Optimal antenna currents using MoM and convex optimization

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Outline

Acknowledgments

- IEEE APS Distinguished Lecturer Program.
- The Swedish Research Council.
- Swedish Foundation for Strategic Research.

Collaboration with:

- Marius Cismasu, Lund University
- Doruk Tayli, Lund University
- Sven Nordebo, Linnæus University
- Lars Jonsson, KTH
- Christian Sohl, SAAB was at LU
- Gerhard Kristensson, LU
- Daniel Sjöberg, LU

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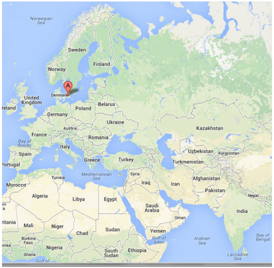
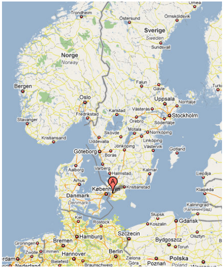


- Lund university was founded in 1666.
- Sweden's Largest University.
- Approximately 40 000 students.
- Department of Electrical and Information Technology; Broadband Communications, Circuits and Systems, Communication, Electromagnetic theory, Networking and Security, Signal Processing.

Lund University



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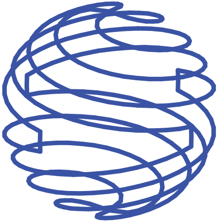
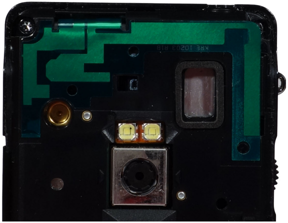
Outline

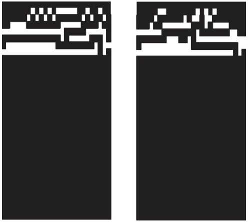
Design of small antennas



Folded spherical helix



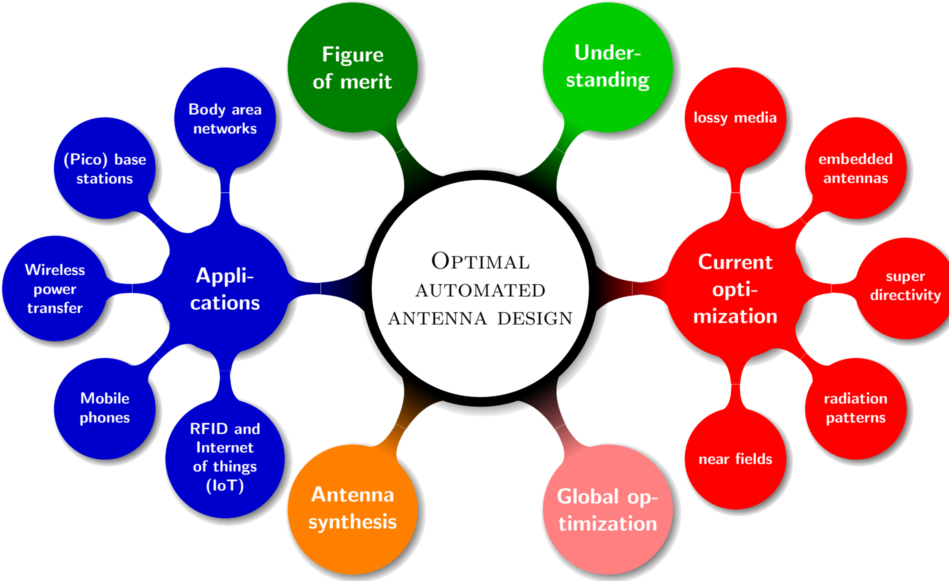
SonyEricsson P1i



Fragmented patches

- There are many advanced methods to design small antennas.
- Often antennas embedded in structures.
- Performance in Q, bandwidth and efficiency.
- How does the performance depend on the design volume?
- What can we learn from performance bounds and optimal currents?
- Can we automate the design of optimal antennas?

Optimal (automated) antenna design



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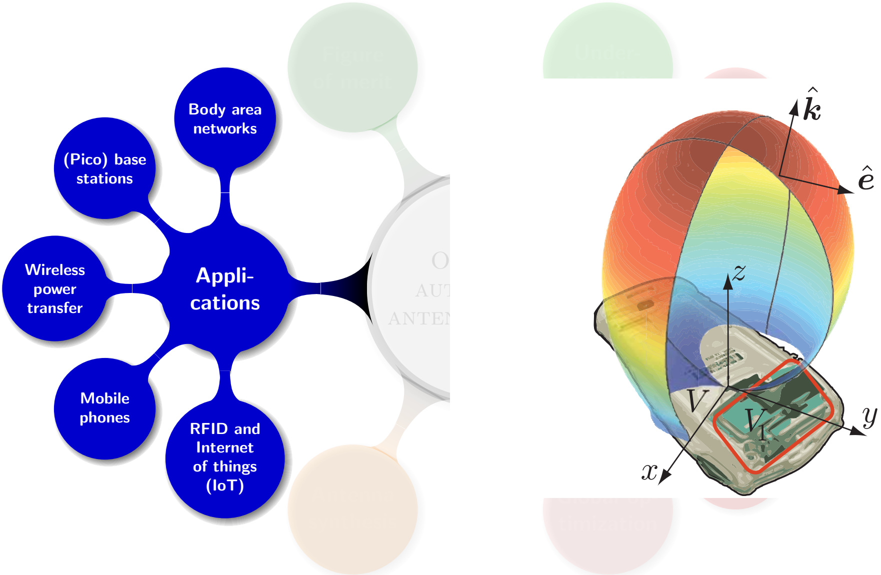
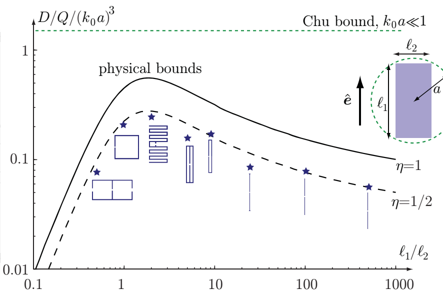


Optimal (automated) antenna design

Figure of merit

Performance of an antenna design in relation to the optimal performance

- **% from the optimal design**
- **a useful number to compare designs**
- **is it worth it improve a design?**
- **...**



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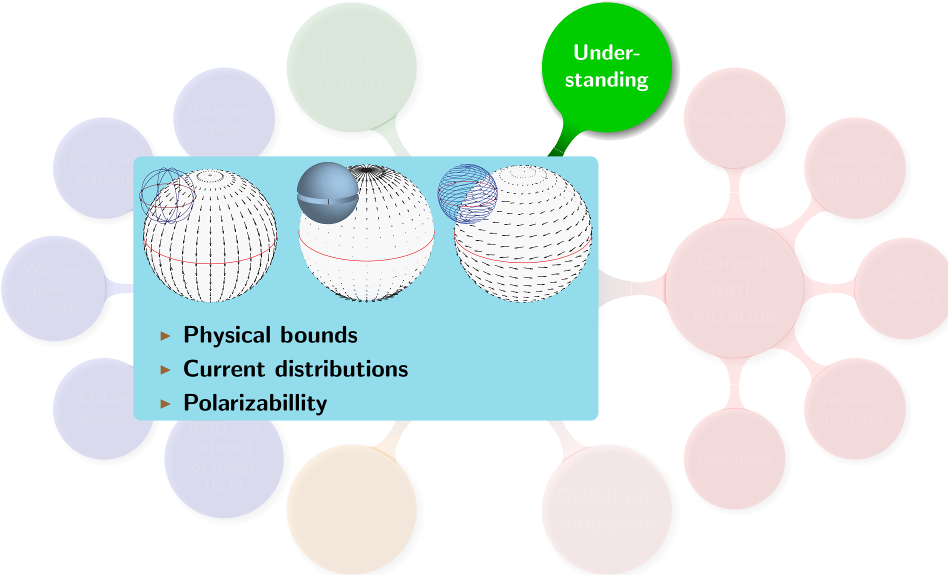
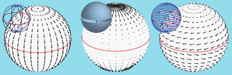
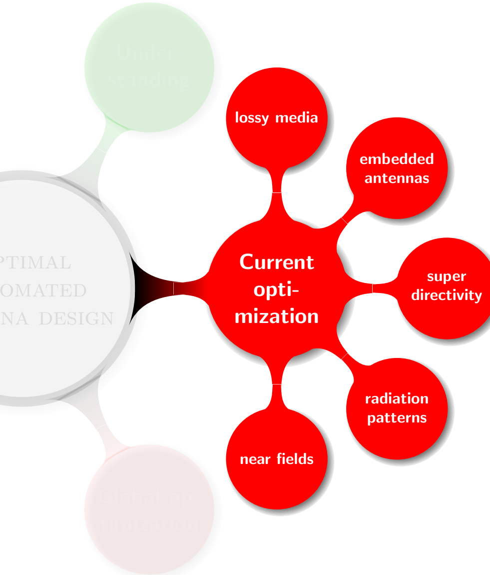


- **Physical bounds**
- **Current distributions**
- **Polarizabillity**

Optimal (automated) antenna design



- **Optimal current distribution**
- **Physical bounds**
- **Convex optimization**
- **Gain over Q (G/Q)**
- **Radiation patterns**
- **Embedded structures**
- **...**



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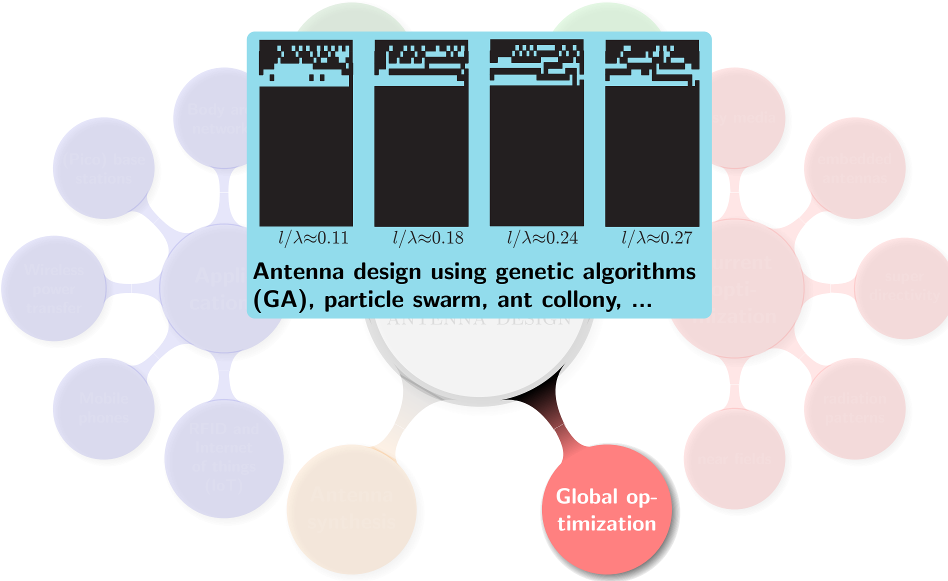

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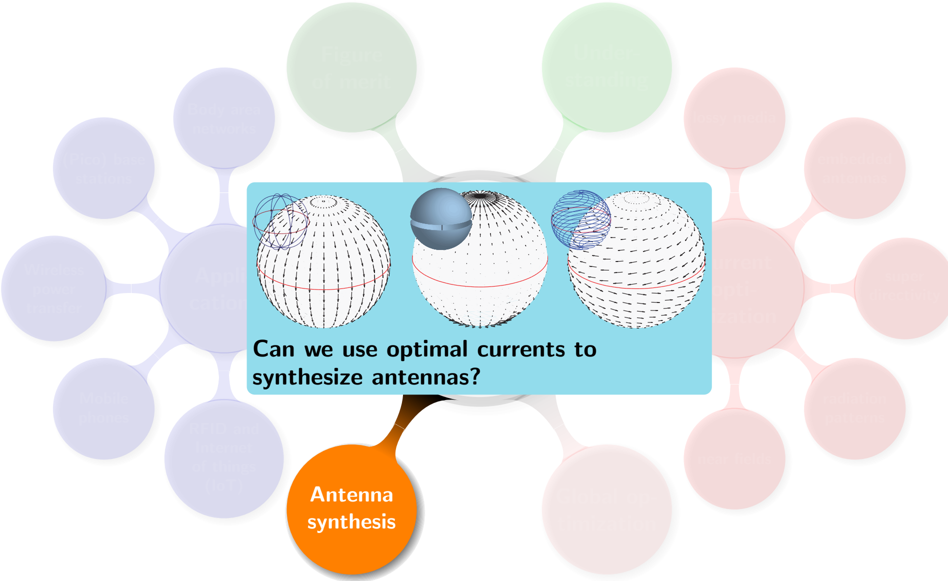

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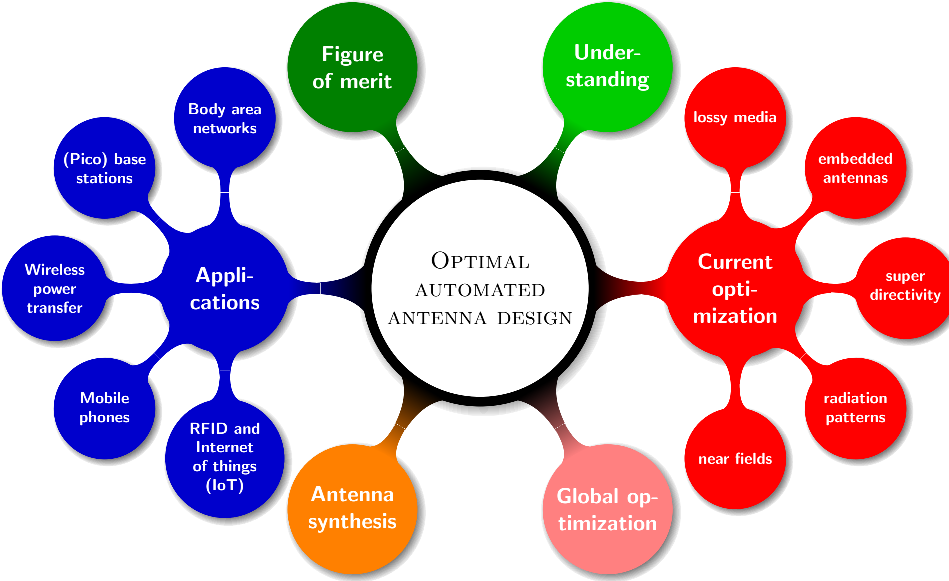
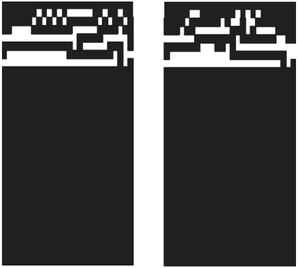
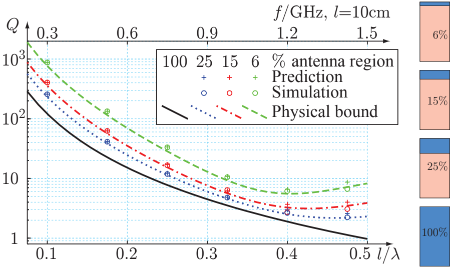


Antenna optimization





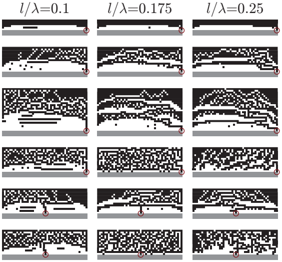
Optimization of structures

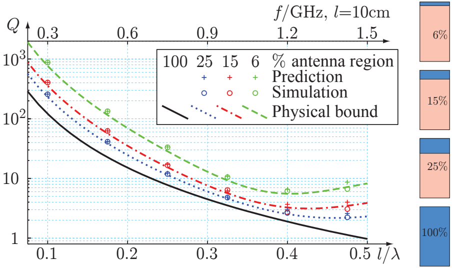
- global optimization.
- new non-intuitive designs.
- convergence?
- stopping criteria?
- optimal?

Optimization of currents

- determine optimal currents for Q, G/Q, ...
- convex optimization.
- physical bounds.
- can we realize the currents?

Antenna optimization





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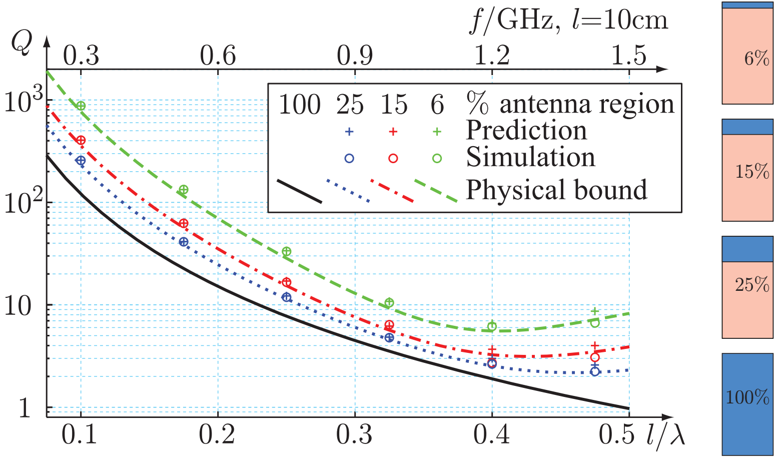
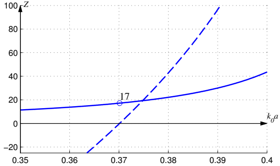
Q from G/Q for a planar PEC ground plane and $100, 25, 15, 6\%$ antenna region



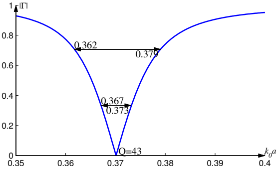
Cismasu, Gustafsson, 'Antenna Bandwidth Optimization with Single Frequency Simulation', IEEE-TAP, 2014.

Outline

Q-factor



$Z(k)$ around $k_0 a$.



$|\Gamma(k)|$ around $k_0 a$.

The Q-factor is defined as the ratio between the stored electric, W_{E}, and magnetic, W_{M}, energies and the dissipated power, *i.e.*,

$$Q = \frac{2\omega \max\{W_{\mathrm{E}}, W_{\mathrm{M}}\}}{P_{\mathrm{rad}} + P_{\mathrm{loss}}}.$$

Fractional bandwidth for single resonances

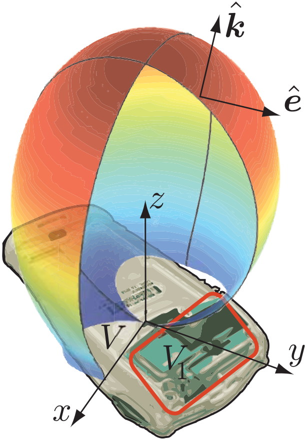
$$B \approx \frac{2}{Q} \frac{\Gamma_0}{\sqrt{1-\Gamma_0^2}}$$

Example

- $B \approx 5\%$ for $Q = 43$ and $\Gamma_0 = 1/\sqrt{2}$.
- $B \approx 2\%$ for $Q = 43$ and $\Gamma_0 = 1/3$.

Physical bounds on antennas

- Tradeoff between performance and size.
- Performance, *e.g.*, in Q, (half-power fractional bandwidth $B \approx 2/Q$), directivity bandwidth product: D/Q, efficiency, capacity,....
- Properties of the best antenna confined to a given (arbitrary) geometry, *e.g.*, spheroid, cylinder, and rectangle.



An overview of physical bounds:

- Circuit models.
- Mode expansion (spheres).
- Forward scattering (arbitrary shape).
- Energy expressions in currents.

Background

- 1947 Wheeler: *Bounds based on circuit models.*
- 1948 Chu: *Bounds on Q and D/Q for spheres.*
- 1964 Collin & Rothchild: *Closed form expressions of Q for arbitrary spherical modes, see also* Harrington, Collin, Fantes, Maclean, Gayi, Hansen, Hujanen, Sten, Best, Yaghjian, Kildal, Karlsson... *(most are based on Chu's approach using spherical modes.)*
- 1999 Foltz & McLean, 2001 Sten, Koivisto, and Hujanen: *Initial bounds for spheroidal volumes.*
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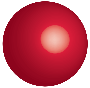
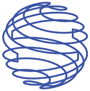

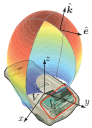

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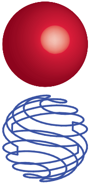

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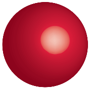

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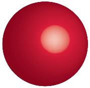
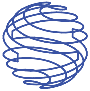


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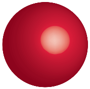
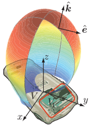
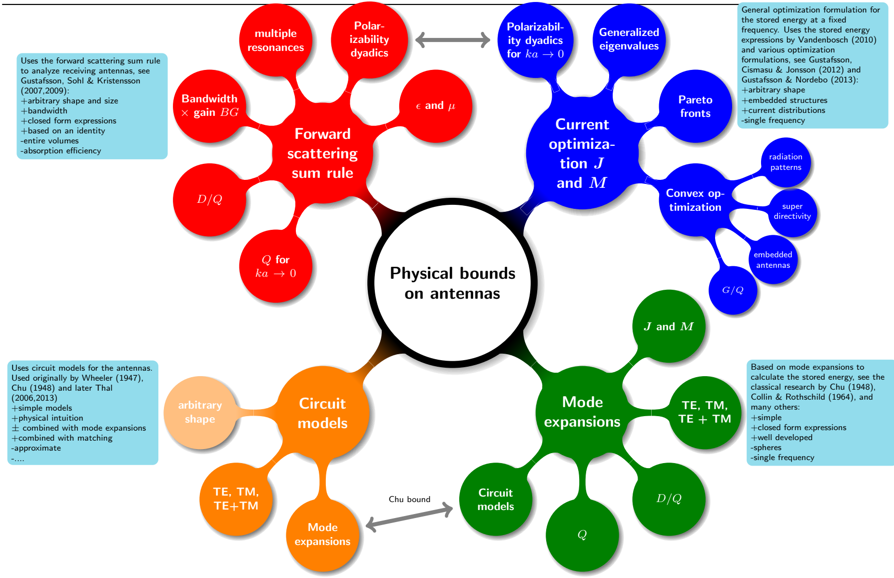
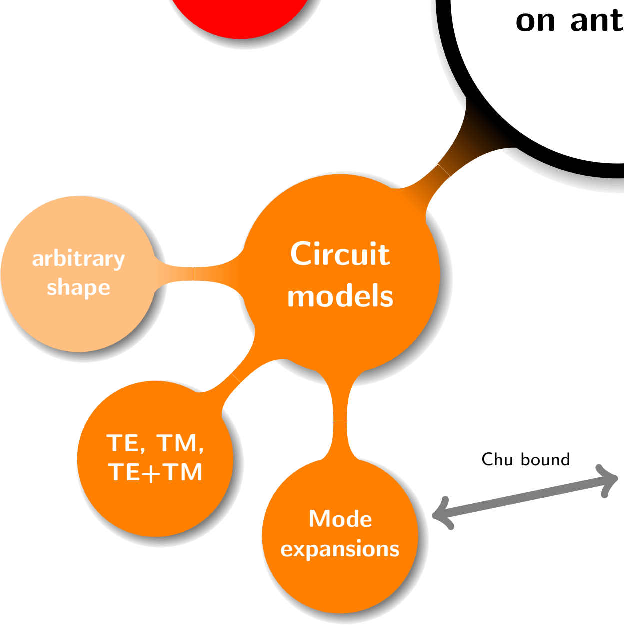


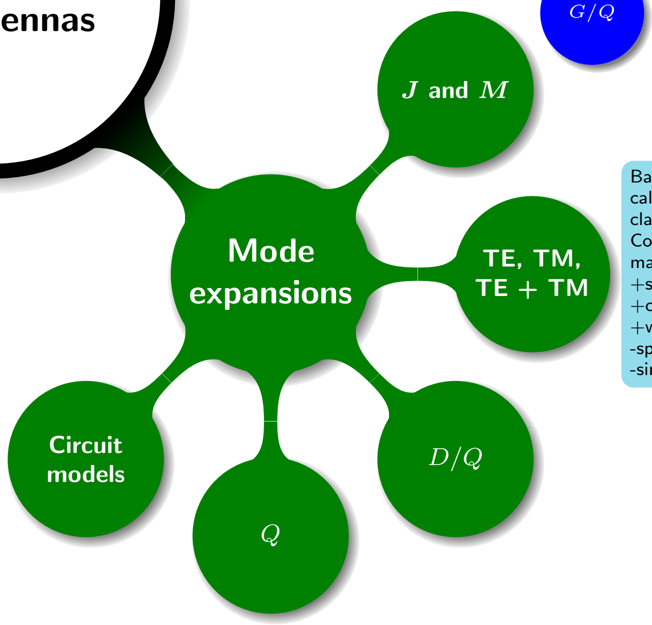



Physical bounds on antennas: methods



Uses circuit models for the antennas.
Used originally by Wheeler (1947), Chu (1948) and later Thal (2006,2013)
+simple models
+physical intuition
± combined with mode expansions
+combined with matching
-approximate
-....

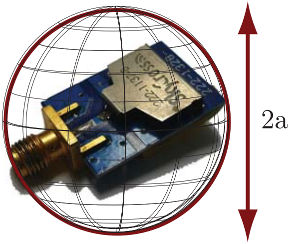


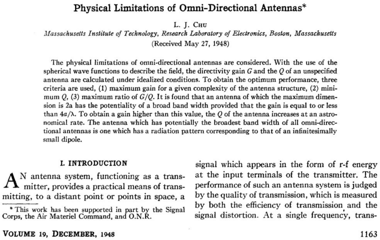


Based on mode expansions to calculate the stored energy, see the classical research by Chu (1948), Collin & Rothschild (1964), and many others:

+simple
+closed form expressions
+well developed
-spheres
-single frequency

Chu bound (1948)





Physical Limitations of Omni-Directional Antennas*

L. J. Chu
Massachusetts Institute of Technology, Research Laboratory of Electronics, Boston, Massachusetts

(Received May 27, 1948)

The physical limitations of omni-directional antennas are considered. With the use of the spherical wave functions to describe the field, the directivity gain G and the Q of an unspecified antenna are calculated under idealized conditions. To obtain the optimum performance, three criteria are used, (1) maximum gain for a given complexity of the antenna structure, (2) minimum Q, (3) maximum ratio of G/Q. It is found that an antenna of which the maximum dimension is 2a has the potentiality of a broad band width provided that the gain is equal to or less than $4a/\lambda$. To obtain a gain higher than this value, the Q of the antenna increases at an astronomical rate. The antenna which has potentially the broadest band width of all omni-directional antennas is one which has a radiation pattern corresponding to that of an infinitesimally small dipole.

I. INTRODUCTION

An antenna system, functioning as a transmitter, provides a practical means of transmitting, to a distant point or points in space, a signal which appears in the form of r-f energy at the input terminals of the transmitter. The performance of such an antenna system is judged by the quality of transmission, which is measured by both the efficiency of transmission and the signal distortion. At a single frequency, trans-

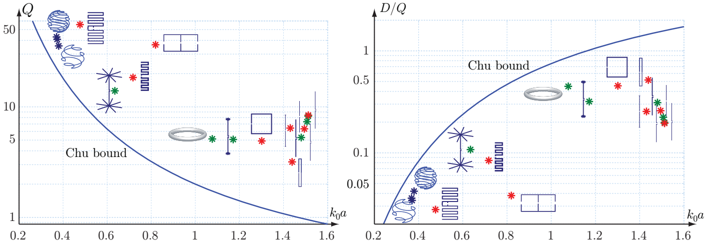
* This work has been supported in part by the Signal Corps, the Air Materiel Command, and O.N.R.

The stored energy and radiated power outside a sphere with radius a give the Chu-bounds for omni-directional antennas, *i.e.*,

$$Q \geq Q_{\mathrm{Chu}} = \frac{1}{(k_0a)^3} + \frac{1}{k_0a} \quad \text{and} \quad \frac{D}{Q} \leq \frac{3}{2Q_{\mathrm{Chu}}} \approx \frac{3}{2}(k_0a)^3$$

for $k_0a \ll 1$, where $k = k_0$ is the resonance wavenumber $k = 2\pi/\lambda = 2\pi f/c_0$.

Chu bound (1948)



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for $k_0a \ll 1$, where $k = k_0$ is the resonance wavenumber $k = 2\pi/\lambda = 2\pi f/c_0$.

Sievenpiper *etal*, *Experimental validation of performance limits and design guidelines for small antennas*, IEEE-TAP, 2012.

Based on the approach by Chu

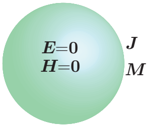
Chu (1948) used circuit models to compute the stored energy. Fine for the dipole mode but technical for higher order modes. There have been a substantial amount of work following the approach by Chu, *e.g.*, (and many more...)

- 1964 Collin & Rothchild: *EM fields for closed form expressions of Q for arbitrary spherical modes.*
- 1969 Fante: *general TE+TM modes.*
- 1996 McLean: *a re-examination of Q.*
- 1999 Foltz & McLean, 2001 Sten, Koivisto, and Hujanen: *extensions to spheroidal volumes.*
- 2001 Sten, Koivisto, and Hujanen: *antennas close to a ground plane.*
- 2003 Geyi: *Q and G/Q for combined TE+TM.*
- 2004 Karlsson: *lossy medium.*
- 2006 Thal: *bounds on Q for small hollow spherical antennas.*

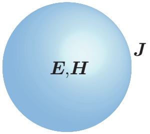
Non-magnetic spheres (Thal bound 2006)

The Chu bound is derived under the assumption of negligible stored energy in the interior of the sphere. Antennas without magnetic material (or magnetic currents) have an internal stored energy.

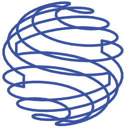
Thal (2006) *Bounds on Q for small non-magnetic spherical antennas.* Electric dipole: $Q \geq 1.5/(ka)^3 = 1.5 Q_{\mathrm{Chu}}$ for $ka \ll 1$, see also Hansen & Collin, Kim *etal*.



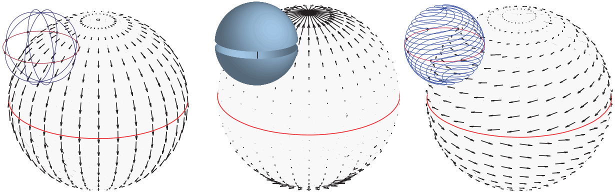
Chu: $\boldsymbol{J}$, $\boldsymbol{M}$ currents.



Thal: $\boldsymbol{J}$ currents.



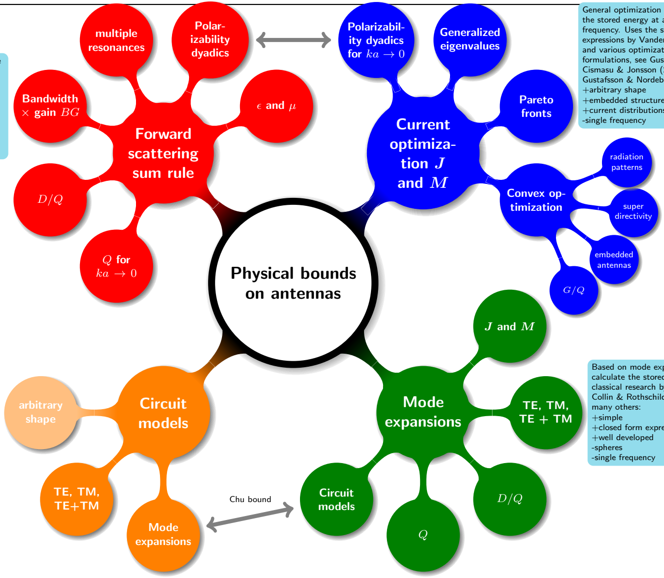
Best (2004) Folded spherical helix $Q \approx 1.5 Q_{\mathrm{Chu}}$.



Illustrations of surface currents $\boldsymbol{J}$ for a dipole, capped dipole, and folded spherical helix.
Gustafsson *etal*'Physical bounds and optimal currents on antennas', IEEE-TAP, 2012.

Physical bounds on antennas: methods

Uses the forward scattering sum rule to analyze receiving antennas, see Gustafsson, Sohl & Kristensson (2007,2009):
+arbitrary shape and size
+bandwidth
+closed form expressions
+based on an identity
-entire volumes
-absorption efficiency



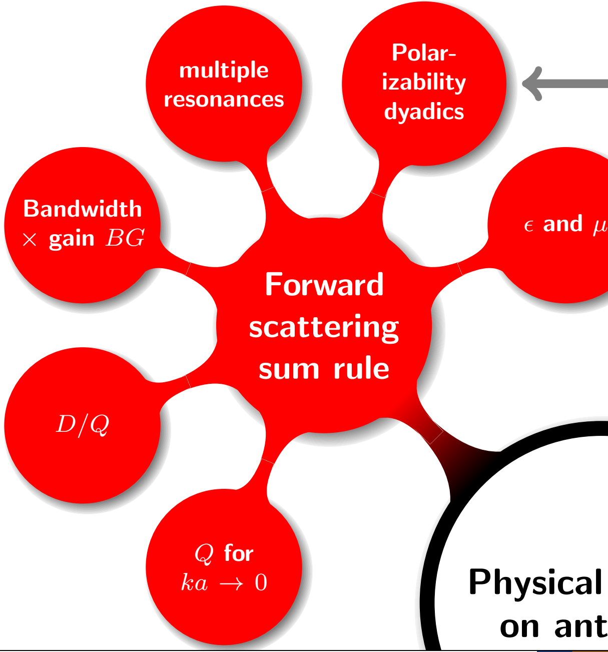
General optimization formulation for the stored energy at a fixed frequency. Uses the stored energy expressions by Vandenbosch (2010) and various optimization formulations, see Gustafsson, Cismasu & Jonsson (2012) and Gustafsson & Nordebo (2013):
+arbitrary shape
+embedded structures
+current distributions
-single frequency

Uses circuit models for the antennas. Used originally by Wheeler (1947), Chu (1948) and later Thal (2006,2013)
+simple models
+physical intuition
$\pm$ combined with mode expansions
+combined with matching
-approximate
-....

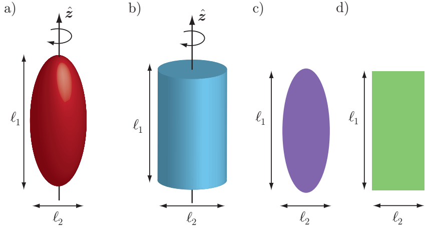
Based on mode expansions to calculate the stored energy, see the classical research by Chu (1948), Collin & Rothschild (1964), and many others:
+simple
+closed form expressions
+well developed
-spheres
-single frequency

Uses the forward scattering sum rule to analyze receiving antennas, see Gustafsson, Sohl & Kristensson (2007,2009):

+arbitrary shape and size
+bandwidth
+closed form expressions
+based on an identity
-entire volumes
-absorption efficiency



Physical bounds on antennas



- Properties of the best antenna confined to a given (arbitrary) geometry, *e.g.*, spheroid, cylinder, elliptic disk, and rectangle.
- Tradeoff between performance and size.
- Performance in
  - Directivity bandwidth product: D/Q (half-power $B \approx 2/Q$).
  - Partial realized gain: $(1 - |\Gamma|^2)G$ over a bandwidth.

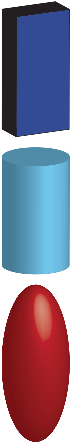
Arbitrary shaped antennas (2007)

The forward scattering identity (lossless, non-magnetic, linearly polarized ($\hat{e}$) antennas)

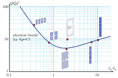
$$\int_0^{\infty} \frac{(1-|\Gamma(k)|^2)D(k;\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{k^4}\,\mathrm{d}k = \frac{\eta}{2}\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}}$$

gives a bound on D/Q (directivity bandwidth product) expressed in the high contrast polarizability dyadic $\boldsymbol{\gamma}_\infty \geq \boldsymbol{\gamma}_{\mathrm{e}}$:

$$\frac{D}{Q} \leq \frac{\eta k_0^3}{2\pi}\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_\infty\cdot\hat{\boldsymbol{e}} \quad \text{and small E-dipoles } Q \geq \frac{6\pi}{k_0^3\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_\infty\cdot\hat{\boldsymbol{e}}}$$

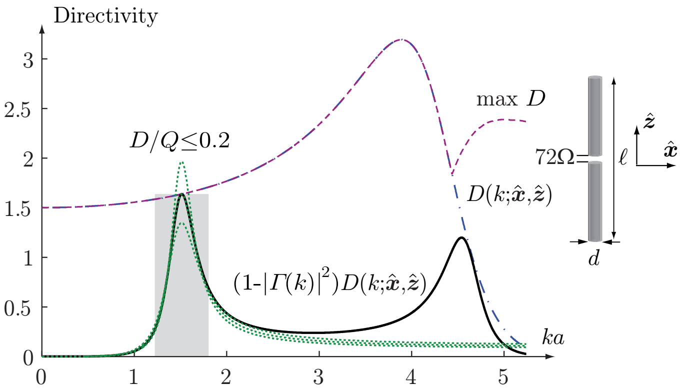


Circumscribing geometries of arbitrary shape.
Performance proportional to the polarizability. ▸ 108
Identical to the Thal bound for spheres.



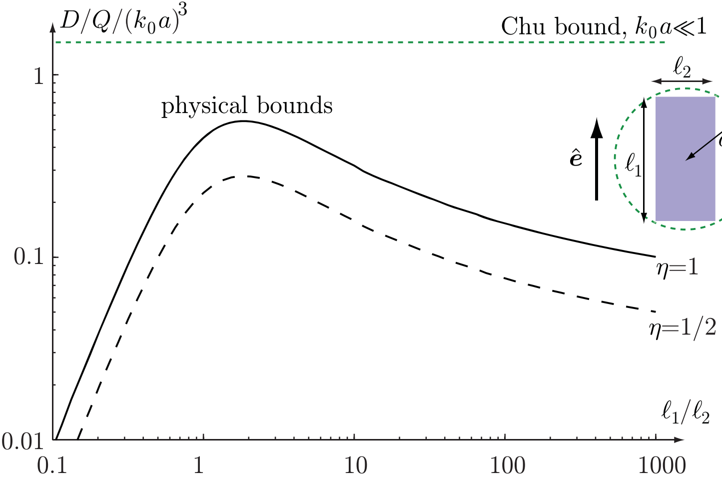
Gustafsson *etal*, Physical limitations on antennas of arbitrary shape, Proc. R. Soc. A , 2007
Gustafsson *etal*, Illustrations of new physical bounds on linearly polarized antennas IEEE TAP. 2009
Gustafsson *etal*, Absorption Efficiency and Physical Bounds on Antennas IJAP, 2010.

Cylindrical dipole



Lossless $\hat{\boldsymbol{z}}$-directed dipole, wire diameter $d = \ell/1000$, matched to $72\,\Omega$. Weighted area under the black curve (partial realized gain) is known. Note, half wavelength dipole for $ka = \pi/2 \approx 1.5$ with directivity $D \approx 1.64 \approx 2.15\,\mathrm{dB_i}$.

Circumscribing rectangles (2007)



Note, $\eta \leq 1/2$ for small electric dipole antennas $k_0a \ll 1$.

Gustafsson *etal*, Illustrations of new physical bounds on linearly polarized antennas IEEE TAP. 2009

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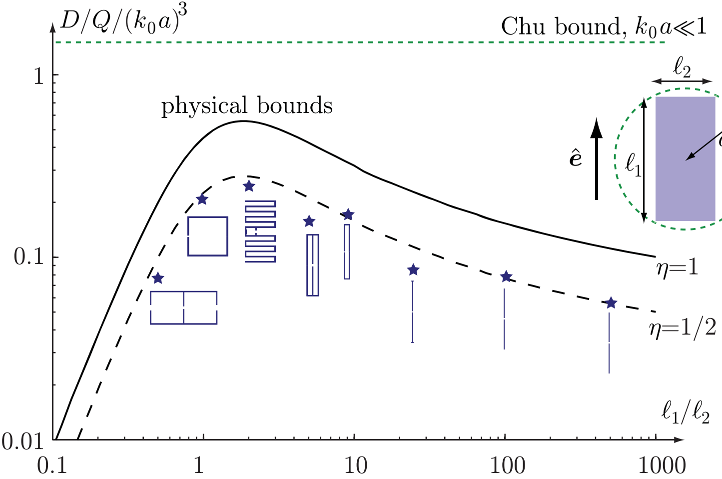

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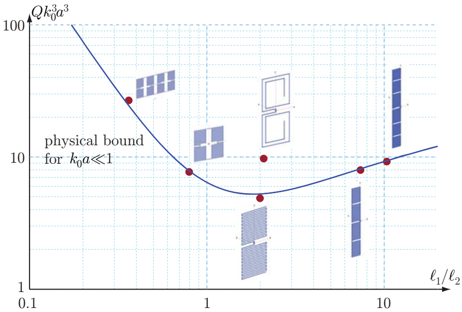
Small planar antennas

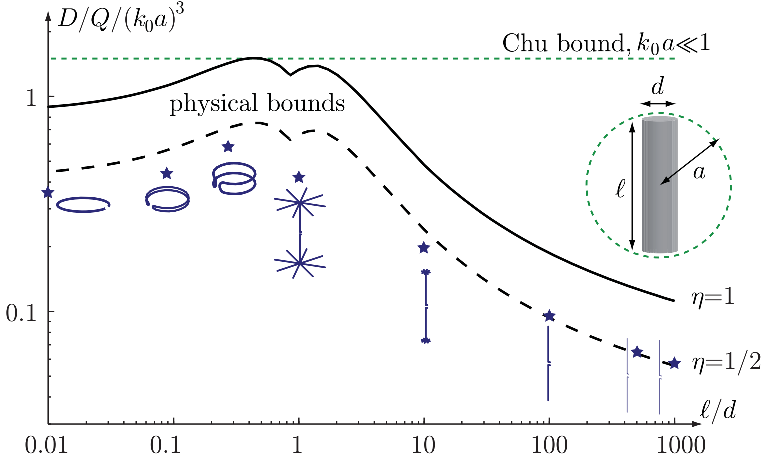


The dependence of $Qk_0^3a^3$ as a function of $\xi = \ell_1/\ell_2$.

- Multiplication of Q with $k_0^3a^3$ removes the dependence of the electrical size.
- A performance bound on $Qk_0^3a^3$ (for $k_0a \ll 1$) that only depends on the shape $\xi = \ell_1/\ell_2$
- Also explains the 'poor' performance of one of the antennas.

S.R. Best 'Optimization of the Bandwidth of Electrically Small Planar Antennas', 2009.

Circumscribing cylinders



Polarizability dyadic and induced dipole moment

The induced dipole moment can be written

$$\boldsymbol{p} = \epsilon_0 \boldsymbol{\gamma}_\mathrm{e} \cdot \boldsymbol{E}$$

where $\boldsymbol{\gamma}_\mathrm{e}$ is the polarizability dyadic.

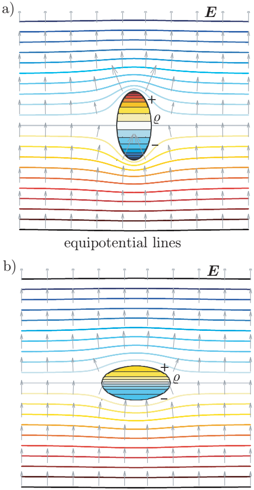
Example (Dielectric sphere)

A dielectric sphere with radius a and relative permittivity ϵ_r has the polarizability dyadic

$$\boldsymbol{\gamma}_\mathrm{e} = 4\pi a^3 \frac{\epsilon_\mathrm{r} - 1}{\epsilon_\mathrm{r} + 2} \mathbf{I} \to \boldsymbol{\gamma}_\infty = 4\pi a^3 \mathbf{I}$$

as $\epsilon_\mathrm{r} \to \infty$.

Analytic expressions for spheroids, elliptic discs, half spheres, hollow half spheres, touching spheres,...



equipotential lines

equipotential lines

High-contrast polarizability dyadics: γ_∞

γ_∞ is determined from the induced normalized surface charge density, ρ, as

$$\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_\infty\cdot\hat{\boldsymbol{e}} = \frac{1}{E_0}\int_{\partial V}\hat{\boldsymbol{e}}\cdot\boldsymbol{r}\rho(\boldsymbol{r})\,\mathrm{dS}$$

where ρ satisfies the integral equation

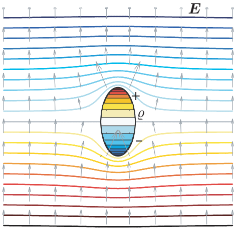
$$\int_{\partial V}\frac{\rho(\boldsymbol{r}')}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{dS}' = E_0\boldsymbol{r}\cdot\hat{\boldsymbol{e}} - V_n$$

with the constraints of zero total charge

$$\int_{\partial V_n}\rho(\boldsymbol{r})\,\mathrm{dS} = 0$$

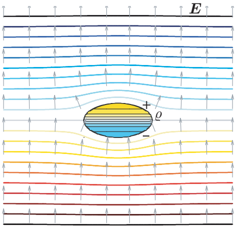
Can also use FEM (Laplace equation).

a)



equipotential lines

b)

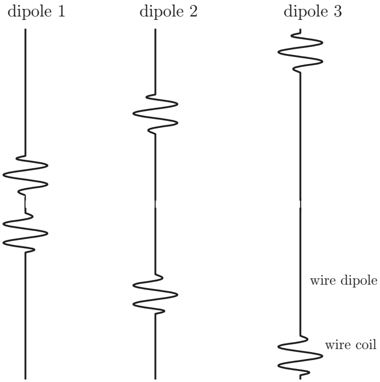


equipotential lines

Polarizability $\gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}}$: Interpretation

Wire dipoles (length $\ell \approx 2a$) with coils

Geometries of the three wire dipoles



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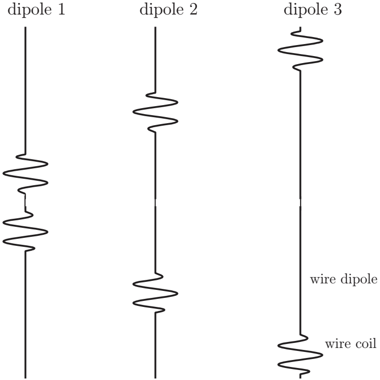
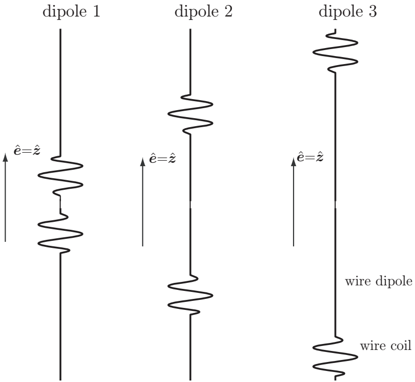


$\gamma/a^3 \approx 0.73$ $\gamma/a^3 \approx 0.95$ $\gamma/a^3 \approx 1.52$

Polarizability $\gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}}$: Interpretation

Wire dipoles (length $\ell \approx 2a$) with coils

External electrostatic field along the dipoles



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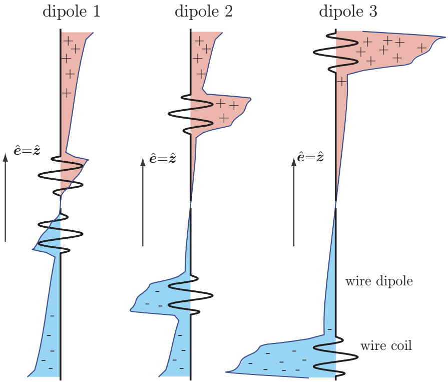
Polarizability $\gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}}$: Interpretation

Wire dipoles (length $\ell \approx 2a$) with coils

Induced charge density on the wire

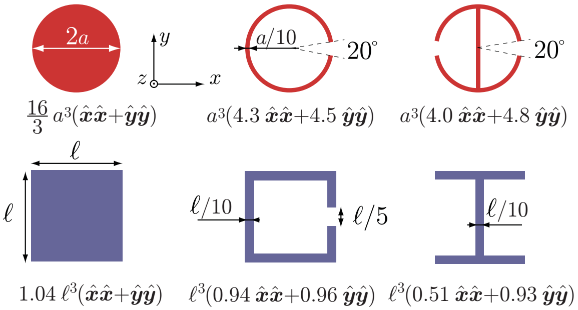


$\gamma/a^3 \approx 0.73 \quad \gamma/a^3 \approx 0.95 \quad \gamma/a^3 \approx 1.52$

Separation of charge for large polarizability.

Properties of the polarizability dyadics

Removal of metal from circular and square plates

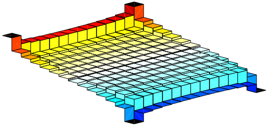


- The polarizability can not increase if you remove material.
- The metal in the center of the structure does not contribute much to the polarizability.
- Volume (and large area) is not necessary for a large polarizability.
- Important to be able to support a large separation of charge.

Numerical evaluation of γ_∞ (single object)

Expand the charge density in basis functions

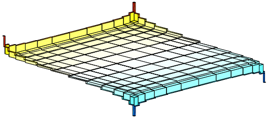
$$\rho(\boldsymbol{r}) = \sum_{n=1}^{N} \rho_n \psi_n(\boldsymbol{r}) = \boldsymbol{\psi}^{\mathsf{T}} \boldsymbol{\rho}$$



Equidistant mesh ($p = 1$)

and solve using Galerkin's method:

$$\begin{cases} \mathbf{W}_{\mathrm{e}}^{(0)} \boldsymbol{\rho} = E_0 \mathbf{f}_{\mathrm{e}} - \mathbf{n} V \\ \mathbf{f}_{\mathrm{e}}^{\mathsf{T}} \boldsymbol{\rho} = E_0/\gamma \\ \mathbf{n}^{\mathsf{T}} \boldsymbol{\rho} = 0 \end{cases} \qquad \begin{pmatrix} \mathbf{W}_{\mathrm{e}}^{(0)} & \mathbf{f}_{\mathrm{e}} & \mathbf{n} \\ \mathbf{f}_{\mathrm{e}}^{\mathsf{T}} & 0 & 0 \\ \mathbf{n}^{\mathsf{H}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\rho} \\ \gamma^{-1} \\ V \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -1 \\ 0 \end{pmatrix}$$



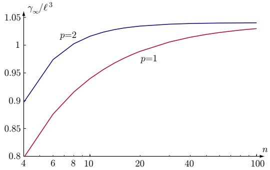
Constant charge ($p = 2$)

where $E_0 = -\gamma$ and ($N \times 1$ matrices)

$$\mathbf{f}_{\mathrm{e}} = \int_{\partial V} (\hat{\boldsymbol{e}} \cdot \boldsymbol{r}) \boldsymbol{\psi}(\boldsymbol{r}) \, \mathrm{dS}\,, \quad \mathbf{n} = \int_{\partial V} \boldsymbol{\psi}(\boldsymbol{r}) \, \mathrm{dS}$$

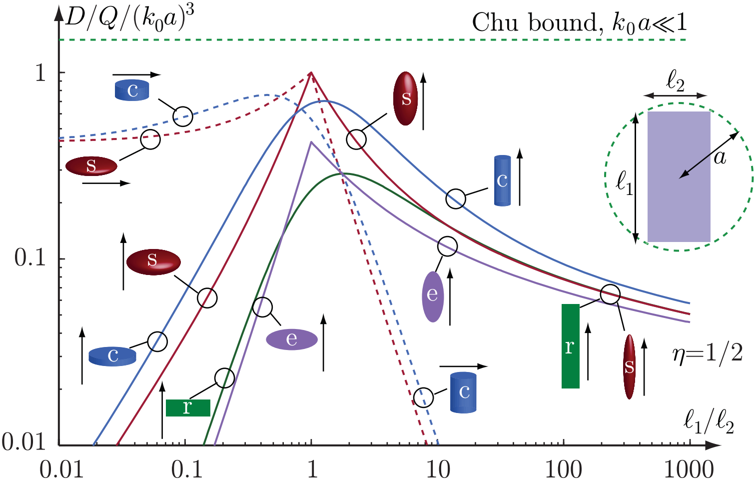
and the $N \times N$ matrix

$$\mathbf{W}_{\mathrm{e}}^{(0)} = \int_{\partial V} \int_{\partial V} \frac{\boldsymbol{\psi}(\boldsymbol{r}) \boldsymbol{\psi}^{\mathsf{T}}(\boldsymbol{r}')}{4\pi |\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{dSdS}'.$$



Convergence

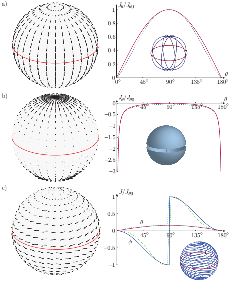
Rectangles, cylinders, elliptic disks, and spheroids (2007)

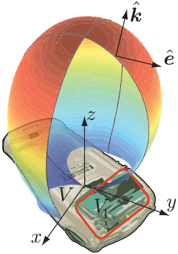


http://www.mathworks.com/matlabcentral/fileexchange/26806-antennaq

Bounds on D/Q (and Q for small antennas)

- Forward scattering (2007).
- Performance in the polarizability.
- Numerical simulations verify the results for electric dipole antennas.
- Similar results for small electric dipole antennas by Yaghjian & Stuart (2010), Vandenbosch (2011), Chalas, Sertel & Volakis (2011), and Gustafsson *etal*(2012).
- Many open questions for mixed modes (TE+TM) and magnetic materials.





What more can we do?

- embedded antennas (mobile phones).
- superdirectivity, efficiency, MIMO...
- current distribution for understanding.

Physical bounds on antennas: methods

Uses the forward scattering sum rule to analyze receiving antennas, see Gustafsson, Sohl & Kristensson (2007,2009):
+arbitrary shape and size
+bandwidth
+closed form expressions
+based on an identity
-entire volumes
-absorption efficiency

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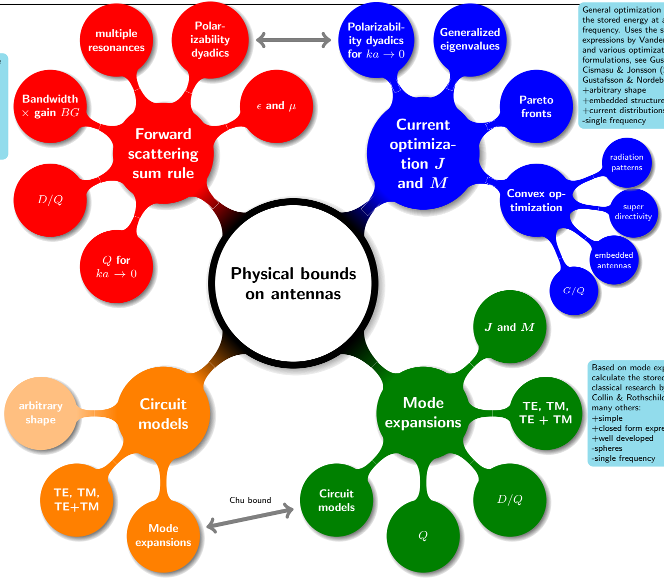

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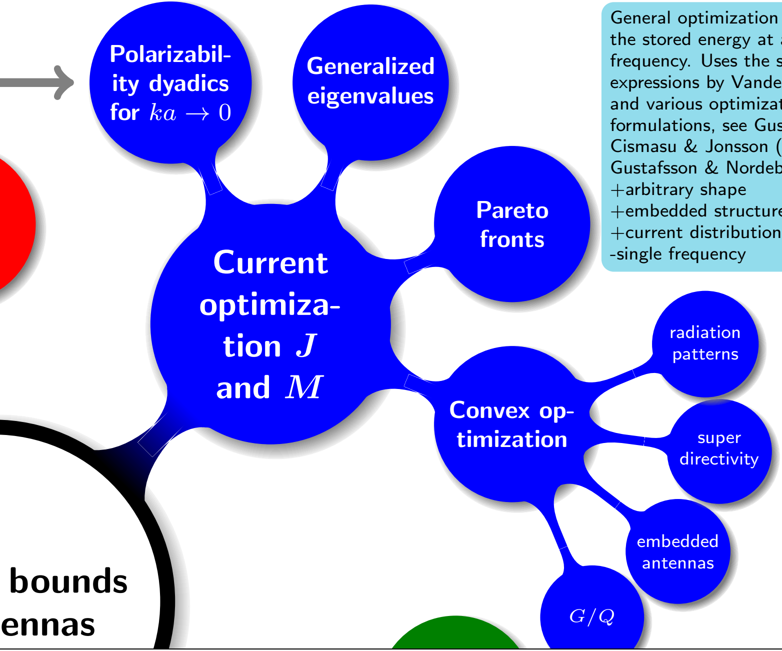

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+arbitrary shape
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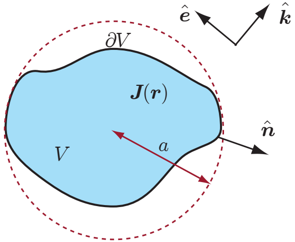
D/Q or (G/Q)

Directivity in the radiation intensity $P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})$ and total radiated power P_{rad}

$$D(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}}) = 4\pi\frac{P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{P_{\mathrm{rad}}}$$



Q-factor

$$Q = \frac{2\omega W}{P_{\mathrm{rad}}} = \frac{2\mathrm{c}_0 k W}{P_{\mathrm{rad}}},$$

where $W = \max\{W_{\mathrm{E}}, W_{\mathrm{M}}\}$ denotes the maximum of the stored electric and magnetic energies. The D/Q quotient cancels P_{rad}

$$\frac{D(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{\omega W} = \frac{2\pi P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{\mathrm{c}_0 k W}.$$

D/Q in the current density $\boldsymbol{J}$ for $ka \to 0$

Partial radiation intensity $P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})$

$$P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}}) = \frac{\eta_0 k^2}{32\pi^2} \left| \int_V \hat{\boldsymbol{e}}^* \cdot \boldsymbol{J}(\boldsymbol{r}) \mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}} \,\mathrm{d}V \right|^2$$

Expand the current density $\boldsymbol{J} = \boldsymbol{J}^{(0)} + k\boldsymbol{J}^{(1)} + \mathcal{O}(k^2)$ for $ka \to 0$ and use the continuity equation $\nabla \cdot \boldsymbol{J} = -\mathrm{j}\omega\rho$, where ρ denotes the charge density, to get

$$P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}}) \approx \frac{\eta_0 k^2 \omega^2}{32\pi^2} \left| \int_V \hat{\boldsymbol{e}}^* \cdot \boldsymbol{r}\rho(\boldsymbol{r}) + \frac{1}{2}\hat{\boldsymbol{h}}^* \times \boldsymbol{r} \cdot \boldsymbol{J}^{(0)}(\boldsymbol{r}) \,\mathrm{d}V \right|^2 .$$

(Quasi) electro- and magnetostatic energies

Low frequency electric energy expressions

$$W^{(\mathrm{E})} \approx \frac{1}{4}\int_{\mathbb{R}^3} \epsilon_0 |\boldsymbol{E}(\boldsymbol{r})|^2 \,\mathrm{dV} = \frac{1}{2}\,\mathrm{Re}\int_V \phi^*(\boldsymbol{r})\rho(\boldsymbol{r})\,\mathrm{dV}$$
$$= \frac{1}{4\epsilon_0}\int_V\int_V \frac{\rho^*(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)}{4\pi|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

where ϕ is the potential and ρ the charge density. Low frequency magneto static energy

$$W^{(\mathrm{M})} \approx \frac{\mu_0}{4}\int_V\int_V \frac{\boldsymbol{J}^*(\boldsymbol{r}_1)\cdot\boldsymbol{J}(\boldsymbol{r}_2)}{4\pi|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

The factor $1/4$ is due to the considered low-frequency time harmonic case. A factor $1/2$ for the static cases.

Small antennas $ka \ll 1$

Expand the current density $\boldsymbol{J} = \boldsymbol{J}^{(0)} + k\boldsymbol{J}^{(1)} + \mathcal{O}(k^2)$ for $ka \to 0$ and use the continuity equation $\nabla\cdot\boldsymbol{J} = -\mathrm{j}\omega\rho$, where ρ denotes the charge density, to get

$$\frac{D}{Q} \leq \max_{\rho,\boldsymbol{J}^{(0)}} \frac{k^3 \left|\int_V \hat{\boldsymbol{e}}^* \cdot \boldsymbol{r}\rho(\boldsymbol{r}) + \frac{1}{2}\hat{\boldsymbol{h}}^* \times \boldsymbol{r}\cdot\boldsymbol{J}^{(0)}(\boldsymbol{r})\,\mathrm{d}V\right|^2}{\max\left\{\iint_V \frac{\rho_1\rho_2^*}{R_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2, \iint_V \frac{\boldsymbol{J}_1^{(0)}\cdot\boldsymbol{J}_2^{(0)*}}{R_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2\right\}},$$

The solution separates into the electric dipole case $\boldsymbol{J}^{(0)} = \boldsymbol{0}$, the magnetic dipole case $\rho = 0$, and combinations of electric and magnetic dipoles. The electric dipole case $\boldsymbol{J}^{(0)} = \boldsymbol{0}$ is

$$\frac{D_{\mathrm{e}}}{Q_{\mathrm{e}}} \leq \max_{\rho} \frac{k^3}{4\pi} \frac{\left|\int \hat{\boldsymbol{e}}^* \cdot \boldsymbol{r}\rho(\boldsymbol{r})\,\mathrm{d}V\right|^2}{\int_V\int_V \frac{\rho(\boldsymbol{r}_1)\rho^*(\boldsymbol{r}_2)}{4\pi|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\mathrm{dV}_1\,\mathrm{dV}_2}.$$

where we note that it only depends on the charge density ρ.

Small antennas $ka \ll 1$

The electric dipole case $\boldsymbol{J}^{(0)} = \boldsymbol{0}$

$$\frac{D}{Q} \leq \max_{\rho} \frac{k^3}{4\pi} \frac{\left|\int \hat{\boldsymbol{e}}^* \cdot \boldsymbol{r}\rho(\boldsymbol{r}) \,\mathrm{d}V\right|^2}{\int_V \int_V \frac{\rho(\boldsymbol{r}_1)\rho^*(\boldsymbol{r}_2)}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \,\mathrm{dV}_1 \,\mathrm{dV}_2}$$

has the solution

$$\frac{D(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})}{Q} \leq \frac{k^3}{4\pi} \hat{\boldsymbol{e}}^* \cdot \boldsymbol{\gamma}_\infty \cdot \hat{\boldsymbol{e}}.$$

where $\boldsymbol{\gamma}_\infty$ is the high contrast polarizability dyadic.

- Depends only on the charge density $\rho = \mathrm{j}\omega^{-1}\nabla \cdot \boldsymbol{J}$.
- Many current distributions $\boldsymbol{J}$ give the same D/Q.
- Q scales as k^{-3} as $\max D = 3/2$.

Alternative optimization formulation for D/Q

Consider the electric dipole case

$$\frac{D}{Q} \leq \max_{\rho} \frac{k^3}{4\pi} \frac{\left|\int \hat{\boldsymbol{e}}^* \cdot \boldsymbol{r}\rho(\boldsymbol{r})\,\mathrm{d}V\right|^2}{\int_V \int_V \frac{\rho(\boldsymbol{r}_1)\rho^*(\boldsymbol{r}_2)}{4\pi|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\mathrm{dV}_1\,\mathrm{dV}_2}$$

A scaling $\rho \to \alpha\rho$ does not change D/Q. We can hence consider the alternative optimization problem (dimensionless)

$$\begin{aligned} &\text{minimize} \quad \int_V \int_V \frac{\rho(\boldsymbol{r}_1)\rho^*(\boldsymbol{r}_2)}{4\pi|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\mathrm{dV}_1\,\mathrm{dV}_2 \\ &\text{subject to} \quad \left|\int \hat{\boldsymbol{e}}^* \cdot \boldsymbol{r}\rho(\boldsymbol{r})\,\mathrm{dV}\right|^2 = P_0 \end{aligned}$$

and by choosing a phase, *e.g.*, $P_0 = 1$

$$\begin{aligned} &\text{minimize} \quad \int_V \int_V \frac{\rho(\boldsymbol{r}_1)\rho^*(\boldsymbol{r}_2)}{4\pi|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\mathrm{dV}_1\,\mathrm{dV}_2 \\ &\text{subject to} \quad \int \hat{\boldsymbol{e}}^* \cdot \boldsymbol{r}\rho(\boldsymbol{r})\,\mathrm{d}V = \sqrt{P_0} \end{aligned}$$

Numerical solution (surface charge density)

Solve by expansion of ρ in basis functions

$$\rho(\boldsymbol{r}) = \sum_{n=1}^{N} \rho_n \psi_n(\boldsymbol{r}) = \boldsymbol{\psi}^{\mathsf{T}} \boldsymbol{\rho}$$

Define the $N \times 1$ matrices

$$\mathbf{f}_{\mathrm{e}} = \int_{\partial V} (\hat{\boldsymbol{e}} \cdot \boldsymbol{r}) \boldsymbol{\psi}(\boldsymbol{r}) \, \mathrm{dS}\,, \quad \mathbf{n} = \int_{\partial V} \boldsymbol{\psi}(\boldsymbol{r}) \, \mathrm{dS}$$

and the $N \times N$ matrix

$$\mathbf{W}_{\mathrm{e}}^{(0)} = \int_{\partial V} \int_{\partial V} \frac{\boldsymbol{\psi}(\boldsymbol{r}) \boldsymbol{\psi}^{\mathsf{T}}(\boldsymbol{r}')}{4\pi |\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{dS} \mathrm{dS}'.$$

Optimization problem with solution

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{\rho}^{\mathsf{H}} \mathbf{W}_{\mathrm{e}}^{(0)} \boldsymbol{\rho} \\ \text{subject to} \quad & \mathbf{f}_{\mathrm{e}}^{\mathsf{H}} \boldsymbol{\rho} = 1 \\ & \mathbf{n}_{\mathrm{e}}^{\mathsf{H}} \boldsymbol{\rho} = 0 \end{aligned} \qquad \begin{pmatrix} \mathbf{W}_{\mathrm{e}}^{(0)} & \mathbf{f}_{\mathrm{e}} & \mathbf{n} \\ \mathbf{f}_{\mathrm{e}}^{\mathsf{T}} & 0 & 0 \\ \mathbf{n}^{\mathsf{H}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\rho} \\ \gamma^{-1} \\ V \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -1 \\ 0 \end{pmatrix}$$

Optimal current distributions on small spheres

- The optimization problem for small (electric) dipole antennas shows that the charge distribution, ρ, is the most important quantity.
- On a sphere, we have surface charge density

$$\rho(\theta,\phi) = \rho_0 \cos\theta$$

  for small optimal antennas with polarization $\hat{\boldsymbol{e}} = \hat{\boldsymbol{z}}$.
- The current density satisfies

$$\nabla\cdot\boldsymbol{J} = -\mathrm{j}k\rho$$

  Many solutions, *e.g.*, all surface currents

$$\boldsymbol{J} = J_{\theta 0}\hat{\boldsymbol{\theta}}\big(\sin\theta - \frac{\beta}{\sin\theta}\big) + \frac{1}{\sin\theta}\frac{\partial A}{\partial\phi}\hat{\boldsymbol{\theta}} - \frac{\partial A}{\partial\theta}\hat{\boldsymbol{\phi}}$$

  where $J_{\theta 0} = -\mathrm{j}ka\rho_0$, β is a constant, and $A = A(\theta,\phi)$

Optimal current distributions on small spheres

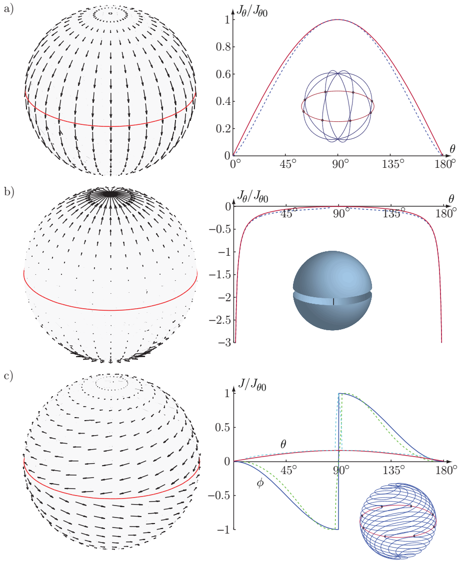
Some solutions:

- Spherical dipole
  $\beta = 0, A = 0.$
- Capped dipole
  $\beta = 1, A = 0.$
- Folded spherical helix
  $\beta = 0, A \neq 0.$

They have identical charge distributions

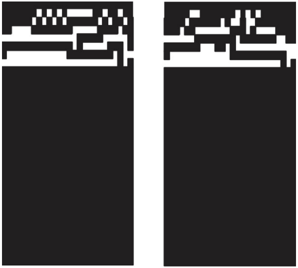
$$\rho(\theta,\phi) = \rho_0 \cos\theta$$

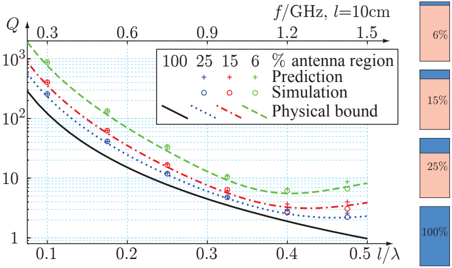
Can mathematical solutions suggest antenna designs?



Outline

Antenna optimization





Optimization of structures

- global optimization.
- new non-intuitive designs.
- convergence?
- stopping criteria?
- optimal?

Optimization of currents

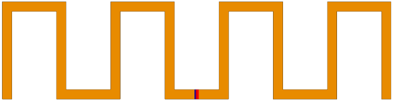
- determine optimal currents for Q, G/Q, ...
- convex optimization.
- physical bounds.
- can we realize the currents?

Antenna and current optimization

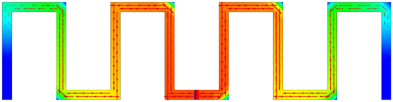
Antennas form a transition between guided waves and propagating waves in free space. Oscillating currents produce radiated EM fields. Antenna design: the *art* to produce the desired current distribution on the structure by shaping and choosing the materials.

- Have a given maximal size of the antenna structure.
- Current optimization: determine an optimal current distribution from all possible currents in the available geometry.

Maximal size of the antenna.

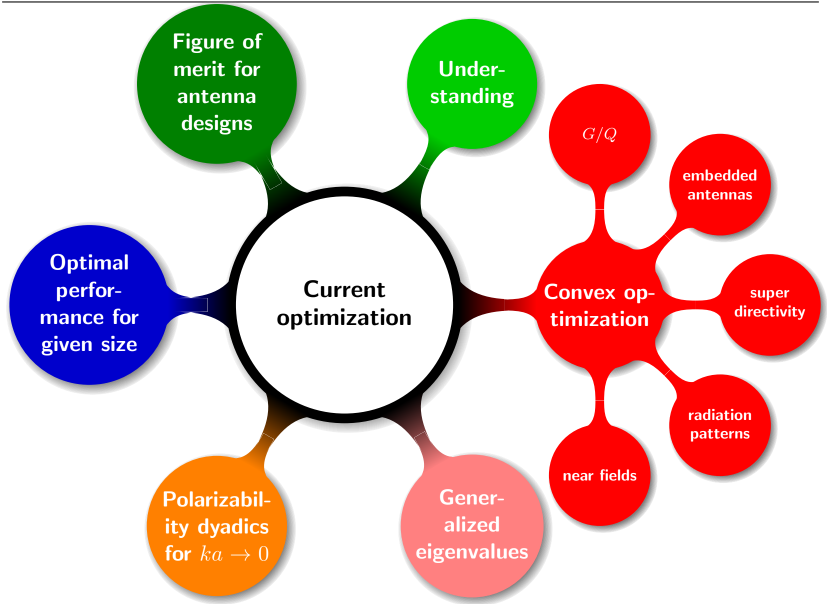


Antenna geometry with feed point.



Current distribution on the antenna.

Current optimization



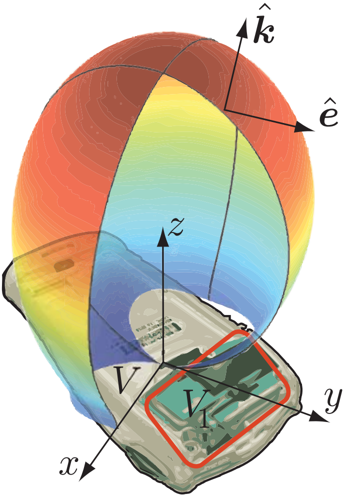
Optimization of currents for antenna analysis

Antenna geometry and parameters:

- Radiating (antenna) structure, V.
- Antenna volume, $V_1 \subset V$.
- Current density $\boldsymbol{J}_1$ in V_1.
- Radiated field, $\boldsymbol{F}(\hat{\boldsymbol{k}})$, in direction $\hat{\boldsymbol{k}}$ and polarization $\hat{\boldsymbol{e}}$.

Physical bounds and optimal currents for:

- maximum $G(\hat{\boldsymbol{k}}, \hat{\mathrm{e}})/Q$.
- superdirectivity.
- embedded antennas.
- efficiency.
- also minimum Q for given radiated fields, sidelobe levels, MIMO...



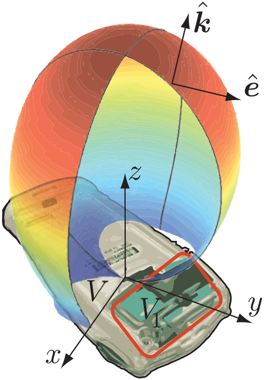
Optimization of antenna currents: examples

Gain over Q

$$\begin{aligned} &\text{minimize} && \text{Stored energy} \\ &\text{subject to} && \text{Radiation intensity} = P_0 \end{aligned}$$

Q for superdirectivity $D \geq D_0$.

$$\begin{aligned} &\text{minimize} && \text{Stored energy} \\ &\text{subject to} && \text{Radiation intensity} = D_0 P_{\text{rad}}/(4\pi) \\ & && \text{Radiated power} \leq P_{\text{rad}} \end{aligned}$$



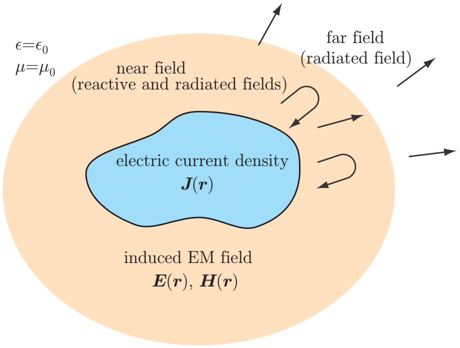
Embedded structures

$$\begin{aligned} &\text{minimize} && \text{Stored energy} \\ &\text{subject to} && \text{Radiation intensity} = P_0 \\ & && \text{Correct induced currents} \end{aligned}$$

Need to:

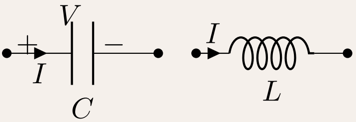
1. Express the *stored energy* in the current density $\boldsymbol{J}$.
2. Solve the optimization problems.

What is (stored) EM energy?



- Time average energy density $\epsilon_0|\boldsymbol{E}|^2/4$ and $\mu_0|\boldsymbol{H}|^2/4$.
- What is stored and radiated?
- How can we express the (stored) energy in the current density?
- Here, currents in free space.

Lumped elements



Time average stored energy in capacitors

$$W^{(\mathrm{E})} = \frac{C|V|^2}{4} = \frac{|I|^2}{4\omega^2 C}$$

and in inductors

$$W^{(\mathrm{M})} = \frac{L|I|^2}{4}$$

Q-factor and stored energy

- The Q-factor for a tuned antenna is

$$Q = \max\{Q^{(\mathrm{E})}, Q^{(\mathrm{M})}\}, \quad Q^{(\mathrm{E})} = \frac{2\omega W^{(\mathrm{E})}}{P_{\mathrm{r}}}, \quad Q^{(\mathrm{M})} = \frac{2\omega W^{(\mathrm{M})}}{P_{\mathrm{r}}}$$

  and $W^{(\mathrm{E})}$ is the stored electric energy, $W^{(\mathrm{M})}$ the stored magnetic energy, and P_{r} the dissipated (radiated for a loss-less antenna) power.
- Fractional bandwidth for single resonance circuits

$$B = \frac{\omega_2 - \omega_1}{\omega_0} \approx \frac{2\Gamma_0}{Q\sqrt{1-\Gamma_0^2}},$$

  where $\omega_0 = (\omega_1 + \omega_2)/2$ and Γ_0 is the threshold of the reflection coefficient.
- The Fano limit for a single resonance circuit, $B \leq 27.29/(Q|\Gamma_{0,\mathrm{dB}}|)$, is an upper bound on the bandwidth after matching.

Electrostatics

- Consider the charge density $\rho(\boldsymbol{r})$ supported in $V \subset \mathbb{R}^3$ in free space. Also assume that the total charge is zero, $\int \rho \, \mathrm{dV} = 0$.
- Have the alternative electric energy expressions

$$W^{(\mathrm{E})} = \frac{1}{2} \int_{\mathbb{R}^3} \epsilon_0 |\boldsymbol{E}(\boldsymbol{r})|^2 \, \mathrm{dV} = \frac{1}{2} \int_V \phi(\boldsymbol{r}) \rho(\boldsymbol{r}) \, \mathrm{dV}$$
$$= \frac{1}{2\epsilon_0} \int_V \int_V \frac{\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \, \mathrm{dV}_1 \, \mathrm{dV}_2$$

where ϕ is the potential and ρ the charge density.

- Alternative interpretations: Energy in the fields or energy in the charges.
- Alternative computation: integral over $\mathbb{R}^3$ or over V.
- Positive definite quadratic form suitable for optimization.

Stored EM energy expressions

- Subtraction of the energy in the radiated field (far field) (Collin+Rothschild 1964, Yaghjian+Best 2005)

$$W_{\mathrm{F}}^{(\mathrm{E})} = \frac{\epsilon_0}{4} \int_{\mathbb{R}_\mathrm{r}^3} |\boldsymbol{E}(\boldsymbol{r})|^2 - \frac{|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2}{r^2} \,\mathrm{dV}$$

- Expressed in the frequency derivative of the reactance (Fante 1969, Yaghjian+Best 2005)

$$W_{\mathrm{F}}^{(\mathrm{E})} = \frac{|I_0|^2}{4} X' - \frac{1}{2\eta_0} \operatorname{Im} \int_{\Omega} \boldsymbol{F}'(\hat{\boldsymbol{r}}) \cdot \boldsymbol{F}^*(\hat{\boldsymbol{r}}) \,\mathrm{d}\Omega$$

- In the current density (Vandenbosch 2010, see also Geyi 2003, Gustafsson+Jonsson 2012)

$$W_{\mathrm{C}}^{(\mathrm{E})} = \frac{\eta_0}{4\omega} \int_V \int_V \nabla_1 \cdot \boldsymbol{J}_1 \nabla_2 \cdot \boldsymbol{J}_2^* \frac{\cos(kr_{12})}{4\pi k r_{12}} - \left(k^2 \boldsymbol{J}_1 \cdot \boldsymbol{J}_2^* - \nabla_1 \cdot \boldsymbol{J}_1 \nabla_2 \cdot \boldsymbol{J}_2^*\right) \frac{\sin(kr_{12})}{8\pi} \,\mathrm{dV}_1 \,\mathrm{dV}_2$$

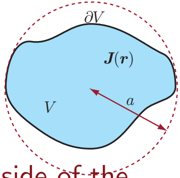
Interpretation by subtraction of the radiated field

The classical approach initiated by Collin & Rothschild 1964, is a subtraction of the power flow, *i.e.*,

$$W_{\mathrm{e}}^{(\mathrm{P})} = \frac{\epsilon_0}{4} \int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}(\boldsymbol{r})|^2 - \eta_0 \,\mathrm{Re}\{\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r}) \cdot \hat{\boldsymbol{r}}\} \,\mathrm{dV}$$

where $\mathbb{R}^3_{\mathrm{r}} = \{\boldsymbol{r} : \lim_{r_0 \to \infty} |\boldsymbol{r}| \leq r_0\}$.
Reinterpret as a subtraction of the far field energy

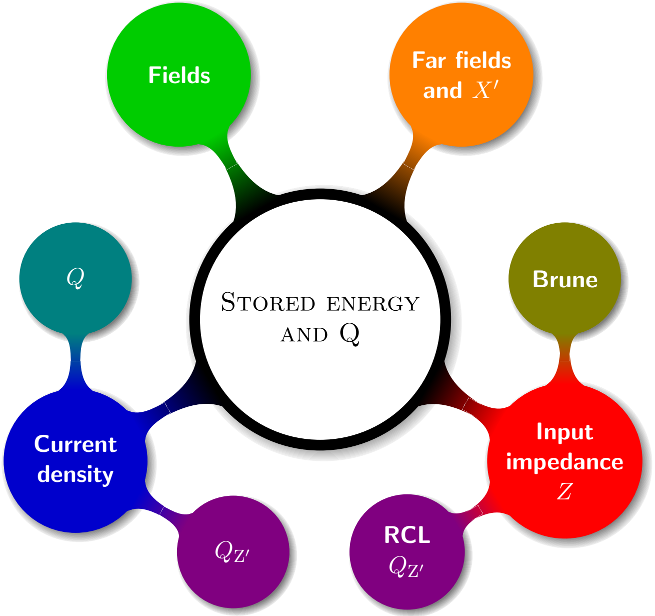
$$W_{\mathrm{e}}^{(\mathrm{F})} = \frac{\epsilon_0}{4} \int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}(\boldsymbol{r})|^2 - \frac{|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2}{r^2} \,\mathrm{dV}$$

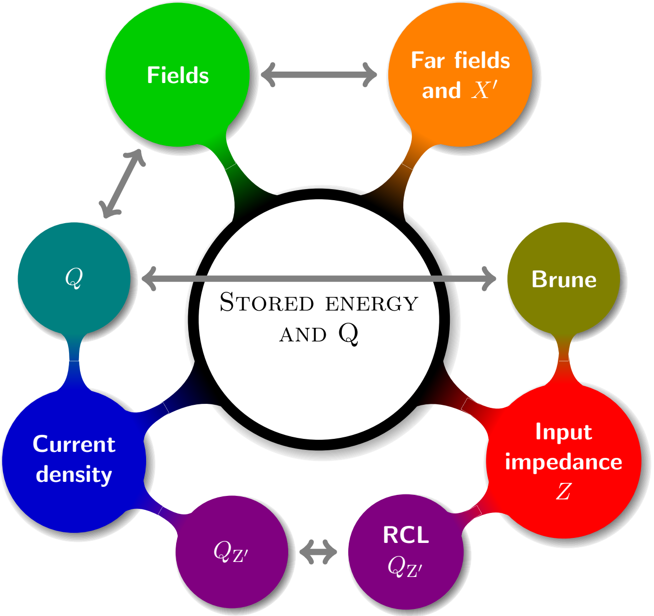


Note, the contributions to the integrals differ only inside of the smallest circumscribing sphere.

$W_{\mathrm{e}}^{(\mathrm{F})}$ and the corresponding magnetic energy $W_{\mathrm{m}}^{(\mathrm{F})}$ are identical to the upcoming integral expressions for 'many' cases.

Gustafsson, Jonsson, 'Stored Electromagnetic Energy and Antenna Q', 2012.


Fields
Far fields and X'
Stored energy and Q
Q
Brune
Current density
Input impedance Z
$Q_{Z'}$
RCL $Q_{Z'}$


Fields
Far fields and X'
Q
Stored energy and Q
Brune
Current density
Input impedance Z
$Q_{Z'}$
RCL $Q_{Z'}$

Stored EM energy from current densities $\boldsymbol{J}$ in V

We use the expressions by Vandenbosch (2010) (and Carpenter (1989), Geyi (2003) for small antennas). Stored electric energy

$$W^{(\mathrm{E})} = \frac{\eta_0}{4\omega}\int_V\int_V \nabla_1\cdot\boldsymbol{J}(\boldsymbol{r}_1)\nabla_2\cdot\boldsymbol{J}^*(\boldsymbol{r}_2)\frac{\cos(kr_{12})}{4\pi kr_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2 + W^{(2)}$$

Stored magnetic energy

$$W^{(\mathrm{M})} = \frac{\eta_0}{4\omega}\int_V\int_V k^2\boldsymbol{J}(\boldsymbol{r}_1)\cdot\boldsymbol{J}^*(\boldsymbol{r}_2)\frac{\cos(kr_{12})}{4\pi kr_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2 + W^{(2)}$$

where $\mathrm{j}\omega\rho = -\nabla\cdot\boldsymbol{J}$, $\phi = \epsilon_0^{-1}g*\rho$, $\boldsymbol{A} = \mu_0 g*\boldsymbol{J}$, $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$

$$W^{(2)} = \frac{\eta_0}{8\omega}\int_V\int_V \left(\nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^* - k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^*\right)\frac{\sin(kr_{12})}{4\pi}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

Stored EM energy from current densities $\boldsymbol{J}$ in V

We use the expressions by Vandenbosch (2010) (and Carpenter (1989), Geyi (2003) for small antennas). Stored electric energy

$$W^{(\mathrm{E})} = \frac{1}{4\epsilon_0}\,\mathrm{Re}\int_V\int_V \rho(\boldsymbol{r}_1)\rho^*(\boldsymbol{r}_2)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2 + W^{(2)}$$

Stored magnetic energy

$$W^{(\mathrm{M})} = \frac{\mu_0}{4}\,\mathrm{Re}\int_V\int_V \boldsymbol{J}(\boldsymbol{r}_1)\cdot\boldsymbol{J}^*(\boldsymbol{r}_2)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2 + W^{(2)}$$

where $\mathrm{j}\omega\rho = -\nabla\cdot\boldsymbol{J}$, $\phi = \epsilon_0^{-1} g * \rho$, $\boldsymbol{A} = \mu_0 g * \boldsymbol{J}$, $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$

$$W^{(2)} = \frac{\eta_0}{8\omega}\int_V\int_V \left(\nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^* - k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^*\right)\frac{\sin(kr_{12})}{4\pi}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

Stored EM energy from current densities $\boldsymbol{J}$ in V

We use the expressions by Vandenbosch (2010) (and Carpenter (1989), Geyi (2003) for small antennas). Stored electric energy

$$W^{(\mathrm{E})} = \frac{1}{4}\,\mathrm{Re}\int_V \phi(\boldsymbol{r})\rho^*(\boldsymbol{r})\,\mathrm{dV} + W^{(2)}$$

Stored magnetic energy

$$W^{(\mathrm{M})} = \frac{1}{4}\,\mathrm{Re}\int_V \boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{J}^*(\boldsymbol{r})\,\mathrm{dV} + W^{(2)}$$

where $\mathrm{j}\omega\rho = -\nabla\cdot\boldsymbol{J}$, $\phi = \epsilon_0^{-1} g * \rho$, $\boldsymbol{A} = \mu_0 g * \boldsymbol{J}$, $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$

$$W^{(2)} = \frac{\eta_0}{8\omega}\int_V\int_V \left(\nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^* - k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^*\right)\frac{\sin(kr_{12})}{4\pi}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

From MoM to stored energy

A standard MoM implementation of the EFIE using the Galerkin procedure computes the impedance matrix $\mathbf{Z}=\mathbf{Z}_{\mathrm{m}}-\mathbf{Z}_{\mathrm{e}}$, where

$$Z_{\mathrm{e},ij}=\frac{-\eta_0}{\mathrm{j}k}\int_V\int_V \nabla_1\cdot\boldsymbol{\psi}_{i1}\nabla_2\cdot\boldsymbol{\psi}_{j2}\frac{\mathrm{e}^{-\mathrm{j}kR_{12}}}{4\pi R_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

and

$$Z_{\mathrm{m},ij}=\mathrm{j}k\eta_0\int_V\int_V \boldsymbol{\psi}_{i1}\cdot\boldsymbol{\psi}_{j2}\frac{\mathrm{e}^{-\mathrm{j}kR_{12}}}{4\pi R_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

and add the non-singular term containing the elements

$$X_{\mathrm{em},ij}=\frac{-\eta_0}{8\pi}\int_V\int_V \big(k^2\boldsymbol{\psi}_{i1}\cdot\boldsymbol{\psi}_{j2} - \nabla_1\cdot\boldsymbol{\psi}_{i1}\nabla_2\cdot\boldsymbol{\psi}_{j2}\big)\sin(kR_{12})\,\mathrm{dV}_1\,\mathrm{dV}_2.$$

to get the electric $\mathbf{X}_{\mathrm{e}}$, and magnetic $\mathbf{X}_{\mathrm{m}}$, reactance matrices

$$\mathbf{X}_{\mathrm{e}}=\mathrm{Im}\{\mathbf{Z}_{\mathrm{e}}\}+\mathbf{X}_{\mathrm{em}}\quad\text{and }\mathbf{X}_{\mathrm{m}}=\mathrm{Im}\{\mathbf{Z}_{\mathrm{m}}\}+\mathbf{X}_{\mathrm{em}}$$

Stored EM energies from current densities $\boldsymbol{J}$ in V II

Also the total radiated power

$$P_{\text{rad}} = \frac{\eta_0}{2}\int_V\int_V \left(k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^* - \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\right)\frac{\sin(kr_{12})}{4\pi k r_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2.$$

Method of Moments approximation (expand $\boldsymbol{J}$ in basis functions)

$$W^{(\mathrm{E})} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}$$ stored E-energy, $\mathbf{X}_{\mathrm{e}}$ electric reactance

$$W^{(\mathrm{M})} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}$$ stored M-energy, $\mathbf{X}_{\mathrm{m}}$ magnetic reactance

$$P_{\text{rad}} \approx \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}$$ radiated power

giving $\mathbf{Z} = \mathbf{R}_{\mathrm{r}} + \mathrm{j}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})$. We also use

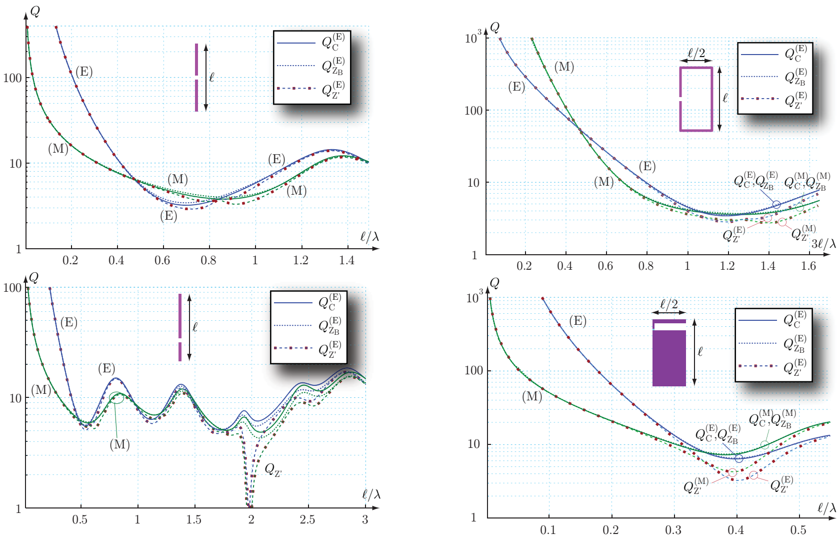
$\boldsymbol{F} \approx \mathbf{F}\mathbf{I}$ far field

$\boldsymbol{E} \approx \mathbf{N}\mathbf{I}$ near field

$\mathbf{I}_2 \approx \mathbf{C}\mathbf{I}_1$ induced current on a PEC

Antenna examples

Q from stored energy expressed in the current density Q_{C}, circuits $Q_{\mathrm{Z_B}}$, and differentiated impedance $Q_{\mathrm{Z'}}$



Antenna examples

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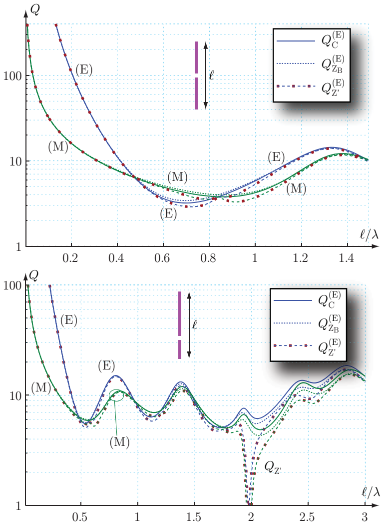
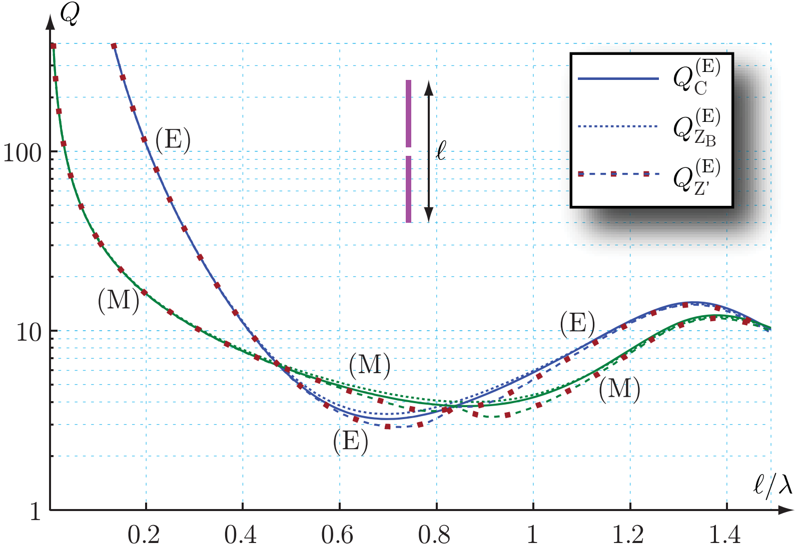


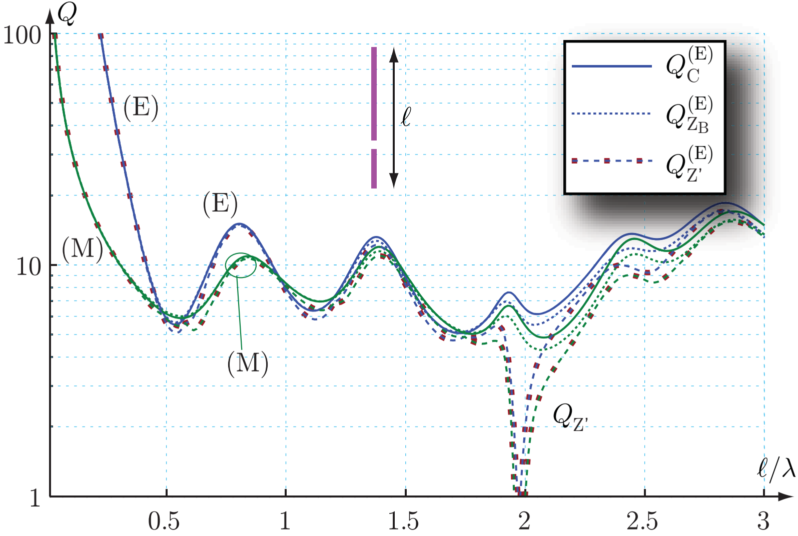
Q computed from

- the currents, Q_{C}.
- a circuit model synthesized from the input impedance using Brune synthesis (1931), $Q_{\mathrm{Z_B}}$.
- differentiation of the (tuned) input impedance,
  $$Q_{\mathrm{Z'}} = \frac{\omega_0 |Z'|}{2R} = \omega_0 |\Gamma'|.$$

All agree for $Q \gg 1$ but the Q from the differentiated impedance ($Q_{\mathrm{Z'}}$) is lower in some regions.
Which one is most accurate/best?



Gustafsson, Jonsson, 'Stored Electromagnetic Energy and Antenna Q', 2012.



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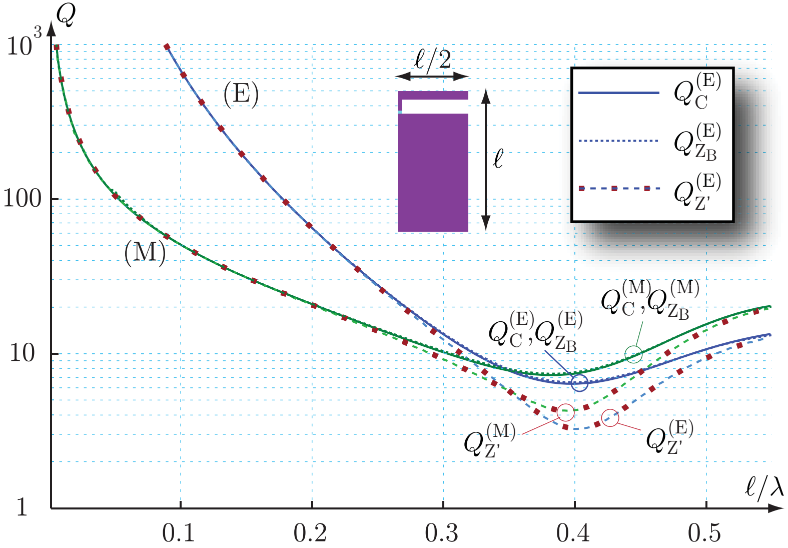

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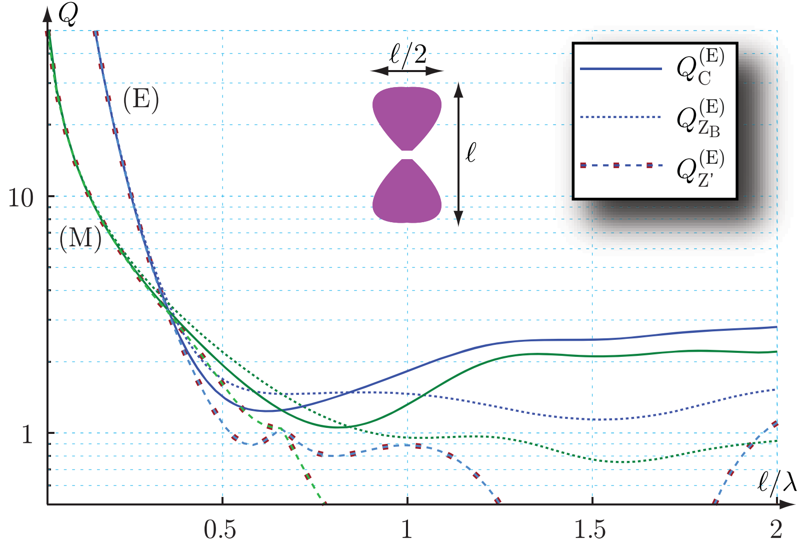

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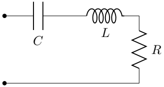
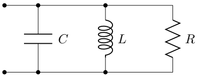
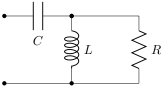
Stored energy from circuit models

Resonance circuits Padé (local) approximation around the resonance frequency (also an all-pass filter), *cf.*, $Q_{Z'} = \frac{\omega_0|Z'|}{2R} = \omega_0|\Gamma'|$

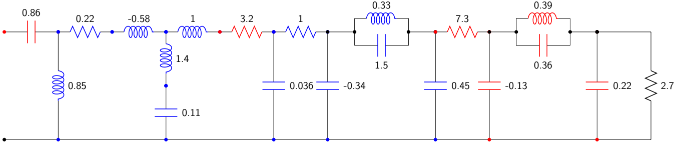




Small open-circuit antennas Model based on the (small) electrical dipole, *cf.*, the Chu bound (1948)



Brune synthesis Brune (1931) synthesized circuit from the input impedance. The negative quantities are replaced by ideal transformers. Here Q-factor $Q_{\mathrm{Z_B}}$



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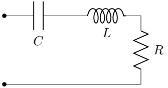
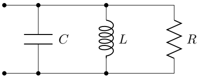
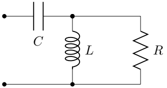
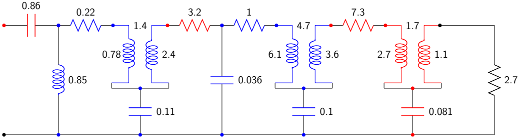




Small open-circuit antennas Model based on the (small) electrical dipole, *cf.*, the Chu bound (1948)



Brune synthesis Brune (1931) synthesized circuit from the input impedance. The negative quantities are replaced by ideal transformers. Here Q-factor $Q_{\mathrm{Z_B}}$



Brune synthesize

Iterative procedure to synthesize circuit models from PR (positive real rational functions) by Brune 1931.

1. Approximate the input impedance with a rational PR function (hard problem).
2. Apply Brune synthesis and compute the stored energy in the capacitors and inductors.

Q-factor and stored energy

- The Q-factor for a tuned antenna is

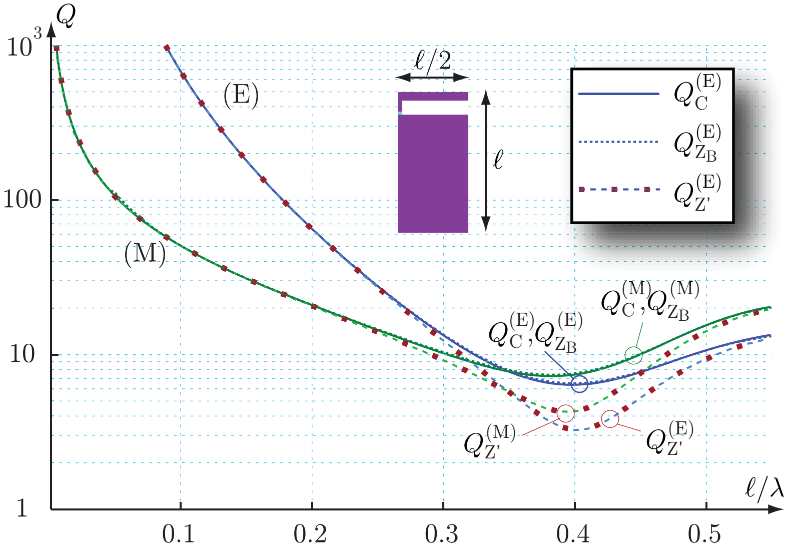
$$Q = \max\{Q^{(\mathrm{E})}, Q^{(\mathrm{M})}\}, \quad Q^{(\mathrm{E})} = \frac{2\omega W^{(\mathrm{E})}}{P_{\mathrm{r}}}, \quad Q^{(\mathrm{M})} = \frac{2\omega W^{(\mathrm{M})}}{P_{\mathrm{r}}}$$

  and $W^{(\mathrm{E})}$ is the stored electric energy, $W^{(\mathrm{M})}$ the stored magnetic energy, and P_{r} the dissipated (radiated for a loss-less antenna) power.
- Fractional bandwidth for single resonance circuits

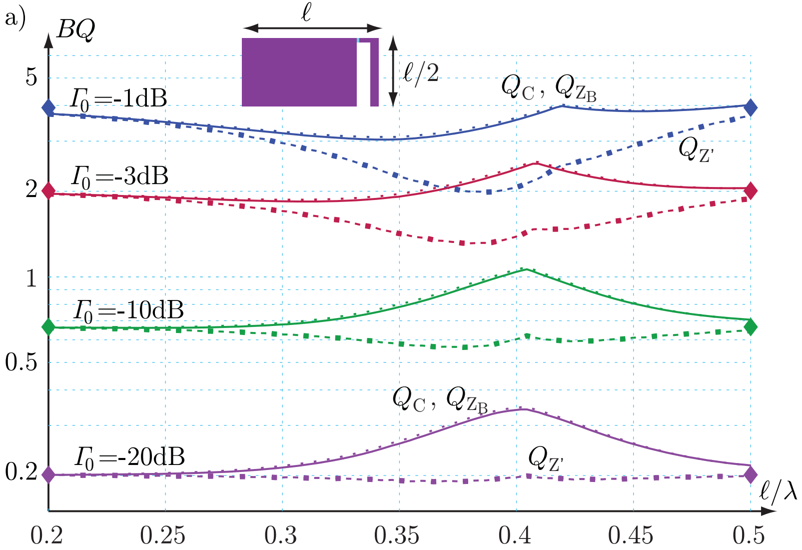
$$B = \frac{\omega_2 - \omega_1}{\omega_0} \approx \frac{2\Gamma_0}{Q\sqrt{1-\Gamma_0^2}},$$

  where $\omega_0 = (\omega_1 + \omega_2)/2$ and Γ_0 is the threshold of the reflection coefficient.
- The Fano limit for a single resonance circuit, $B \leq 27.29/(Q|\Gamma_{0,\mathrm{dB}}|)$, is an upper bound on the bandwidth after matching.

Bandwidth

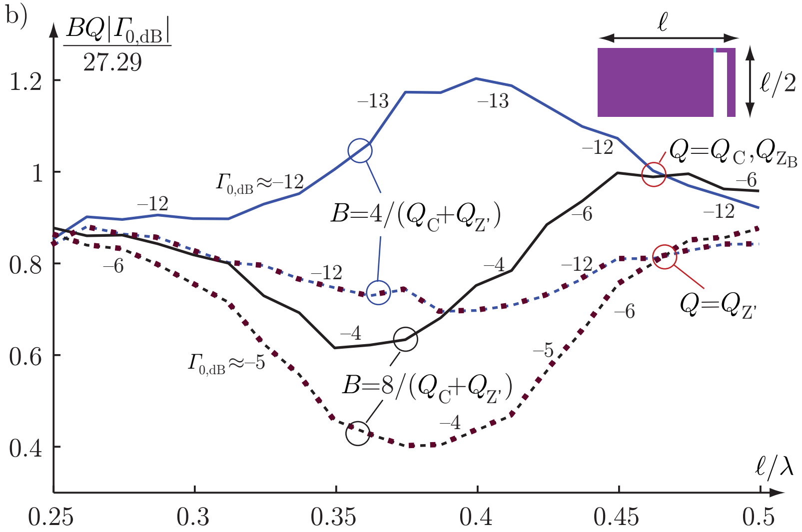

Q
10^3
100
10
1
0.1
0.2
0.3
0.4
0.5
ℓ/λ
(E)
(M)
ℓ/2
ℓ
Q_C^(E)
Q_ZB^(E)
Q_Z'^(E)
Q_C^(M), Q_ZB^(M)
Q_C^(E), Q_ZB^(E)
Q_Z'^(M)
Q_Z'^(E)

Bandwidth

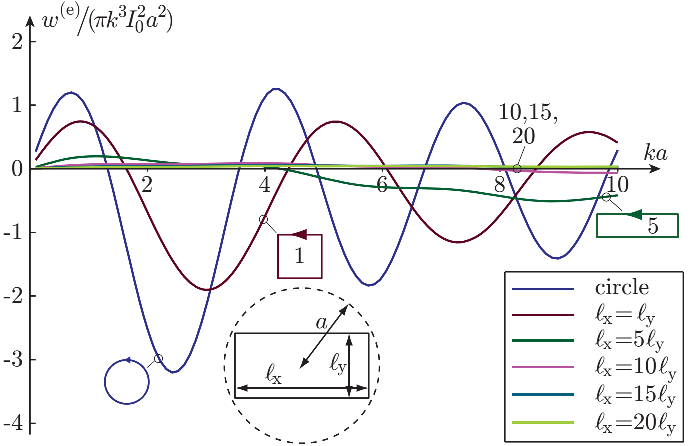


a)
BQ
ℓ
$\ell/2$
5
2
1
0.5
0.2
Γ_0=-1dB
Γ_0=-3dB
Γ_0=-10dB
Γ_0=-20dB
Q_{C} , $Q_{\mathrm{Z_B}}$
$Q_{\mathrm{Z'}}$
Q_{C} , $Q_{\mathrm{Z_B}}$
$Q_{\mathrm{Z'}}$
ℓ/λ
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Bandwidth

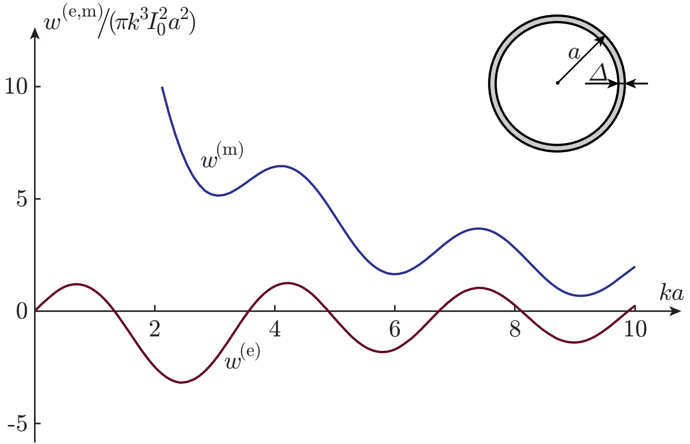

b)
$\frac{BQ|\Gamma_{0,\mathrm{dB}}|}{27.29}$
1.2
1
0.8
0.6
0.4
0.25
0.3
0.35
0.4
0.45
0.5
ℓ/λ
ℓ
$\ell/2$
–13
–13
–12
–12
–12
–6
–6
–6
–12
–12
–12
–4
–4
–4
–5
–6
$\Gamma_{0,\mathrm{dB}}\approx -12$
$\Gamma_{0,\mathrm{dB}}\approx -5$
$B=4/(Q_{\mathrm{C}}+Q_{\mathrm{Z'}})$
$B=8/(Q_{\mathrm{C}}+Q_{\mathrm{Z'}})$
$Q=Q_{\mathrm{C}}, Q_{\mathrm{Z_B}}$
$Q=Q_{\mathrm{Z'}}$

Negative stored energy of loop current

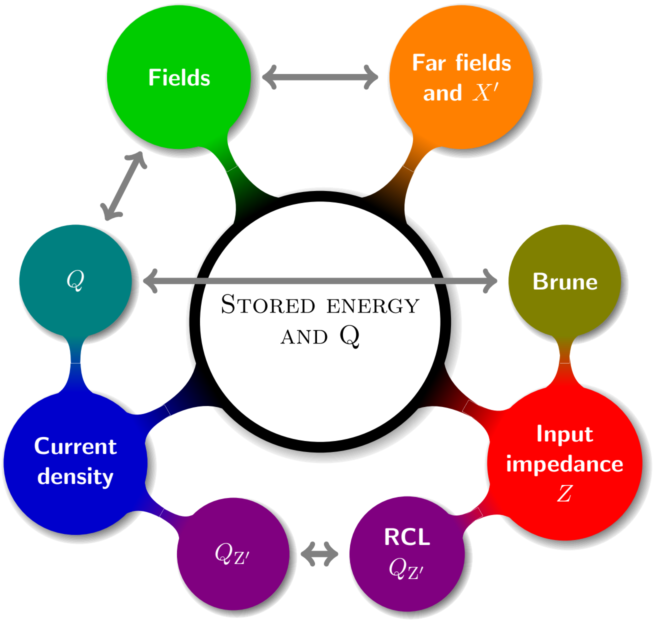


The presented stored energy expressions and produce negative values for large antennas, *i.e.*, they are not positive semidefinite.

Negative stored energy of loop current



The presented stored energy expressions and produce negative values for large antennas, *i.e.*, they are not positive semidefinite.


Fields
Far fields and X'
Q
Stored energy and Q
Brune
Current density
Input impedance Z
$Q_{Z'}$
RCL $Q_{Z'}$

Summary: Stored EM energies

- Introduced by Vandenbosch in *Reactive energies, impedance, and Q factor of radiating structures*, IEEE-TAP 2010.
- In the limit $ka \to 0$ by Geyi, IEEE-TAP 2003 and also similar expressions by Carpenter 1989.
- Verification for wire antennas in Hazdra *etal*, IEEE-AWPL 2011.
- Some issues with 'negative stored energy' for large structures in Gustafsson *etal*, IEEE-TAP 2012. See also Gustafsson and Jonsson, *Stored Electromagnetic Energy and Antenna Q*, 2012.
- Time-domain version by Vandenbosch 2013.
- $Q_{\mathrm{Z}'}$ formulation by Capek *etal*, IEEE-TAP 2014.

One of the most powerful new tools in EM and antenna theory. Still many open questions, and probably no consensus (yet).

- How do we interpret the stored energy? Subtracted far-field...
- How do we verify the expressions? Circuit models (Brune), unique,...
- Dialectics, losses, ... There are some suggestions...

Outline

Optimization of antenna currents: G/Q

Consider the optimization problem (**Gain over Q**)

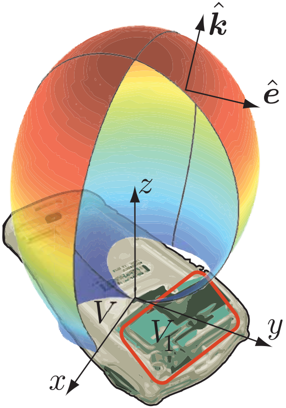
$$\begin{aligned} &\text{minimize} && \text{Stored energy} \\ &\text{subject to} && \text{Partial far field} = F_0 \end{aligned}$$

Use the MoM approximation of the energy to get

$$W = \max\{W^{(\mathrm{E})}, W^{(\mathrm{M})}\} = \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\}$$

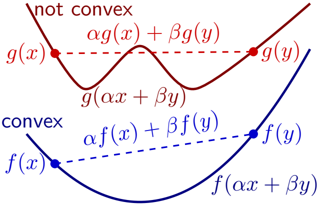
or $\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq W$ and $\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq W$ and the partial far field $F_0 = \mathbf{F}^{\mathsf{T}}\mathbf{I}$. Totally the (convex) optimization problem

$$\begin{aligned} &\text{minimize}_{\mathbf{I},W} && W \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq W \\ & && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq W \\ & && \mathbf{F}^{\mathsf{H}}\mathbf{I} = F_0 \end{aligned}$$



Convex optimization

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0,\ i = 1, ..., N_1 \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$



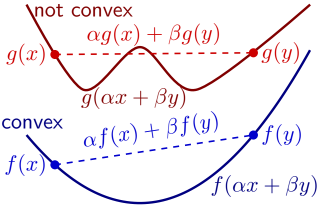
where $f_i(x)$ are convex, *i.e.*, $f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$ for $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

Properties

- Solved with efficient standard algorithms.
- No risk of getting trapped in a local minimum.
- A problem is 'solved' if formulated as a convex optimization problem.

Convex optimization

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \; i = 1, ..., N_1 \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$



where $f_i(x)$ are convex, *i.e.*, $f_i(\alpha \mathbf{x} + \beta \mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$ for $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

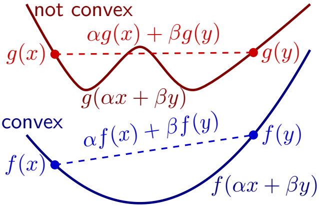
Smooth convex functions a single variable have a non-negative second derivative $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = f''(x) \geq 0$, *e.g.*,

$$f(x) = ax^2 + bx + c \quad \text{with } f''(x) = 2a$$

is convex if $a \geq 0$.
The affine function $f(x) = bx + c$ is convex (and concave).

Convex optimization

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \le 0,\ i = 1, \dots, N_1 \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$



where $f_i(x)$ are convex, *i.e.*, $f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \le \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$ for $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha, \beta \ge 0$.

Common convex functions used here:

linear forms: $f(\mathbf{x}) = \mathbf{b}\mathbf{x}$ for $1 \times N$ matrices $\mathbf{b}$.

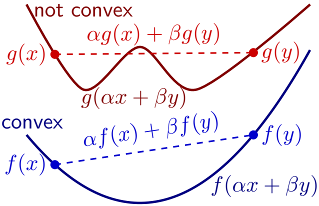
quadratic forms: $f(\mathbf{x}) = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x}$ for symmetric positive semidefinite (PSD) $N \times N$ matrices $\mathbf{A}$. (also $\mathbf{x}^{\mathsf{H}}\mathbf{A}\mathbf{x}$)

norms: $f(\mathbf{x}) = ||\mathbf{A}\mathbf{x}||$

max: $\max\{f_1(\mathbf{x}), f_2(\mathbf{x})\}$ of convex functions $f_1(\mathbf{x}), f_2(\mathbf{x})$

Convex optimization

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \; i = 1, ..., N_1 \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$



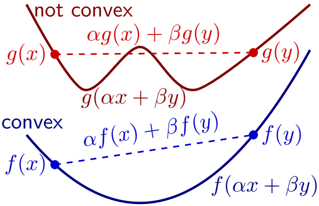
where $f_i(x)$ are convex, *i.e.*, $f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$ for $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

The G/Q optimization problem is convex

$$\begin{aligned} \text{minimize}_{\mathbf{I},W} \quad & W \\ \text{subject to} \quad & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq W \\ & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq W \\ & \mathbf{F}^{\mathsf{H}}\mathbf{I} = F_0 \end{aligned}$$

Convex optimization

$$
\begin{aligned}
\text{minimize} \quad & f_0(\mathbf{x}) \\
\text{subject to} \quad & f_i(\mathbf{x}) \leq 0,\ i = 1, ..., N_1 \\
& \mathbf{A}\mathbf{x} = \mathbf{b}
\end{aligned}
$$



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$$
\begin{aligned}
\text{minimize}_{\mathbf{I},W} \quad & W \\
\text{subject to} \quad & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq W \\
& \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq W \\
& \mathbf{F}^{\mathsf{H}}\mathbf{I} = F_0
\end{aligned}
$$

W is a linear form

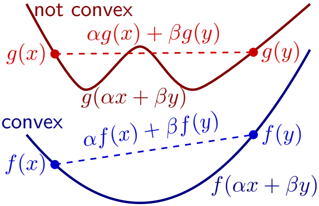
$\mathbf{X}_{\mathrm{e}}$ is positive semidefinite

$\mathbf{X}_{\mathrm{m}}$ is positive semidefinite

$\mathbf{F}^{\mathsf{H}}\mathbf{I}$ is a linear form

Convex optimization

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0,\ i = 1, \dots, N_1 \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$



where $f_i(x)$ are convex, *i.e.*, $f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$ for $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

Antenna performance expressed in the current density $\boldsymbol{J}$, *e.g.*,

- Radiated field $\boldsymbol{F}(\hat{\boldsymbol{k}}) = -\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} \times \int_V \boldsymbol{J}(\boldsymbol{r}) \mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{dV}$ is affine.
- Radiated power, stored electric and magnetic energies, and Ohmic losses are positive semi-definite quadratic forms in $\boldsymbol{J}$.

Convex optimization for antennas

The stored energy, radiated power, and radiated fields are simple matrix operators in the current densities.

Convex optimization offer many possibilities to analyze radiating structures. Quantities are:

Examples of quantities commonly found in electromagnetics that are linear, quadratic, norms, and logarithmic in the current density $\boldsymbol{J}$ are

linear forms near fields $\mathbf{N}_{\mathrm{e}}^{\mathsf{H}}\mathbf{I}$ and $\mathbf{N}_{\mathrm{m}}^{\mathsf{H}}\mathbf{I}$, far field $\mathbf{F}^{\mathsf{H}}\mathbf{I}$, and induced currents $\mathbf{C}^{\mathsf{H}}\mathbf{I}$.

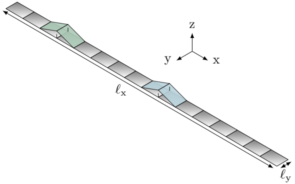
quadratic forms radiated power $\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}$, absorbed power, stored electric energy $\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}$, stored magnetic energy $\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}$, ohmic losses $\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\Omega}\mathbf{I}$.

norms field strengths $||\mathbf{N}^{\mathsf{H}}\mathbf{I}||_2$, far-field levels $||\mathbf{F}^{\mathsf{H}}\mathbf{I}||_2$

max stored energy for tuned antennas $W = \max\{W^{(\mathrm{E})}, W^{(\mathrm{M})}\}$

Currents for maximal G/Q on a strip dipole

- Strip dipole with length $\ell_{\rm x} = \ell$ and width $\ell_{\rm y} = \ell/100$.
- Maximize G/Q in the $\hat{\boldsymbol{z}}$ direction for the $\hat{\boldsymbol{x}}$ polarization.



- start with the coarse discretization $N_{\rm x} \times N_{\rm y} = 16 \times 1$ identical rectangular elements.
- The translational symmetry gives the Toeplitz matrizes

$$\mathbf{X}_{\rm e} = \mathrm{toeplitz}(\mathbf{X}_{\rm e1}) \quad \text{and } \mathbf{X}_{\rm m} = \mathrm{toeplitz}(\mathbf{X}_{\rm m1})$$

  where $\mathbf{X}_{\rm e1}$ denotes the first row of $\mathbf{X}_{\rm e}$.
- The far-field matrix $\mathbf{F}$ is an imaginary valued constant column matrix for this case.

MoM MATLAB data

```
eta0 = 299792458 * 4e-7*pi; % free space impedance
kl = 0.47 * 2*pi;           % wavenumber, lambda/2
Nx = 16;                    % number of elements
N = Nx-1;                   % number of unknowns
dx = 1/Nx;                  % rectangle length
dy = 0.02;                  % ractangle width
Xe1 = 0.1*[4.657 -1.832 -0.3783 -0.06258 -0.0239 ...
 -0.0121 -0.00734 -0.00503 -0.00379 -0.00305 ...
 -0.00258 -0.00225 -0.00199 -0.00178 -0.00159];
Xe = toeplitz(Xe1);         % E-energy
Xm1 = 1e-3*[7.14 3.413 1.148 0.6564 0.421 0.273 ...
 0.169 0.0897 0.028 -0.0205 -0.0587 -0.088 -0.11 ...
 -0.124 -0.133];
Xm = toeplitz(Xm1);         % M-energy
Rr1 = 1e-4*[2.72 2.711 2.683 2.638 2.57 2.5 2.4 ...
 2.29 2.17 2.04 1.9 1.75 1.6 1.45 1.29];
Rr = toeplitz(Rr1)+eye(N)*2e-6;
F = eta0*1i*kl/4/pi*ones(1,N)*dx*dy; % far field
```

Convex optimization in MATLAB using CVX

```
cvx_begin
   variable J(N) complex;      % current density
   variable W;                 % stored energy
   minimize W
   subject to
     quad_form(J,Xe) <= W;     % stored E energy
     quad_form(J,Xm) <= W;     % stored M energy
     F'*J == -\ju;          % far-field
cvx_end
```

of the G/Q problem

$$\begin{aligned} \text{minimize}_{\mathbf{I},W} \quad & W \\ \text{subject to} \quad & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq W \\ & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq W \\ & \mathbf{F}^{\mathsf{H}}\mathbf{I} = F_0 \end{aligned}$$

MATLAB results

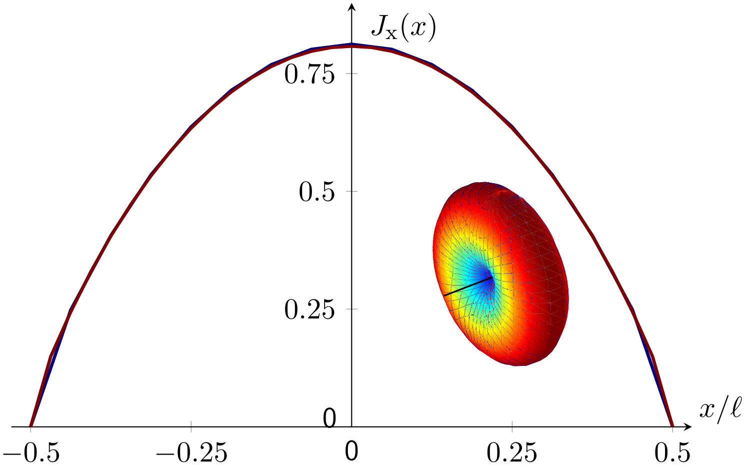
CVX solves the optimization problem and computes

```
J= [0.2483    0.4061    0.5352    0.6371    0.7146
    0.7691    0.8016    0.8123    0.8016    0.7691
    0.7146    0.6371    0.5352    0.4061    0.2483];
W=0.0555;
```

That we use to compute the Q-factors and directivity

```
We = real(J'*Xe*J)/2;    % stored E energy
Wm = real(J'*Xm*J)/2;    % stored M energy
Pr = real(J'*Rr*J)/2;    % radiated power
W = max(We,Wm);
Q = W/Pr;
Qe = We/Pr;              % E Q-factor
Qm = Wm/Pr;              % M Q-factor
P = abs(F*J)^2/2/eta0;   % radiation intensity
GoQ = 2*pi*abs(F*J)^2/W/eta0; % G/Q
D = 4*pi*P/Pr;           % Directivity
```

MATLAB results, $\boldsymbol{J}$ for $N_{\mathrm{x}} = 16$ and $N_{\mathrm{x}} = 32$.



The computed current is real valued and similar to the classical $\cos(x\pi/\ell)$ shape for this case. Parameters: $D \approx 1.64$, $Q_{\mathrm{e}} \approx 5.5$, $Q \approx 5.4$

CVX

Developed by M. C. Grant and S. Boyd.

download from http://cvxr.com/cvx/

See the CVX Users' Guide and Video introduction by S. Boyd.

- CVX is a modeling system for constructing and solving disciplined convex programs.
- CVX supports a number of standard problem types, including linear and quadratic programs (LPs/QPs), second-order cone programs (SOCPs), and semidefinite programs (SDPs).
- CVX can also solve much more complex convex optimization problems.
- CVX is implemented in Matlab. Model specifications are constructed using common Matlab operations and functions.
- two free SQLP solvers, SeDuMi and SDPT3. CVX also supports the commercial solvers Gurobi and MOSEK.

Currents for maximal G/Q

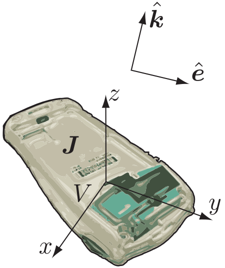
Determine a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V that maximizes the partial-gain Q-factor quotient $G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})/Q$.

- Partial radiation intensity $P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})$

$$\frac{G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})}{c_0 k \max\{W_\mathrm{e}, W_\mathrm{m}\}}.$$

- Scale $\boldsymbol{J}$ and reformulate $\max. P$ as $\max. \mathrm{Re}\{\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F}\}$.
- Convex optimization problem.

$$\begin{aligned} &\text{maximize} && \mathrm{Re}\{\mathbf{F}\mathbf{I}\} \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_\mathrm{e}\mathbf{I} \leq 1 \\ &&& \mathbf{I}^{\mathsf{H}}\mathbf{X}_\mathrm{m}\mathbf{I} \leq 1 \end{aligned}$$



Determines a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V with maximal partial radiation intensity and unit stored EM energy.

Maximal $G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{x}})/Q$ for planar rectangles

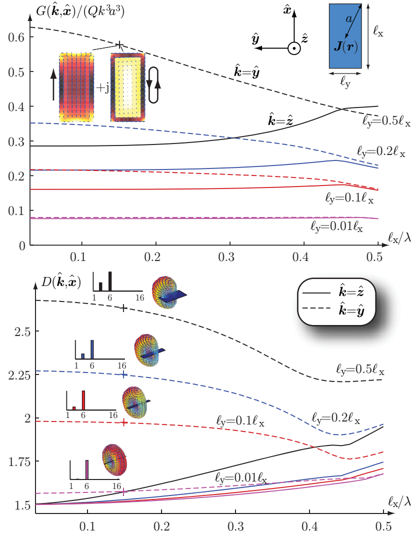
Solution of the convex optimization problem

$$\begin{aligned} \text{max.} \quad & \mathrm{Re}\{\mathbf{F}^{\mathsf{H}}\mathbf{I}\} \\ \text{s.t.} \quad & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq 1 \\ & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq 1 \end{aligned}$$

or similarly

$$\begin{aligned} \text{min.} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{s.t.} \quad & \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \end{aligned}$$

for current densities confined to planar rectangles with side lengths ℓ_{x} and $\ell_{\mathrm{y}} = \{0.01, 0.1, 0.2, 0.5\}\ell_{\mathrm{x}}$.



Note $\ell_{\mathrm{x}}/\lambda = k\ell_{\mathrm{x}}/(2\pi)$, giving $\ell_{\mathrm{x}} = \lambda/2 \rightarrow k\ell_{\mathrm{x}} = \pi \rightarrow ka \geq \pi/2$.

Maximal $G(\hat{\boldsymbol{k}},\hat{\boldsymbol{x}})/Q$ for planar rectangles

Solution of the convex optimization problem

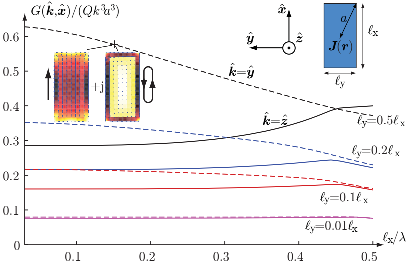
$$\begin{aligned} &\text{max.} && \mathrm{Re}\{\mathbf{F}^{\mathsf{H}}\mathbf{I}\} \\ &\text{s.t.} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq 1 \\ & && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq 1 \end{aligned}$$

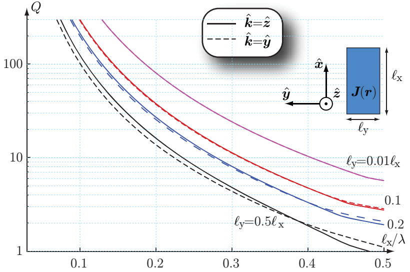
or similarly

$$\begin{aligned} &\text{min.} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{s.t.} && \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \end{aligned}$$

for current densities confined to planar rectangles with side lengths ℓ_{x} and $\ell_{\mathrm{y}} = \{0.01, 0.1, 0.2, 0.5\}\ell_{\mathrm{x}}$.

Note $\ell_{\mathrm{x}}/\lambda = k\ell_{\mathrm{x}}/(2\pi)$, giving $\ell_{\mathrm{x}} = \lambda/2 \to k\ell_{\mathrm{x}} = \pi \to ka \geq \pi/2$.





D/Q (or G/Q) bounds

- Similar to the forward scattering bounds (2007) for TM.
- Can design 'optimal' electric dipole mode (TM) antennas.
- TE modes and TE+TM are not well understood.
- Typical (but not optimal) matlab code using CVX

```
cvx_begin
   variable J(n) complex;          % current density
   dual variables We Wm
   maximize(real(F'*J))            % far-field
   subject to
      We: quad_form(J,Xe) <= 1;    % stored E energy
      Wm: quad_form(J,Xm) <= 1;    % stored M energy
cvx_end
```

We now reformulate the complex optimization problem to analyze superdirectivity, antennas with a prescribed radiation pattern, losses, and antennas embedded in a PEC structure.

D/Q (or G/Q) bounds

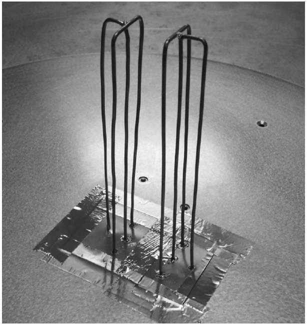
- Similar to the forward scattering bounds (2007) for TM.
- Can design 'optimal' electric dipole mode (TM) antennas.
- TE modes and TE+TM are not well understood.
- Better matlab code (`sqrtXe=sqrtm(Xe)`) using CVX

```
cvx_begin
   variable J(n) complex;          % current density
   dual variables We Wm
   maximize(real(F'*J))            % far-field
   subject to
      We: norm(sqrtXe*J) <= 1;     % stored E energy
      Wm: norm(sqrtXm*J) <= 1;     % stored M energy
cvx_end
```

We now reformulate the complex optimization problem to analyze superdirectivity, antennas with a prescribed radiation pattern, losses, and antennas embedded in a PEC structure.

Superdirectivity

- A superdirective antenna has a directivity that is much higher than for a typical reference antenna.
- Often low efficiency (low gain) and narrow bandwidth.
- There is an interest in small superdirective antennas, *e.g.*, Best *etal.* 2008 and Arceo & Balanis 2011,



Best, *etal.*, An Impedance-Matched 2-Element Superdirective Array, IEEE-TAP, 2008

Here, we add the constraint $D \geq D_0$ to the convex optimization problem for G/Q to determine the minimum Q for superdirective lossless antennas. We can also add constraints on the losses.

Superdirectivity: $\min. G/Q$ s.t. $D \geq D_0$

The directivity is given by $D = 4\pi P/P_{\mathrm{r}}$ that implies that the partial directivity is at least D_0 if

$$D_0 \leq D = \frac{4\pi|\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F}(\hat{\boldsymbol{k}})|^2}{2\eta_0 P_{\mathrm{r}}} \Rightarrow P_{\mathrm{r}} \leq \frac{2\pi|\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F}(\hat{\boldsymbol{k}})|^2}{\eta_0 D_0}$$

This is added as the convex constraint $\frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \leq 2\pi/(\eta_0 D_0)$.

$$\begin{aligned} \text{minimize}_{\mathbf{I}} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \mathbf{F}\mathbf{I} = -\mathrm{j} \\ & \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \leq \frac{4\pi}{\eta_0 D_0} \end{aligned}$$

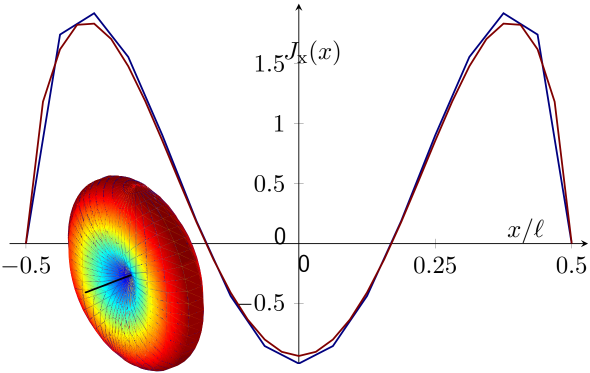
Superdirectivity: min. G/Q s.t. $D \geq D_0$

$$\begin{aligned} \text{minimize}_{\mathbf{I}} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \mathbf{F}^{\mathsf{H}}\mathbf{I} = -\mathrm{j} \\ & \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \leq \frac{4\pi}{\eta_0 D_0} \end{aligned}$$

with the CVX code

```
D0 = 2;                                 % directivity
cvx_begin
   variable J(N) complex;               % current density
   variable W;                          % stored energy
   minimize W
   subject to
     quad_form(J,Xe) <= W;              % stored E energy
     quad_form(J,Xm) <= W;              % stored M energy
     F'*J == -1i;                   % far-field
     quad_form(J,Rr) <= 4*pi/D0/eta0; % radiated power
cvx_end
```

Superdirectivity: min. G/Q s.t. $D \geq 2$



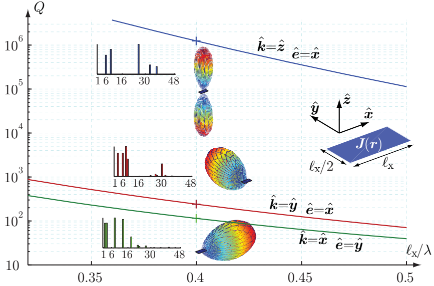
Computed current density with $N_x = 16$ (and $N_x = 32$) giving $D = 2$, $Q_e \approx 197$, and $Q_m \approx 17$.

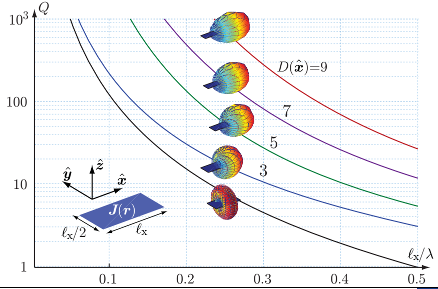
Superdirectivity: min. G/Q s.t. $D \geq D_0$

Add the constraint $P_{\mathrm{rad}} \leq 4\pi D_0^{-1}$ the get the convex optimization problem

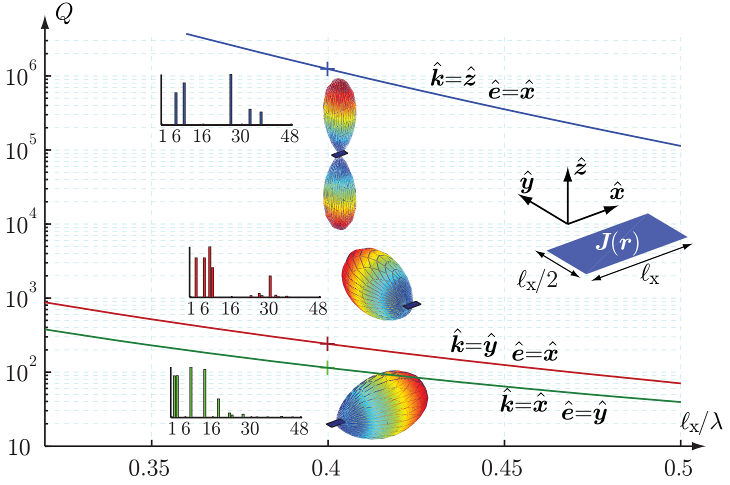
$$\begin{aligned} \text{min.} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{s.t.} \quad & \mathbf{F}^{\mathsf{H}}\mathbf{I}\} = 1 \\ & \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \leq k^3 D_0^{-1} \end{aligned}$$

Example for current densities confined to planar rectangles with side lengths ℓ_{x} and $\ell_{\mathrm{y}} = 0.5\ell_{\mathrm{x}}$.





Superdirectivity with $D \geq D_0 = 10$



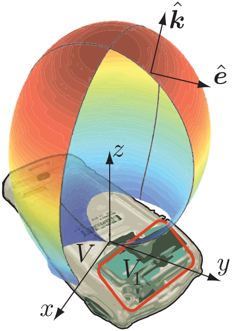
Note, it gives a bound on Q as D is known.

Currents for a desired radiated field

Determine a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V that radiates the field $\boldsymbol{F}_0(\hat{\boldsymbol{k}})$.

Many possible formulations. Deviation from the desired field $\boldsymbol{F}_0(\hat{\boldsymbol{k}})$:

$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \int_{\Omega} |\boldsymbol{F}(\hat{\boldsymbol{k}}) - \boldsymbol{F}_0(\hat{\boldsymbol{k}})|^2 \, \mathrm{d}\Omega_{\hat{\boldsymbol{k}}} < (4\pi\delta)^2 \end{aligned}$$



Determines a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V with unit stored EM energy that radiates the field $\boldsymbol{F}(\hat{\boldsymbol{k}})$ with an 'error' level δ.

Currents for a desired radiated field

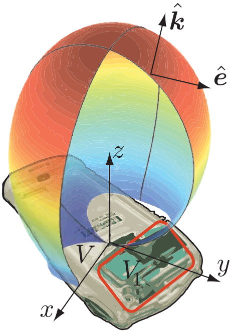
Determine a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V that radiates the field $\boldsymbol{F}_0(\hat{\boldsymbol{k}})$.

Alternative formulation: Maximization of the projected field on the desired field $\boldsymbol{F}_0(\hat{\boldsymbol{k}})$:

$$\begin{aligned} &\text{maximize} && \mathrm{Re}\{\mathbf{I}_0^{\mathsf{H}}\mathbf{V}\mathbf{I}\} \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq 1 \\ &&& \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq 1 \end{aligned}$$

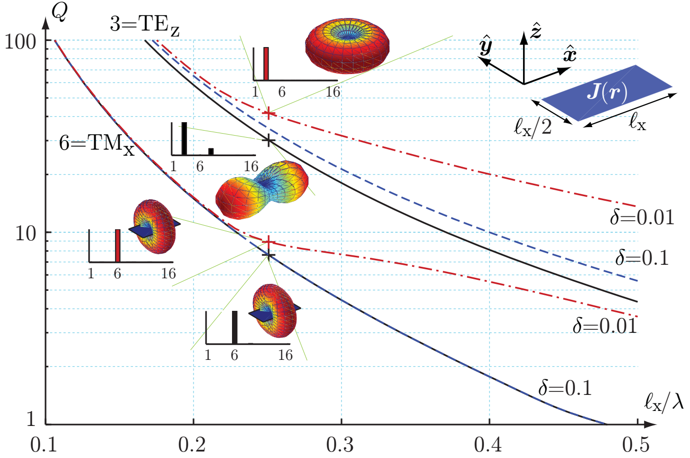
where

$$\boldsymbol{J}_0(\boldsymbol{r}) = \int_{\Omega} \boldsymbol{F}_0(\hat{\boldsymbol{k}}) \mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}} \, \mathrm{d}\Omega_{\hat{\boldsymbol{k}}}.$$



Determines a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V unit stored EM energy that maximizes the projection $\mathrm{Re}\int_{\Omega} \boldsymbol{F}_0^*(\hat{\boldsymbol{k}}) \cdot \boldsymbol{F}(\hat{\boldsymbol{k}}) \, \mathrm{d}\Omega$.

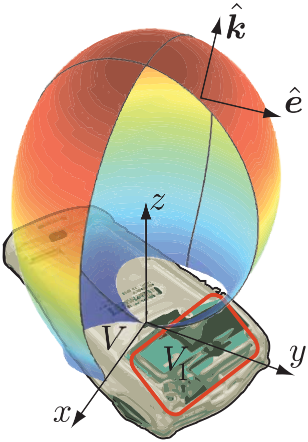
Optimal performance for a given radiated field



It is good to have approximate but not exact dipole fields.

Optimal performance for embedded antennas

- Common with antennas embedded in metallic structures.
- The induced currents radiate but they are not arbitrary.
- Linear map from the antenna region adds a (convex) constraint.
- Here, we assume that the surrounding structure is PEC and add a constraint to account for the induced currents on the surrounding structure in the G/Q formulation.



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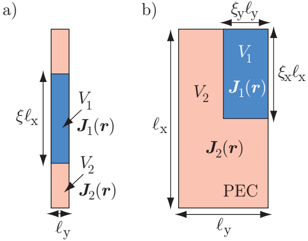


Currents for maximal G/Q for embedded antennas

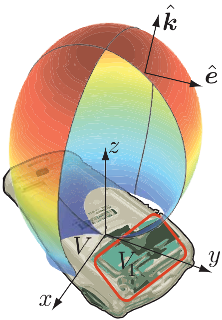
Determine an optimal current density $\boldsymbol{J}_1(\boldsymbol{r})$ in the volume V_1. Assume that V is PEC outside V_1.

Can minimize the stored energy for given radiated field

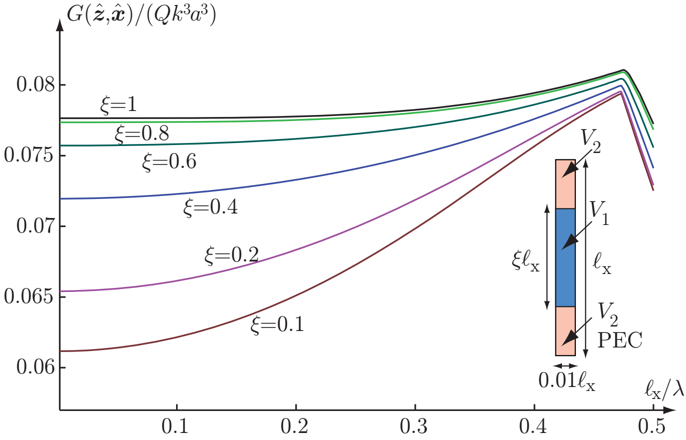
$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \mathrm{Re}\{\mathbf{F}\mathbf{I}\} = 1 \\ &&& \mathbf{I}_2 = \mathbf{C}\mathbf{I}_1 \end{aligned}$$

or maximize the radiated field for given stored energy

$$\begin{aligned} &\text{maximize} && \mathrm{Re}\{\mathbf{F}\mathbf{I}\} \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \le 1 \\ &&& \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \le 1 \\ &&& \mathbf{I}_2 = \mathbf{C}\mathbf{I}_1 \end{aligned}$$

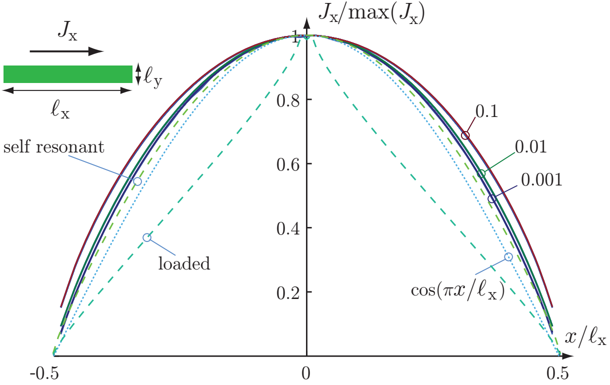


Center fed strip dipole



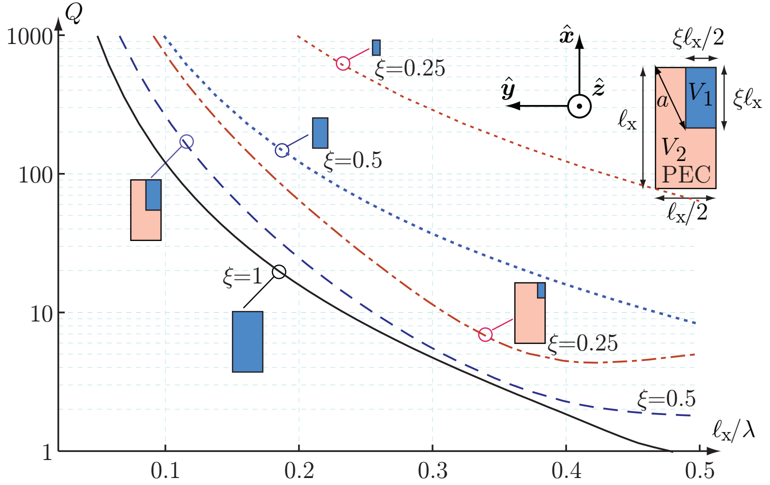
Almost independent of the feed width at the resonance just below $\ell_x = 0.5\lambda$.

Center fed strip dipole

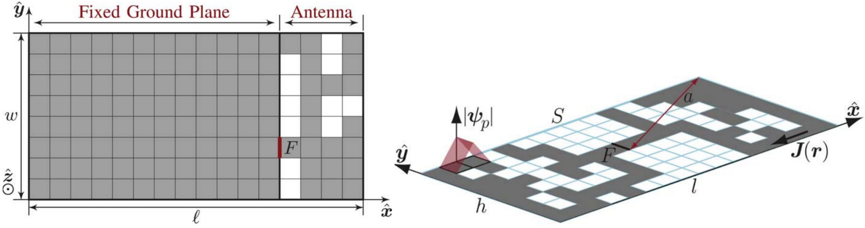


Almost independent of the feed width at the resonance just below $\ell_x = 0.5\lambda$.

Embedded antennas in planar PEC rectangles



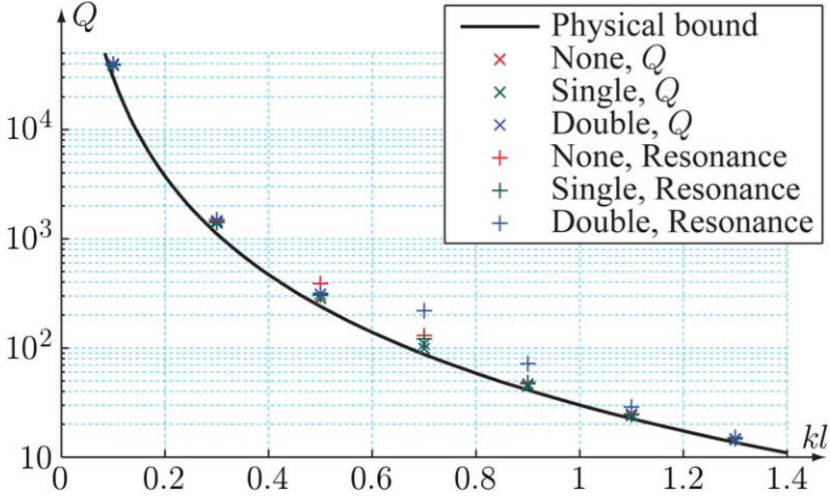
Antenna optimization



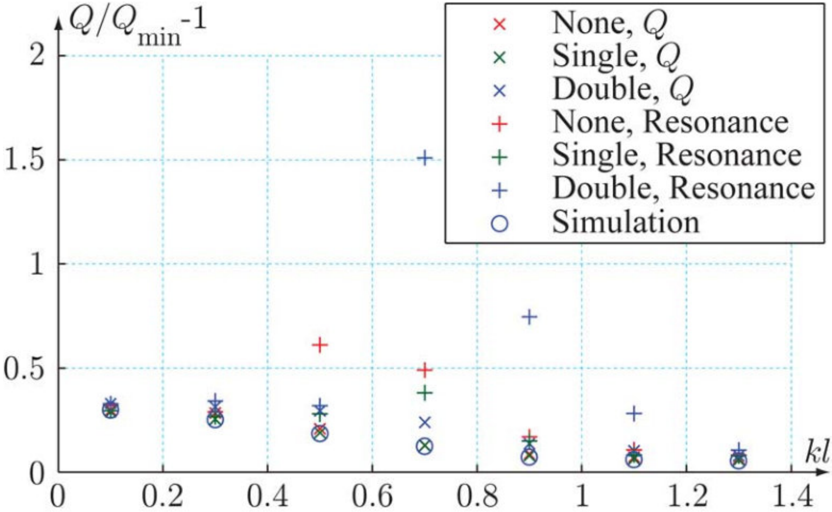
Have pre-computed matrices for the stored energy, radiated power, far-field, ...

- model the antenna as fragmented rectangular patches (many other possibilities).
- removing a patch corresponds to elimination of rows and columns from the matrices.
- use a genetic algorithm (or any other suitable optimization algorithm) to maximize G/Q or minimize Q.

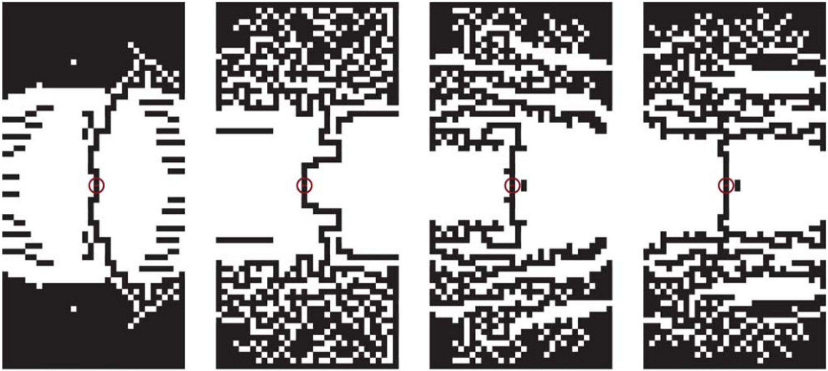
Planar rectangle



Planar rectangle

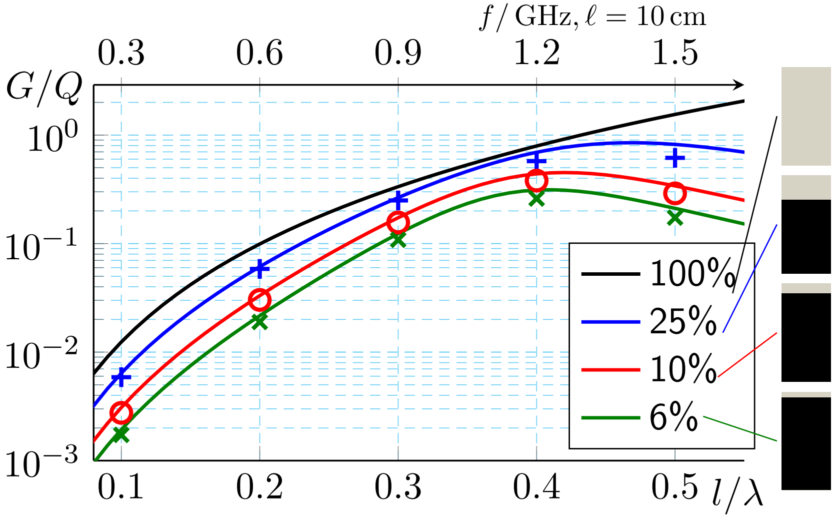


Planar rectangle



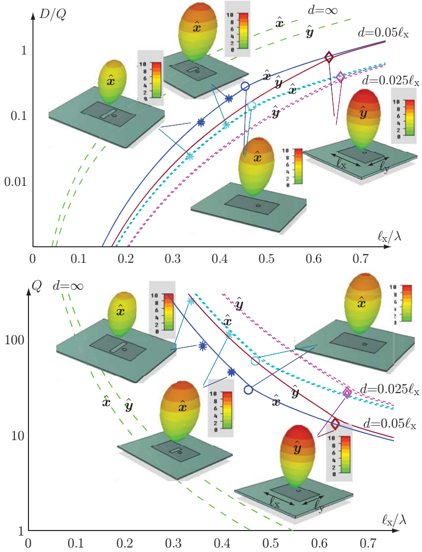
kl=0.1 kl=0.5 kl=0.9 kl=1.3

Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region



Antennas above ground planes

- Common with antennas above ground planes.
- Add mirror currents for the stored energy and radiated field.
- Preliminary results for rectangular structures at height d above the ground plane.
- Comparison with patch and slot loaded patches.



Why convex optimization?

Problems can often be considered as solved if formulated as convex optimization problems.

Consider the G/Q problem

$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \mathrm{Re}\{\mathbf{F}^{\mathsf{H}}\mathbf{I}\} = 1 \end{aligned}$$

There are many (optimization) algorithms that can be used to solve this problem.

- Can *e.g.*, use any of the solvers included in CVX.
  - Very simple to use.
  - Good for small problems but less efficient for larger problems.
- A dedicated solver for quadratic programs.
  - More efficient for larger problems.
- Random search algorithms, eg genetic algorithms (GA), particle swarms,....
  - Very inefficient. Note you do not (should not) use (GA, ...) to solve *e.g.*, $\mathbf{Ax} = \mathbf{b}$ $(\mathrm{min.}||\mathbf{Ax} - \mathbf{b}||)$.
- Here, we use a dual formulation
  - Computational efficient for large problems.
  - MATLAB implementation using fminbnd.
  - Illustrates the properties of dual problems and the posteriori error estimates.

Relaxation and dual problem

An illustrative method is to use the inequality

$$W = \max\{W^{(\mathrm{E})}, W^{(\mathrm{M})}\} \geq \alpha W^{(\mathrm{E})} + (1-\alpha) W^{(\mathrm{M})} = W_\alpha \quad \text{for } 0 \leq \alpha \leq 1$$

or with the matrices $\mathbf{X}_{\mathrm{e}}, \mathbf{X}_{\mathrm{m}}$

$$W = \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \geq W_\alpha = \mathbf{I}_\alpha^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}_\alpha$$

or for the quotient G/Q

$$\frac{G}{Q} = \frac{2\pi P}{\omega \max\{W_{\mathrm{e}\alpha}, W_{\mathrm{m}\alpha}\}} \leq \frac{2\pi P}{\omega W_\alpha} = \frac{2\pi P}{\omega(\alpha W_{\mathrm{e}\alpha} + (1-\alpha W_{\mathrm{m}\alpha}))} = \frac{G_\alpha}{Q_\alpha}$$

Note $P = 1$ is fixed in the optimization problem.

Relaxation and dual problem

The inequality relaxes the G/Q optimization problem

$$\begin{array}{ll} \text{minimize} & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \geq \mathbf{I}_{\alpha}^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}_{\alpha} \\ \text{subject to} & \mathrm{Re}\{\mathbf{F}^{\mathsf{H}}\mathbf{I}\} = 1 \end{array}$$

into

$$\begin{array}{ll} \text{maximize}_{\alpha}\text{minimize}_{\mathbf{I}_{\alpha}} & W_{\alpha} = \mathbf{I}_{\alpha}^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}_{\alpha} \\ \text{subject to} & \mathrm{Im}\{\mathbf{F}^{\mathsf{H}}\mathbf{I}_{\alpha}\} = 1 \\ & 0 \leq \alpha \leq 1 \end{array}$$

where for the quotient G/Q (note $P = 1$)

$$\frac{G}{Q} = \frac{2\pi P}{\omega \max\{W_{\mathrm{e}\alpha}, W_{\mathrm{m}\alpha}\}} \leq \frac{2\pi P}{\omega W_{\alpha}} = \frac{2\pi P}{\omega(\alpha W_{\mathrm{e}\alpha} + (1-\alpha W_{\mathrm{m}\alpha}))} = \frac{G_{\alpha}}{Q_{\alpha}}$$

Relaxation and dual problem

The dual problem

$$\begin{aligned} &\text{maximize}_{\alpha}\text{minimize}_{\mathbf{I}_\alpha} && W_\alpha = \mathbf{I}_\alpha^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}_\alpha \\ &\text{subject to} && \mathrm{Im}\{\mathbf{F}^{\mathsf{H}}\mathbf{I}_\alpha\} = 1 \\ &&& 0 \leq \alpha \leq 1 \end{aligned}$$

is solved as a linear system (MoM equation) for fixed α with

$$\mathbf{I}_\alpha = \frac{\big(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}\big)^{-1}\mathbf{F}}{\mathbf{F}^{\mathsf{H}}\big(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}\big)^{-1}\mathbf{F}}$$

giving the optimization problem

$$\underset{0\leq\alpha\leq1}{\text{maximize}}\, W_\alpha$$

or

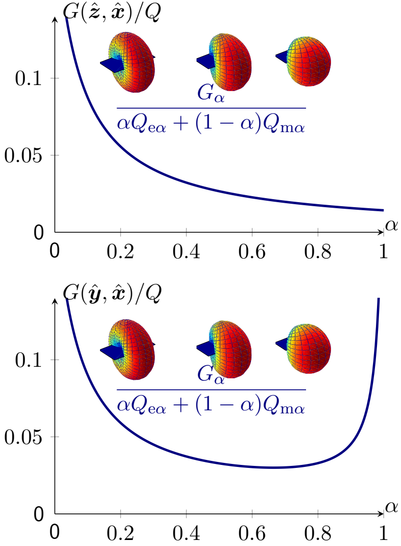
$$\underset{0\leq\alpha\leq1}{\text{minimize}}\,\frac{G_\alpha}{Q_\alpha} = \mathbf{F}^{\mathsf{H}}\big(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}\big)^{-1}\mathbf{F}$$

Why convex optimization: illustration

The upper bound on $G/Q|_{\mathrm{ub}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (blue) curve

$$\left.\frac{G}{Q}\right|_{\mathrm{ub}} \leq \frac{G_\alpha}{\alpha Q_{\mathrm{e}\alpha} + (1-\alpha) Q_{\mathrm{m}\alpha}}$$

This is efficiently solved by golden section search and parabolic interpolation.

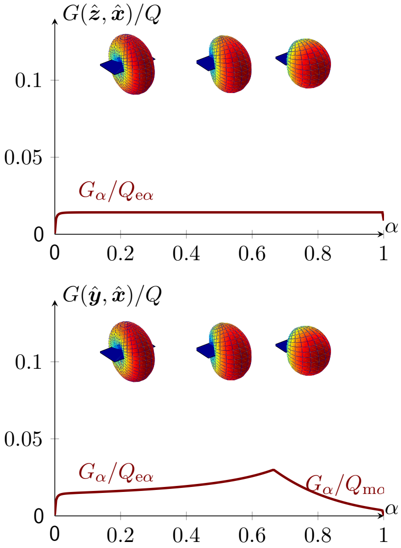


$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

Why convex optimization: illustration

We also compute the actual G/Q for the current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{\mathrm{e}\alpha}, Q_{\mathrm{m}\alpha}\}} \leq \left.\frac{G}{Q}\right|_{\mathrm{ub}}$$



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We also compute the actual G/Q for the current $\mathbf{I}_\alpha$ to get

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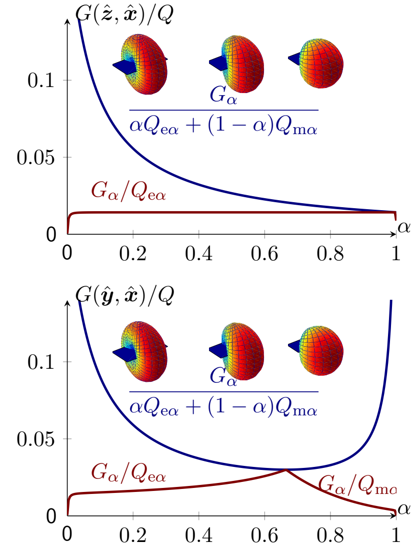
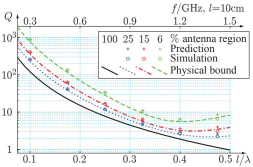


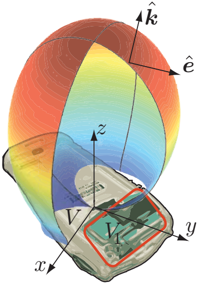
$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

Outline

Summary

- Physical bounds from spheres (Chu 1948) and arbitrary shapes (Gustafsson *etal* 2007) to embedded antennas...
- Stored energy in the current density.
- Optimization of the antenna structure (global optimization) and the antenna currents (convex optimization).
- Convex optimization for bounds and optimal currents: G/Q, superdirective, embedded, ...
- Closed form solutions for small antennas.
- Non-Foster to overcome $B \sim 1/Q$ ...

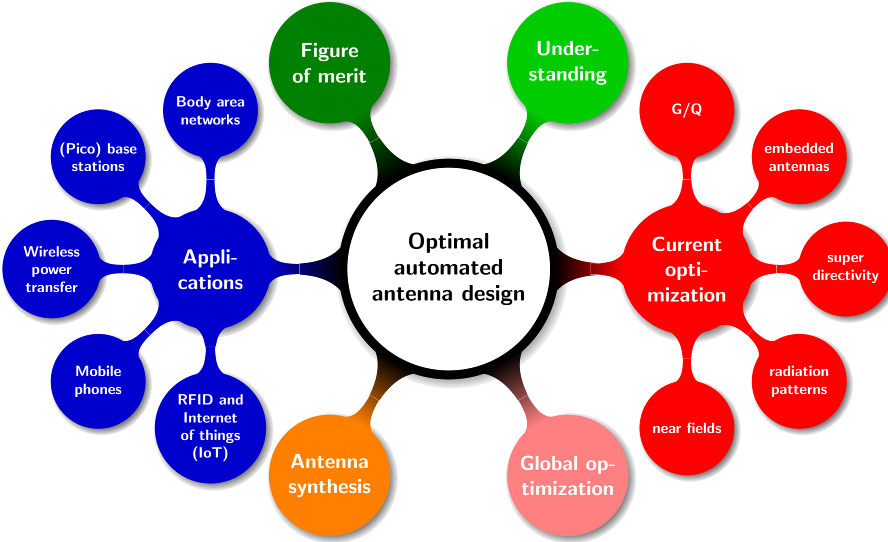




Initial results for efficiency, more realistic geometries (phones), SAR, MIMO. Investigating dielectrics, volume currents, magnetic currents, ...

Gustafsson and Nordebo, *Optimal antenna currents for Q, superdirectivity, and radiation patterns using convex optimization*, IEEE-TAP, 61(3), 1109-1118, 2013

Optimal automated antenna design



References

Current optimization and physical bounds

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Polarizability dyadic and induced dipole moment

The induced dipole moment can be written

$$\boldsymbol{p} = \epsilon_0 \boldsymbol{\gamma}_{\mathrm{e}} \cdot \boldsymbol{E}$$

where $\boldsymbol{\gamma}_{\mathrm{e}}$ is the polarizability dyadic.

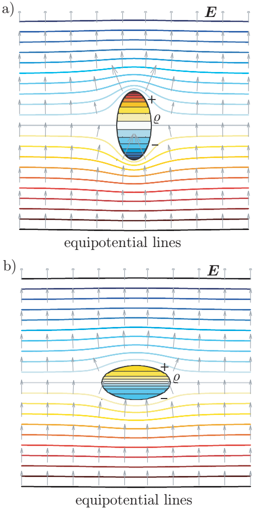
Example (Dielectric sphere)

A dielectric sphere with radius a and relative permittivity ϵ_{r} has the polarizability dyadic

$$\boldsymbol{\gamma}_{\mathrm{e}} = 4\pi a^3 \frac{\epsilon_{\mathrm{r}} - 1}{\epsilon_{\mathrm{r}} + 2} \mathbf{I} \to \boldsymbol{\gamma}_\infty = 4\pi a^3 \mathbf{I}$$

as $\epsilon_{\mathrm{r}} \to \infty$.

Analytic expressions for spheroids, elliptic discs, half spheres, hollow half spheres, touching spheres,...



High-contrast polarizability dyadics: γ_∞

γ_∞ is determined from the induced normalized surface charge density, ρ, as

$$\hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_\infty \cdot \hat{\boldsymbol{e}} = \frac{1}{E_0} \int_{\partial V} \hat{\boldsymbol{e}} \cdot \boldsymbol{r} \rho(\boldsymbol{r}) \, \mathrm{dS}$$

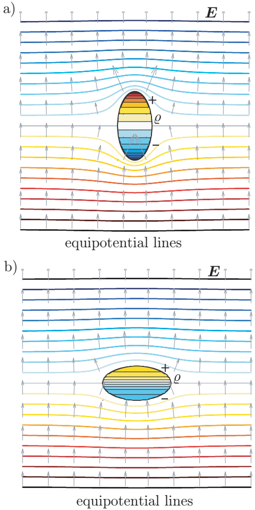
where ρ satisfies the integral equation

$$\int_{\partial V} \frac{\rho(\boldsymbol{r}')}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{dS}' = E_0 \boldsymbol{r} \cdot \hat{\boldsymbol{e}} - V_n$$

with the constraints of zero total charge

$$\int_{\partial V_n} \rho(\boldsymbol{r}) \, \mathrm{dS} = 0$$

Can also use FEM (Laplace equation).



Removal of metal from a square plate and circular disk