Herglotz functions, sum rules, and fundamental limitations on electromagnetic systems

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Complex analysis and convex optimization for EM design

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Herglotz functions and applications in electromagnetics

Typical applications are design of antennas, scatterers, filters, periodic structures, materials, ...

Desired to design physically small structures with good (the best) performance.
What does it have to do with Herglotz functions?
Here, we focus on the use of passive systems to derive sum rules and physical bounds. The are other approaches to derive fundamental limitations on the performance of metamaterial and devices:

- Often trade off between size and performance
- Based on assumptions such as linearity and passivity

Determine the stored energy in the (antenna) system:

\[ \pi \left( \int_{-\infty}^{\infty} s(t) e^{it\omega} \, dt \right) = 0 \]

Not more energy out than in for all times and signals.

Stieltjes functions [4]

Can it be used in scattering, metamaterial, ...?

Passive systems:

- Well-defined for small radiating structures
- Based on assumptions such as linearity and passivity

There are many passive systems:

- B
- Q-factor. Ratio of stored and dissipated energy.

There are many passive systems:

- Imply causality.

Physics based models [1,2]. Need a model of the internal structure.

Passivity is (here) a time domain system concept.

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Passivity is (here) a time domain system concept.

Material models such as Reection and transmission coefficients of periodic structures.

Forward scattering.

Integral identities (sum rules)

Herglotz and PR functions [3]

Bounds on frequency derivative (lossless) [1,2].

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Integral identities (sum rules)
Physical bounds on EM systems

- Integral representations
- Admittance
- Passive systems
- Scattering
- Sum rules
- Stored energy

Other approaches to derive these bounds include:
- Metamaterial and devices: Often trade-off between size and performance based on assumptions such as linearity and passivity.
- Integral identities for Herglotz functions with some positive measure.
- Closed-form expressions for fundamental limitations on performance.
- Convex optimization and Stieltjes functions for scattering, metamaterial, and devices.
- Larger structures, dispersive and inhomogeneous media. What is stored and radiated?

- Physical bounds on EM systems imply causality.
- Q-factor. Ratio of stored and dissipated energy.
- There are many passive systems:
  - Kramers-Kronig relations
  - Representation with some positive measure
  - State-space models
  - Physics-based models
- Admittance passive systems and internal structure.
- Antenna and material reflection coefficients.
- Material models such as...
Fundamental limitations on the performance

- based on assumptions such as linearity and passivity
- often tradeoff between size and performance
- metamaterial and devices:
  - temporal dispersion
  - thickness of absorbers
  - size of antennas
  - scattering and absorption in scatterers

Here, we focus on the use of passive systems to derive sum rules and physical bounds. There are other approaches to derive bounds.
Fundamental limitations on the performance

- based on assumptions such as linearity and passivity
- often tradeoff between size and performance
- metamaterial and devices:
  - temporal dispersion
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Here, we focus on the use of passive systems to derive sum rules and physical bounds. There are other approaches to derive bounds.
Physical bounds on EM systems

\[ \int_{-\infty}^{T} v(t)u(t) \, dt \geq 0 \]

System with input \( u \), output \( v = \theta \ast u \)

\[ \int_{-\infty}^{T} |u(t)|^2 - |v(t)|^2 \, dt \geq 0 \]

Integral representations

Sum rules

Stored energy
Not more energy out than in for all times and signals.

- Passivity is (here) a time domain system concept.
- Imply causality.
- Not sufficient with passive material (devices).
- There are many passive systems:
  - Admittance passive
    - Material models such as $s\epsilon(s)$ (bi-anisotropic).
    - Input impedance $Z_{in}(s)$.
    - Forward scattering.
  - Scattering passive
    - Antenna and material reflection coefficients.
    - Reflection and transmission coefficients of periodic structures.

1. Youla et al (1959)
2. Zemanian (1963, 1965)
3. Wohlers and Beltrami (1965)
Physical bounds on EM systems

\[ h(z) = Lz + \int_{-\infty}^{\infty} \frac{1}{\xi - z} \, d\nu(\xi) \]

Integral representations

Admittance

Passive systems

Scattering

Sum rules

Stored energy
Representation with some positive measure.

Kramers-Kronig relations

- Causal and $L^2$ signals (finite energy)
- Not suited for systems (not $L^2$)
- Bounds on frequency derivative (lossless) [1,2].

Passive systems

- Herglotz and PR functions [3]
- Integral identities (sum rules)

Stieltjes functions [4]

Convex optimization [5]

1. Landau, Lifshitz, Electrodynamics of Continuous Media.
3. Herglotz, Cauer, Nevanlinna, Pick, ...
5. Nordebo et al., IEEE-TAP (2014)
Physical bounds on EM systems

\[ \frac{2}{\pi} \int_{0}^{\infty} \frac{\text{Im} h(x)}{x^2} \, dx = a_1 - b_1 \]
Integral identities for Herglotz functions (passive systems) [1]

- Relates the low and high frequency asymptotes with the dynamic response

- Closed form expressions

- Investigated for many systems: matching [2], absorbers [3], scattering [4,5], antennas [6], high impedance surfaces [7], temporal dispersion [8], and extra ordinary transmission.

2. Bode, Fano (1950)
7. Gustafsson and Sjöberg, IEEE-TAP (2011)
8. Gustafsson and Sjöberg, NJP (2010)
9. see also table in King 2009.
Physical bounds on EM systems

- Sum rules
- Integral representations
- Admittance
- Passive systems
- Scattering

Physics based models, state-space models
Determine the stored energy in the (antenna) system

- Physics based models [1,2]. Need a model of the internal structure.
- State-space models [3]. Synthesizes a model of the internal structure.
- Q-factor. Ratio of stored and dissipated energy.
- $B \sim 2/Q$, $B$ fractional bandwidth [4].
- Well-defined for small radiating structures ($a \ll \lambda$).
- Larger structures, dispersive and inhomogeneous media. What is stored and radiated?
- Can it be used in scattering, metamaterial, ...?

3. Willems (1972)
Physical bounds on EM systems

- Integral representations
- Admittance
- Passive systems
- Scattering
- Stored energy

Sum rules and physical bounds. There are other approaches to derive admissibility and passivity based on assumptions such as linearity and passivity. Passive systems are well-defined for small radiating structures. Admittance and passive energy stored. Determining the stored energy in the (antenna) system:

\[ L z(x) = \frac{1}{Q} \]  

\[ \nu = x(1,0) \]  

\[ ν = \frac{1}{2} \]  

\[ x = |u| \]  

\[ ξ ≥ 0 \]  

- Convex optimization [5]
- Stieltjes functions [4]
Metamaterials are materials engineered to have properties that have not yet been found in nature.

Sometimes (often) applications that are difficult to realize:

- negative refraction:
- perfect absorbers:
- cloaking:
- artificial magnetism:

\[ \epsilon(\omega) < 0 \text{ for some frequency}. \]

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Metamaterials are materials engineered to have properties that have not yet been found in nature.

Sometimes (often) applications that are difficult to realize:

- negative refraction: \( n(\omega) < 0 \)
- perfect absorbers: \( r(\omega) \approx 0 \)
- cloaking: \( \sigma(\omega) \approx 0 \)
- artificial magnetism: \( \mu(\omega) > 1 \)
Metamaterials are materials engineered to have properties that have not yet been found in nature.

Sometimes (often) applications that are difficult to realize:

- **negative refraction**: constitutive relations \( n(\omega) < 0 \)
- **perfect absorbers**: reflection coefficient \( r(\omega) \approx 0 \)
- **cloaking**: cross section \( \sigma(\omega) \approx 0 \)
- **artificial magnetism**: constitutive relations \( \mu(\omega) > 1 \)

Start with the constitutive relations \( \epsilon(\omega) < 0 \) for some frequency.
Passive constitutive relations

The linear, causal, time translational invariant, continuous, non-magnetic, and isotropic constitutive relations are

\[ D(t) = \epsilon_0 \epsilon_\infty E(t) + \epsilon_0 \int_{\mathbb{R}} \chi_{ee}(t - t') E(t') \, dt' \]

where \( \chi_{ee}(t) = 0 \) for \( t < 0 \), the spatial coordinate is suppressed, and \( \epsilon_\infty > 0 \) is the instantaneous response. The material model is passive if

\[ 0 \leq \int_{-\infty}^T E(t) \cdot \frac{\partial D(t)}{\partial t} \, dt \]

for all times \( T \) and fields \( E \).

- Similarly for the magnetic fields.
- Fourier transform to get the frequency-domain model
  \[ D(\omega) = \epsilon_0 \epsilon(\omega) E(\omega) \]
  for the angular frequency \( \omega \).
- Herglotz function \( h(\omega) = \omega \epsilon(\omega) \) for passive models.
Implications of $h = \omega \epsilon(\omega)$ (Herglotz) for metamaterials?

- Are there $\epsilon(\omega_0) = \epsilon_m$ (e.g., $\epsilon_m = -1$) for a fixed frequency $\omega_0$? Yes, easy to synthesize.
- What about for a range of frequencies around $\omega_0$ (bandwidth)? Limited range with $\epsilon(\omega_0) \approx \epsilon_m$.

Analyze Herglotz functions $h(\omega)$:
- $h(\omega) = \omega \epsilon_\infty + o(\omega)$ as $\omega \to \infty$.
- $h(\omega) \approx \omega \epsilon_m$ for $\omega \in [\omega_1, \omega_2]$.

(Gustafsson and Sjöberg 2010), also (Landau, Lifshitz, and Pitaevskīĭ 1984; Nordebo et al. 2014; Skaar and Seip 2006).

Want $\epsilon(\omega_0) < 0$ for the perfect lens and cloaking.
Canonical form

Construct a new Herglotz function

\[ h(\omega) = \frac{\omega}{\omega_0} (\epsilon(\omega) - \epsilon_m) \sim \frac{\omega}{\omega_0} (\epsilon_\infty - \epsilon_m) \]

as \( \omega \to \infty \), where \( \epsilon_\infty \geq \epsilon_m \). Want

\[ h(\omega) \approx 0 \quad \text{for} \quad \omega \in [\omega_1, \omega_2] \]

Have a Herglotz function \( h(\omega) \) with \( h(\omega) \sim b_1 \omega \) as \( \omega \to \infty \).

How are the bandwidth \( \omega_2 - \omega_1 \), amplitude \( |h(\omega)| \leq \Delta \) over \( \omega \in [\omega_1, \omega_2] \), and coefficient \( b_1 \) related?

Compose with a Herglotz function which is unity (Im-part) if \( |z| < \Delta \).
Herglotz pulse function

$$h_{\Delta}(z) = \frac{1}{\pi} \int_{|\xi| \leq \Delta} \frac{1}{\xi - z} \, d\xi = \frac{1}{\pi} \ln \frac{z - \Delta}{z + \Delta} \sim \begin{cases} i \frac{2\Delta}{\pi z} & \text{as } z \to 0 \\ -\frac{2\Delta}{\pi z} & \text{as } z \to \infty \end{cases}$$

where $2\Delta$ is the width of the pulse. The composed Herglotz function ($h(\omega) \sim b_1 \omega$ as $\omega \to \infty$) has the asymptotic expansions

$$h_1(\omega) = h_{\Delta}(h(\omega)) \sim \begin{cases} \mathcal{O}(1) & \text{as } \omega \to 0 \\ -\frac{2\Delta}{\omega \pi b_1} & \text{as } \omega \to \infty \end{cases}$$
Use that a Herglotz function with this asymptotic expansion satisfies the integral identity

\[
\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{1}{\pi} \int_{\varepsilon < |x| < \frac{1}{\varepsilon}} \text{Im} \ h_1(x + iy) \, dx = \frac{2\Delta}{\pi b_1}
\]

and the symmetry \( h_1(z) = -h_1(-z^*)^* \) to get

\[
\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{\frac{1}{\varepsilon}} \text{Im} \ h_1(x + iy) \, dx \overset{\text{def}}{=} \frac{2}{\pi} \int_0^{\infty} \text{Im} \ h_1(x) \, dx = \frac{2\Delta}{\pi b_1}
\]

This simplified notation is sometimes used in this presentation. Totally, we have the sum rule (integral identity)

\[
\int_{0^+}^{\infty} \text{Im} \ h_\Delta(h(\omega)) \, d\omega = \frac{\Delta}{b_1} = \frac{\Delta\omega_0}{\epsilon_\infty - \epsilon_m}
\]
Example: Drude model

The Drude model (common model for metals and metamaterials)

\[ \epsilon(\omega) = 1 + \frac{1}{-i\omega(0.01 - i\omega)}, \]

- Interested in the behavior of 
  \( \epsilon(\omega) \approx -1 = \epsilon_m \)
- \( \epsilon(0.7) \approx -1 = \epsilon_m. \)
- Difference 
  \[ |\epsilon(\omega) - \epsilon_m| \leq \Delta = 0.4 \text{ for approximately } 0.6 \leq \omega \leq 0.8. \]
- Sum rule 
  \[ \frac{\Delta}{\epsilon_{\infty} - \epsilon_m} = \frac{0.4}{1-(-1)} = 0.2 \]

\[ \Im h_{\Delta 1}(\omega) \] with \( \Delta = 0.4. \)

Note, \( \Im h_{\Delta}(z) \approx 1 \text{ for } |z| < \Delta \) and \( \Im z \approx 0. \)


\text{(Gustafsson and Sjöberg 2010)}
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\( \text{Im} h_{\Delta_1}(\omega) \) with \( \Delta = 0.4 \).

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(Gustafsson and Sjöberg 2010)
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- Sum rule \(\frac{\Delta}{\epsilon_{\infty} - \epsilon_m} = \frac{0.4}{1 - (-1)} = 0.2\)

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Note, \(\text{Im} \; h_{\Delta}(z) \approx 1\) for \(|z| < \Delta\) and \(\text{Im} \; z \approx 0\).

\((\text{Gustafsson and Sjöberg 2010})\)
Consider an interval $B = [\omega_1, \omega_2]$ and estimate the sum rule

$$(\omega_2 - \omega_1) \min_{\omega_1 \leq \omega \leq \omega_2} \text{Im} h_\Delta(h(\omega)) \leq \int_{0^+}^{\infty} \text{Im} h_\Delta(h(\omega)) \, d\omega = \frac{\Delta}{b_1}$$

and use $\text{Im} h_\Delta(z) \geq 1/2$ for $|z| \leq \Delta$ to get

$$\Delta = \max_{\omega_1 \leq \omega \leq \omega_2} |h(\omega)| \geq \frac{\omega_2 - \omega_1}{2} b_1$$

Reintroduce $h = \omega(\epsilon - \epsilon_m)/\omega_0$, $b_1 = (\epsilon_\infty - \epsilon_m)/\omega_0$ and use the fractional bandwidth $B = (\omega_2 - \omega_1)/\omega_0$, $\omega_0 = (\omega_1 + \omega_2)/2$, $\epsilon_\infty =$ instantaneous, and $\epsilon_m =$ target values to get

$$\max_{\omega \in B} |\epsilon(\omega) - \epsilon_m| \geq \frac{B}{1 + B/2} (\epsilon_\infty - \epsilon_m) \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{cases}$$

where we also used that $\text{Im} h_\Delta(z) = 1$ for the lossless case.

Smoothly from the lossless case the lossy case as losses increases.
Similar bounds for other material cases

Interval $B = [\omega_1, \omega_2]$ with fractional bandwidth $B = (\omega_2 - \omega_1)/\omega_0$, $\omega_0 = (\omega_1 + \omega_2)/2$

$\epsilon_s =$ static, $\epsilon_\infty =$ instantaneous, $\epsilon_m =$ target values.

1. $\epsilon_m < \epsilon_\infty$:

$$\max_{\omega \in B} |\epsilon(\omega) - \epsilon_m| \geq \frac{B}{1 + B/2} (\epsilon_\infty - \epsilon_m) \left\{ \begin{array}{ll} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{array} \right.,$$

2. without static conductivity

$$\max_{\omega \in B} \frac{|\epsilon(\omega) - \epsilon_m|}{|\epsilon(\omega) - \epsilon_\infty|} \geq \frac{B}{1 + B/2} \frac{\epsilon_s - \epsilon_m}{\epsilon_s - \epsilon_\infty} \left\{ \begin{array}{ll} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{array} \right.,$$

3. artificial magnetism $\mu_m > \mu_s$

$$\max_{\omega \in B} \frac{|\mu(\omega) - \mu_m|}{|\mu(\omega) - \mu_\infty|} \geq \frac{B}{1 + B/2} \frac{\mu_m - \mu_s}{\mu_s - \mu_\infty} \left\{ \begin{array}{ll} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{array} \right.,$$


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Sum rules in Hilbert Transforms by King 2009, examples
Table 19.1. *Summary of sum rules for the dielectric constant*

<table>
<thead>
<tr>
<th>Number</th>
<th>Sum rule</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>[ \int_0^\infty \frac{\varepsilon_i(\omega)d\omega}{\omega} = \frac{\pi}{2} {\varepsilon_r(0) - \varepsilon_0} \text{ (insulators)} ]</td>
<td>Gorter and Kronig (1936)</td>
</tr>
<tr>
<td>(2)</td>
<td>[ \int_0^\infty \frac{{\varepsilon_i(\omega) - \sigma(0)/\omega}d\omega}{\omega} = \frac{\pi}{2} {\varepsilon_r(0) - \varepsilon_0} ]</td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>[ \int_0^\infty [\varepsilon_r(\omega) - \varepsilon_0]d\omega = 0 \text{ (insulators)} ]</td>
<td>Saslow (1970); Scaife (1972)</td>
</tr>
<tr>
<td>(4)</td>
<td>[ \int_0^\infty [\varepsilon_r(\omega) - \varepsilon_0]d\omega = -\frac{\pi \sigma(0)}{2} ]</td>
<td>Saslow (1970)</td>
</tr>
<tr>
<td>(5)</td>
<td>[ \int_0^\infty \omega \varepsilon_i(\omega)d\omega = \frac{\pi \varepsilon_0 \omega_p^2}{2} ]</td>
<td>Landau and Lifshitz (1960); Stern (1963)</td>
</tr>
<tr>
<td>(6)</td>
<td>[ \int_0^\infty [\varepsilon_r(\omega) - \varepsilon_0] \cos \omega t \ d\omega = \int_0^\infty \varepsilon_i(\omega) \sin \omega t \ d\omega, \ t &gt; 0 ]</td>
<td>Cole and Cole (1942); Scaife (1972); King (1978a)</td>
</tr>
<tr>
<td>(7)</td>
<td>[ \int_0^\infty [\varepsilon_r(\omega) - \varepsilon_0]^2d\omega = \int_0^\infty \varepsilon_i(\omega)^2d\omega \text{ (insulators)} ]</td>
<td></td>
</tr>
<tr>
<td>(8)</td>
<td>[ \int_0^\infty [\varepsilon_r(\omega) - \varepsilon_0][[\varepsilon_r(\omega) - \varepsilon_0]^2 - 3\varepsilon_i(\omega)^2]d\omega = 0 \text{ (insulators)} ]</td>
<td></td>
</tr>
<tr>
<td>(9)</td>
<td>[ \int_0^\infty \omega \varepsilon_i(\omega)(\varepsilon_r(\omega) - \varepsilon_0)d\omega = 0 \text{ (insulators)} ]</td>
<td>Villani and Zimerman (1973b)</td>
</tr>
</tbody>
</table>
### Table 19.2. Summary of sum rules for the refractive index

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<tr>
<td>(1)</td>
<td>$\int_0^\infty [n(\omega) - 1] d\omega = 0$</td>
<td>Saslow (1970); Altarelli et al. (1972); Smith (1985)</td>
</tr>
<tr>
<td>(2)</td>
<td>$\int_0^\infty \omega \kappa(\omega) d\omega = \frac{\pi}{4} \omega_p^2$</td>
<td>Kronig (1926)</td>
</tr>
<tr>
<td>(3)</td>
<td>$\int_0^\infty \frac{\kappa(\omega) d\omega}{\omega} = \frac{\pi}{2} {n(0) - 1}$ (insulators)</td>
<td>Moss (1961)</td>
</tr>
<tr>
<td>(4)</td>
<td>$\int_0^\infty \omega \kappa(\omega) n(\omega) d\omega = \frac{\pi}{4} \omega_p^2$</td>
<td>Villani and Zimerman (1973a)</td>
</tr>
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<td>(5)</td>
<td>$\int_0^\infty [n(\omega) - 1] \cos \omega t , d\omega = \int_0^\infty \kappa(\omega) \sin \omega t , d\omega, \quad t &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>(6)</td>
<td>$\int_0^\infty \omega \kappa(\omega) [3n(\omega)^2 - \kappa(\omega)^2] d\omega = \frac{3\pi}{4} \omega_p^2$</td>
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<td>(7)</td>
<td>$\int_0^\infty \omega \kappa(\omega) {n(\omega) - 1} d\omega = 0$</td>
<td>Stern (1963); Altarelli et al. (1972)</td>
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<td>(8)</td>
<td>$\int_0^\infty \omega^m \kappa(\omega) [3n(\omega) - 1]^2 - \kappa(\omega)^2] d\omega = 0, \quad m = 1, 3$</td>
<td>Villani and Zimerman (1973b)</td>
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<td>(9)</td>
<td>$\int_0^\infty \omega^m [n(\omega) - 1][{n(\omega) - 1}^2 - 3\kappa(\omega)^2] d\omega = 0, \quad m = 2, 4$</td>
<td>Villani and Zimerman (1973b)</td>
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</tbody>
</table>
Sum rules and physical bounds on passive systems

General simple approach

1. Identify a linear and passive system.
2. Construct a Herglotz (or similarly a positive real) function $h(z)$ that models the parameter of interest.
3. Investigate the asymptotic expansions of $h(z)$ as $z \to 0$ and $z \to \infty$.
4. Use integral identities for Herglotz functions to relate the dynamic properties to the asymptotic expansions.
5. Bound the integral.

Examples: Matching networks (Bode 1945; Fano 1950), Radar absorbers (Rozanov 2000), Antennas (Gustafsson 2010a; Gustafsson, Sohl, and Kristensson 2007; Gustafsson, Sohl, and Kristensson 2009), Scattering (Bernland, Gustafsson, and Nordebo 2011; Sohl, Gustafsson, and Kristensson 2007). High-impedance surfaces (Gustafsson and Sjöberg 2011), Metamaterials (Gustafsson and Sjöberg 2010). Extraordinary transmission (Gustafsson 2009). Periodic structures (Gustafsson et al. 2012),...
Definition (Passivity)

A system \( (v = h \ast u) \) is admittance-passive if

\[
\mathcal{W}_{\text{adm}}(T) = \Re \int_{-\infty}^{T} v^*(t)u(t) \, dt \geq 0
\]

and scatter-passive if

\[
\mathcal{W}_{\text{scat}}(T) = \int_{-\infty}^{T} |u(t)|^2 - |v(t)|^2 \, dt \geq 0,
\]

for all \( T \in \mathbb{R} \) and smooth functions of compact support \( u \).

Passivity is a systems concept. Not sufficient with passive materials (devices). Need less energy in the output signal than in the input signal for all times and signals.

The transfer function, \( H(s) \) is holomorphic (analytic) for \( \Re s > 0 \), and can be related to a positive real (PR) (or Herglotz) function.

(Wohlers and Beltrami 1965; Youla, Castriota, and Carlin 1959; Zemanian 1963; Zemanian 1965)
Passive systems: examples

▶ Reflection and transmission of periodic slabs (scattering)

\[ E_i \xrightarrow{\epsilon_r} E_t = T E_i \]
\[ E_r = \Gamma E_i \]

▶ Constitutive relations (admittance)

\[ D(t) = \epsilon_0 \epsilon_\infty E(t) + \epsilon_0 \int_{\mathbb{R}} \chi_{ee}(t - t') E(t') \, dt' \]

▶ Scattering (forward (admittance) and modes (scattering))

\[ \epsilon_0 \quad \mu_0 \]
\[ \epsilon(r) \quad \mu(r) \]
Definition (Herglotz functions, $h(z)$)

A Herglotz (Nevanlinna, Pick, or R-) function $h(z)$ is holomorphic for $\text{Im } z > 0$ and

$$\text{Im } h(z) \geq 0 \quad \text{for } \text{Im } z > 0$$

Representation for $\text{Im } z > 0$, cf., the Hilbert transform

$$h(z) = A_h + Lz + \int_{-\infty}^{\infty} \frac{1}{\xi - z} - \frac{\xi}{1 + \xi^2} \, d\nu(\xi)$$

where $A_h \in \mathbb{R}$, $L \geq 0$, and $\int_{\mathbb{R}} \frac{1}{1+\xi^2} \, d\nu(\xi) < \infty$. 

Gustav Herglotz 1881-1953
Rolf Nevanlinna 1895-1980
Georg Alexander Pick 1859-1942
Wilhelm Cauer 1900-1945
Integral identities for Herglotz functions

Herglotz functions with the symmetry \( h(z) = -h^*(-z^*) \) (real-valued in the time domain) have asymptotic expansions (\( N_0 \geq 0 \) and \( N_\infty \geq 0 \))

\[
\begin{align*}
  h(z) &= \sum_{n=0}^{N_0} a_{2n-1} z^{2n-1} + o(z^{2N_0-1}) \quad \text{as } z \to 0 \\
  h(z) &= \sum_{n=0}^{N_\infty} b_{1-2n} z^{1-2n} + o(z^{1-2N_\infty}) \quad \text{as } z \to \infty
\end{align*}
\]

where \( \to \) denotes limits in the Stoltz domain \( 0 < \theta \leq \arg(z) \leq \pi - \theta \). They satisfy the identities (\( 1 - N_\infty \leq n \leq N_0 \))

\[
\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{1/\varepsilon} \frac{\text{Im} \ h(x + iy)}{x^{2n}} \, dx = a_{2n-1} - b_{2n-1} = \begin{cases} 
-b_{2n-1} & n < 0 \\
 a_{-1} - b_{-1} & n = 0 \\
 a_{1} - b_{1} & n = 1 \\
 a_{2n-1} & n > 1 
\end{cases}
\]

Derivation of the integral identities

Similar integral identities can be derived under various assumptions.

- Passive system with PR (Herglotz) functions. Limits in the Stoltz domain.
- Holomorphic functions in a region that includes the frequency axis except for simple poles at the frequency axis. Limits in $\mathbb{C}_+$.
- $L^p$ functions at the frequency axis with $1 < p < \infty$ and with limits along the frequency axis.

Easy to show passivity but difficult to show the last two properties.
Integral identities for Herglotz functions

**Known low-frequency expansion** \((a_1 \geq 0)\):

\[
h(z) \sim \begin{cases} 
  a_1 z & \text{as } z \to 0 \\ 
  b_1 z & \text{as } z \to \infty
\end{cases}
\]

which gives the \(n = 1\) identity (we drop the limits for simplicity)

\[
\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{1/\varepsilon} \frac{\text{Im} h(x + iy)}{x^2} \, dx \overset{\text{def}}{=} \frac{2}{\pi} \int_0^\infty \frac{\text{Im} h(x)}{x^2} \, dx = a_1 - b_1 \leq a_1
\]

**Known high-frequency expansion (short times)** \((b_{-1} \leq 0)\):

\[
h(z) \sim \begin{cases} 
  a_{-1}/z & \text{as } z \to 0 \\ 
  b_{-1}/z & \text{as } z \to \infty
\end{cases}
\]

which gives the \(n = 0\) identity

\[
\frac{2}{\pi} \int_0^\infty \text{Im} h(x) \, dx = a_{-1} - b_{-1} \leq -b_{-1}.
\]
Some sum rules for passive systems

Sum rules for passive systems

- Spherical modes
- Extraordinary transmission
- Forward scattering
- Artificial $\mu$
- Dispersion $\varepsilon$ near 0
- Negative refraction
- Antennas
- Absorbers
- High-impedance surfaces
- Cross section
- Blockage
- Extraordinary transmission
- Matching
- Periodic structures

Mats Gustafsson, (21), EIT, Lund University, Sweden
Some sum rules for passive systems

- **Spherical modes**
  - $|\Gamma| \ll 1$
  - $\Gamma(0) = 1$

- **Scattering**
  - $|\Gamma| \ll 1$
  - $\Gamma(0) = -1$

- **Antennas**
  - $|\Gamma| \ll 1$
  - $\Gamma(0) = 1$

- **Absorbers**
  - $|Y| \approx 0$
  - $Y(0) = \infty$

- **High-impedance surfaces**
  - $|T| \ll 1$
  - $T(0) = 1$

- **Periodic structures**
  - $|\Gamma| \ll 1$
  - $\Gamma(0) = 1$

- **Extraordinary transmission**
  - $|Y| \approx 0$
  - $Y(0) = \infty$

- **Cross section**
  - $\text{Re} \ Y_{\text{big}}$
  - $Y(0) = 0$

- **Blockage**
  - $|T| \ll 1$
  - $T(0) = 1$

- **Artificial $\mu$**
  - $|Y| \approx 0$
  - $Y(0) = \infty$

- **Dispersion**
  - $|Y| \approx 0$
  - $Y(\infty) = \infty$

- **Negative refraction**
  - $|Y| \approx 0$
  - $Y(\infty) = \infty$

- **Matching**
  - $|Y| \approx 0$
  - $Y(0) = \infty$

Mats Gustafsson, (22), EIT, Lund University, Sweden
A physical bound for absorbers

- A structure (above a ground plane) that absorbs incident EM waves.
- Pyramids, homogeneous, periodic, metamaterials,…
- Often desired to be thin and absorb energy over large bandwidths.

Tradeoff between thickness $d$ fractional bandwidth $B$ and wavelength $\lambda$;

\[
\lambda_2 - \lambda_1 = B \lambda_0 \leq \frac{2\pi^2 d \mu_s}{\ln \Gamma_0^{-1}} \leq \frac{172 d \mu_s}{|\Gamma_0, dB|}
\]

$\Gamma_0 = \max_{\lambda_1 \leq \lambda \leq \lambda_2} |\Gamma(\lambda)|$ and $\mu_s$ is the maximal static relative permeability of the absorber.

(Rozanov 2000)
1. Identify the reflection coefficient, $\Gamma$, as a passive system ($|\Gamma| \leq 1$).

2. Analyze the low- (and high) frequency behavior:

$$\Gamma(k) \sim -1 - ik(2d \cos \theta + \gamma/A), \quad k \to 0$$

where $\gamma$ is the polarizability per unit cell. A well-defined static quantity which is easily determined.

3. Construct the Herglotz function $h = -i \ln(\Gamma/B)$ and the sum rule

$$\frac{2}{\pi} \int_0^\infty \frac{1}{k^2} \ln \frac{1}{|\Gamma(k)|} \, dk \leq 2d \cos \theta + \gamma/A \leq 2\mu_S d$$

(Gustafsson and Sjöberg 2011; Rozanov 2000)
Rewrite in the wavelength $\lambda = \frac{2\pi}{k}$ and estimate the integral, e.g.,

$$\frac{1}{\pi^2} (\lambda_2 - \lambda_1) \ln \frac{1}{|\Gamma_0|} d\lambda \leq \frac{1}{\pi^2} \int_{\lambda_1}^{\lambda_2} \ln \frac{1}{|\Gamma(\lambda)|} d\lambda \leq \frac{1}{\pi^2} \int_{0}^{\infty} \ln \frac{1}{|\Gamma(\lambda)|} d\lambda \leq 2\mu_s d$$

with $\Gamma_0 = \max_{[\lambda_1,\lambda_2]} |\Gamma(\lambda)|$.

Bandwidth limited by the thickness $d$ and (static) permeability $\mu_s$.

Applied to array antennas in (Doane, Sertel, and Volakis 2013; Jonsson, Kolitsidas, and Hussain 2013).
Magnitude of scattering parameters

The absorber is an example with scattering parameters $|\Gamma(0)| = 1$ where it is desired to have $|\Gamma(\omega)| \leq \Gamma_0$.

Map to Herglotz/PR functions using $\ln(h/B)$ and use the $n = 1$ identity.

Similar approaches for matching (Bode 1945; Fano 1950), transmission blockage (Gustafsson et al. 2009; Sjöberg, Gustafsson, and Larsson 2010), mode scattering (Bernland 2012; Bernland, Gustafsson, and Nordebo 2011).
Magnitude of Herglotz/PR functions

- Scattering parameters $\Gamma(0) = \mp 1$ and desire $\Gamma(j\omega) \approx \pm 1$
- Admittance $P(0) = \infty$ and desire $|P(j\omega)| \leq P_0$.

Temporal dispersion (Gustafsson and Sjöberg 2010), high-impedance surfaces (Gustafsson and Sjöberg 2011), extraordinary transmission (Gustafsson, Sjöberg, and Vakili 2011), and superluminal transmission (Gustafsson 2012).
Basically four (or two) cases

admittance want
large $\Re Y(\omega_0)$ with $Y(0) = 0$.
forward scattering (cross section)
Use identity
S-parameter want
$|S(\omega_0)| \leq \delta$ with $|S(0)| = 1$.
absorber, matching, blockage, modes, ...
Use log+identity

admittance want
small $|Y(\omega_0)| \leq \delta$ with $Y(0) = \infty$.
high impedance surface, temporal dispersion.
Use pulse+identity
S-parameter want
$S(\omega_0) \approx 1$ with $S(0) = -1$.
high impedance surface,
extraordinary transmission
Use Cayley+pulse+identity

Many physical bounds based on sum rules can be formulated as these
4 (or 2) cases. Convex optimization can be used for some other cases.
Passive systems and sum rules

$|S(0)| = 1$
to
$|S(\omega_0)| \leq \delta$

$P(0) = 0$
to
$\Re P(\omega_0) \geq \delta$

$|P(\omega_0)| \leq \delta$

$P(0) = \infty$
to
$\Re P(\omega_0) \geq \delta$

$\epsilon \approx 0$

$\mu_{\text{artificial}}$

$n \approx -1$

Array antennas

Absorbers

Blockage

Matching

High impedance surface

High impedance surface

EOT

Cross sections

Cayley

Forward scattering

Antenna $D/Q$
Challenges with the sum rules technique

Use of passive systems, Herglotz functions, and sum rules is a very powerful technique to derive bounds for EM design problems. Some challenges:

▶ Can easily change a problem to a physically equivalent problem where the approach does not work so well. For example: change of a PEC ground plane to Cu changes the expansion to $-d + \gamma \omega$ as $\omega \to 0$, where $d < 1$. Cannot use the integral identities.

▶ Multi-parameter cases, have often more than one parameter, e.g., frequency and incident angle. Can the parameters be treated together?

▶ Have sometimes active systems or systems that are made of passive materials but not time-domain passive.

Some of these problems can be analyzed using convex optimization techniques.
How do we relate the antenna with Herglotz functions?

\[
\hat{\varepsilon} \ \hat{k} \quad \hat{\varepsilon} = \varepsilon(r) \mu(r) \quad \hat{\varepsilon} \ \hat{k}
\]

Assumptions:

- Finite scattering object composed of a linear, passive, and time translational invariant materials.
- Incident linearly polarized plane wave.

Passive system with \( h(k) \sim \gamma k \) as \( k \to 0 \) and \( \sigma_{\text{ext}} = \text{Im} h \).

From physics:

- The propagation speed is limited by the speed of light.
- Optical theorem (energy conservation).
- Induced dipole moment in the static limit.

(Gustafsson 2010b; Purcell 1969; Sohl, Gustafsson, and Kristensson 2007)
Forward scattering sum rule

Use the \( n = 1 \) identity with

\[
a_1 = \gamma = \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e}) \quad \text{and} \quad b_1 = 0, \quad \text{i.e.,}
\]

\[
\frac{2}{\pi} \int_0^\infty \frac{\sigma_{\text{ext}}(k)}{k^2} \, dk = \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e})
\]

or written in the free-space wavelength \( \lambda = 2\pi/k \)

\[
\frac{1}{\pi^2} \int_0^\infty \sigma_{\text{ext}}(\lambda) \, d\lambda = \hat{e} \cdot \gamma_e \cdot \hat{e} + (\hat{k} \times \hat{e}) \cdot \gamma_m \cdot (\hat{k} \times \hat{e})
\]
Forward scattering measurement of a dipole antenna.

- Loaded, short, and open circuit.
- Length 15 cm and 0.5 GHz to 6 GHz.

<table>
<thead>
<tr>
<th></th>
<th>loaded</th>
<th>short</th>
<th>open</th>
</tr>
</thead>
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</tr>
<tr>
<td>meas:</td>
<td>605</td>
<td>670</td>
<td>322</td>
</tr>
</tbody>
</table>

Antenna forward scattering bounds (rectangles)

\[ D/Q/(k_0a)^3 \]

Chu bound, \( k_0a \ll 1 \)

physical bounds

(Gustafsson, Sohl, and Kristensson 2007; Gustafsson, Sohl, and Kristensson 2009)
Antenna forward scattering bounds (rectangles)

\[ \frac{D/Q}{(k_0 a)^3} \]

Chu bound, \( k_0 a \ll 1 \)

physical bounds

\((Gustafsson, Sohl, and Kristensson 2007; Gustafsson, Sohl, and Kristensson 2009)\)
Sum rules (integral identities) and stored energy are commonly used to construct physical bounds. Both are related to passive systems.

The stored energy is used to determine the Q-factor and hence an estimate of the bandwidth. This changes the perspective from optimization of the frequency behavior of the system to optimization of the states that models the system at a fixed frequency.
Antenna current optimization and physical bounds [1].
Single frequency antenna optimization, e.g., minimize $Q$ [2].

Express the stored energies as semi-positive quadratic forms in the current (density) $I$:

$$W_e \sim I^H X_e I \geq 0 \quad \text{and} \quad W_m \sim I^H X_m I \geq 0$$

 Stored energy and antenna current optimization

- Antenna current optimization and physical bounds [1].
- Single frequency antenna optimization, e.g., minimize $Q$ [2].

Express the stored energies as semi-positive quadratic forms in the current (density) $I$:

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Express the stored energies as semi-positive quadratic forms in the current (density) $I$:

$$W_e \sim I^H X_e I \geq 0 \quad \text{and} \quad W_m \sim I^H X_m I \geq 0$$

What is stored energy? (lumped circuits)

- Capacitor $i = C \frac{dv}{dt}$.
- Multiply with $v$ to get the power $iv = \frac{d}{dt} \frac{C|v|^2}{2}$ and integrate

$$\int_{-\infty}^{T} iv \, dt = \frac{C}{2} (|v(T)|^2 - |v(-\infty)|^2)$$

- Time harmonic case

$v(t) = \text{Re}\{V e^{j\omega t}\}$ gives the time average stored electric energy

$W_e = C|V|^2/4$

and in inductors

$W_m = \frac{L|I|^2}{4}$
What is stored energy? (lumped circuits)

- Capacitor $i = C \frac{dv}{dt}$.
- Multiply with $v$ to get the power $iv = \frac{d}{dt} \frac{C|v|^2}{2}$ and integrate
  \[
  \int_{-\infty}^{T} iv \, dt = \frac{C}{2} (|v(T)|^2 - |v(-\infty)|^2)
  \]
- Time harmonic case
  $v(t) = \text{Re}\{V e^{j\omega t}\}$ gives the time average stored electric energy
  $W_e = C|V|^2/4$

Stored energy from the $\frac{d}{dt}$-differentiated quadratic form. Frequency domain
$\frac{d}{dt} \rightarrow s$ implies term proportional to the frequency, $s = j\omega$.

Lumped elements

Time average stored energy in capacitors

$W_e = \frac{C|V|^2}{4} = \frac{|I|^2}{4\omega^2 C}$

and in inductors

$W_m = \frac{L|I|^2}{4}$
Rational PR functions

Consider a rational PR function \( (a_n, b_n \in \mathbb{R}) \)

\[
Z(s) = \frac{\sum_{n=0}^{N_b} b_n s^n}{\sum_{n=0}^{N_a} a_n s^n}
\]

Often good for low frequencies but there are passive systems that are not well represented by rational functions.

Rewrite as a state-space model

\[
\begin{aligned}
\frac{d}{dt} x &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
\]

with

\[
Z(s) = \frac{Y(s)}{U} = C(sI - A)^{-1}B + D
\]
Rational PR functions

Consider a rational PR function \((a_n, b_n) \in \mathbb{R})

\[
Z(s) = \sum_{N} b_n s^n \sum_{N} a_n s^n
\]

Often go good for low frequencies but there are passive systems that are not well represented by rational functions.

Rewrite as a state-space model

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

with

\[
Z(s) = \frac{Y(s)}{U} = C(sI - A)^{-1} - B + D
\]
Rational PR functions

Consider a rational PR function \((a_n, b_n \in \mathbb{R})\)

\[
Z(s) = \frac{\sum_{n=0}^{N_b} b_n s^n}{\sum_{n=0}^{N_a} a_n s^n}
\]

Often good for low frequencies but there are passive systems that are not well represented by rational functions.

Rewrite as a state-space model

\[
\begin{align*}
\frac{dx}{dt} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

with

\[
Z(s) = \frac{Y(s)}{U} = C(sI - A)^{-1}B + D
\]
Consider a state-space model with input $u$ and output $y$

\[
\begin{align*}
\frac{d}{dt} x &= Ax + Bu \\
y &=Cx + Du
\end{align*}
\]

- Stored energy has been analyzed for state-space models.
- Need minimal representations and reciprocity (symmetry). Necessary but not sufficient.
- Still some open questions.
- What about for non-rational PR (or Herglotz) functions?
- Can we define a (useful) stored energy for a Herglotz (or PR) function?

J.C. Willems, Dissipative dynamical systems I,II (1972), ....., (2013)
Summary

- Herglotz/PR functions, physical bounds, and passive systems.
- Sufficient with passivity and low- and/or high-frequency expansion to derive sum rules and physical bounds.
- Antennas, scatterers, absorbers, high impedance surfaces, temporal dispersion, ...
- The bounds are tight for many cases.
- Need active (non-foster), non-linear and/or time varying devices to overcome the bounds.
- Also convex optimization using the integral representation and minimization of stored energy.
References I


