Completeness of the Characteristic Mode Expansion

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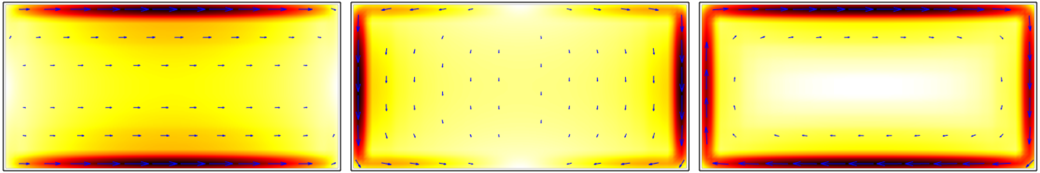
Characteristic modes (CM)

- Developed in the 70s by Garbacz, Turpin, Harrington, Mautz [HM71].
- Provides physical understanding and complements simulation and optimization driven antenna design.
- Modes (electric current) determined by the geometry (here lossless).
- Scattering properties, resonances, antenna feed placement, ...
- Generalized eigenvalue problem $\mathbf{X}\mathbf{I}_n = \lambda_n \mathbf{R}_\mathrm{r}\mathbf{I}_n$, where $\mathbf{Z} = \mathbf{R}_\mathrm{r} + \mathrm{j}\mathbf{X}$ denotes the MoM impedance matrix.
- Orthogonal far fields $\mathbf{I}_m^\mathsf{H}\mathbf{R}_\mathrm{r}\mathbf{I}_n = \delta_{mn}$ and reactance $\mathbf{I}_m^\mathsf{H}\mathbf{X}\mathbf{I}_n = \lambda_n\delta_{mn}$.



First three modes for a rectangle. Only a few dominant modes (small $|\lambda_n|$) for electrically small structures.

Expansion of a current density in characteristic modes

Expansion of a current density

$$\mathbf{I}=\sum_{n=1}^{N}\alpha_n\mathbf{I}_n \quad \text{and } \boldsymbol{J}(\boldsymbol{r})=\sum_{n=1}^{N}\alpha_n\boldsymbol{J}_n(\boldsymbol{r})$$

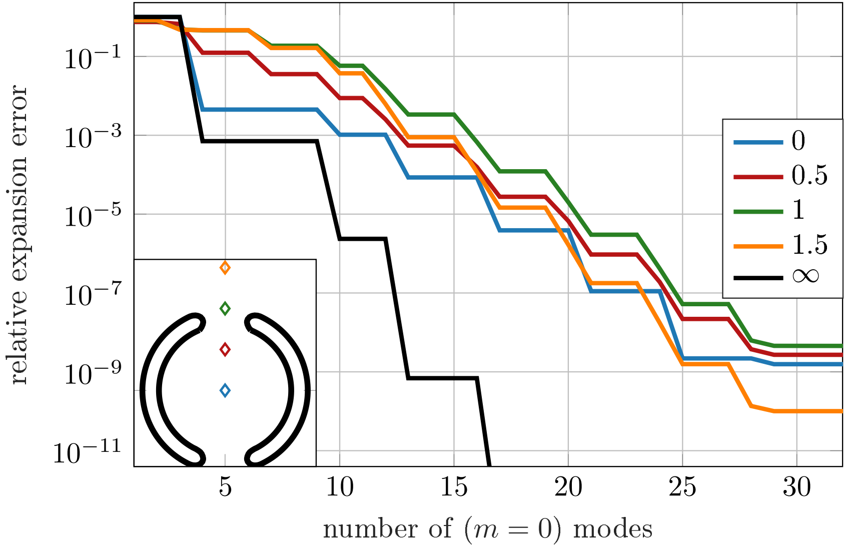
Diagonalization of the MoM equation $\mathbf{Z}\mathbf{I}=\mathbf{V}$

$$\sum_n \alpha_n\mathbf{I}_n^{\mathsf{T}}\mathbf{Z}\mathbf{I}_n=\alpha_n(1+\mathrm{j}\lambda_n)=\mathbf{I}_n^{\mathsf{T}}\mathbf{V}\to\alpha_n=\frac{\mathbf{I}_n^{\mathsf{T}}\mathbf{V}}{1+\mathrm{j}\lambda_n}$$

When can one expand currents in CM, convergence, theory, practice?

- Always possible for matrices. This is a general property for finite dimensional cases.
- What happens in the (continuous) infinite operator case?

Spherical (BoR) torus with radius $ka = 4$



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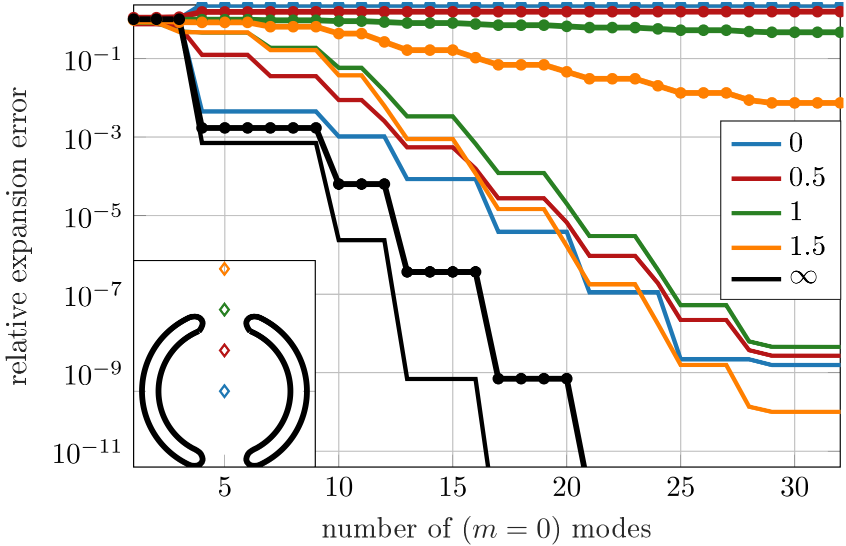

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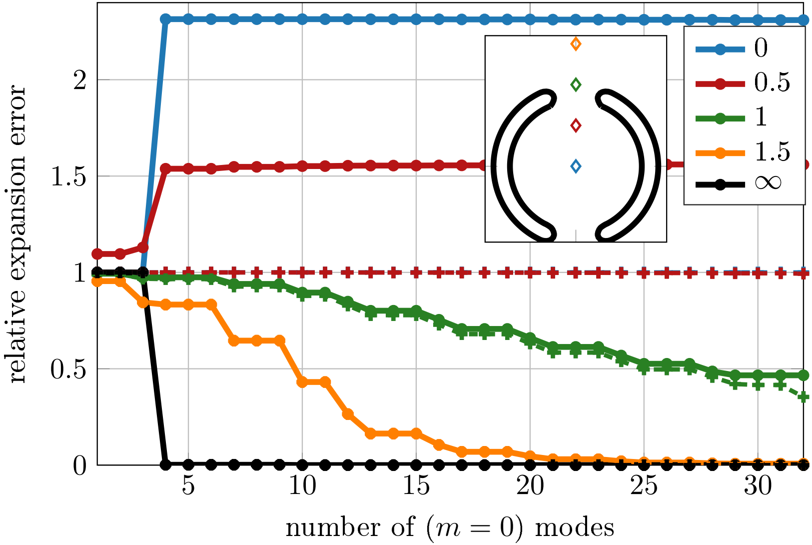


Efficient and accurate evaluation of $\mathbf{R}_{\mathrm{r}}$

The real-valued part of the impedance matrix with elements

$$Z_{pq} = \mathrm{j}k\eta_0 \int_{\Omega}\int_{\Omega} \boldsymbol{\psi}_p(\boldsymbol{r}_1)\cdot\mathbf{G}(\boldsymbol{r}_1,\boldsymbol{r}_2)\cdot\boldsymbol{\psi}_q(\boldsymbol{r}_2)\,\mathrm{dS}_1\,\mathrm{dS}_2$$

is decomposed by expanding the Green dyadic in regular $\mathbf{u}_{\upsilon}^{(1)}$ and out-going $\mathbf{u}_{\upsilon}^{(4)}$ vector spherical waves

$$\mathbf{G}(\boldsymbol{r}_1,\boldsymbol{r}_2) = -\mathrm{j}k\sum_{\upsilon}\mathbf{u}_{\upsilon}^{(1)}(k\boldsymbol{r}_<)\,\mathbf{u}_{\upsilon}^{(4)}(k\boldsymbol{r}_>),$$

where $\upsilon(\tau,\sigma,m,l)$, $\boldsymbol{r}_< = \boldsymbol{r}_1$ and $\boldsymbol{r}_> = \boldsymbol{r}_2$ if $|\boldsymbol{r}_1| < |\boldsymbol{r}_2|$ and so on. Factorization $\mathbf{R}_{\mathrm{r}} = \mathbf{S}^{\mathsf{T}}\mathbf{S} = \mathbf{S}^{\mathsf{H}}\mathbf{S}$, where $\mathbf{S}$ has the elements

$$S_{\upsilon p} = k\eta_0^{1/2}\int_{\Omega}\boldsymbol{\psi}_p(\boldsymbol{r})\cdot\mathbf{u}_{\upsilon}^{(1)}(k\boldsymbol{r})\,\mathrm{dS}.$$

See [Tay+18] and compare with the T-matrix [Kri16], FMM [CRW93], and far-field expansion [GN13] methods.

Computationally efficient CM evaluation

Factorize the radiation matrix $\mathbf{R}_{\mathrm{r}} = \mathbf{S}^{\mathsf{T}}\mathbf{S}$ to rewrite the $N_{\psi} \times N_{\psi}$ generalized eigenvalue problem $\mathrm{eig}(\mathbf{X}, \mathbf{R}_{\mathrm{r}})$

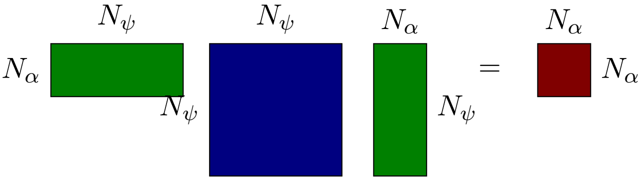
$$\mathbf{X}\mathbf{I}_n = \lambda_n \mathbf{R}_{\mathrm{r}} \mathbf{I}_n = \lambda_n \mathbf{S}^{\mathsf{T}}\mathbf{S}\mathbf{I}_n$$

by moving $\mathbf{X}$ to the right-hand side (or use $\mathbf{X} + \alpha\mathbf{R}_{\mathrm{r}}$ if $\mathbf{X}$ is not invertible)

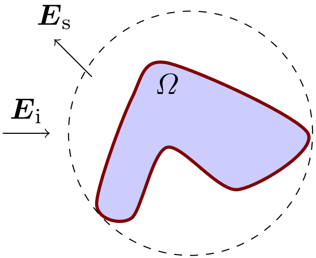
$$\mathbf{I}_n = \lambda_n \mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}}\mathbf{S}\mathbf{I}_n \implies \mathbf{S}\mathbf{I}_n = \lambda_n \mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}}\mathbf{S}\mathbf{I}_n \implies \mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}}\mathbf{f} = \lambda_n^{-1}\mathbf{f}_n$$

with $\mathbf{f}_n = \mathbf{S}\mathbf{I}_n$. Producing the $N_{\alpha} \times N_{\alpha}$ eigenvalue problem [Tay+18]

$$\lambda_n^{-1} = \mathrm{eig}(\mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}}) \quad \text{with } \mathbf{I}_n = \lambda_n \mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}}\mathbf{f}_n$$



CM from diagonalization of the T-matrix (I)



Expand in regular $\mathbf{u}_{\upsilon}^{(1)}$ and outgoing $\mathbf{u}_{\upsilon}^{(4)}$ spherical waves

$$\boldsymbol{E}_{\mathrm{i}} = k\eta_0^{1/2} \sum_{\upsilon} a_{\upsilon}\, \mathbf{u}_{\upsilon}^{(1)}(k\boldsymbol{r}) \text{ and } \boldsymbol{E}_{\mathrm{s}} = k\eta_0^{1/2} \sum_{\upsilon} f_{\upsilon}\, \mathbf{u}_{\upsilon}^{(4)}(k\boldsymbol{r})$$

outside a circumscribing sphere [Han88]. Transition matrix

$$\mathbf{f} = \mathbf{T}\mathbf{a}$$

characterizes the scattering properties of the object [Kri16].

Expansion of the incident field in regular spherical waves is written $\mathbf{V} = \mathbf{S}^{\mathsf{T}}\mathbf{a}$, *i.e.*,

$$V_n = \int_{\Omega} \boldsymbol{\Psi}_n(\hat{\boldsymbol{r}}) \cdot \boldsymbol{E}_{\mathrm{i}}(\boldsymbol{r})\, \mathrm{dV} = \sum_{\upsilon} a_{\upsilon} k\eta_0^{1/2} \int_{\Omega} \boldsymbol{\Psi}_n(\hat{\boldsymbol{r}}) \cdot \mathbf{u}_{\upsilon}^{(1)}(k\boldsymbol{r})\, \mathrm{dV} = \sum_{\upsilon} a_{\upsilon} S_{\upsilon n}$$

CM from diagonalization of the T-matrix (II)

Expansion of the incident field in regular spherical waves $\mathbf{V} = \mathbf{S}^{\mathsf{T}}\mathbf{a}$ substituted into the MoM formula

$$\mathbf{Z}\mathbf{I} = (\mathbf{R}_{\mathrm{r}} + \mathrm{j}\mathbf{X})\mathbf{I} = (\mathbf{S}^{\mathsf{T}}\mathbf{S} + \mathrm{j}\mathbf{X})\mathbf{I} = \mathbf{V} = \mathbf{S}^{\mathsf{T}}\mathbf{a}$$

and multiplication with $\mathbf{S}\mathbf{X}^{-1}$ produces

$$(\mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}}\mathbf{S} + \mathrm{j}\mathbf{S})\mathbf{I} = (\mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}} + \mathrm{j}\mathbf{1})\mathbf{S}\mathbf{I} = (\mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}} + \mathrm{j}\mathbf{1})\mathbf{f} = \mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}}\mathbf{a}.$$

Here it is seen that the transition matrix $\mathbf{T}$ mapping regular spherical waves to outgoing spherical waves, $\mathbf{f} = \mathbf{T}\mathbf{a}$, is diagonalized by characteristic modes, *i.e.*, $\mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}} = \mathbf{F}\mathbf{\Lambda}^{-1}\mathbf{F}^{\mathsf{T}}$ diagonalizes the T-matrix as

$$(\mathbf{\Lambda}^{-1} + \mathrm{j}\mathbf{1})\mathbf{F}^{\mathsf{T}}\mathbf{f} = \mathbf{\Lambda}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{a} \Rightarrow \mathbf{f} = \mathbf{F}(\mathbf{\Lambda}^{-1} + \mathrm{j}\mathbf{1})^{-1}\mathbf{\Lambda}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{a} = \mathbf{F}(\mathbf{1} + \mathrm{j}\mathbf{\Lambda})^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{a}$$

This is a short and simple algebraic formulation of Garbacz, Turpin, Harrington, and Mautz original suggestion [HM71].

Spherical PEC shell

The T-matrix and characteristic modes are well known for a PEC spherical shell. Diagonal T-matrix determined from the Mie series giving

$$t_{\tau l} = -\frac{\mathrm{R}_{\tau l}^{(1)}(ka)}{\mathrm{R}_{\tau l}^{(4)}(ka)} = -\frac{\mathrm{R}_{\tau l}^{(1)}(ka)}{\mathrm{R}_{\tau l}^{(1)}(ka) + \mathrm{j}\,\mathrm{R}_{\tau l}^{(2)}(ka)} = -\frac{1}{1 - \mathrm{j}\,\mathrm{R}_{\tau l}^{(2)}(ka)/\,\mathrm{R}_{\tau l}^{(1)}(ka)}$$

for $\tau = 1, 2$ and $l = 1, 2, ...$ corresponding to the CM values

$$\lambda_{\tau l} = -\frac{\mathrm{R}_{\tau l}^{(2)}(ka)}{\mathrm{R}_{\tau l}^{(1)}(ka)}$$

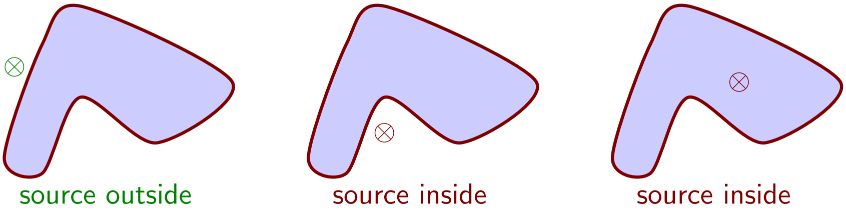
where $\mathrm{R}_{\tau l}^{(p)}$ denote the radial functions [Han88] defined as $\mathrm{R}_{1l}^{(p)} = z_l^{(p)}$ and $\mathrm{R}_{2l}^{(p)}(\kappa) = (\kappa z_l^{(p)}(\kappa))'/\kappa$ with $z_l^{(1)} = \mathrm{j}_l$, $z_l^{(2)} = \mathrm{n}_l$, and $z_l^{(3,4)} = \mathrm{h}_l^{(1,2)}$ denoting spherical Bessel, Neumann, and Hankel function, respectively.

CM excitation $\mathbf{Z}\mathbf{I}_n = \mathbf{V}_n$

Excitation (incident field) corresponding to a characteristic mode $\mathbf{I}_n = \lambda_n \mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}}\mathbf{f}_n$ can be written

$$\mathbf{Z}\mathbf{I}_n = (\mathbf{S}^{\mathsf{T}}\mathbf{S} + \mathrm{j}\mathbf{X})\lambda_n \mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}}\mathbf{f}_n = \mathbf{S}^{\mathsf{T}}(\mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathsf{T}} + \mathrm{j}\mathbf{1})\mathbf{f}_n\lambda_n = \mathbf{S}^{\mathsf{T}}\mathbf{f}_n(1 + \mathrm{j}\lambda_n) = \mathbf{V}_n$$

Characteristic modes can be used to expand current densities corresponding to incident waves which can be expanded in regular spherical waves ($\mathbf{a}_n = (1 + \mathrm{j}\lambda_n)\mathbf{f}_n$) such as plane waves or near fields with sources supported outside a sphere circumscribing the object.

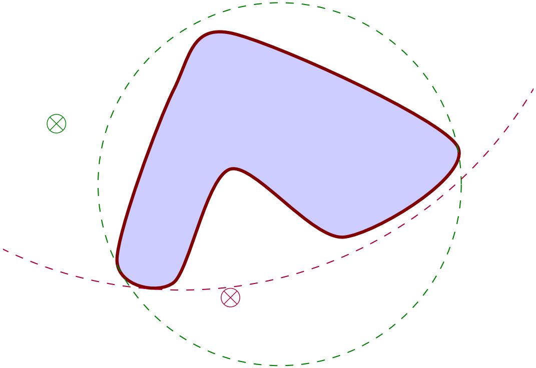


Expansion in regular waves

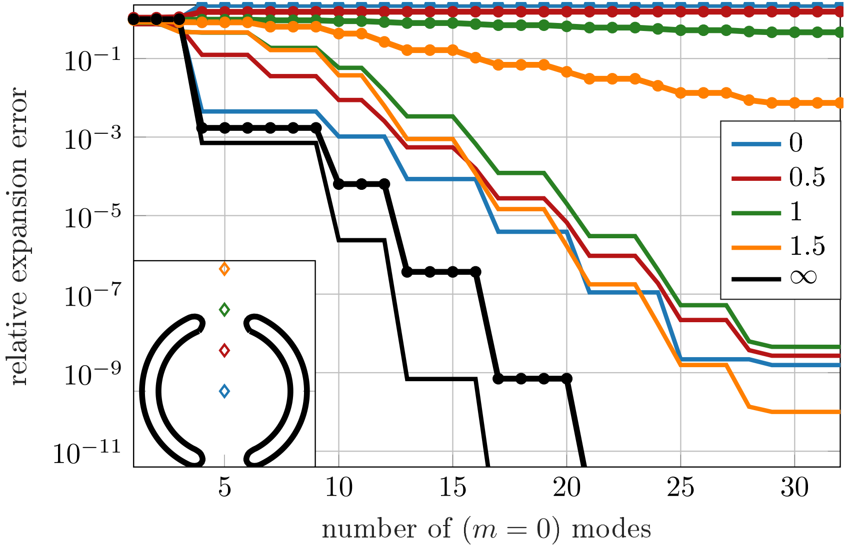
Can use large spheres to show that the relevant region divide into sources inside and outside the convex hull of the object.

Can expand in CM if the source region can separated from the object with a hyperplane.

- Size of sphere in ka indicate how many modes are needed to expand the field.
- Related to the number of CMs.
- Also need to consider expansion of the source.



Spherical (BoR) torus with radius $ka = 4$



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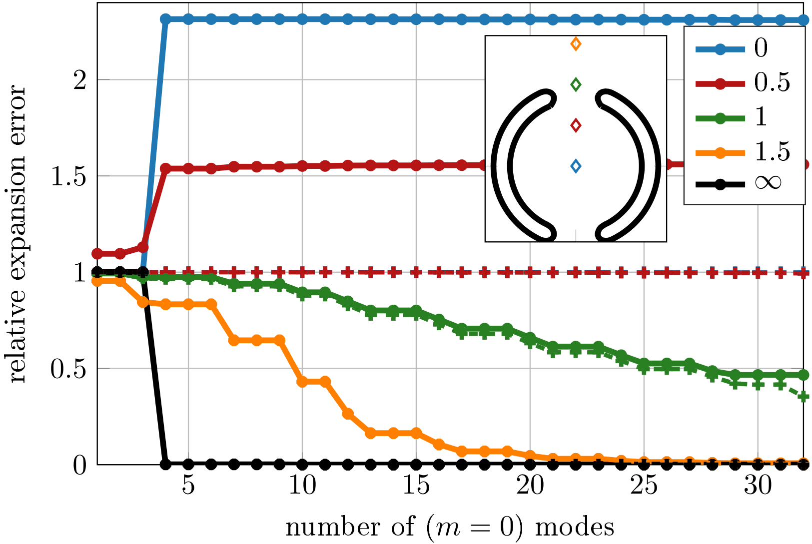
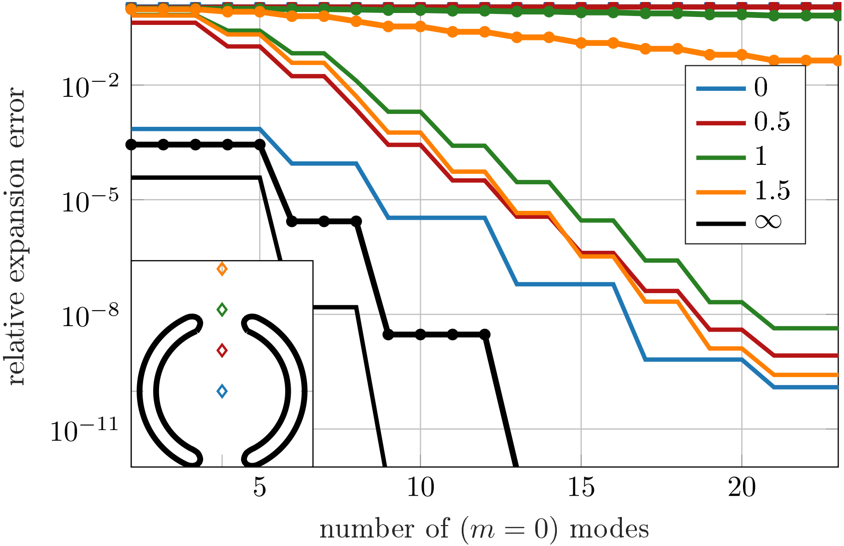
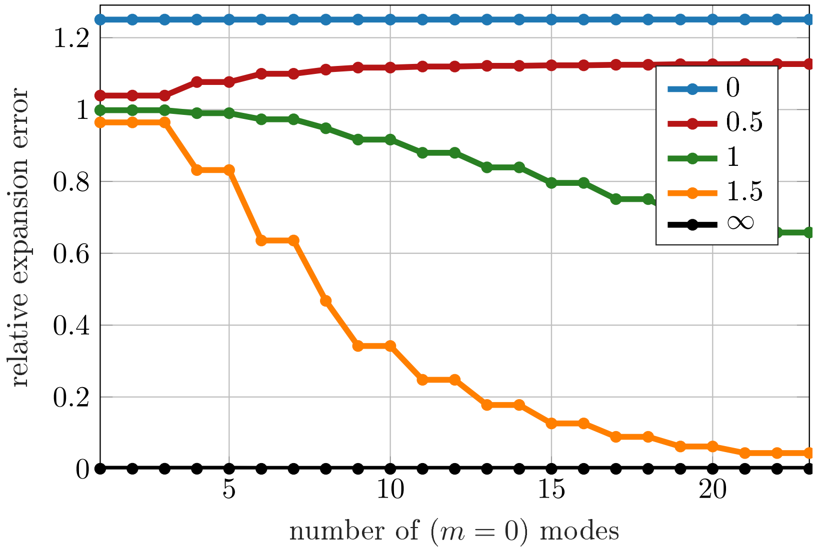


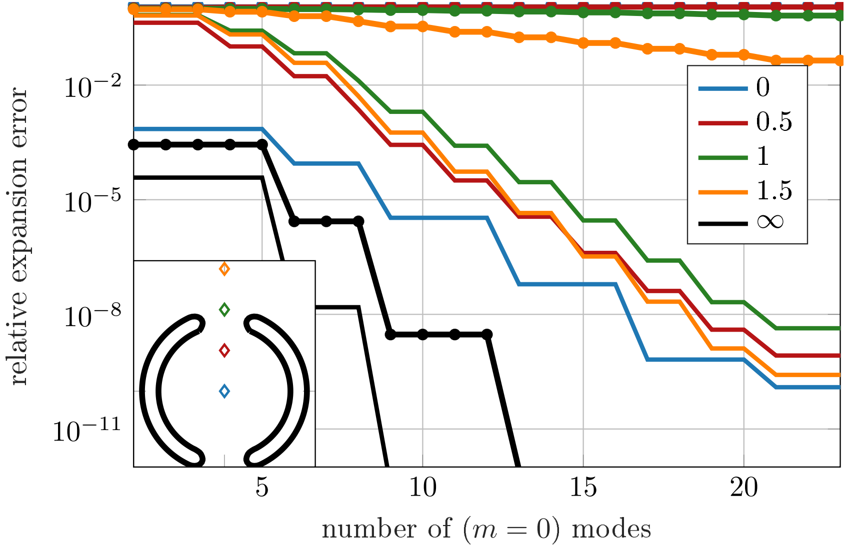
Spherical (BoR) torus with radius $ka = 2$



Spherical (BoR) torus with radius $ka = 2$



Spherical (BoR) torus with radius $ka = 2$ (hole size)



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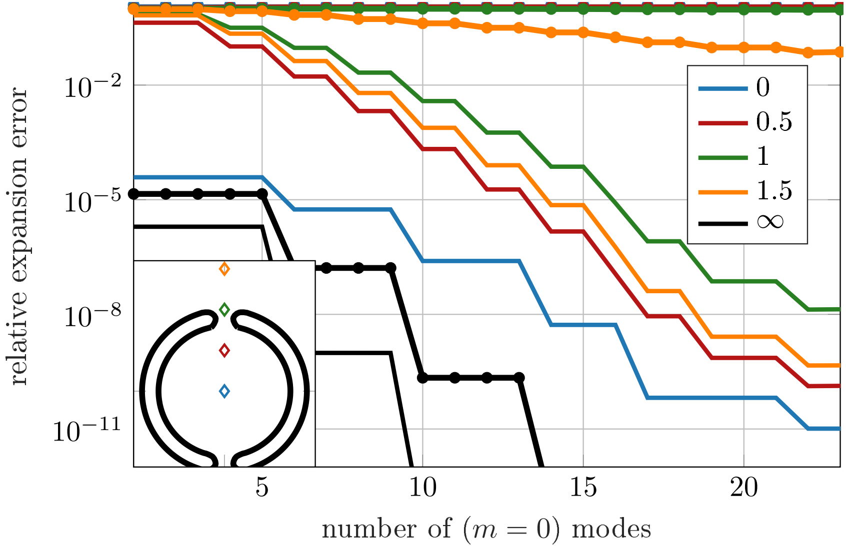
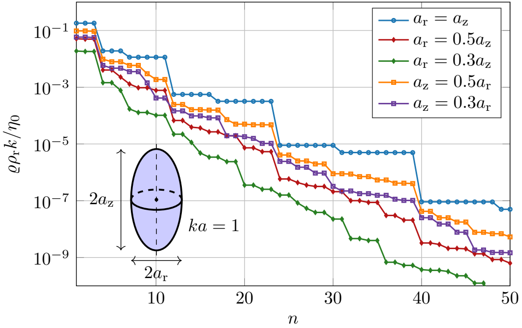


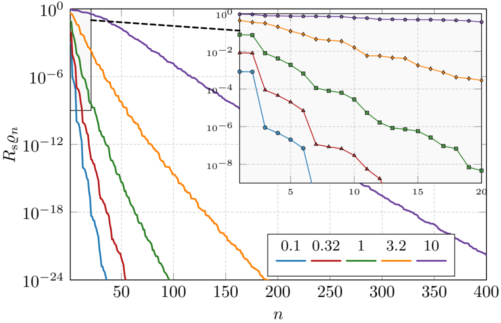
Radiation modes

Radiation modes ϱ_n are defined by the eigenvalue problem [EG18; GC19; Sch16]

$$\mathbf{R}_{\mathrm{r}}\mathbf{I}_n = \varrho_n \mathbf{R}_{\Omega}\mathbf{I}_n \quad \text{or from an SVD of } \mathbf{S}\boldsymbol{\Upsilon}^{-1}$$

where spherical wave $\mathbf{R}_{\mathrm{r}} = \mathbf{S}^{\mathsf{H}}\mathbf{S}$ and Cholesky $\mathbf{R}_{\Omega} = \boldsymbol{\Upsilon}^{\mathsf{H}}\boldsymbol{\Upsilon}$ decompositions are used.





Exponentially decreasing ϱ_n. Are there any $\varrho_n = 0$?

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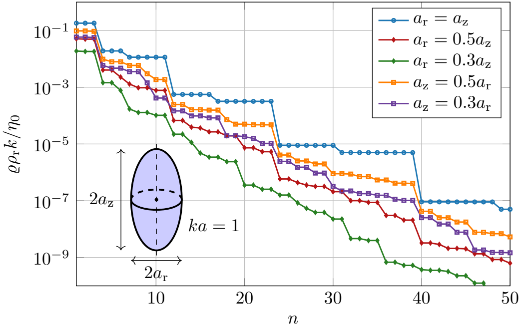
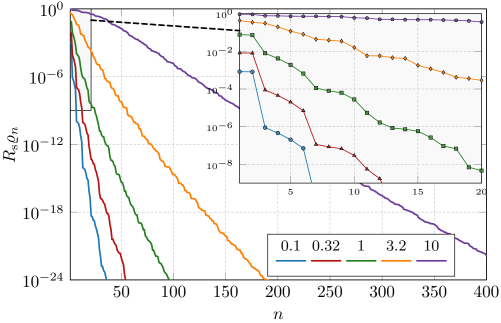




Exponentially decreasing ϱ_n. Are there any $\varrho_n = 0$? Left yes, right probably not.

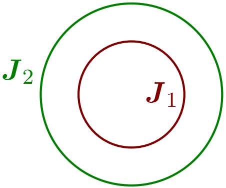
Non-radiating currents

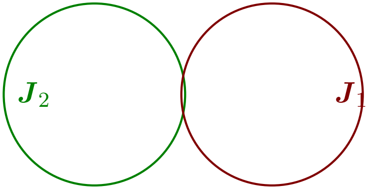
The nullspace of $\mathbf{R}_{\mathrm{r}} = \mathbf{S}^{\mathsf{H}}\mathbf{S}$ $(\mathbf{SI} = \mathbf{0})$ is related to non-radiating currents

- Volume distributed current density

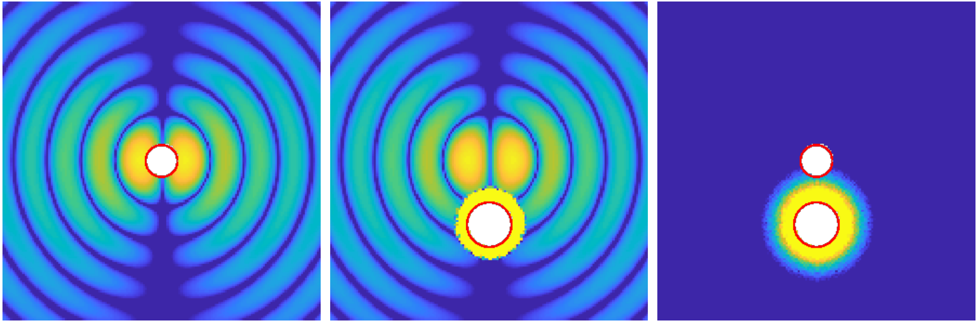
$$\boldsymbol{J}_{\mathrm{NR}}(\boldsymbol{r}) = \nabla \times (\nabla \times \boldsymbol{F}) - k^2\boldsymbol{F}$$

  with a differentiable vector field $\boldsymbol{F}$ produces a field $\boldsymbol{E} = -\mathrm{j}\omega\mu\boldsymbol{F}$. Note $\boldsymbol{F} = \mathbf{0}$ imply $\boldsymbol{J}_{\mathrm{NR}} = \boldsymbol{E} = \mathbf{0}$ [DW73; Kri16].
- Surface currents [Dev04] are more involved.
  - Cavity solutions with PEC boundary create a field with support inside the cavity. Low-dimensional space at the cavity resonance.
  - Cancellation from two sources with identical radiation.



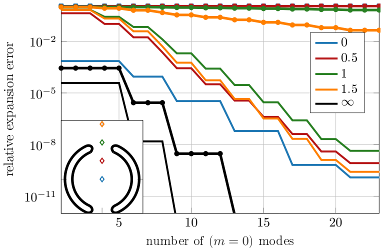


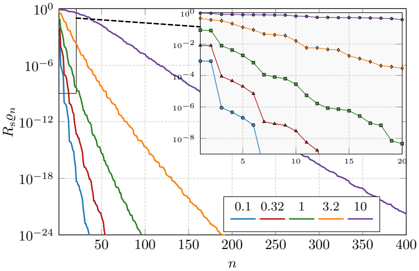
Non-radiating currents



Conclusions

- Decomposition of the radiation matrix $\mathbf{R}_{\mathrm{r}} = \mathbf{S}^{\mathsf{T}}\mathbf{S}$
- Diagonalization of the transition matrix
- Excitation in the form of regular waves
- Convergence can be bad for sources inside of a structures
- Non-radiating currents
- Rapid decay of the radiation modes





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