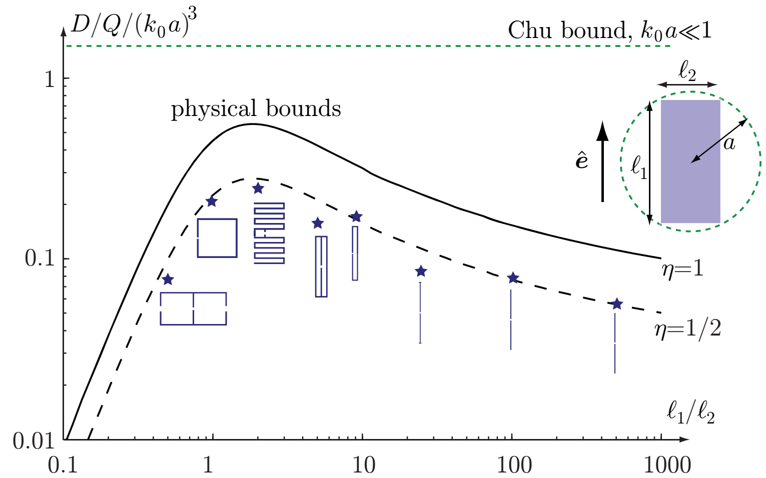
- $(1-|\Gamma(k)|^2)D(k;\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})$: partial realized gain, *cf.*, Friis transmission formula.
- $\Gamma(k)$: reflection coefficient
- $D(k;\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})$: directivity
- $k = 2\pi/\lambda = 2\pi f/c_0$: wavenumber
- $\hat{\boldsymbol{k}}$: direction of radiation
- $\hat{\boldsymbol{e}}$: polarization of the electric field, $\boldsymbol{E} = E_0\hat{\boldsymbol{e}}$.
- $\boldsymbol{\gamma}_{\mathrm{e}}$: electro-static polarizability dyadic of the structure.
- $\boldsymbol{\gamma}_{\mathrm{m}}$: magneto-static polarizability dyadic (assume $\boldsymbol{\gamma}_{\mathrm{m}} = \mathbf{0}$)
- $0 \leq \eta < 1$: generalized (all spectrum) absorption efficiency ($\eta \approx 1/2$ for small antennas).

Cylindrical dipole



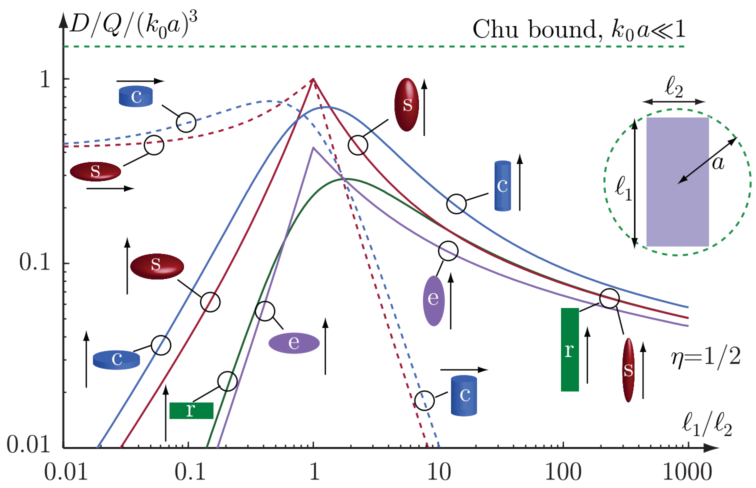
Lossless $\hat{\boldsymbol{z}}$-directed dipole, wire diameter $d = \ell/1000$, matched to $72\,\Omega$. Weighted area under the black curve (partial realized gain) is known. Note, half wavelength dipole for $ka = \pi/2 \approx 1.5$ with directivity $D \approx 1.64 \approx 2.15\,\mathrm{dB_i}$.

Circumscribing rectangles



Note, $\eta \approx 1/2$ for small optimal antennas $k_0a \ll 1$.

Rectangles, cylinders, elliptic disks, and spheroids



http://www.mathworks.com/matlabcentral/fileexchange/26806-antennaq

High-contrast polarizability dyadics: $\boldsymbol{\gamma}_\infty$

$\boldsymbol{\gamma}_\infty$ is determined from the induced normalized surface charge density, ρ, as

$$\boldsymbol{\gamma}_\infty \cdot \hat{\boldsymbol{e}} = \int_{\partial V} \boldsymbol{r}\rho(\boldsymbol{r})\,\mathrm{dS}$$

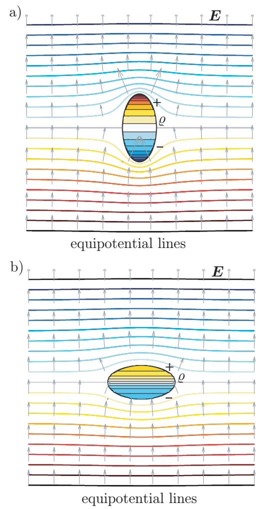
where ρ satisfies the integral equation

$$\int_{\partial V} \frac{\rho(\boldsymbol{r}')}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{dS}' = \boldsymbol{r}\cdot\hat{\boldsymbol{e}} + C_n$$

with the constraints of zero total charge

$$\int_{\partial V_n} \rho(\boldsymbol{r})\,\mathrm{dS} = 0$$

Can also use FEM (Laplace equation).



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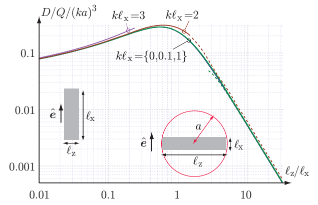
Bounds based on the stored energy

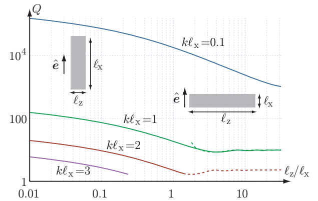
- Yaghjian and Stuart, *Lower Bounds on the Q of Electrically Small Dipole Antennas*, TAP 2010. Bounds on Q for small dipole antennas (in the limit $ka \to 0$).
- Vandenbosch, *Simple procedure to derive lower bounds for radiation Q of electrically small devices of arbitrary topology*, TAP 2011. Bounds on Q for small (non-magnetic) antennas (in the limit $ka \to 0$).
- Chalas, Sertel, and Volakis, *Computation of the Q Limits for Arbitrary-Shaped Antennas Using Characteristic Modes*, APS 2011. Bounds on Q not restricted to small ka.

Here, we reformulate the D/Q bound as an optimization problem that is solved using a variation approach and/or Lagrange multipliers, see Physical Bounds and Optimal Currents on Antennas, IEEE-TAP (in press).

Bounds on D/Q or Q

- Chu derived bounds on Q and D/Q for dipole antennas.
- Most papers analyze Q for small spherical dipole antennas. Results are independent of the direction and polarization so $D = 3/2$ and it is sufficient to determine Q for this case.
- The D/Q results are advantageous for general shapes as:
  - they provide a methodology to quantify the performance for different directions and polarizations.
  - they can separate linear and circular polarization.
  - $D/(Qk^3a^3)$ appears to depend relatively weakly on ka in contrast to Qk^3a^3.

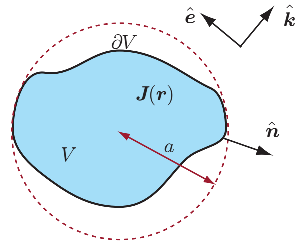




D/Q

Directivity in the radiation intensity $P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})$ and total radiated power P_{rad}

$$D(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}}) = 4\pi \frac{P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})}{P_{\mathrm{rad}}}$$



Q-factor

$$Q = \frac{2\omega W}{P_{\mathrm{rad}}} = \frac{2c_0 k W}{P_{\mathrm{rad}}},$$

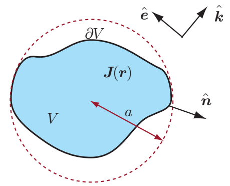
where $W = \max\{W_{\mathrm{e}}, W_{\mathrm{m}}\}$ denotes the maximum of the stored electric and magnetic energies. The D/Q quotient cancels P_{rad}

$$\frac{D(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})}{c_0 k W}.$$

D/Q in the current density $\boldsymbol{J}$

Radiation intensity $P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})$

$$P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}}) = \frac{\zeta_0 k^2}{32\pi^2}\left|\int_V \hat{\boldsymbol{e}}^*\cdot\boldsymbol{J}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{d}V\right|^2,$$



Stored electric energy $\widetilde{W}^{(\mathrm{e})}_{\mathrm{vac}} = \frac{\mu_0}{16\pi k^2}w^{(\mathrm{e})}$

$$w^{(\mathrm{e})} = \int_V\int_V \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\frac{\cos(kR_{12})}{R_{12}} - \frac{k}{2}(k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^* - \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*)\sin(kR_{12})\,\mathrm{dV}_1\,\mathrm{dV}_2,$$

where $\boldsymbol{J}_1 = \boldsymbol{J}(\boldsymbol{r}_1)$, $\boldsymbol{J}_2 = \boldsymbol{J}(\boldsymbol{r}_2)$, $R_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$.

$$\frac{D(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q} = k^3\frac{\left|\int_V \hat{\boldsymbol{e}}^*\cdot\boldsymbol{J}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{d}V\right|^2}{\max\{w^{(\mathrm{e})}(\boldsymbol{J}), w^{(\mathrm{m})}(\boldsymbol{J})\}},$$

Small antennas $ka\to 0$

Expand for $ka\to 0$

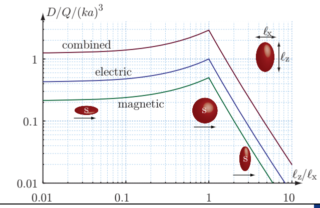
$$\frac{D}{Q}\leq\max_{\rho,\boldsymbol{J}^{(0)}}\frac{k^3\left|\int_V \hat{\boldsymbol{e}}^*\cdot\boldsymbol{r}\rho(\boldsymbol{r}) + \frac{1}{2}\hat{\boldsymbol{h}}^*\times\boldsymbol{r}\cdot\boldsymbol{J}^{(0)}(\boldsymbol{r})\,\mathrm{d}V\right|^2}{\max\left\{\iint_V\frac{\rho_1\rho_2^*}{R_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2, \iint_V\frac{\boldsymbol{J}_1^{(0)}\cdot\boldsymbol{J}_2^{(0)*}}{R_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2\right\}},$$

Electric dipole $\boldsymbol{J}^{(0)} = \boldsymbol{0}$

$$\frac{D_{\mathrm{e}}}{Q_{\mathrm{e}}}\leq\max_{\rho}\frac{k^3}{4\pi}\frac{\left|\int\hat{\boldsymbol{e}}^*\cdot\boldsymbol{r}\rho(\boldsymbol{r})\,\mathrm{d}V\right|^2}{\int_V\int_V\frac{\rho(\boldsymbol{r}_1)\rho^*(\boldsymbol{r}_2)}{4\pi|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\mathrm{dV}_1\,\mathrm{dV}_2}.$$

With the solution

$$\frac{D_{\mathrm{e}}(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q_{\mathrm{e}}}\leq\frac{k^3}{4\pi}\hat{\boldsymbol{e}}^*\cdot\boldsymbol{\gamma}_\infty\cdot\hat{\boldsymbol{e}}.$$



It verifies our previous bound.

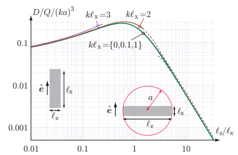
Non-electrically small antennas

Reformulate the D/Q bound as the minimization problem

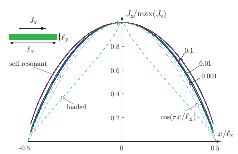
$$\min_{\boldsymbol{J}}\int_V\int_V\nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\frac{\cos(kR_{12})}{R_{12}} - \frac{k}{2}(k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^* - \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*)\sin(kR_{12})\,\mathrm{dV}_1\,\mathrm{dV}_2,$$

subject to the constraint

$$\left|\int_V\hat{\boldsymbol{e}}^*\cdot\boldsymbol{J}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{d}V\right| = 1.$$



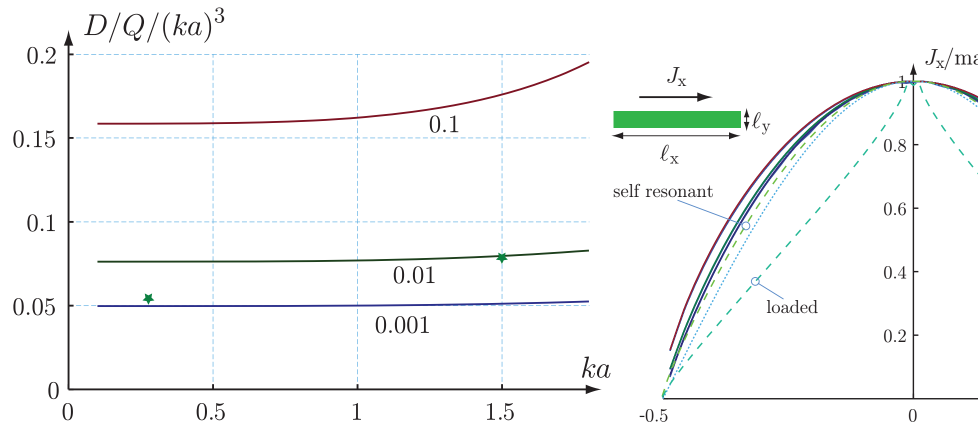
D/Q bound for rectangles.



Optimal currents on a strip.

Solve using Lagrange multipliers. It gives bounds and the optimal current distribution $\boldsymbol{J}$.

Strip dipole $\xi = d/\ell = \{0.001, 0.01, 0.1\}$



The stars indicate the performance of strip dipoles with $\xi = 0.01$. Almost no dependence on ka for $D/(Qk^3a^3)$. More dependence on ka for D and Qk^3a^3. Note the directivity of the half-wave dipole.

Optimal current distributions on small spheres

- The optimization problem for small dipole antennas show that the charge distribution is the most important quantity.
- On a sphere, we have

$$\rho(\theta,\phi) = \rho_0 \cos\theta$$

for optimal antennas with polarization $\hat{\boldsymbol{e}} = \hat{\boldsymbol{z}}$.

- The current density satisfies

$$\nabla \cdot \boldsymbol{J} = -\mathrm{j}k\rho$$

Many solutions, *e.g.*, all surface currents

$$\boldsymbol{J} = J_{\theta 0}\hat{\boldsymbol{\theta}}\big(\sin\theta - \frac{\beta}{\sin\theta}\big) + \frac{1}{\sin\theta}\frac{\partial A}{\partial\phi}\hat{\boldsymbol{\theta}} - \frac{\partial A}{\partial\theta}\hat{\boldsymbol{\phi}}$$

where $J_{\theta 0} = -\mathrm{j}ka\rho_0$, β is a constant, and $A = A(\theta,\phi)$

Optimal current distributions on small spheres

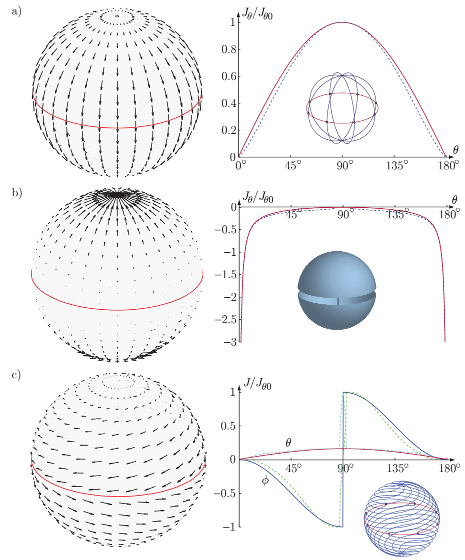
Some solutions:

- Spherical dipole, $\beta = 0, A = 0$.
- Capped dipole, $\beta = 1, A = 0$.
- Folded spherical helix, $\beta = 0, A \neq 0$.

They all have almost identical charge distributions

$$\rho(\theta,\phi) = \rho_0 \cos\theta$$

Can mathematical solutions suggest antenna designs?



Outline

Conclusions

- Forward scattering and/or optimization to determine bounds on D/Q for arbitrary shaped antennas.
- Closed form solution for small antennas.
  - Performance in the polarizability of the antenna structure.
  - Forwards scattering and optimization approach coincide for $ka \to 0$.
- Lagrange multipliers to solve the optimization problem for larger structures.
- $D/(Qk^3a^3)$ nearly independent on ka for $0 < ka < 1.5$.
- Optimal current distributions.