Physical bounds on antennas of arbitrary shape

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Outline

1 Motivation and background

2 Antenna bounds based on forward scattering

3 Antenna bounds and optimal currents based on stored energy

4 Conclusions

Background

- 1948 Chu: Bounds on Q and D/Q for spheres.
- 1964 Collin & Rothchild: Closed form expressions of Q for arbitrary spherical modes, see also Harrington, Collin, Fantes, Maclean, Gayi, Hansen, Hujanen, Sten, Thiele, Best, Yaghjian, ...
- 2006 Thal: Bounds on Q for small hollow spherical antennas.
- 2007 Gustafsson, Sohl, Kristensson: Bounds on D/Q for arbitrary geometries (and Q for small antennas).
- 2010 Yaghjian & Stuart: Bounds on Q for dipole antennas in the limit $ka \rightarrow 0$.
- 2011 Vandenbosch: Bounds on Q for small (non-magnetic) antennas in the limit $ka \rightarrow 0$.
- 2011 Chalas, Sertel, and Volakis: Bounds on Q using characteristic modes.
- 2011 Gustafsson, Cismasu, Jonsson: Optimal charge and current distributions on antennas.

Physical bounds on antennas

- Properties of the best antenna confined to a given (arbitrary) geometry, e.g., spheroid, cylinder, elliptic disk, and rectangle.
- Tradeoff between performance and size.
- Performance in
  - Directivity bandwidth product: $D/Q$ (half-power $B \approx 2/Q$).
  - Partial realized gain: $(1 - |\Gamma|^2)G$ over a bandwidth.
Calculation of the stored energy and radiated power outside a sphere with radius \(a\) gives the Chu-bounds (1948) for omni-directional antennas, i.e.,

\[
Q \geq Q_{\text{Chu}} = \frac{1}{k_0 a} + \frac{1}{k_0 a} \quad \text{and} \quad \frac{D}{Q} \leq \frac{3}{2} \left( k_0 a \right)^3
\]

for \(k_0 a \ll 1\), where \(k = k_0\) is the resonance wavenumber \(k = 2\pi/\lambda = 2\pi f/c_0\).

**New physical bounds on antennas (2007)**

Given a geometry, \(V\), e.g., sphere, rectangle, spheroid, or cylinder. Determine how \(D/Q\) (directivity bandwidth product) for optimal antennas depends on size and shape of the geometry.

Solution:

\[
\frac{D}{Q} \leq \frac{\eta k_0^3}{2\pi} \left( \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e}) \right)
\]

is based on

- Antenna forward scattering
- Mathematical identities for Herglotz functions

**Antenna identity (sum rule)**

Lossless linearly polarized antennas

\[
\int_0^\infty \frac{(1 - |\Gamma(k)|^2)D(k; \hat{k}, \hat{e})}{k^4} \, dk = \frac{\eta}{2} \left( \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e}) \right)
\]

- \((1 - |\Gamma(k)|^2)D(k; \hat{k}, \hat{e})\): partial realized gain, cf., Friis transmission formula.
- \(\Gamma(k)\): reflection coefficient
- \(D(k; \hat{k}, \hat{e})\): directivity
- \(k = 2\pi/\lambda = 2\pi f/c_0\): wavenumber
- \(\hat{k}\): direction of radiation
- \(\hat{e}\): polarization of the electric field, \(E = E_0\hat{e}\).
- \(\gamma_e\): electro-static polarizability dyadic of the structure.
- \(\gamma_m\): magneto-static polarizability dyadic (assume \(\gamma_m = 0\)).
- \(0 \leq \eta < 1\): generalized (all spectrum) absorption efficiency \((\eta \approx 1/2\) for small antennas).
Cylindrical dipole

Lossless \(\hat{z}\)-directed dipole, wire diameter \(d = \ell/1000\), matched to 72\(\Omega\). Weighted area under the black curve (partial realized gain) is known. Note, half wavelength dipole for \(ka = \pi/2 \approx 1.5\) with directivity \(D \approx 1.64 \approx 2.15\) dB.

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Circumscribing rectangles

High-contrast polarizability dyadics: \(\gamma_\infty\)

\(\gamma_\infty\) is determined from the induced normalized surface charge density, \(\rho\), as

\[
\gamma_\infty \cdot \hat{e} = \int_{\partial\mathcal{V}} \rho(\mathbf{r}) \ dS
\]

where \(\rho\) satisfies the integral equation

\[
\int_{\partial\mathcal{V}} \frac{\rho(\mathbf{r}')}{{4\pi}|r - r'|} \ dS' = \mathbf{r} \cdot \hat{e} + C_n
\]

with the constraints of zero total charge

\[
\int_{\partial\mathcal{V}_n} \rho(\mathbf{r}) \ dS = 0
\]

Can also use FEM (Laplace equation).

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Rectangles, cylinders, elliptic disks, and spheroids

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http://www.mathworks.com/matlabcentral/fileexchange/26806-antennaq
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Bounds on $D/Q$ or $Q$

- Chu derived bounds on $Q$ and $D/Q$ for dipole antennas.
- Most papers analyze $Q$ for small spherical dipole antennas. Results are independent of the direction and polarization so $D = 3/2$ and it is sufficient to determine $Q$ for this case.
- The $D/Q$ results are advantageous for general shapes as:
  - they provide a methodology to quantify the performance for different directions and polarizations.
  - they can separate linear and circular polarization.
  - $D/(Qk^3a^3)$ appears to depend relatively weakly on $ka$ in contrast to $Qk^3a^3$.

Bounds based on the stored energy

- Vandenbosch, *Simple procedure to derive lower bounds for radiation Q of electrically small devices of arbitrary topology*, TAP 2011. Bounds on $Q$ for small (non-magnetic) antennas (in the limit $ka \to 0$).

Here, we reformulate the $D/Q$ bound as an optimization problem that is solved using a variation approach and/or Lagrange multipliers, see *Physical Bounds and Optimal Currents on Antennas*, IEEE-TAP (in press).

Directivity in the radiation intensity $P(\hat{k}, \hat{e})$ and total radiated power $P_{rad}$

$$D(\hat{k}, \hat{e}) = 4\pi \frac{P(\hat{k}, \hat{e})}{P_{rad}}$$

Q-factor

$$Q = \frac{2\omega W}{P_{rad}} = \frac{2c_0kW}{P_{rad}},$$

where $W = \max\{W_e, W_m\}$ denotes the maximum of the stored electric and magnetic energies. The $D/Q$ quotient cancels $P_{rad}$

$$\frac{D(\hat{k}, \hat{e})}{Q} = \frac{2\pi P(\hat{k}, \hat{e})}{c_0kW}.$$
$D/Q$ in the current density $\mathbf{J}$

Radiation intensity $P(\hat{k}, \hat{e})$

$$P(\hat{k}, \hat{e}) = \frac{\Omega k^2}{32\pi^2} \left| \int_V \hat{e}^* \cdot \mathbf{J}(r)e^{ik\mathbf{r} \cdot \hat{r}} dV \right|^2,$$

Stored electric energy $W_{\text{vac}}^{(e)} = \frac{\mu_0}{16\pi\varepsilon_0} u^{(e)}$

$$u^{(e)} = \int_V \int_V \nabla_1 \cdot \mathbf{J}_1 \nabla_2 \cdot \mathbf{J}_2 \cos(kR_{12}) - \frac{k}{2}(k^2 \mathbf{J}_1 \cdot \mathbf{J}_2^*) \sin(kR_{12}) dV_1 dV_2,$$

where $\mathbf{J}_1 = \mathbf{J}(r_1)$, $\mathbf{J}_2 = \mathbf{J}(r_2)$, $R_{12} = |r_1 - r_2|$. 

$$\frac{D(\hat{k}, \hat{e})}{Q} = k^3 \left[ \int_V \hat{e}^* \cdot \mathbf{J}(r)e^{ik\mathbf{r} \cdot \hat{r}} dV \right]^2 \max\{u^{(e)}(\mathbf{J}), u^{(m)}(\mathbf{J})\},$$

Non-electrically small antennas

Reformulate the $D/Q$ bound as the minimization problem

$$\min_{\mathbf{J}} \int_V \int_V \nabla_1 \cdot \mathbf{J}_1 \nabla_2 \cdot \mathbf{J}_2 \cos(kR_{12}) - \frac{k}{2}(k^2 \mathbf{J}_1 \cdot \mathbf{J}_2^*) \sin(kR_{12}) dV_1 dV_2,$$

subject to the constraint

$$\int_V \hat{e}^* \cdot \mathbf{J}(r)e^{ik\mathbf{r} \cdot \hat{r}} dV = 1.$$

Solve using Lagrange multipliers. It gives bounds and the optimal current distribution $\mathbf{J}$.

Small antennas $ka \to 0$

Expand for $ka \to 0$

$$\frac{D}{Q} \leq \max_{\rho} \frac{k^3}{4\pi} \int_V \hat{e}^* \cdot \mathbf{J}(r) dV^2 \frac{|J^{(0)}(r)|}{\max_{\rho} \left\{ \int \int \hat{e}^* \cdot \mathbf{J}(r) dV \right\}},$$

Electric dipole $J^{(0)} = 0$

$$\frac{D_e}{Q_e} = \max_{\rho} \frac{k^3}{4\pi} \int_V \hat{e}^* \cdot \mathbf{J}(r) dV^2 \frac{|J^{(0)}(r)|}{\max_{\rho} \left\{ \int \int \hat{e}^* \cdot \mathbf{J}(r) dV \right\}}.$$

With the solution

$$\frac{D_e(\hat{k}, \hat{e})}{Q_e} = \ell^3 \hat{e}^* \cdot \gamma_{\infty} \cdot \hat{e}.$$ It verifies our previous bound.

Strip dipole $\xi = d/\ell = \{0.001, 0.01, 0.1\}$

The stars indicate the performance of strip dipoles with $\xi = 0.01$. Almost no dependence on $ka$ for $D/(Qk^3a^3)$.

Almost no dependence on $ka$ for $D$ and $Qk^3a^3$. Note the directivity of the half-wave dipole.
The optimization problem for small dipole antennas show that the charge distribution is the most important quantity. On a sphere, we have
\[ \rho(\theta, \phi) = \rho_0 \cos \theta \]
for optimal antennas with polarization \( \hat{e} = \hat{z} \).

The current density satisfies
\[ \nabla \cdot J = -j k \rho \]

Many solutions, e.g., all surface currents
\[ J = J_{\theta 0} \left( \sin \theta - \frac{\beta}{\sin \theta} \right) + \frac{1}{\sin \theta} \frac{\partial A}{\partial \phi} - \frac{\partial A}{\partial \theta} \hat{\phi} \]
where \( J_{\theta 0} = -j k a \rho_0 \), \( \beta \) is a constant, and \( A = A(\theta, \phi) \).

Some solutions:
- Spherical dipole, \( \beta = 0, A = 0 \).
- Capped dipole, \( \beta = 1, A = 0 \).
- Folded spherical helix, \( \beta = 0, A \neq 0 \).

They all have almost identical charge distributions
\[ \rho(\theta, \phi) = \rho_0 \cos \theta \]

Can mathematical solutions suggest antenna designs?

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Conclusions
- Forward scattering and/or optimization to determine bounds on \( D/Q \) for arbitrary shaped antennas.
- Closed form solution for small antennas.
  - Performance in the polarizability of the antenna structure.
  - Forwards scattering and optimization approach coincide for \( ka \to 0 \).
- Lagrange multipliers to solve the optimization problem for larger structures.
  - \( D/(Q k^3 a^3) \) nearly independent on \( ka \) for \( 0 < ka < 1.5 \).
- Optimal current distributions.