An overview of current optimization and physical bounds on antennas

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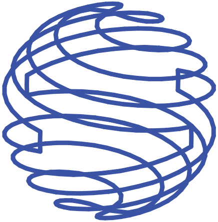
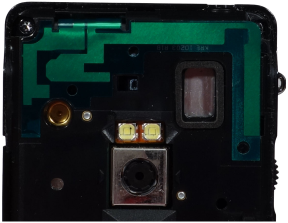
PIERS, Guangzhou, China, August 25, 2014

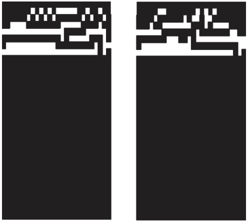
Design of small antennas



Folded spherical helix



SonyEricsson P1i

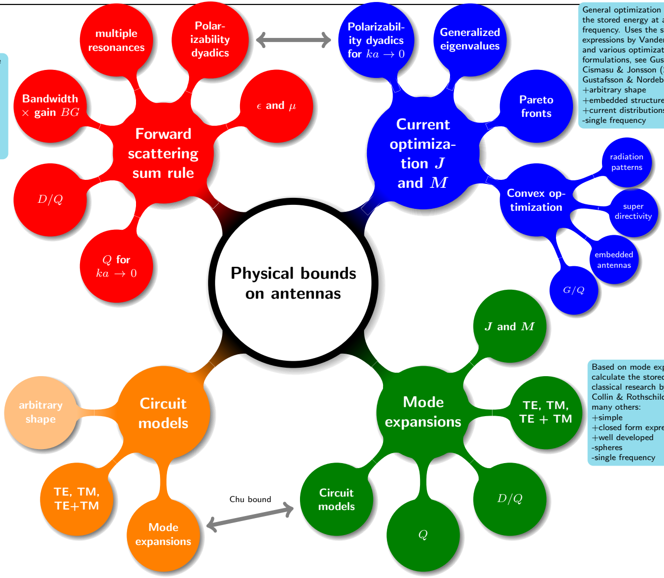


Fragmented patches

- There are many advanced methods to design small antennas.
- Often antennas embedded in structures.
- Performance in Q, bandwidth and efficiency.
- How does the performance depend on the design volume?
- What can we learn from performance bounds and optimal currents?
- Can we automate the design of optimal antennas?

Physical bounds on antennas: methods

Uses the forward scattering sum rule to analyze receiving antennas, see Gustafsson, Sohl & Kristensson (2007,2009):
+arbitrary shape and size
+bandwidth
+closed form expressions
+based on an identity
-entire volumes
-absorption efficiency



General optimization formulation for the stored energy at a fixed frequency. Uses the stored energy expressions by Vandenbosch (2010) and various optimization formulations, see Gustafsson, Cismasu & Jonsson (2012) and Gustafsson & Nordebo (2013):
+arbitrary shape
+embedded structures
+current distributions
-single frequency

Uses circuit models for the antennas. Used originally by Wheeler (1947), Chu (1948) and later Thal (2006,2013)
+simple models
+physical intuition
$\pm$ combined with mode expansions
+combined with matching
-approximate
-....

Based on mode expansions to calculate the stored energy, see the classical research by Chu (1948), Collin & Rothschild (1964), and many others:
+simple
+closed form expressions
+well developed
-spheres
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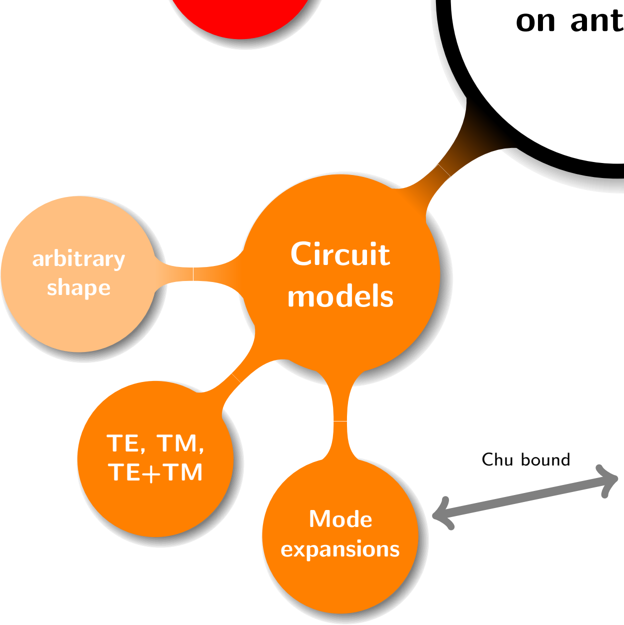

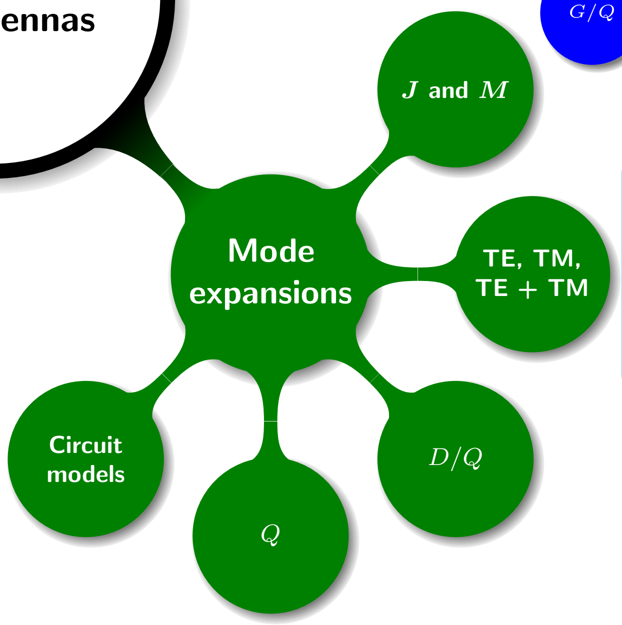

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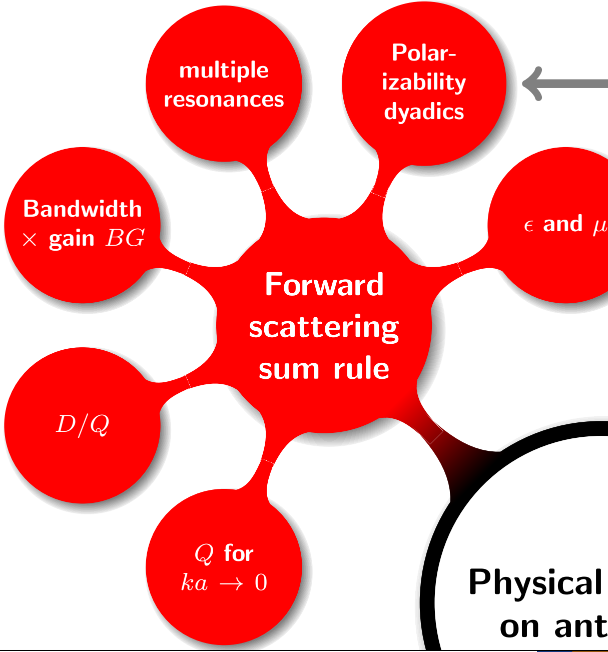

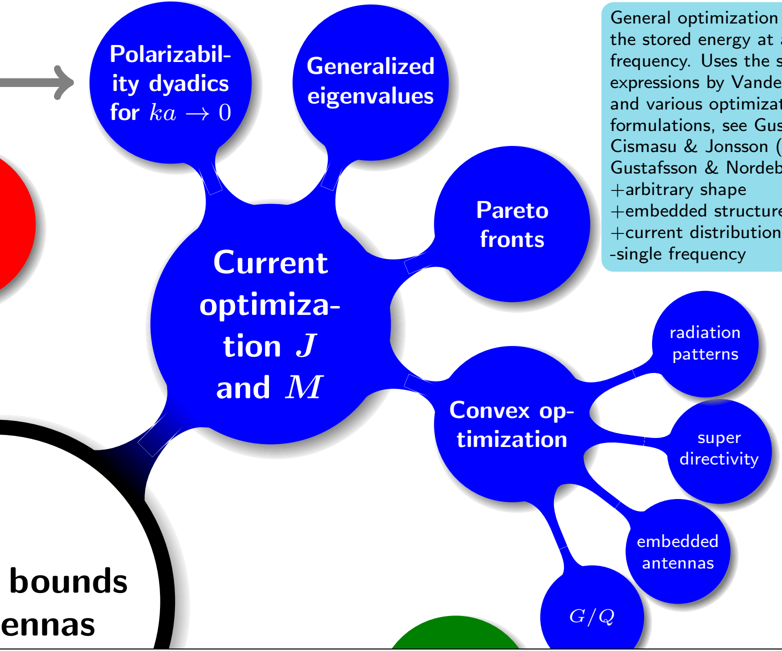

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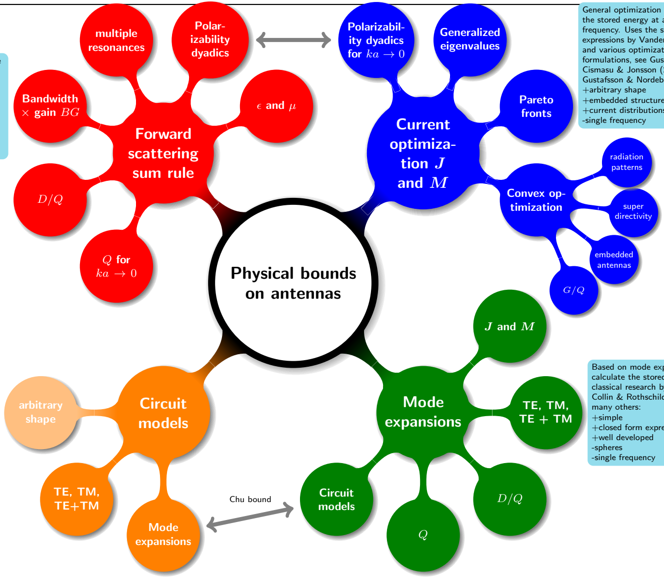

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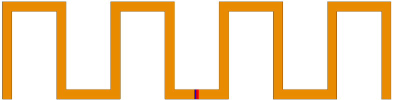
Antenna and current optimization

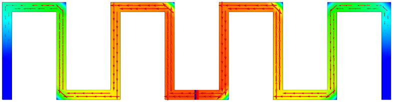
Antenna design: produce the desired current distribution on the structure by shaping and choosing the materials.

- Have a given maximal size of the antenna structure.
- Antenna optimization: determine the shape and material properties for optimal performance.
- Current optimization: determine an optimal current distribution from all possible currents in the available geometry.

Maximal size of the antenna.

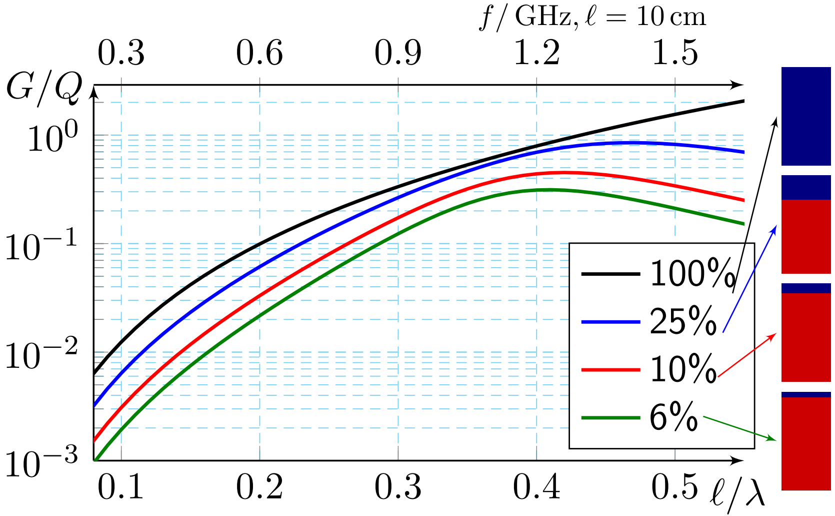


Antenna geometry with feed point.



Current distribution on the antenna.

Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region



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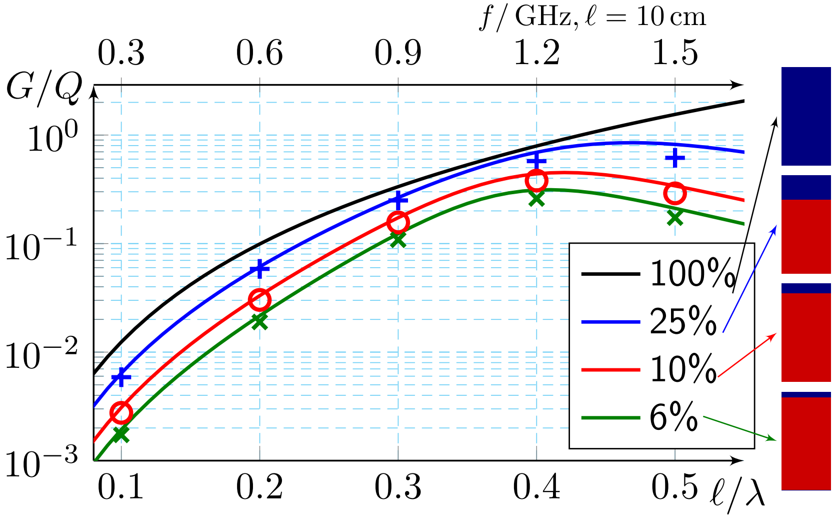
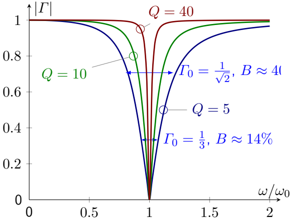


Q-factor and single frequency evaluation



Reflection coefficient $|\Gamma|$ for a RCL circuit with Q-factors $Q=\{5,10,40\}$. Fractional bandwidths for $\Gamma_0=\{1/\sqrt{2},1/3\}$.

The Q-factor is defined as the ratio between the stored electric, W_e, and magnetic, W_m, energies and the dissipated power, *i.e.*,

$$Q=\frac{2\omega\max\{W_\mathrm{e},W_\mathrm{m}\}}{P_\mathrm{rad}+P_\mathrm{loss}}.$$

Fractional bandwidth for single resonances

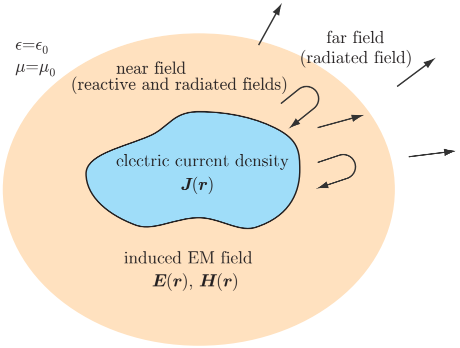
$$B\approx\frac{2}{Q}\frac{\Gamma_0}{\sqrt{1-\Gamma_0^2}}$$

Single frequency evaluation

use the Q-factor to estimate the bandwidth. Need to:

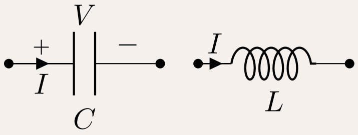
1. Compute the *stored energy*.
2. Solve the optimization problems.

What is (stored) EM energy?



- Time average energy density $\epsilon_0|\boldsymbol{E}|^2/4$ and $\mu_0|\boldsymbol{H}|^2/4$.
- What is stored and radiated?
- How can we express the (stored) energy in the current (density)?
- First, currents in free space.

Lumped elements

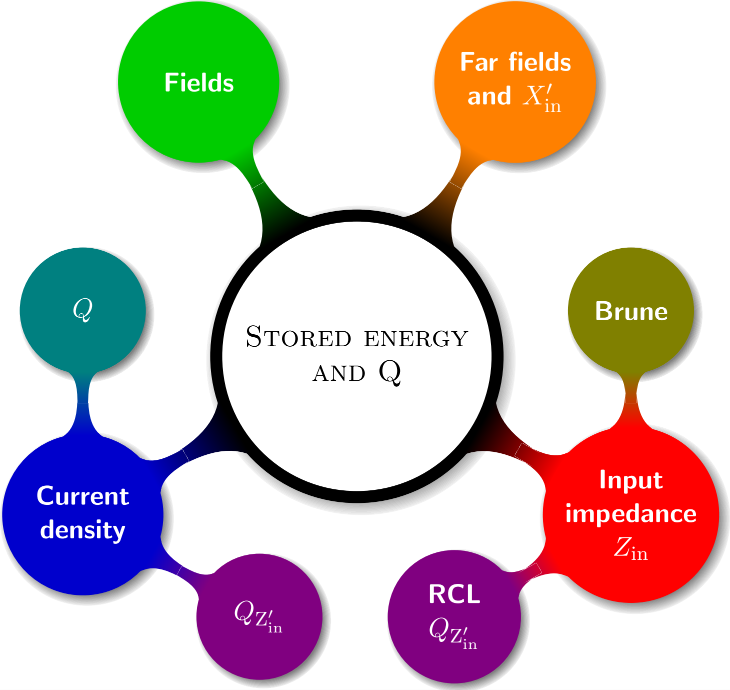


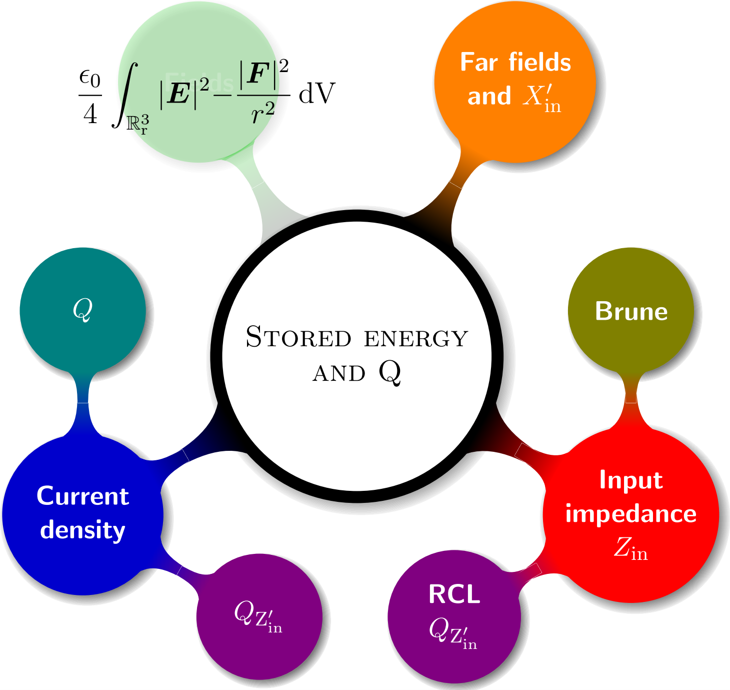
Time average stored energy in capacitors

$$W_{\mathrm{e}} = \frac{C|V|^2}{4} = \frac{|I|^2}{4\omega^2 C}$$

and in inductors

$$W_{\mathrm{m}} = \frac{L|I|^2}{4}$$


Fields
Far fields and X'_{in}
Stored energy and Q
Q
Brune
Current density
Input impedance Z_{in}
$Q_{Z'_{\mathrm{in}}}$
RCL $Q_{Z'_{\mathrm{in}}}$



$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_r}|\boldsymbol{E}|^2-\frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
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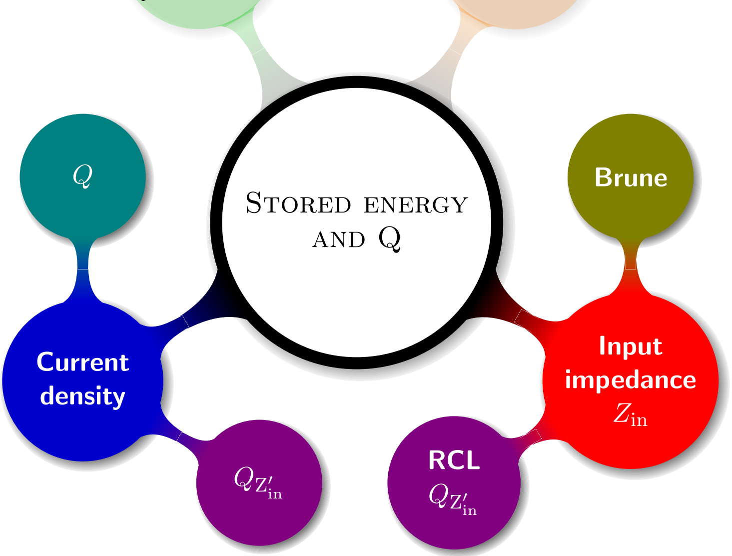
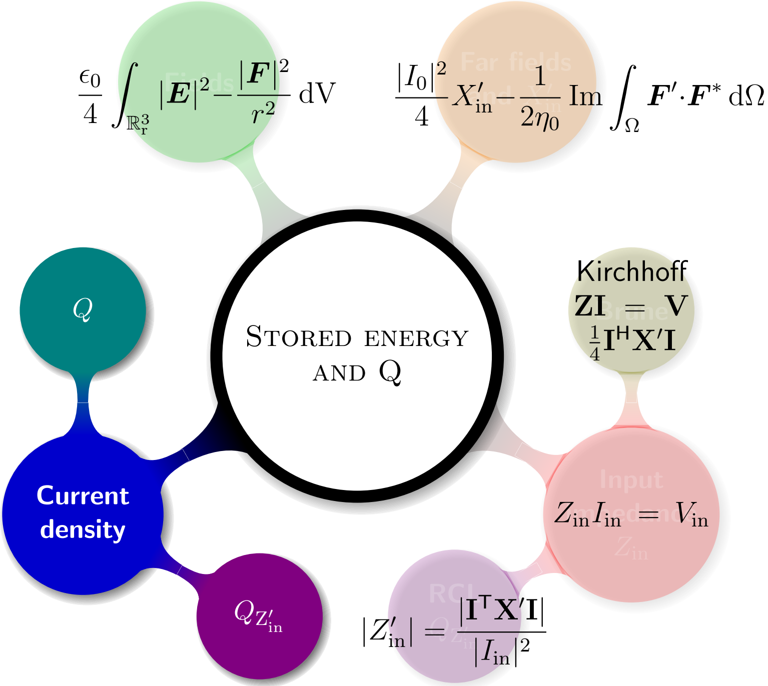
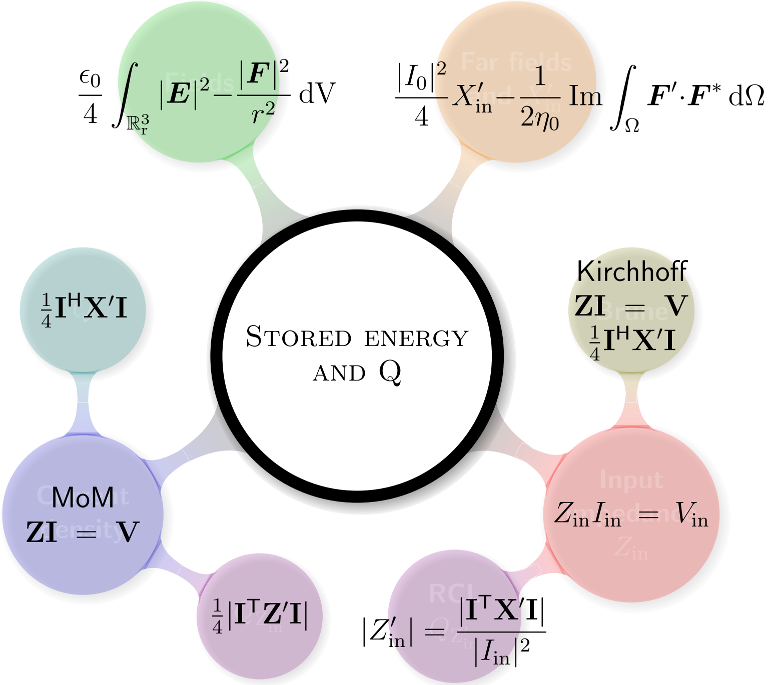
$$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV} \qquad \frac{|I_0|^2}{4}X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega}\boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$$





Stored energy and Q
$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_r} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
$\frac{|I_0|^2}{4}X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega}\boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
Kirchhoff
$\mathbf{ZI}=\mathbf{V}$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
$Z_{\mathrm{in}}I_{\mathrm{in}} = V_{\mathrm{in}}$
$|Z'_{\mathrm{in}}| = \frac{|\mathbf{I}^{\mathsf{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$
$Q_{Z'_{\mathrm{in}}}$
Current density
Q



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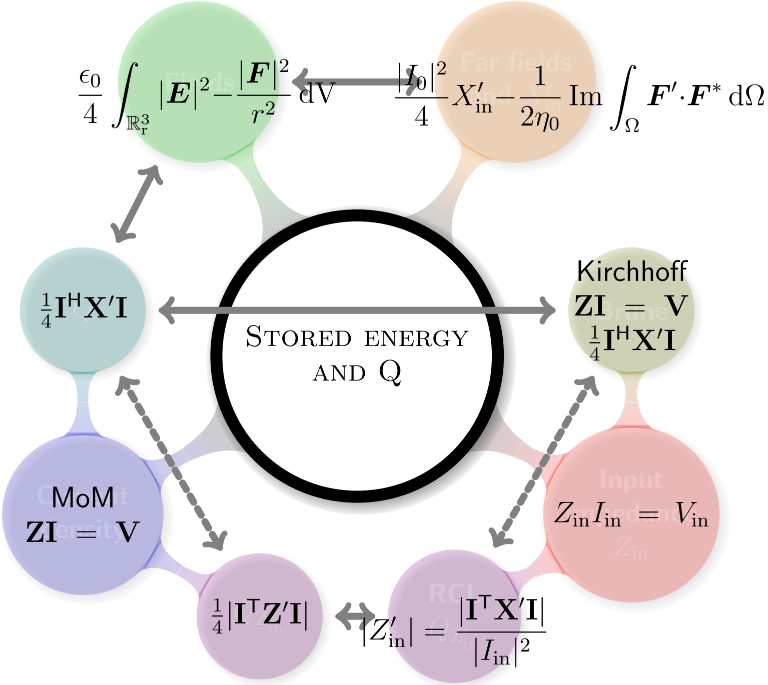

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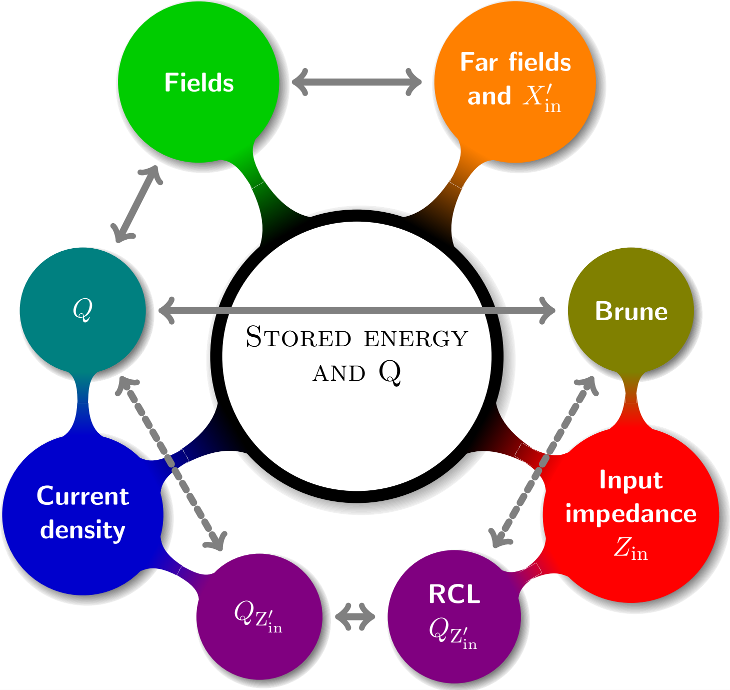
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Far fields and X'_{in}
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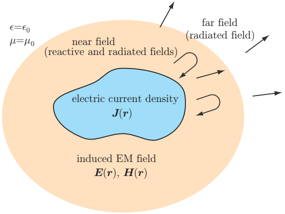
Subtracted far field approach

$$W_{\mathrm{F}}^{(\mathrm{E})} = \frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}(\boldsymbol{r})|^2 - \frac{|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2}{r^2}\,\mathrm{dV}$$



Have shown that $W_{\mathrm{F}}^{(\mathrm{E})} = W_{\mathrm{C}}^{(\mathrm{E})} + W_{\mathrm{c},\mathbf{0}}$:

$$W_{\mathrm{C}}^{(\mathrm{E})} = \frac{\eta_0}{4\omega}\int_V\int_V \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\frac{\cos(kr_{12})}{4\pi kr_{12}} - \left(k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^* - \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\right)\frac{\sin(kr_{12})}{8\pi}$$

with $\boldsymbol{J}_n = \boldsymbol{J}(\boldsymbol{r}_n)$, $n = 1, 2$ and a **coordinate dependent** part

$$W_{\mathrm{c},\mathbf{0}} = \frac{\eta_0}{4\omega}\int_V\int_V \mathrm{Im}\left\{k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^* - \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\right\}\frac{r_1^2 - r_2^2}{8\pi r_{12}}k_1(kr_{12})$$

where $\,_1(z) = (\sin(z) - z\cos(z))/z^2$ is a spherical Bessel function.

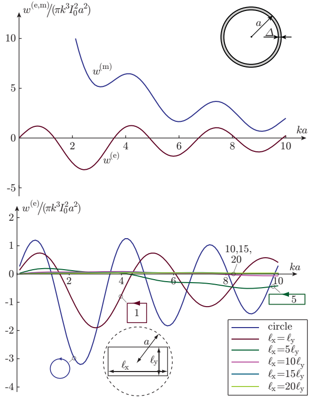
Gustafsson & Jonsson: Stored electromagnetic energy and antenna Q, 2012

Subtracted far field: comments

- Coordinate dependent for far-fields $\boldsymbol{F}$ with

$$W_{\mathrm{c},0} - W_{\mathrm{c},} = \frac{\epsilon_0}{4}\boldsymbol{d}\cdot\int_{\Omega}\hat{\boldsymbol{r}}|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2\,\mathrm{d}\Omega \neq 0$$

- Identical coordinate independent part as for the stored energy introduced by Vandenbosch 2010 (Geyi 2003 $ka \ll 1$).
- Can produce negative values for lager structures.
- Difficult to generalize to antennas embedded in lossy media (no far field).



We now introduce an alternative approach to analyze antennas in lossy (dispersive) media.

Yaghjian & Best IEEE-TAP 2005, Gustafsson & Jonsson 2012.

Frequency derivatives of impedance/admittance matrices

Impedance and admittance matrices relate voltages and currents

$$\mathbf{Z}\mathbf{I} = \mathbf{V} \quad \text{or } \mathbf{I} = \mathbf{Z}^{-1}\mathbf{V} = \mathbf{Y}\mathbf{V}$$

The (angular) frequency derivative of the admittance matrix is

$$\mathbf{Y}' = \frac{\partial \mathbf{Y}}{\partial \omega} = \frac{\partial \mathbf{Z}^{-1}}{\partial \omega} = -\mathbf{Z}^{-1}\mathbf{Z}'\mathbf{Z}^{-1} = -\mathbf{Y}\mathbf{Z}'\mathbf{Y}$$

No complex conjugate. Better to use quadratic forms with the transpose $\mathbf{V}^{\mathsf{T}}\mathbf{Y}'\mathbf{V}$ than Hermitian transpose $\mathbf{V}^{\mathsf{H}}\mathbf{Y}'\mathbf{V} = \mathbf{V}^{\mathsf{T}*}\mathbf{Y}'\mathbf{V}$.

For the case of a (frequency independent (MoM)) voltage source

$$Y_{\text{in}} = \frac{1}{Z_{\text{in}}} = \frac{\mathbf{V}^{\mathsf{T}}\mathbf{Y}\mathbf{V}}{V_{\text{in}}^2} \quad \text{and } V_{\text{in}}^2 Y_{\text{in}}' = \mathbf{V}^{\mathsf{T}}\mathbf{Y}'\mathbf{V} = -\mathbf{I}^{\mathsf{T}}\mathbf{Z}'\mathbf{I}$$

Z_{in} for antennas using MoM

Use a method of moments (MoM) formulation of the electric field integral equation (EFIE). Impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$

$$\frac{Z_{mn}}{\eta} = \mathrm{j} \int_S \int_S \left(k^2 \boldsymbol{\psi}_{m1} \cdot \boldsymbol{\psi}_{n2} - \nabla_1 \cdot \boldsymbol{\psi}_{m1} \nabla_2 \cdot \boldsymbol{\psi}_{n2}\right) \frac{\mathrm{e}^{-\mathrm{j}kR_{12}}}{4\pi k R_{12}} \, \mathrm{dS}_1 \, \mathrm{dS}_2$$

where $\boldsymbol{\psi}_{n1} = \boldsymbol{\psi}_n(\boldsymbol{r}_1)$, $\boldsymbol{\psi}_{n2} = \boldsymbol{\psi}_n(\boldsymbol{r}_2)$, $n = 1, ..., N$, and $R_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$.

The current density is $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} I_n \boldsymbol{\psi}_n(\boldsymbol{r})$ with the expansion coefficients determined from

$$\mathbf{Z}\mathbf{I} = \mathbf{V} \quad \text{or } \mathbf{I} = \mathbf{Z}^{-1}\mathbf{V} = \mathbf{Y}\mathbf{V}$$

where $\mathbf{V}$ is a column matrix with the excitation coefficients. The input admittance is

$$Y_{\text{in}} = 1/Z_{\text{in}} = \mathbf{V}^{\mathsf{T}} \mathbf{Y} \mathbf{V} / V_{\text{in}}^2$$

where $Z_{\text{in}} = R_{\text{in}} + \mathrm{j}X_{\text{in}}$ is the input impedance.

$Q_{Z'_{\text{in}}}$ and Q for antennas (fields)

Differentiate the MoM impedance matrix

$$\frac{k\,\partial Z_{mn}}{\eta\,\partial k} = \int_V\int_V \mathrm{j}\left(k^2\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2} + \nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\right)\frac{\mathrm{e}^{-\mathrm{j}kR_{12}}}{4\pi kR_{12}}$$
$$+\left(k^2\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2} - \nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\right)\frac{\mathrm{e}^{-\mathrm{j}kR_{12}}}{4\pi}\,\mathrm{dS}_1\,\mathrm{dS}_2$$

Differentiated input admittance

$$V_{\text{in}}^2 Y'_{\text{in}} = (\mathbf{V}^{\mathsf{T}}\mathbf{Y}\mathbf{V})' = \mathbf{V}^{\mathsf{T}}\mathbf{Y}'\mathbf{V} = -\mathbf{I}^{\mathsf{T}}\mathbf{Z}'\mathbf{I}.$$

The stored energy determined from $\mathbf{X}' = \operatorname{Im}\mathbf{Z}'$

$$W_{\mathrm{e}\mathbf{X}'} + W_{\mathrm{m}\mathbf{X}'} = \frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$$

is identical to the stored energy expressions introduced by Vandenbosch (IEEE-TAP 2010).

Q and $Q_{Z'_{\rm in}}$ for free-space self-resonant antennas

Assume for simplicity a **self-resonant** antenna (circuit)

$$Q_{Z'_{\rm in}} = \frac{\omega |Z'_{\rm in}|}{2R_{\rm in}} = \frac{\omega |\mathbf{I}^{\mathsf{T}}\mathbf{Z}'\mathbf{I}|}{2\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

and using MoM with the stores energy by Vandenbosch

$$Q = \frac{2\omega \max\{W_{\mathrm{e}\mathbf{X}'}, W_{\mathrm{m}\mathbf{X}'}\}}{P_{\rm d}} = \frac{\omega \mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}}{2\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

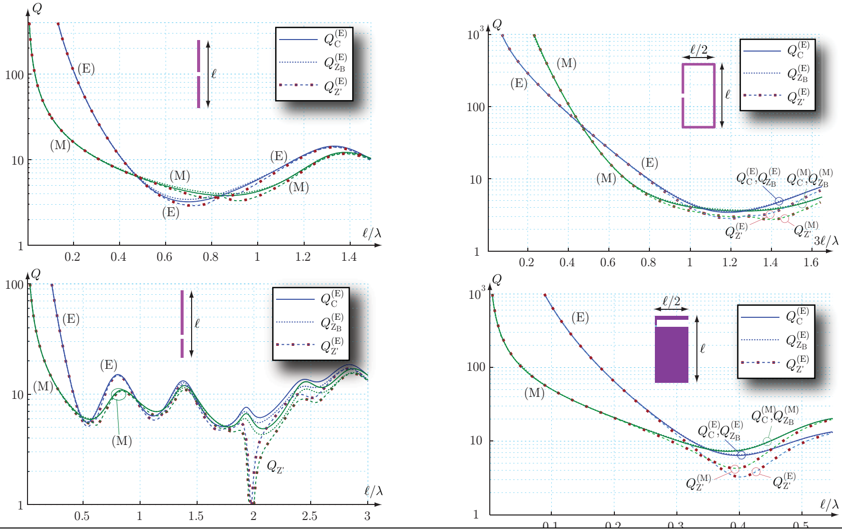
Transpose for $Q_{Z'_{\rm in}}$ and Hermitian transpose for Q

- $\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I} \geq 0$ for positive semidefine matrices $\mathbf{X}'$.
- $|\mathbf{I}^{\mathsf{T}}\mathbf{Z}'\mathbf{I}| = 0$ for some $\mathbf{I}$ $(\mathrm{rank} > 1)$.

See also Capek+*etal.* IEEE-TAP 2014 for $Q_{Z'_{\rm in}}$ using $\mathbf{I}^{\mathsf{H}}$ and $\mathbf{I}'$.

Antenna examples (free space)

Q from stored energy expressed in the current density Q_{C}, Brune circuit $Q_{Z_{\mathrm{in}}^{\mathrm{B}}}$, and differentiated input impedance $Q_{Z'_{\mathrm{in}}}$



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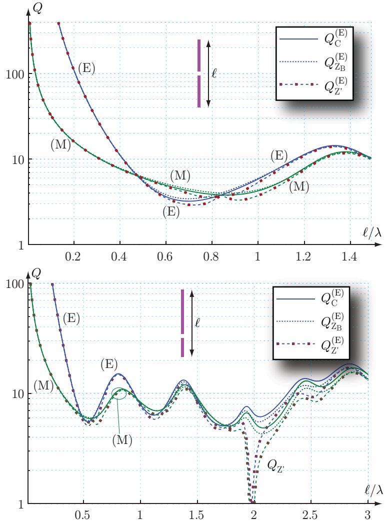


Q computed from

- the currents, Q_{C}.
- a circuit model synthesized from the input impedance using Brune synthesis (1931), $Q_{Z_{\mathrm{in}}^{\mathrm{B}}}$.
- differentiation of the (tuned) input impedance,

$$Q_{Z'_{\mathrm{in}}} = \frac{\omega_0 |Z'_{\mathrm{in}}|}{2R_{\mathrm{in}}} = \omega_0 |\mathit{\Gamma}'|.$$

All agree for $Q \gg 1$ but the Q from the differentiated impedance ($Q_{Z'_{\mathrm{in}}}$) is lower in some regions.
Which one is most accurate/best?

Dispersive media

The frequency derivative of the EFIE impedance matrix $\mathbf{Z}$ is

$$\omega\frac{\partial\mathbf{Z}}{\partial\omega} = k\frac{\partial(\mathbf{Z}/\eta)}{\partial k}\frac{\eta\omega}{k}\frac{\partial k}{\partial\omega} + \omega\frac{\mathbf{Z}}{\eta}\frac{\partial\eta}{\partial\omega}$$

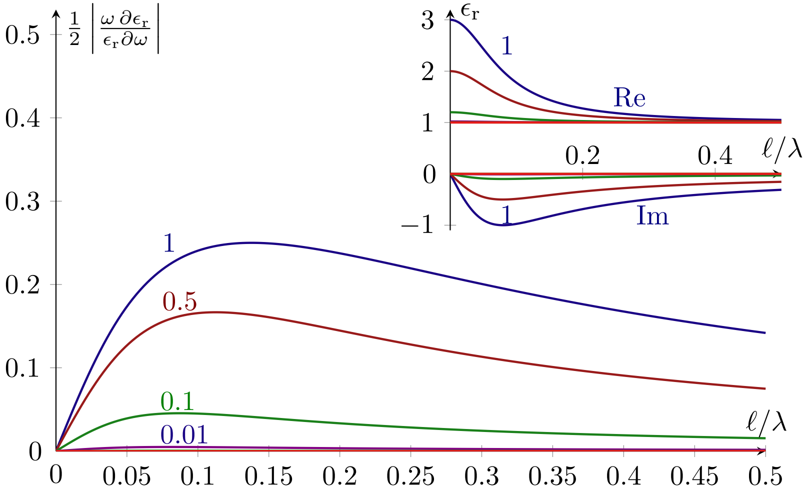
for a temporally dispersive background medium with $k = \omega\sqrt{\epsilon\mu}$ and $\eta = \sqrt{\mu/\epsilon}$. The derivative simplifies to

$$\omega\frac{\partial\mathbf{Z}}{\partial\omega} = k\frac{\partial(\mathbf{Z}/\eta)}{\partial k}\eta\left(\frac{\omega\partial\epsilon}{2\epsilon\partial\omega} + 1\right) - \frac{\mathbf{Z}}{2}\frac{\omega\partial\epsilon}{\epsilon\partial\omega}$$

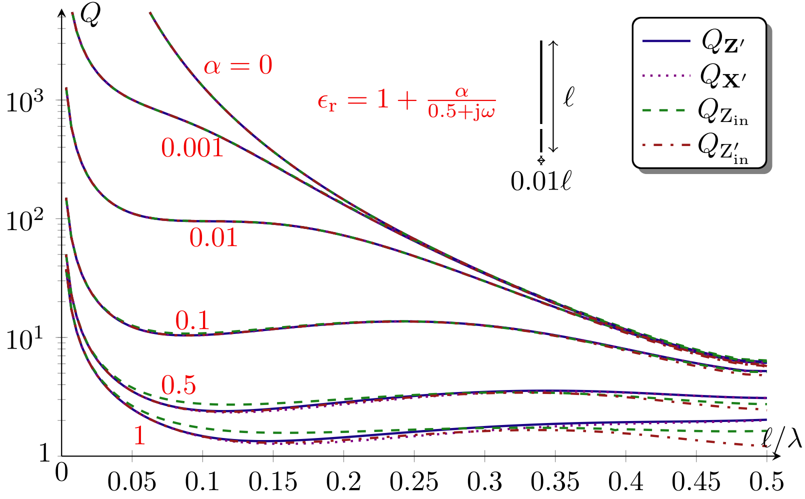
for the common case of a non-magnetic medium, $\mu_{\mathrm{r}} = 1$.

Multiplication of the previously calculated derivative (with respect to the wavenumber k in the medium) with a factor that only depends on the medium. The factor $\omega\epsilon' = (\omega\epsilon)' - \epsilon$ is similar to the classical approach used to define the energy density in dispersive media.

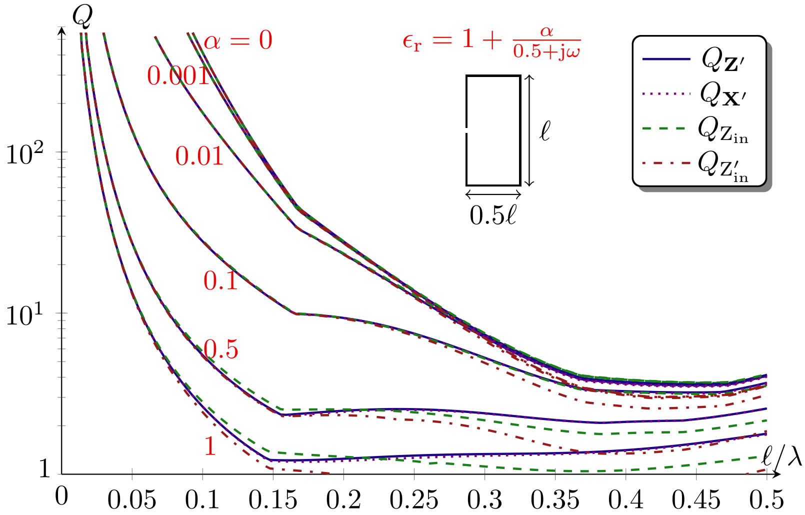
Numerical examples: Debye media



Numerical examples: Debye media



Numerical examples: Debye media



Stored EM energy matrices

Method of Moments approximation (expand $\boldsymbol{J}$ in basis functions)

$$W_{\mathrm{e}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \quad \text{stored E-energy, } \mathbf{X}_{\mathrm{e}} \text{ electric reactance}$$

$$W_{\mathrm{m}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \quad \text{stored M-energy, } \mathbf{X}_{\mathrm{m}} \text{ magnetic reactance}$$

$$P_{\mathrm{rad}} \approx \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \quad \text{radiated power}$$

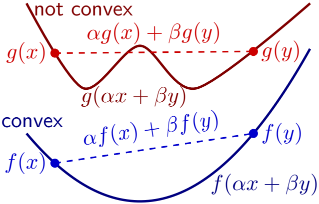
giving $\mathbf{Z} = \mathbf{R}_{\mathrm{r}} + \mathrm{j}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})$. We also use

$\boldsymbol{F} \approx \mathbf{F}^{\mathsf{H}}\mathbf{I}$(far field), $\boldsymbol{E} \approx \mathbf{N}^{\mathsf{H}}\mathbf{I}$(near field), $\mathbf{I}_2 \approx \mathbf{C}^{\mathsf{H}}\mathbf{I}_1$(induced current)

Pre-computed matrices used in the optimization.

Convex optimization

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \; i = 1, ..., N_1 \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$



where $f_i(x)$ are convex, *i.e.*, $f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$ for $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

Solved with efficient standard algorithms. No risk of getting trapped in a local minimum. A problem is 'solved' if formulated as a convex optimization problem.

Antenna performance expressed in the current density $\boldsymbol{J}$, *e.g.*,

- Radiated field $\boldsymbol{F}(\hat{\boldsymbol{k}}) = -\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} \times \int_V \boldsymbol{J}(\boldsymbol{r}) \mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}} \, \mathrm{dV}$ is affine.
- Radiated power, stored electric and magnetic energies, and Ohmic losses are positive semi-definite quadratic forms in $\boldsymbol{J}$.

Currents for maximal G/Q

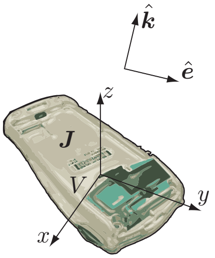
Determine a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V that maximizes the partial-gain Q-factor quotient $G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})/Q$.

- Partial radiation intensity $P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})$

$$\frac{G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{\mathrm{c}_0 k \max\{W_{\mathrm{e}}, W_{\mathrm{m}}\}}.$$

- Scale $\boldsymbol{J}$ and reformulate $P=1$ as $\hat{\boldsymbol{e}}^{*}\cdot\boldsymbol{F} = \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1$.
- Convex optimization problem:

$$\begin{aligned} &\text{minimize} && W \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq W \\ & && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq W \\ & && \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \end{aligned}$$



Determines a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V with minimal stored EM energy and unit partial radiation intensity.

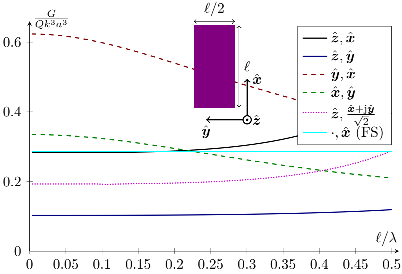
Maximal $G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{x}})/Q$ for planar rectangles

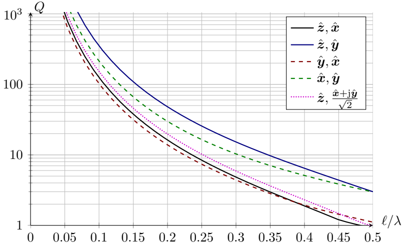
$$\begin{aligned} \text{minimize} \quad & W \\ \text{subject to} \quad & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq W \\ & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq W \\ & \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \end{aligned}$$

Solution for current densities confined to planar rectangles with side lengths ℓ_{x} and $\ell_{\mathrm{y}} = 0.5\ell_{\mathrm{x}}$.

CVX code: http://cvxr.com/cvx/

```
cvx_begin
  variable J(n) complex;
  minimize W
  subject to
     quad_form(J,Xe) <= W;
     quad_form(J,Xm) <= W;
     F'*J == 1;
cvx_end
```

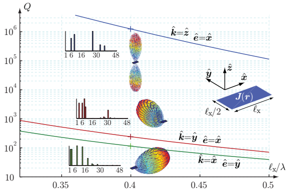




Example of current optimization formulations

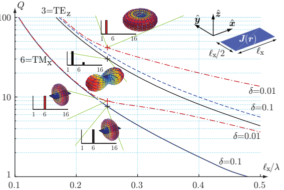
Super directivity:

$$\begin{aligned} \text{minimize}_{\mathbf{I}} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \\ & \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \leq 4\pi/(\eta_0 D_0) \end{aligned}$$



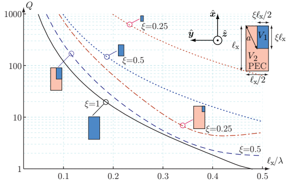
Prescribed far field:

$$\begin{aligned} \text{minimize} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \int_{\Omega} |\boldsymbol{F}(\hat{\boldsymbol{k}}) - \boldsymbol{F}_0(\hat{\boldsymbol{k}})|^2 \, \mathrm{d}\Omega_{\hat{\boldsymbol{k}}} < \delta \end{aligned}$$



Embedded antennas:

$$\begin{aligned} \text{minimize} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \\ & \mathbf{I}_2 = \mathbf{C}^{\mathsf{H}}\mathbf{I}_1 \end{aligned}$$



Why convex optimization?

Solved if formulated as a convex optimization problem.

Consider the G/Q problem

$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \end{aligned}$$

Many (optimization) algorithms can be used to solve this problem.

- Can *e.g.*, use any of the solvers included in CVX.
  - Very simple to use.
  - Good for small problems but less efficient for larger problems.
- A dedicated solver for quadratic programs.
  - More efficient for larger problems.
- Random search, eg genetic algorithms (GA), particle swarms,....
  - Very inefficient. Note you do not (should not) use (GA, ...) to solve *e.g.*, $\mathbf{A}\mathbf{x} = \mathbf{b}$ $(\min. ||\mathbf{A}\mathbf{x} - \mathbf{b}||)$.
- We also use a dual formulation
  - Computational efficient for large problems.
  - Illustrates dual problems and posteriori error estimates.

Relaxation and dual problem

An illustrative method is to use the inequality

$$W = \max\{W_\mathrm{e}, W_\mathrm{m}\} \geq \alpha W_\mathrm{e} + (1-\alpha) W_\mathrm{m} = W_\alpha \quad \text{for } 0 \leq \alpha \leq 1$$

or with the matrices $\mathbf{X}_\mathrm{e}, \mathbf{X}_\mathrm{m}$

$$W = \max\{\mathbf{I}^\mathsf{H}\mathbf{X}_\mathrm{e}\mathbf{I}, \mathbf{I}^\mathsf{H}\mathbf{X}_\mathrm{m}\mathbf{I}\} \geq W_\alpha = \mathbf{I}_\alpha^\mathsf{H}(\alpha\mathbf{X}_\mathrm{e} + (1-\alpha)\mathbf{X}_\mathrm{m})\mathbf{I}_\alpha$$

or for the quotient G/Q

$$\frac{G}{Q} = \frac{2\pi P}{\omega \max\{W_{\mathrm{e}\alpha}, W_{\mathrm{m}\alpha}\}} \leq \frac{2\pi P}{\omega W_\alpha} = \frac{2\pi P}{\omega(\alpha W_{\mathrm{e}\alpha} + (1-\alpha W_{\mathrm{m}\alpha}))} = \frac{G_\alpha}{Q_\alpha}$$

Note $P = 1$ is fixed in the optimization problem.

Relaxation and dual problem

The inequality relaxes the G/Q optimization problem

$$\begin{aligned} &\text{minimize} \quad \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \geq \mathbf{I}_{\alpha}^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}_{\alpha} \\ &\text{subject to} \quad \mathbf{F}^{\mathsf{H}}\mathbf{I} = 1 \end{aligned}$$

into

$$\begin{aligned} &\text{maximize}_{\alpha}\text{minimize}_{\mathbf{I}_{\alpha}} \quad W_{\alpha} = \mathbf{I}_{\alpha}^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}_{\alpha} \\ &\text{subject to} \quad \mathbf{F}^{\mathsf{H}}\mathbf{I}_{\alpha} = 1 \\ &\qquad 0 \leq \alpha \leq 1 \end{aligned}$$

where for the quotient G/Q (note $P = 1$)

$$\frac{G}{Q} = \frac{2\pi P}{\omega \max\{W_{\mathrm{e}\alpha}, W_{\mathrm{m}\alpha}\}} \leq \frac{2\pi P}{\omega W_{\alpha}} = \frac{2\pi P}{\omega(\alpha W_{\mathrm{e}\alpha} + (1-\alpha W_{\mathrm{m}\alpha}))} = \frac{G_{\alpha}}{Q_{\alpha}}$$

Relaxation and dual problem

The dual problem

$$\begin{aligned} &\text{maximize}_{\alpha}\text{minimize}_{\mathbf{I}_\alpha} && W_\alpha = \mathbf{I}_\alpha^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}_\alpha \\ &\text{subject to} && \mathbf{F}^{\mathsf{H}}\mathbf{I}_\alpha = 1 \\ &&& 0 \leq \alpha \leq 1 \end{aligned}$$

is solved as a linear system (MoM equation) for fixed α with

$$\mathbf{I}_\alpha = \frac{\big(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}\big)^{-1}\mathbf{F}}{\mathbf{F}^{\mathsf{H}}\big(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}\big)^{-1}\mathbf{F}}$$

giving the optimization problem

$$\underset{0\leq\alpha\leq 1}{\text{maximize}}\, W_\alpha$$

or

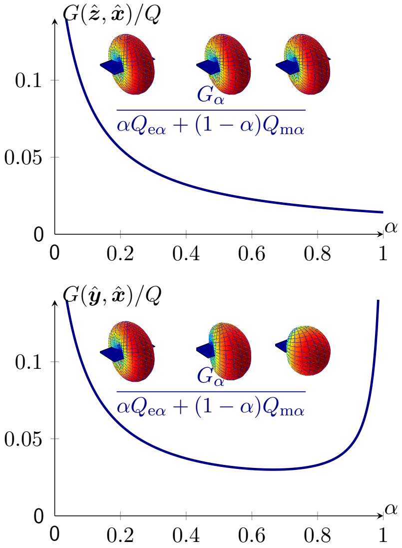
$$\underset{0\leq\alpha\leq 1}{\text{minimize}}\, \frac{G_\alpha}{Q_\alpha} = \mathbf{F}^{\mathsf{H}}\big(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}\big)^{-1}\mathbf{F}$$

Why convex optimization: illustration

The upper bound on $G/Q|_{\mathrm{ub}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (blue) curve

$$\left.\frac{G}{Q}\right|_{\mathrm{ub}} \leq \frac{G_\alpha}{\alpha Q_{\mathrm{e}\alpha} + (1-\alpha) Q_{\mathrm{m}\alpha}}$$

This is efficiently solved by golden section search and parabolic interpolation.

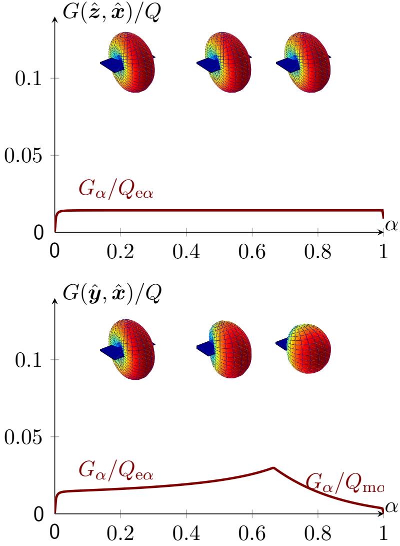


$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

Why convex optimization: illustration

We also compute the actual G/Q for the current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{\mathrm{e}\alpha}, Q_{\mathrm{m}\alpha}\}} \leq \frac{G}{Q}\Big|_{\mathrm{ub}}$$



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Why convex optimization: illustration

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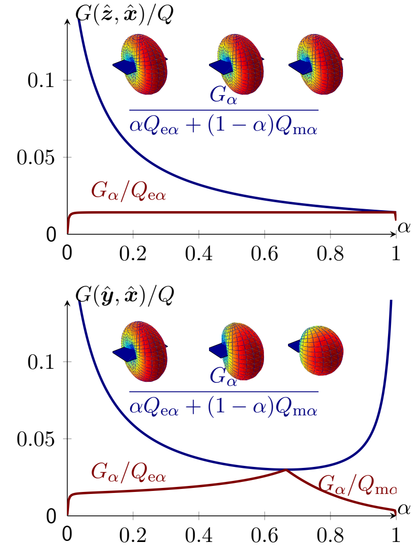


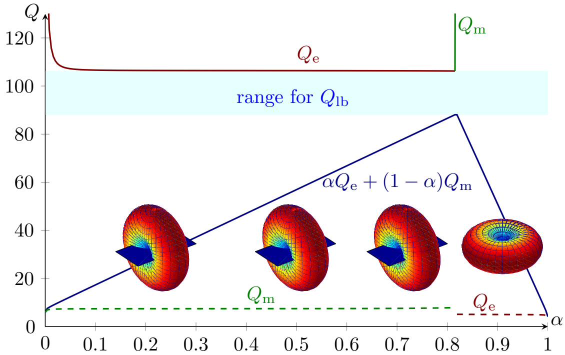
$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

Minimization of Q

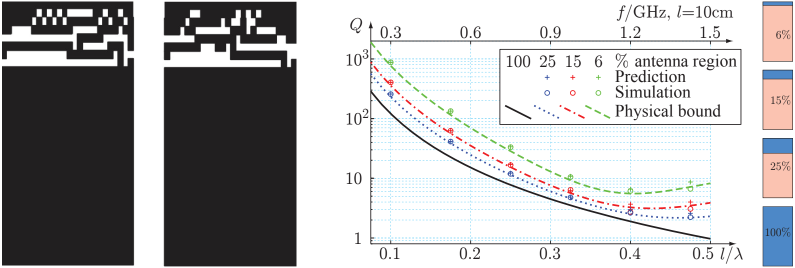
The corresponding formulation for Q is not convex. For $0 \leq \alpha \leq 1$

$$Q = \frac{\max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}} = \max\{Q_{\mathrm{e}}, Q_{\mathrm{m}}\}$$

$$\geq \alpha Q_{\mathrm{e}} + (1-\alpha)Q_{\mathrm{m}} = \frac{\mathbf{I}^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}}$$



Conclusions



- Current optimization for physical bounds.
- Stored energy from MoM reactance matrices (basically, already computed in most MoM codes for surface currents).
- Promising results for temporally dispersive media.
- Convex optimization (efficiently solved with a few $\mathbf{Ax} = \mathbf{b}$).

Minimization of Q

Compare maximization of G/Q with minimization of Q. Use the same inequality for $0 \leq \alpha \leq 1$

$$Q = \frac{\max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}} = \max\{Q_{\mathrm{e}}, Q_{\mathrm{m}}\}$$

$$\geq \alpha Q_{\mathrm{e}} + (1-\alpha)Q_{\mathrm{m}} = \frac{\mathbf{I}^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}}$$

The lower bound on Q, Q_{lb}, is a minimization problem for a Rayleigh quotient solved as a generalized eigenvalue problem

$$\min(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}, \mathbf{R}_{\mathrm{r}})$$

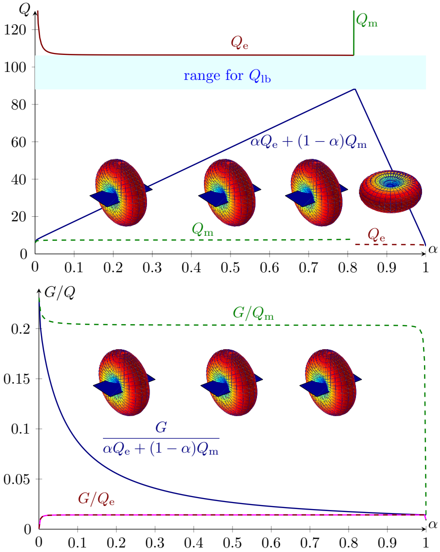
Let $Q_{\mathrm{e}\alpha}$ and $Q_{\mathrm{m}\alpha}$ denote the corresponding electric and magnetic Q-factors to get the estimate

$$\alpha Q_{\mathrm{e}\alpha} + (1-\alpha)Q_{\mathrm{m}\alpha} \leq Q_{\mathrm{lb}} \leq \max\{Q_{\mathrm{e}\alpha}, Q_{\mathrm{m}\alpha}\}$$

for the lower bound Q_{lb}.

Numerical illustration of $\min.Q$ and $\max.G/Q$

- The formulation for $\min.Q$ has a duality gap, *i.e.*, we have an interval for Q_{lb} here $88 \leq Q_{\mathrm{lb}} \leq 106$.
- The optimization problem $\min.Q$ is not convex.
- The formulation for $\max.G/Q$ has no duality gap.
- This is common for many convex optimization problems.



Why convex optimization? Simple algorithm

Consider the G/Q problem

$$\begin{aligned}\text{minimize} \quad & W = \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \mathrm{Im}\{\mathbf{F}^{\mathsf{H}}\mathbf{I}\} = 1\end{aligned}$$

There are many (optimization) algorithms that can be used to solve this problem. An illustrative method is to use $(0 \leq \alpha \leq 1)$

$$W = \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \geq W_\alpha = \mathbf{I}_\alpha^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}_\alpha$$

and (hence $G/Q \leq G_\alpha/Q_\alpha$) to relax to the dual problem

$$\begin{aligned}\text{maximize}_\alpha\text{minimize}_{\mathbf{I}_\alpha} \quad & W_\alpha = \mathbf{I}_\alpha^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}_\alpha \\ \text{subject to} \quad & \mathrm{Im}\{\mathbf{F}^{\mathsf{H}}\mathbf{I}_\alpha\} = 1 \\ & 0 \leq \alpha \leq 1\end{aligned}$$

that is solved as a linear system (MoM equation) for fixed α giving

$$\max_{0\leq\alpha\leq 1} W_\alpha \quad \text{with } \mathbf{I}_\alpha = \frac{\big(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}\big)^{-1}\mathbf{F}}{\mathbf{F}^{\mathsf{H}}\big(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}\big)^{-1}\mathbf{F}} \qquad \text{(relaxed problem)}$$