Stored energy and current optimization

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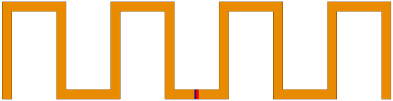
Antenna and current optimization

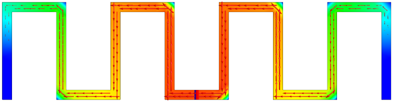
Antenna design: produce the desired current distribution on the structure by shaping and choosing the materials.

- Have a given maximal size of the antenna structure.
- Antenna optimization: determine the shape and material properties for optimal performance.
- Current optimization: determine an optimal current distribution from all possible currents in the available geometry.

Maximal size of the antenna.

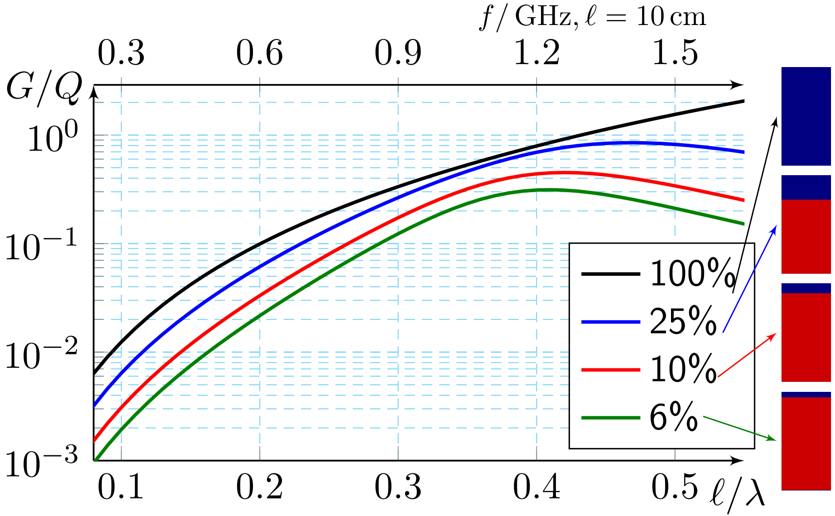


Antenna geometry with feed point.

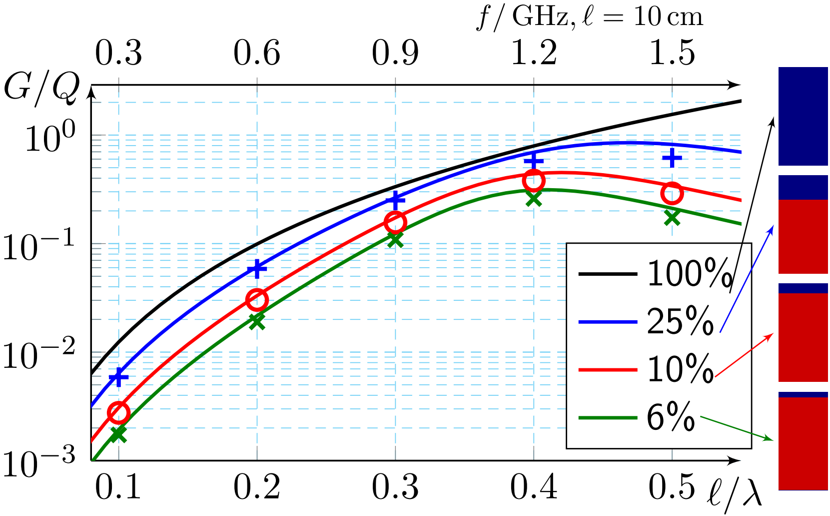


Current distribution on the antenna.

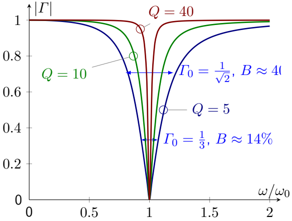
Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region



Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region



Q-factor and single frequency evaluation



Reflection coefficient $|\Gamma|$ for a RCL circuit with Q-factors $Q=\{5,10,40\}$. Fractional bandwidths for $\Gamma_0=\{1/\sqrt{2},1/3\}$.

The Q-factor is defined as the ratio between the stored electric, W_{e}, and magnetic, W_{m}, energies and the dissipated power, *i.e.*,

$$Q=\frac{2\omega\max\{W_{\mathrm{e}},W_{\mathrm{m}}\}}{P_{\mathrm{rad}}+P_{\mathrm{loss}}}.$$

Fractional bandwidth for single resonances

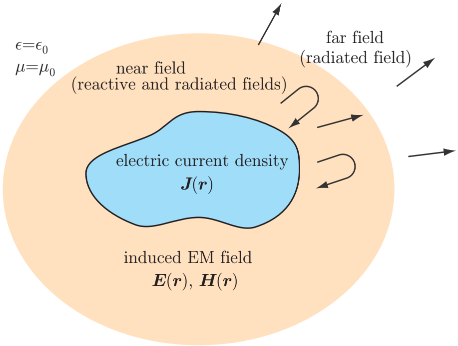
$$B\approx\frac{2}{Q}\frac{\Gamma_0}{\sqrt{1-\Gamma_0^2}}$$

Single frequency evaluation

use the Q-factor to estimate the bandwidth. Need to:

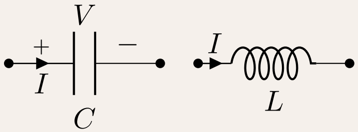
1. Compute the *stored energy*.
2. Solve the optimization problems.

What is (stored) EM energy?



- Time average energy density $\epsilon_0|\boldsymbol{E}|^2/4$ and $\mu_0|\boldsymbol{H}|^2/4$.
- What is stored and radiated?
- How can we express the (stored) energy in the current (density)?
- First, currents in free space.

Lumped elements

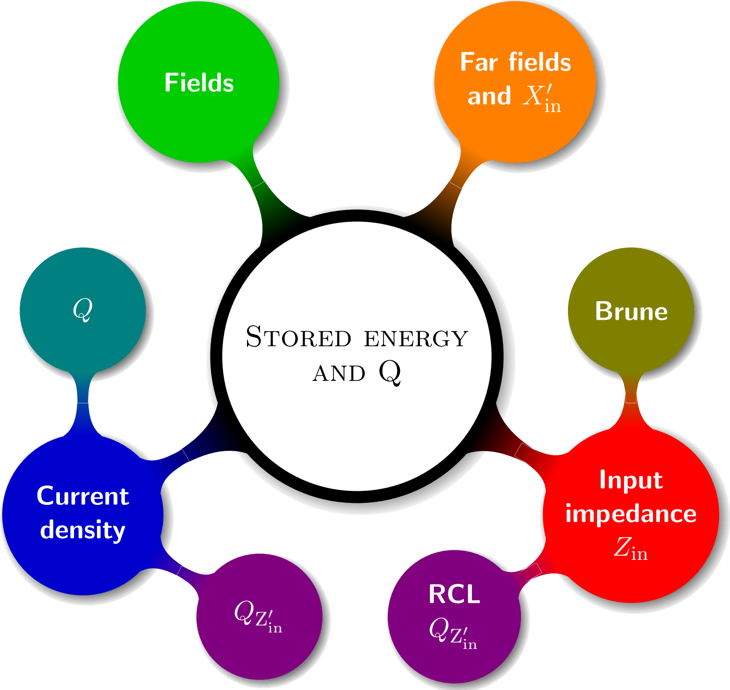


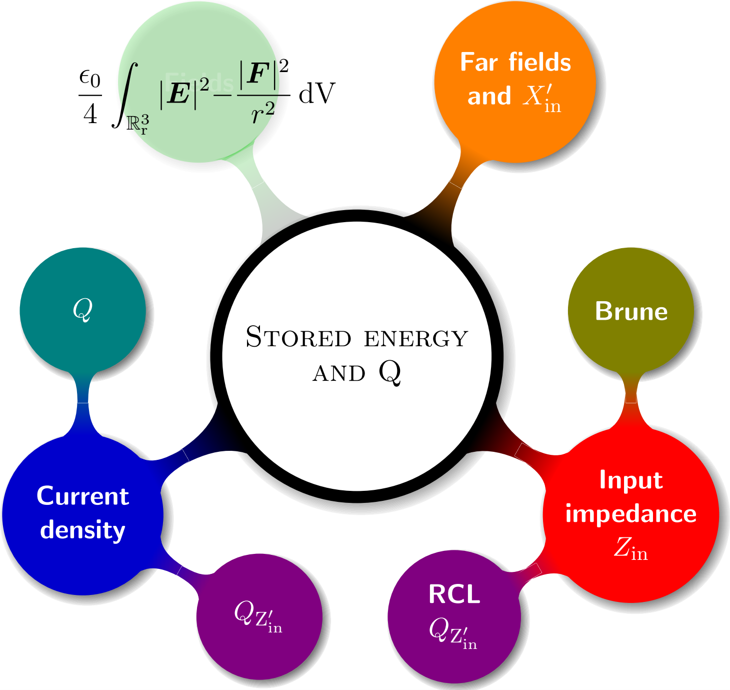
Time average stored energy in capacitors

$$W_{\mathrm{e}} = \frac{C|V|^2}{4} = \frac{|I|^2}{4\omega^2 C}$$

and in inductors

$$W_{\mathrm{m}} = \frac{L|I|^2}{4}$$


Stored energy and Q
Fields
Far fields and X'_{in}
Brune
Input impedance Z_{in}
RCL $Q_{Z'_{\text{in}}}$
$Q_{Z'_{\text{in}}}$
Current density
Q


$$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_r} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$$
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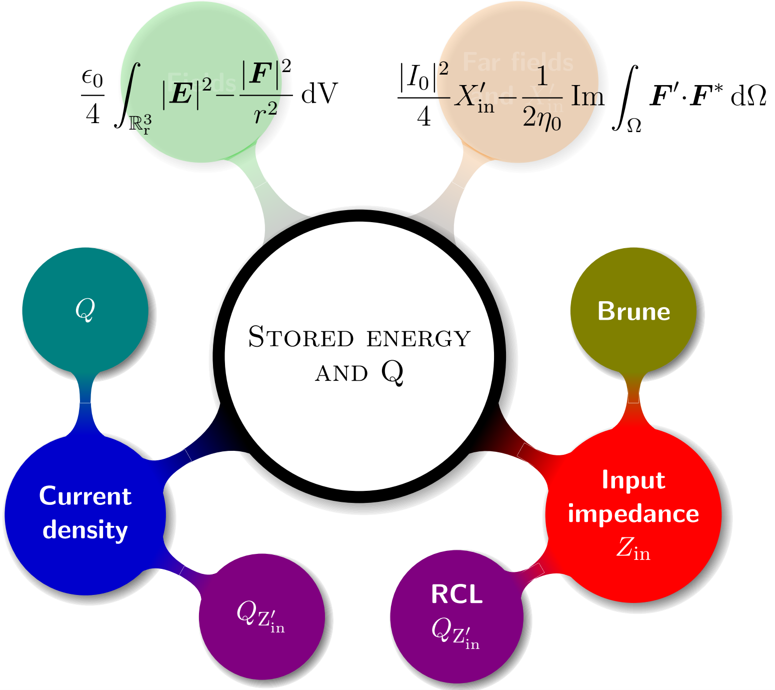
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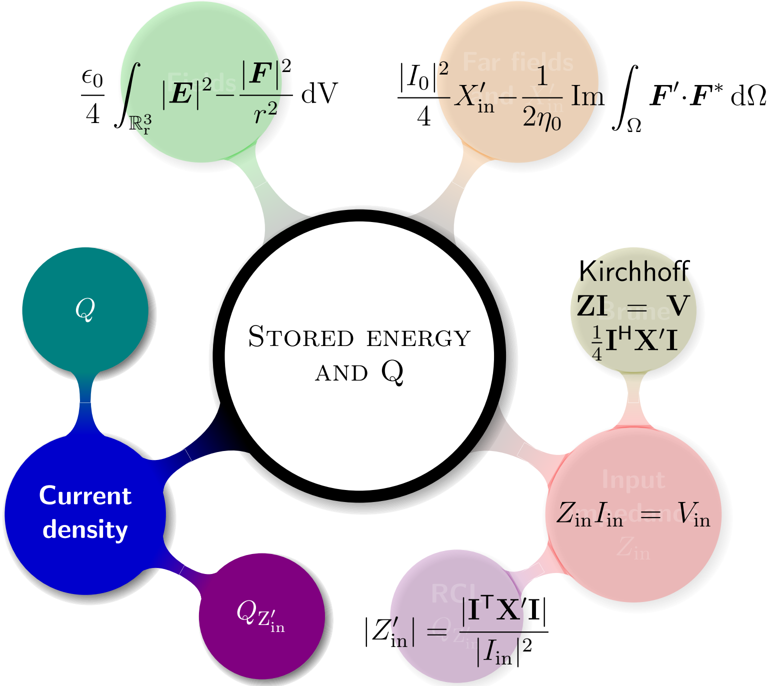

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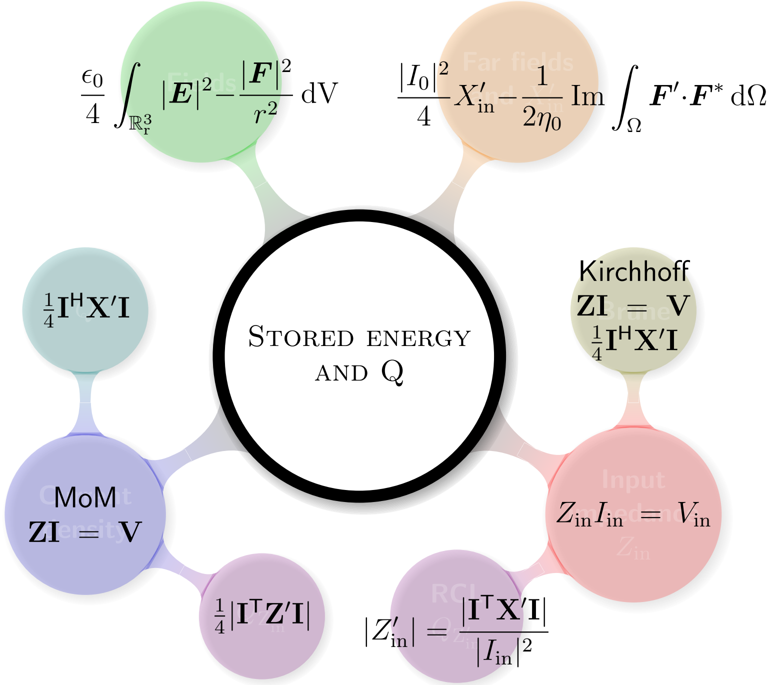
Kirchhoff
$\mathbf{ZI}=\mathbf{V}$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
$Z_{\mathrm{in}}I_{\mathrm{in}}=V_{\mathrm{in}}$
$|Z'_{\mathrm{in}}|=\frac{|\mathbf{I}^{\mathsf{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$
Q
Current density
$Q_{Z'_{\mathrm{in}}}$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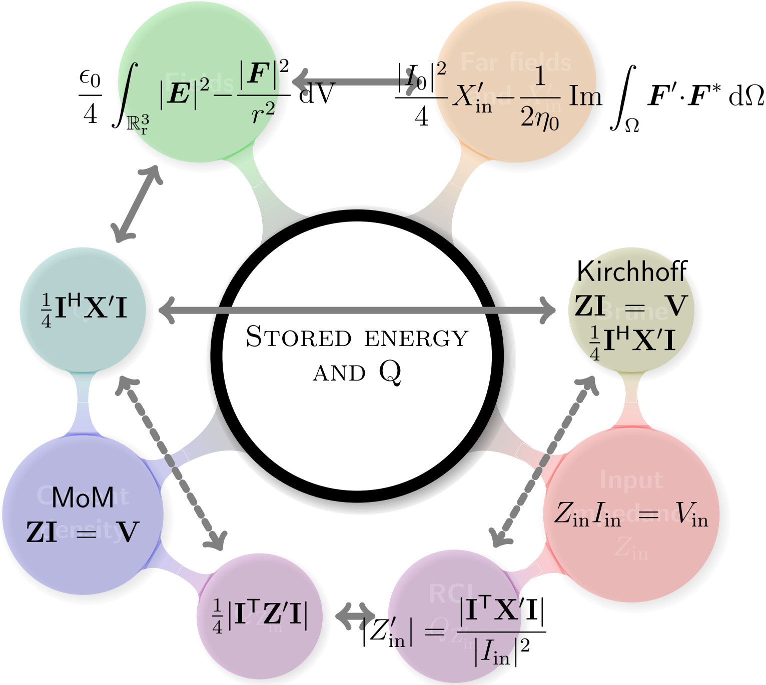
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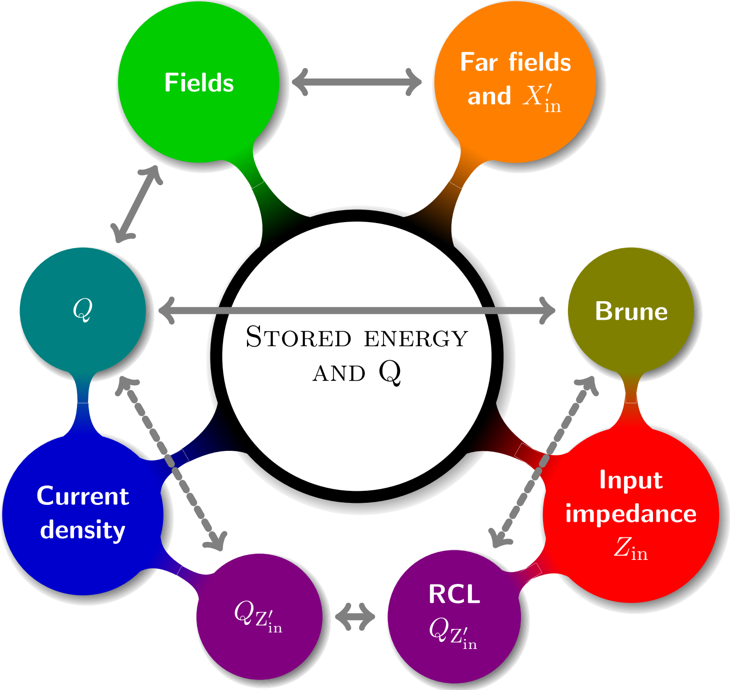
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Frequency derivatives of impedance/admittance matrices

Impedance and admittance matrices relate voltages and currents

$$\mathbf{Z}\mathbf{I} = \mathbf{V} \quad \text{or } \mathbf{I} = \mathbf{Z}^{-1}\mathbf{V} = \mathbf{Y}\mathbf{V}$$

The (angular) frequency derivative of the admittance matrix is

$$\mathbf{Y}' = \frac{\partial \mathbf{Y}}{\partial \omega} = \frac{\partial \mathbf{Z}^{-1}}{\partial \omega} = -\mathbf{Z}^{-1}\mathbf{Z}'\mathbf{Z}^{-1} = -\mathbf{Y}\mathbf{Z}'\mathbf{Y}$$

No complex conjugate. Better to use quadratic forms with the transpose $\mathbf{V}^{\mathsf{T}}\mathbf{Y}'\mathbf{V}$ than Hermitian transpose $\mathbf{V}^{\mathsf{H}}\mathbf{Y}'\mathbf{V} = \mathbf{V}^{\mathsf{T}*}\mathbf{Y}'\mathbf{V}$.

For the case of a (frequency independent (MoM)) voltage source

$$Y_{\text{in}} = \frac{1}{Z_{\text{in}}} = \frac{\mathbf{V}^{\mathsf{T}}\mathbf{Y}\mathbf{V}}{V_{\text{in}}^2} \quad \text{and } V_{\text{in}}^2 Y_{\text{in}}' = \mathbf{V}^{\mathsf{T}}\mathbf{Y}'\mathbf{V} = -\mathbf{I}^{\mathsf{T}}\mathbf{Z}'\mathbf{I}$$

Z_{in} for antennas using MoM

Use a method of moments (MoM) formulation of the electric field integral equation (EFIE). Impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$

$$\frac{Z_{mn}}{\eta} = \mathrm{j} \int_S \int_S \left(k^2 \boldsymbol{\psi}_{m1} \cdot \boldsymbol{\psi}_{n2} - \nabla_1 \cdot \boldsymbol{\psi}_{m1} \nabla_2 \cdot \boldsymbol{\psi}_{n2}\right) \frac{\mathrm{e}^{-\mathrm{j}kR_{12}}}{4\pi k R_{12}} \,\mathrm{dS}_1 \,\mathrm{dS}_2$$

where $\boldsymbol{\psi}_{n1} = \boldsymbol{\psi}_n(\boldsymbol{r}_1)$, $\boldsymbol{\psi}_{n2} = \boldsymbol{\psi}_n(\boldsymbol{r}_2)$, $n = 1, ..., N$, and $R_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$.
The current density is $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} I_n \boldsymbol{\psi}_n(\boldsymbol{r})$ with the expansion coefficients determined from

$$\mathbf{Z}\mathbf{I} = \mathbf{V} \quad \text{or } \mathbf{I} = \mathbf{Z}^{-1}\mathbf{V} = \mathbf{Y}\mathbf{V}$$

where $\mathbf{V}$ is a column matrix with the excitation coefficients.
The input admittance is

$$Y_{\mathrm{in}} = 1/Z_{\mathrm{in}} = \mathbf{V}^{\mathsf{T}}\mathbf{Y}\mathbf{V}/V_{\mathrm{in}}^2$$

where $Z_{\mathrm{in}} = R_{\mathrm{in}} + \mathrm{j}X_{\mathrm{in}}$ is the input impedance.

$Q_{\mathrm{Z}'_{\mathrm{in}}}$ and Q for antennas (fields)

Differentiate the MoM impedance matrix

$$\frac{k\,\partial Z_{mn}}{\eta\,\partial k} = \int_V\int_V \mathrm{j}\left(k^2\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2} + \nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\right)\frac{\mathrm{e}^{-\mathrm{j}kR_{12}}}{4\pi kR_{12}} + \left(k^2\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2} - \nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\right)\frac{\mathrm{e}^{-\mathrm{j}kR_{12}}}{4\pi}\,\mathrm{dS}_1\,\mathrm{dS}_2$$

Differentiated input admittance

$$V_{\mathrm{in}}^2 Y_{\mathrm{in}}' = (\mathbf{V}^{\mathsf{T}}\mathbf{Y}\mathbf{V})' = \mathbf{V}^{\mathsf{T}}\mathbf{Y}'\mathbf{V} = -\mathbf{I}^{\mathsf{T}}\mathbf{Z}'\mathbf{I}.$$

The stored energy determined from $\mathbf{X}' = \mathrm{Im}\,\mathbf{Z}'$

$$W_{\mathrm{e}\mathbf{X}'} + W_{\mathrm{m}\mathbf{X}'} = \frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$$

is identical to the stored energy expressions introduced by Vandenbosch (IEEE-TAP 2010).

Q and $Q_{Z'_{\mathrm{in}}}$ for free-space self-resonant antennas

Assume for simplicity a **self-resonant** antenna (circuit)

$$Q_{Z'_{\mathrm{in}}} = \frac{\omega |Z'_{\mathrm{in}}|}{2R_{\mathrm{in}}} = \frac{\omega |\mathbf{I}^{\mathsf{T}}\mathbf{Z}'\mathbf{I}|}{2\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

and using MoM with the stores energy by Vandenbosch

$$Q = \frac{2\omega \max\{W_{\mathrm{e}\mathbf{X}'}, W_{\mathrm{m}\mathbf{X}'}\}}{P_{\mathrm{d}}} = \frac{\omega \mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}}{2\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

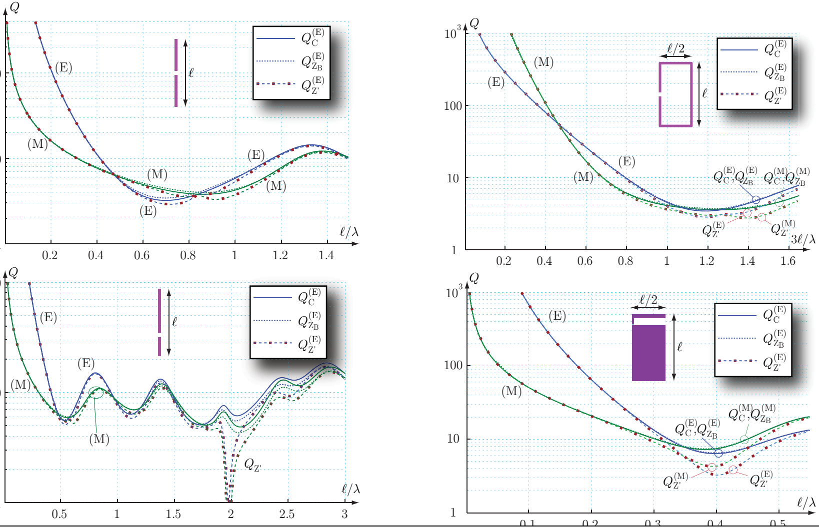
Transpose for $Q_{Z'_{\mathrm{in}}}$ and Hermitian transpose for Q

- $\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I} \geq 0$ for positive semidefine matrices $\mathbf{X}'$.
- $|\mathbf{I}^{\mathsf{T}}\mathbf{Z}'\mathbf{I}| = 0$ for some $\mathbf{I}$ $(\mathrm{rank} > 1)$.

See also Capek+*etal.* IEEE-TAP 2014 for $Q_{Z'_{\mathrm{in}}}$ using $\mathbf{I}^{\mathsf{H}}$ and $\mathbf{I}'$.

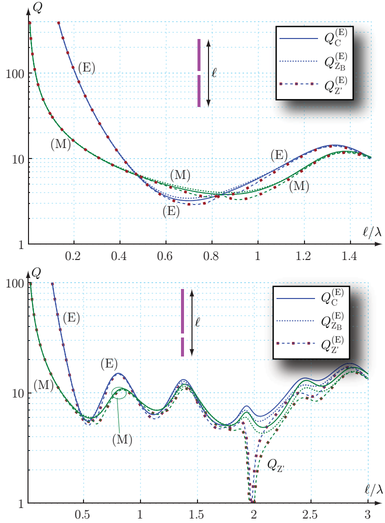
Antenna examples (free space)

Q from stored energy expressed in the current density Q_{C}, Brune circuit $Q_{Z^{\mathrm{B}}_{\mathrm{in}}}$, and differentiated input impedance $Q_{Z'_{\mathrm{in}}}$



Antenna examples (free space)

Q from stored energy expressed in the current density Q_{C}, Brune circuit $Q_{Z_{\mathrm{in}}^{\mathrm{B}}}$, and differentiated input impedance $Q_{Z'_{\mathrm{in}}}$



Q computed from

- the currents, Q_{C}.
- a circuit model synthesized from the input impedance using Brune synthesis (1931), $Q_{Z_{\mathrm{in}}^{\mathrm{B}}}$.
- differentiation of the (tuned) input impedance,

$$Q_{Z'_{\mathrm{in}}} = \frac{\omega_0 |Z'_{\mathrm{in}}|}{2R_{\mathrm{in}}} = \omega_0 |\Gamma'|.$$

All agree for $Q \gg 1$ but the Q from the differentiated impedance ($Q_{Z'_{\mathrm{in}}}$) is lower in some regions.
Which one is most accurate/best?

Dispersive media

The frequency derivative of the EFIE impedance matrix $\mathbf{Z}$ is

$$\omega\frac{\partial \mathbf{Z}}{\partial\omega} = k\frac{\partial(\mathbf{Z}/\eta)}{\partial k}\frac{\eta\omega}{k}\frac{\partial k}{\partial\omega} + \omega\frac{\mathbf{Z}}{\eta}\frac{\partial\eta}{\partial\omega}$$

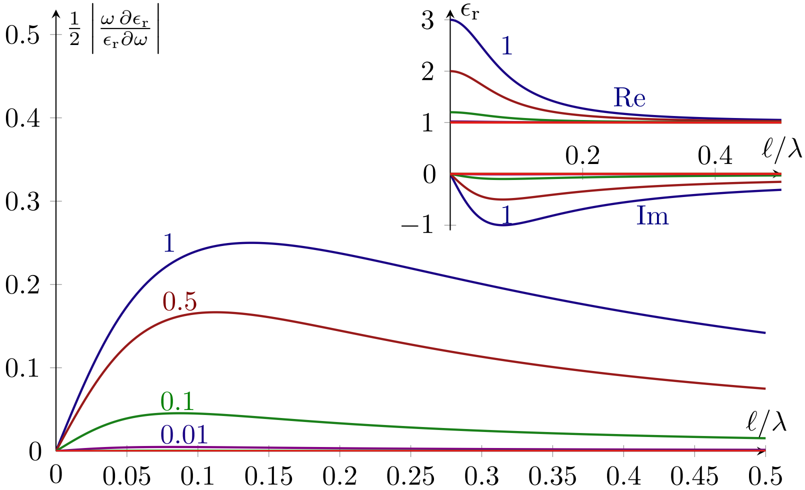
for a temporally dispersive background medium with $k=\omega\sqrt{\epsilon\mu}$ and $\eta=\sqrt{\mu/\epsilon}$. The derivative simplifies to

$$\omega\frac{\partial \mathbf{Z}}{\partial\omega} = k\frac{\partial(\mathbf{Z}/\eta)}{\partial k}\eta\left(\frac{\omega\partial\epsilon}{2\epsilon\partial\omega}+1\right) - \frac{\mathbf{Z}}{2}\frac{\omega\partial\epsilon}{\epsilon\partial\omega}$$

for the common case of a non-magnetic medium, $\mu_{\mathrm{r}}=1$.

Multiplication of the previously calculated derivative (with respect to the wavenumber k in the medium) with a factor that only depends on the medium. The factor $\omega\epsilon' = (\omega\epsilon)' - \epsilon$ is similar to the classical approach used to define the energy density in dispersive media.

Numerical examples: Debye media



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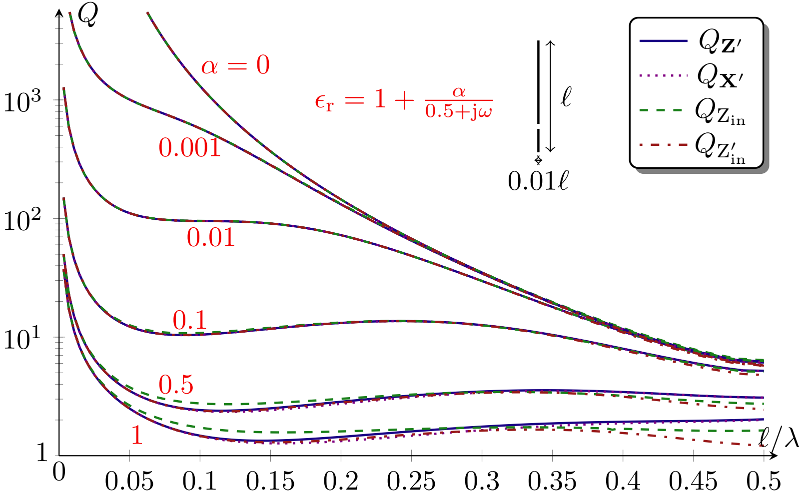

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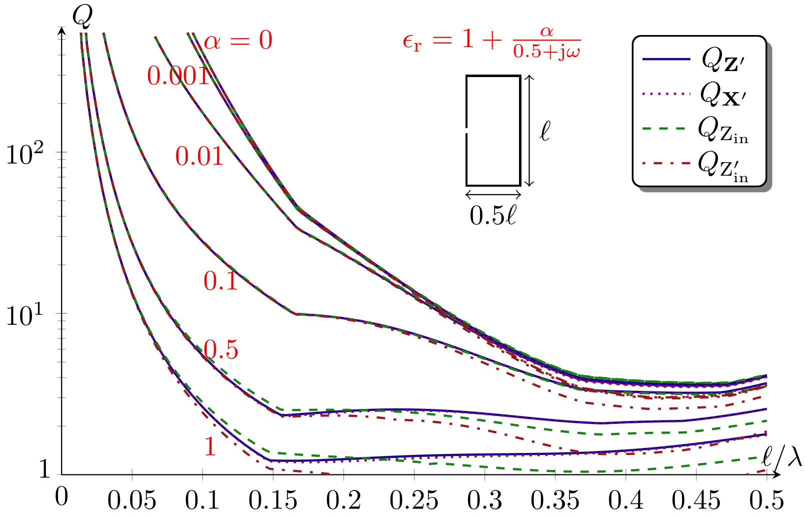


Stored EM energy matrices

Method of Moments approximation (expand $\boldsymbol{J}$ in basis functions)

$$W_{\mathrm{e}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \quad \text{stored E-energy, } \mathbf{X}_{\mathrm{e}} \text{ electric reactance}$$

$$W_{\mathrm{m}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \quad \text{stored M-energy, } \mathbf{X}_{\mathrm{m}} \text{ magnetic reactance}$$

$$P_{\mathrm{rad}} \approx \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \quad \text{radiated power}$$

giving $\mathbf{Z} = \mathbf{R}_{\mathrm{r}} + \mathrm{j}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})$. We also use

$\boldsymbol{F} \approx \mathbf{F}^{\mathsf{H}}\mathbf{I}$(far field), $\boldsymbol{E} \approx \mathbf{N}^{\mathsf{H}}\mathbf{I}$(near field), $\mathbf{I}_2 \approx \mathbf{C}^{\mathsf{H}}\mathbf{I}_1$(induced current)

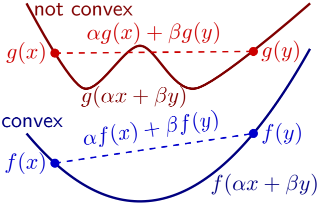
Pre-computed matrices used in the optimization.

Convex optimization

minimize $\quad f_0(\mathbf{x})$

subject to $\quad f_i(\mathbf{x}) \leq 0,\ i = 1, ..., N_1$

$\quad \mathbf{A}\mathbf{x} = \mathbf{b}$



where $f_i(x)$ are convex, *i.e.*, $f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$ for $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

Solved with efficient standard algorithms. No risk of getting trapped in a local minimum. A problem is 'solved' if formulated as a convex optimization problem.

Antenna performance expressed in the current density $\boldsymbol{J}$, *e.g.*,

- Radiated field $\boldsymbol{F}(\hat{\boldsymbol{k}}) = -\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} \times \int_V \boldsymbol{J}(\boldsymbol{r}) \mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\, \mathrm{dV}$ is affine.
- Radiated power, stored electric and magnetic energies, and Ohmic losses are positive semi-definite quadratic forms in $\boldsymbol{J}$.

Currents for maximal G/Q

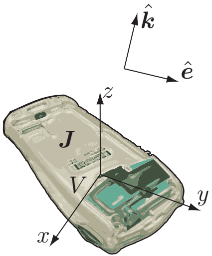
Determine a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V that maximizes the partial-gain Q-factor quotient $G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})/Q$.

- Partial radiation intensity $P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})$

$$\frac{G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})}{\mathrm{c}_0 k \max\{W_\mathrm{e}, W_\mathrm{m}\}}.$$

- Scale $\boldsymbol{J}$ and reformulate $P = 1$ as $\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F} = \mathbf{F}^\mathsf{H}\mathbf{I} = 1$.
- Convex optimization problem:

$$\begin{aligned} &\text{minimize} && W \\ &\text{subject to} && \mathbf{I}^\mathsf{H}\mathbf{X}_\mathrm{e}\mathbf{I} \le W \\ &&& \mathbf{I}^\mathsf{H}\mathbf{X}_\mathrm{m}\mathbf{I} \le W \\ &&& \mathbf{F}^\mathsf{H}\mathbf{I} = 1 \end{aligned}$$



Determines a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V with minimal stored EM energy and unit partial radiation intensity.

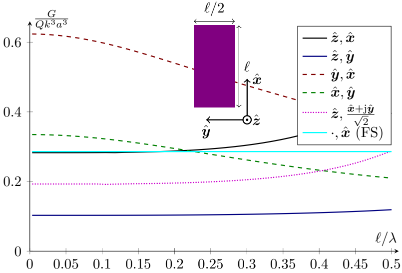
Maximal $G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{x}})/Q$ for planar rectangles

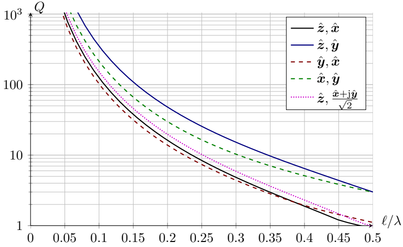
Solution of the convex optimization problem

$$
\begin{aligned}
&\text{minimize} && W \\
&\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \le W \\
& && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \le W \\
& && \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1
\end{aligned}
$$

for current densities confined to planar rectangles with side lengths ℓ_{x} and $\ell_{\mathrm{y}} = 0.5\ell_{\mathrm{x}}$.

Note $\ell_{\mathrm{x}}/\lambda = k\ell_{\mathrm{x}}/(2\pi)$, giving $\ell_{\mathrm{x}} = \lambda/2 \to k\ell_{\mathrm{x}} = \pi \to ka \ge \pi/2$.





Why convex optimization?

Solved if formulated as a convex optimization problem.

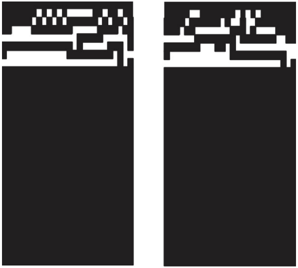
Consider the G/Q problem

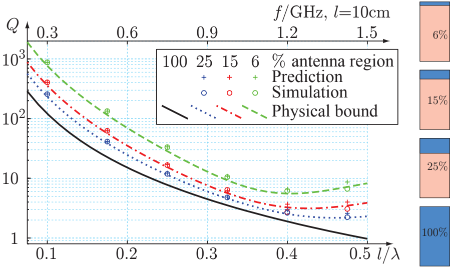
$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \end{aligned}$$

Many (optimization) algorithms can be used to solve this problem.

- Can *e.g.*, use any of the solvers included in CVX.
  - Very simple to use.
  - Good for small problems but less efficient for larger problems.
- A dedicated solver for quadratic programs.
  - More efficient for larger problems.
- Random search, eg genetic algorithms (GA), particle swarms,....
  - Very inefficient. Note you do not (should not) use (GA, ...) to solve *e.g.*, $\mathbf{Ax} = \mathbf{b}$ $(\min. ||\mathbf{Ax} - \mathbf{b}||)$.
- We also use a dual formulation
  - Computational efficient for large problems.
  - Illustrates dual problems and posteriori error estimates.

Conclusions





- Current optimization for physical bounds.
- Stored energy from MoM reactance matrices (basically, already computed in most MoM codes for surface currents).
- Promising results for temporally dispersive media.
- Convex optimization (efficiently solved with a few $\mathbf{Ax} = \mathbf{b}$).

Why convex optimization? Simple algorithm

Consider the G/Q problem

$$\begin{aligned} &\text{minimize} && W = \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \mathrm{Im}\{\mathbf{F}^{\mathsf{H}}\mathbf{I}\} = 1 \end{aligned}$$

There are many (optimization) algorithms that can be used to solve this problem. An illustrative method is to use $(0 \leq \alpha \leq 1)$

$$W = \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \geq W_\alpha = \mathbf{I}_\alpha^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}_\alpha$$

and (hence $G/Q \leq G_\alpha/Q_\alpha$) to relax to the dual problem

$$\begin{aligned} &\text{maximize}_\alpha \text{minimize}_{\mathbf{I}_\alpha} && W_\alpha = \mathbf{I}_\alpha^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}_\alpha \\ &\text{subject to} && \mathrm{Im}\{\mathbf{F}^{\mathsf{H}}\mathbf{I}_\alpha\} = 1 \\ & && 0 \leq \alpha \leq 1 \end{aligned}$$

that is solved as a linear system (MoM equation) for fixed α giving

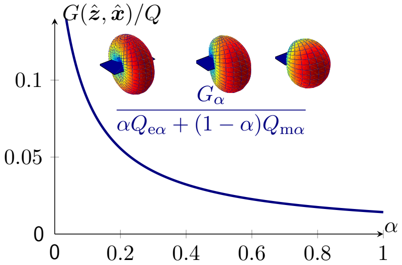
$$\underset{0\leq\alpha\leq 1}{\text{maximize}}\, W_\alpha \quad \text{with } \mathbf{I}_\alpha = \frac{\big(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}\big)^{-1}\mathbf{F}}{\mathbf{F}^{\mathsf{H}}\big(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}\big)^{-1}\mathbf{F}} \qquad \text{(relaxed problem)}$$

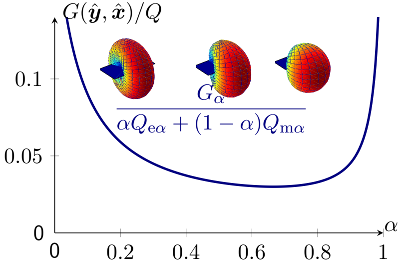
Why convex optimization: illustration

The upper bound on $G/Q|_{\mathrm{ub}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (blue) curve

$$\frac{G}{Q}\bigg|_{\mathrm{ub}} \leq \frac{G_\alpha}{\alpha Q_{\mathrm{e}\alpha} + (1-\alpha) Q_{\mathrm{m}\alpha}}$$

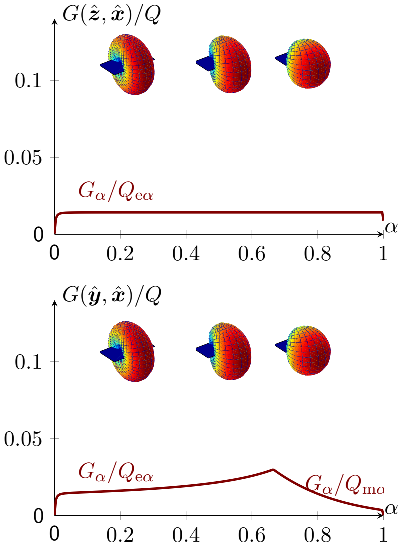
This is efficiently solved by golden section search and parabolic interpolation.





$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

Why convex optimization: illustration



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

We also compute the actual G/Q for the current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{e\alpha}, Q_{m\alpha}\}} \leq \left.\frac{G}{Q}\right|_{\mathrm{ub}}$$

Why convex optimization: illustration

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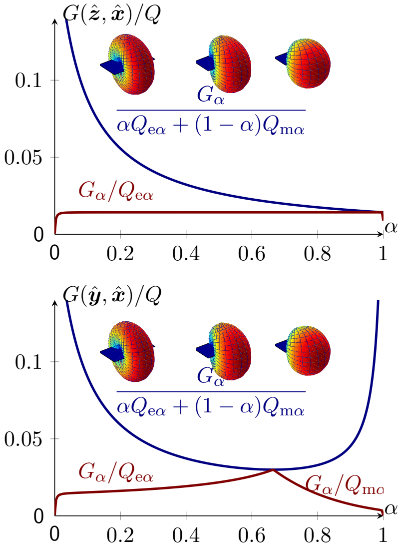

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Minimization of Q

The corresponding formulation for Q is not convex. For $0 \leq \alpha \leq 1$

$$Q = \frac{\max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}} = \max\{Q_{\mathrm{e}}, Q_{\mathrm{m}}\}$$

$$\geq \alpha Q_{\mathrm{e}} + (1-\alpha)Q_{\mathrm{m}} = \frac{\mathbf{I}^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}}$$