Herglotz functions, sum rules, and fundamental limitations on electromagnetic systems

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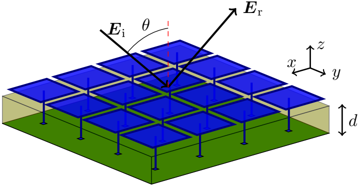
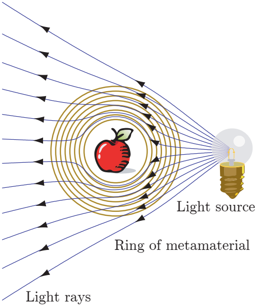
Complex analysis and convex optimization for EM design

Department of Electrical and Information Technology
Lund University

Herglotz functions and applications in electromagnetics

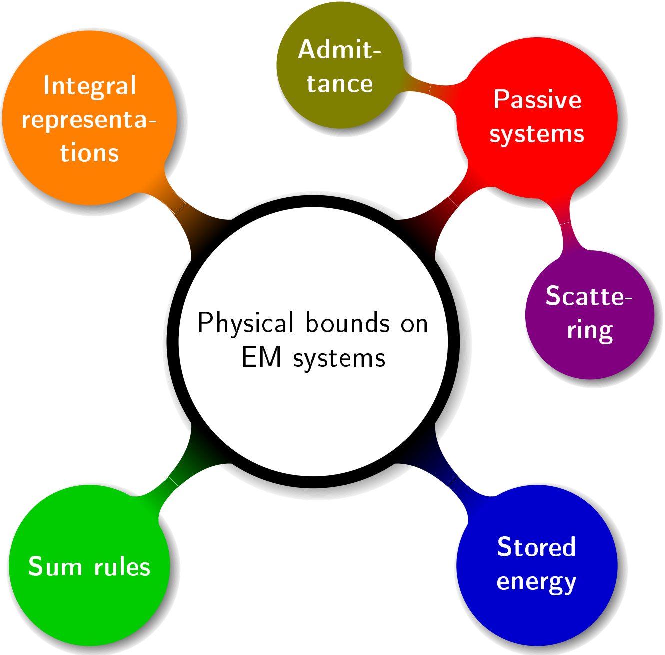
Typical applications are design of antennas, scatterers, filters, periodic structures, materials, ...

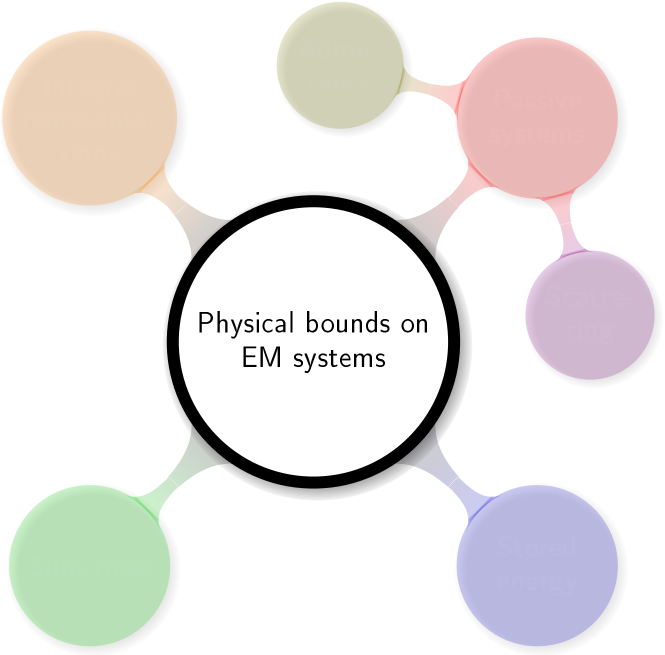




Desired to design physically small structures with good (the best) performance.

What does it have to do with Herglotz functions?


Integral representa- tions
Admit- tance
Passive systems
Scatte- ring
Physical bounds on EM systems
Sum rules
Stored energy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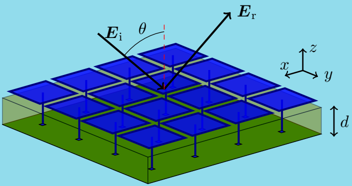
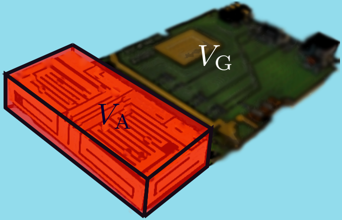
Fundamental limitations on the performance

- based on assumptions such as linearity and passivity
- often tradeoff between size and performance
- metamaterial and devices:
  - temporal dispersion
  - thickness of absorbers
  - size of antennas
  - scattering and absorption in scatterers





Here, we focus on the use of passive systems to derive sum rules and physical bounds. The are other approaches to derive bounds.

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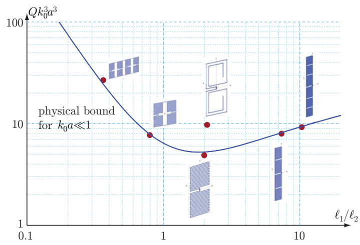

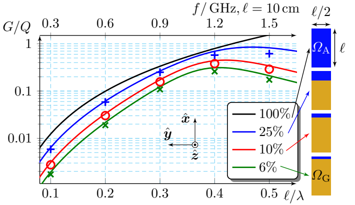

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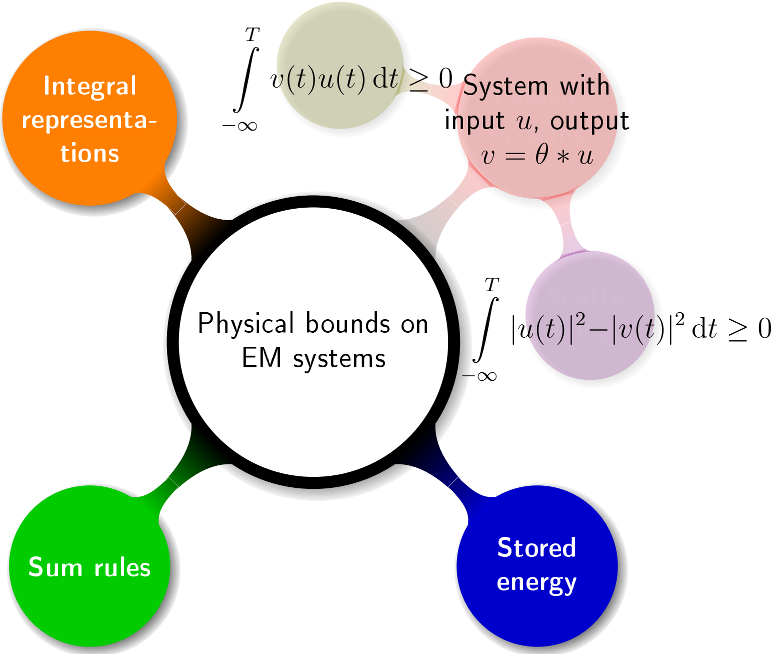

Integral representa-tions
$\int_{-\infty}^{T} v(t)u(t)\,\mathrm{d}t \geq 0$
System with input u, output $v = \theta * u$
Physical bounds on EM systems
$\int_{-\infty}^{T} |u(t)|^2 - |v(t)|^2\,\mathrm{d}t \geq 0$
Sum rules
Stored energy

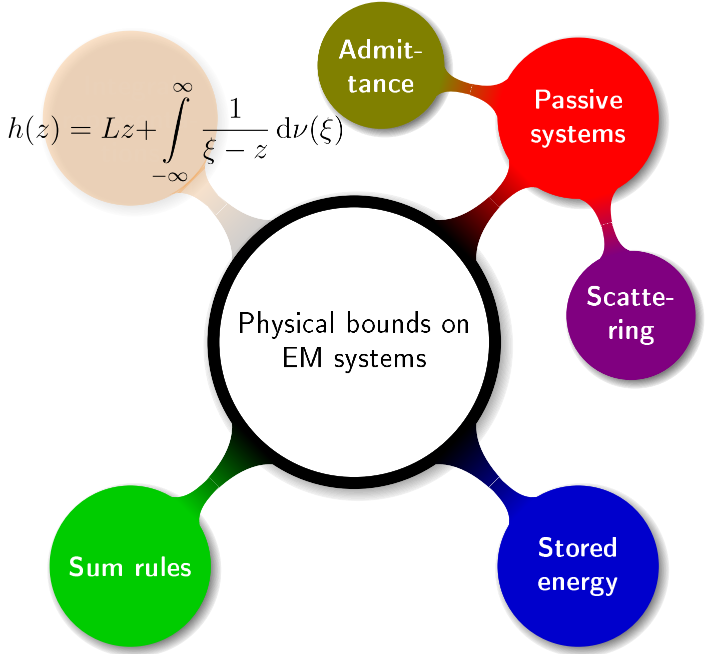
Not more energy out than in for all times and signals.

- Passivity is (here) a time domain system concept.
- Imply causality.
- Not sufficient with passive material (devices).
- There are many passive systems:
  Admittance passive
  - Material models such as $s\epsilon(s)$ (bi-anisotropic).
  - Input impedance $Z_{\text{in}}(s)$.
  - Forward scattering.

  Scattering passive
  - Antenna and material reflection coefficients.
  - Reflection and transmission coefficients of periodic structures.

1. Youla *etal*(1959)
2. Zemanian (1963,1965)
3. Wohlers and Beltrami (1965)

$^2\,\mathrm{d}t \geq 0$


$h(z) = Lz + \int_{-\infty}^{\infty} \frac{1}{\xi - z} \,\mathrm{d}\nu(\xi)$
Admit-
tance
Passive
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Scatte-
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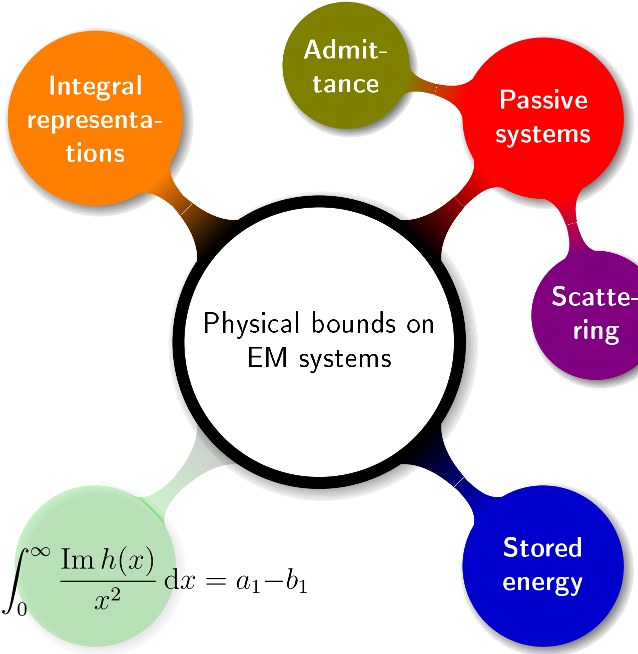
$$h(z) = Lz + \int_{-\infty}^{\infty} \frac{1}{\xi - z} \, \mathrm{d}\nu(\xi)$$

Admit-
tance

Passive
systems

- Representation with some positive measure.
- Kramers-Kronig relations
  - Causal and L^2 signals (finite energy)
  - Not suited for systems (not L^2)
  - Bounds on frequency derivative (lossless) [1,2].
- Passive systems
  - Herglotz and PR functions [3]
  - Integral identities (sum rules)
- Stieltjes functions [4]
- Convex optimization [5]

1. Landau, Lifshitz, Electrodynamics of Continuous Media.
2. King, The Hilbert transform I,II (2009)
3. Herglotz, Cauer, Nevanlinna, Pick, ...
4. Milton, The Theory of Composites (2002)
5. Nordebo *etal*IEEE-TAP (2014)



$$\frac{2}{\pi}\int_0^\infty \frac{\operatorname{Im} h(x)}{x^2}\,\mathrm{d}x = a_1 - b_1$$

Integral representa tions

- Integral identities for Herglotz functions (passive systems) [1]
- Relates the low and high frequency asymptotes with the dynamic response
- Closed form expressions
- Investigated for many systems: matching [2], absorbers [3], scattering [4,5], antennas [6], high impedance surfaces [7], temporal dispersion [8], and extra ordinary transmission.

1. Bernland *etal*, J.Phys.A (2011)
2. Bode, Fano (1950)
3. Rozanov, IEEE-TAP (2000)
4. Purcell, J. Astrophys (1969)
5. Sohl *etal*, J.Phys.A (2007), J.Phys.D (2007), J.Appl.Phys (2008)
6. Gustafsson *etal*, , Proc.R.Soc.A (2007), IEEE-TAP (2009)
7. Gustafsson and Sjöberg, IEEE-TAP (2011)
8. Gustafsson and Sjöberg, NJP (2010)
9. see also table in King 2009.

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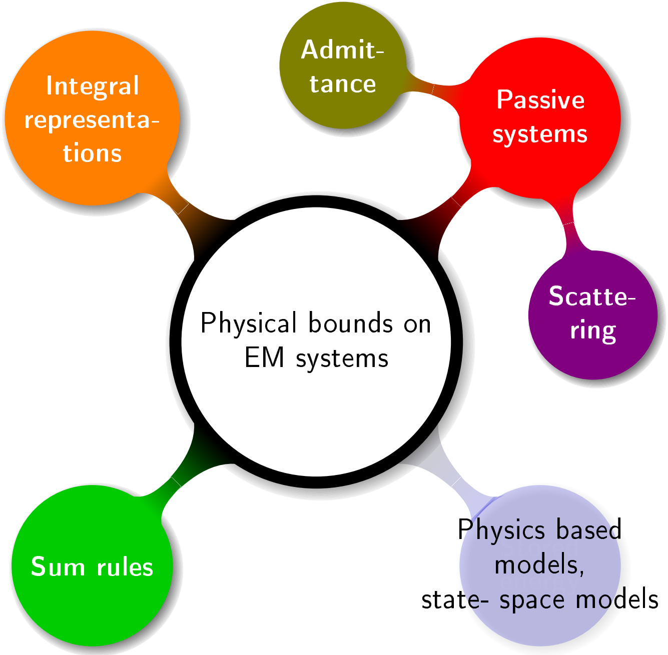
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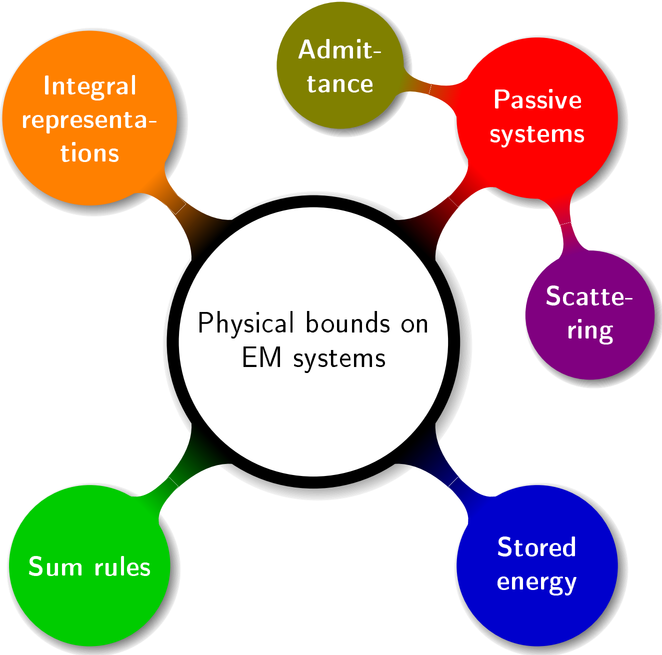
Physics based models, state- space models

Admit-

Determine the stored energy in the (antenna) system

- Physics based models [1,2]. Need a model of the internal structure.
- State-space models [3]. Synthesizes a model of the internal structure.
- Q-factor. Ratio of stored and dissipated energy.
- $B \sim 2/Q$, B fractional bandwidth [4].
- Well-defined for small radiating structures ($a \ll \lambda$).
- Larger structures, dispersive and inhomogeneous media. What is stored and radiated?
- Can it be used in scattering, metamaterial, ...?

1. Vandenbosch, IEEE-TAP (2010)
2. Gustafsson *etal*, IEEE-TAP (2012,2015)
3. Willems (1972)
4. Yaghjian, Best, IEEE-TAP (2005)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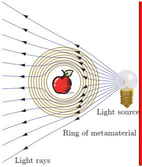
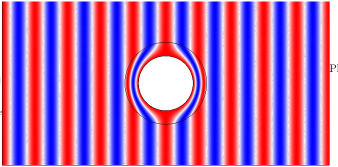
Metamaterials

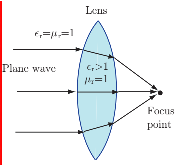
Metamaterials are materials engineered to have properties that have not yet been found in nature.

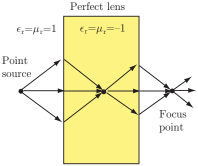
Sometimes (often) applications that are difficult to realize:

- negative refraction:
- perfect absorbers:
- cloaking:
- artificial magnetism:









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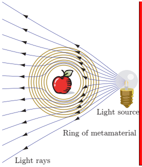
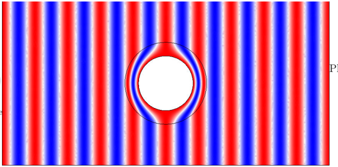
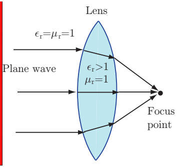
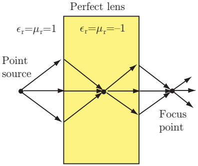
Sometimes (often) applications that are difficult to realize:

- negative refraction: constitutive relations $n(\omega) < 0$
- perfect absorbers: reflection coefficient $r(\omega) \approx 0$
- cloaking: cross section $\sigma(\omega) \approx 0$
- artificial magnetism: constitutive relations $\mu(\omega) > 1$









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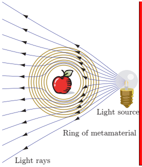
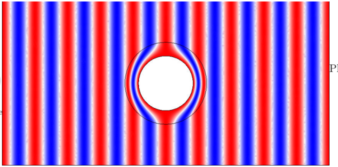
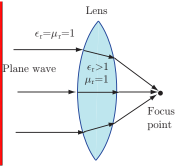
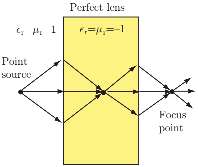








Start with the constitutive relations $\epsilon(\omega) < 0$ for some frequency.

Passive constitutive relations

The linear, causal, time translational invariant, continuous, non-magnetic, and isotropic constitutive relations are

$$\boldsymbol{D}(t) = \epsilon_0\epsilon_\infty \boldsymbol{E}(t) + \epsilon_0 \int_{\mathbb{R}} \chi_{\mathrm{ee}}(t-t')\boldsymbol{E}(t')\,\mathrm{d}t'$$

where $\chi_{\mathrm{ee}}(t) = 0$ for $t < 0$, the spatial coordinate is suppressed, and $\epsilon_\infty > 0$ is the instantaneous response. The material model is passive if

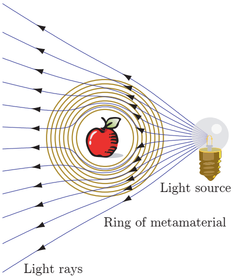
$$0 \leq \int_{-\infty}^{T} \boldsymbol{E}(t) \cdot \frac{\partial \boldsymbol{D}(t)}{\partial t}\,\mathrm{d}t$$

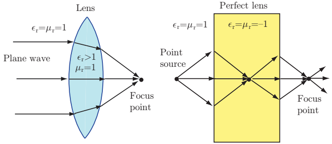
for all times T and fields $\boldsymbol{E}$.

- Similarly for the magnetic fields.
- Fourier transform to get the frequency-domain model $\boldsymbol{D}(\omega) = \epsilon_0\epsilon(\omega)\boldsymbol{E}(\omega)$ for the angular frequency ω.
- Herglotz function $h(\omega) = \omega\epsilon(\omega)$ for passive models.

Implications of $h = \omega\epsilon(\omega)$ (Herglotz) for metamaterials?

- Are there $\epsilon(\omega_0) = \epsilon_{\mathrm{m}}$ (*e.g.*, $\epsilon_{\mathrm{m}} = -1$) for a fixed frequency ω_0? Yes, easy to synthesize.
- What about for a range of frequencies around ω_0 (bandwidth)? Limited range with $\epsilon(\omega_0) \approx \epsilon_{\mathrm{m}}$.





Analyze Herglotz functions $h(\omega)$:

- $h(\omega) = \omega\epsilon_\infty + o(\omega)$ as $\omega \hat{\to} \infty$.
- $h(\omega) \approx \omega\epsilon_{\mathrm{m}}$ for $\omega \in [\omega_1, \omega_2]$.

(Gustafsson and Sjöberg 2010), also (Landau, Lifshitz, and Pitaevskiĭ 1984; Nordebo et al. 2014; Skaar and Seip 2006).

Want $\epsilon(\omega_0) < 0$ for the perfect lens and cloaking.

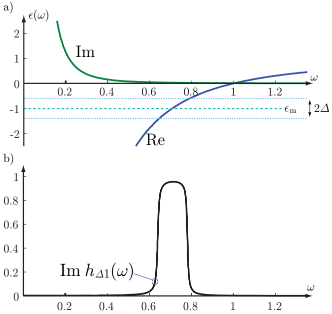
Canonical form

Construct a new Herglotz function

$$h(\omega) = \frac{\omega}{\omega_0}(\epsilon(\omega) - \epsilon_{\mathrm{m}}) \sim \frac{\omega}{\omega_0}(\epsilon_\infty - \epsilon_{\mathrm{m}})$$

as $\omega \hat{\to} \infty$, where $\epsilon_\infty \geq \epsilon_{\mathrm{m}}$. Want

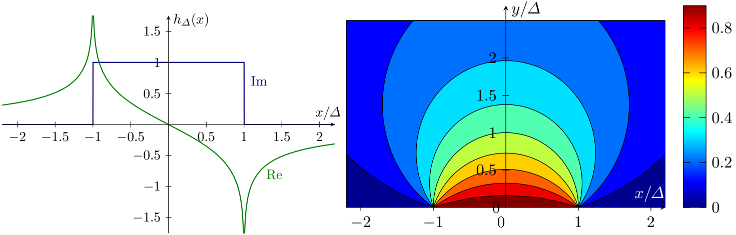
$$h(\omega) \approx 0 \quad \text{for} \quad \omega \in [\omega_1, \omega_2]$$



Have a Herglotz function $h(\omega)$ with $h(\omega) \sim b_1\omega$ as $\omega \hat{\to} \infty$.
How are the bandwidth $\omega_2 - \omega_1$, amplitude $|h(\omega)| \leq \Delta$ over $\omega \in [\omega_1, \omega_2]$, and coefficient b_1 related?

Compose with a Herglotz function which is unity (Im-part) if $|z| < \Delta$.

Herglotz pulse function



Herglotz pulse function

$$h_\Delta(z) = \frac{1}{\pi}\int_{|\xi|\le\Delta} \frac{1}{\xi - z}\,\mathrm{d}\xi = \frac{1}{\pi}\ln\frac{z-\Delta}{z+\Delta} \sim \begin{cases} \mathrm{i} & \text{as} \quad z\to 0 \\ -\frac{2\Delta}{\pi z} & \text{as} \quad z\to\infty \end{cases}$$

where 2Δ is the width of the pulse. The composed Herglotz function ($h(\omega)\sim b_1\omega$ as $\omega\hat{\to}\infty$) has the asymptotic expansions

$$h_1(\omega) = h_\Delta(h(\omega)) \sim \begin{cases} \mathcal{O}(1) & \text{as} \quad \omega\hat{\to}0 \\ -\frac{2\Delta}{\omega\pi b_1} & \text{as} \quad \omega\hat{\to}\infty \end{cases}$$

Integral identities and sum rules

Use that a Herglotz function with this asymptotic expansion satisfies the integral identity

$$\lim_{\varepsilon\to 0^+}\lim_{y\to 0^+}\frac{1}{\pi}\int_{\varepsilon<|x|<\frac{1}{\varepsilon}}\operatorname{Im} h_1(x+\mathrm{i}y)\,\mathrm{d}x=\frac{2\varDelta}{\pi b_1}$$

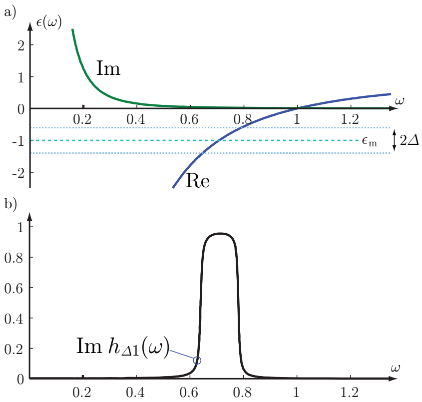
and the symmetry $h_1(z)=-h_1(-z^*)^*$ to get

$$\lim_{\varepsilon\to 0^+}\lim_{y\to 0^+}\frac{2}{\pi}\int_{\varepsilon}^{\frac{1}{\varepsilon}}\operatorname{Im} h_1(x+\mathrm{i}y)\,\mathrm{d}x\overset{\text{def}}{=}\frac{2}{\pi}\int_{0^+}^{\infty}\operatorname{Im} h_1(x)\,\mathrm{d}x=\frac{2\varDelta}{\pi b_1}$$

This simplified notation is sometimes used in this presentation. Totally, we have the sum rule (integral identity)

$$\int_{0^+}^{\infty}\operatorname{Im} h_{\varDelta}(h(\omega))\,\mathrm{d}\omega=\frac{\varDelta}{b_1}=\frac{\varDelta\omega_0}{\epsilon_\infty-\epsilon_{\mathrm{m}}}$$

Example: Drude model



$\mathrm{Im}\, h_{\Delta 1}(\omega)$ with $\Delta = 0.4$.
Note, $\mathrm{Im}\, h_{\Delta}(z) \approx 1$ for $|z| < \Delta$ and $\mathrm{Im}\, z \approx 0$.
(Gustafsson and Sjöberg 2010)

The Drude model (common model for metals and metamaterials)

$$\epsilon(\omega) = 1 + \frac{1}{-\mathrm{i}\omega(0.01 - \mathrm{i}\omega)},$$

- Interested in the behavior of $\epsilon(\omega) \approx -1 = \epsilon_{\mathrm{m}}$
- $\epsilon(0.7) \approx -1 = \epsilon_{\mathrm{m}}$.
- Difference $|\epsilon(\omega) - \epsilon_{\mathrm{m}}| \leq \Delta = 0.4$ for approximately $0.6 \leq \omega \leq 0.8$.
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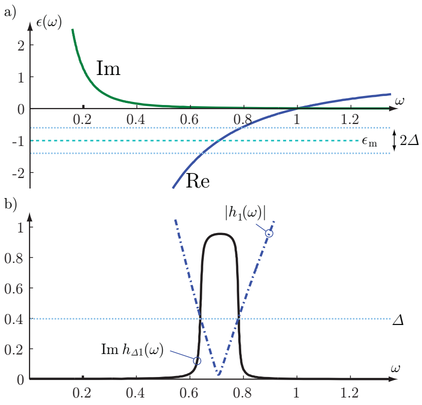

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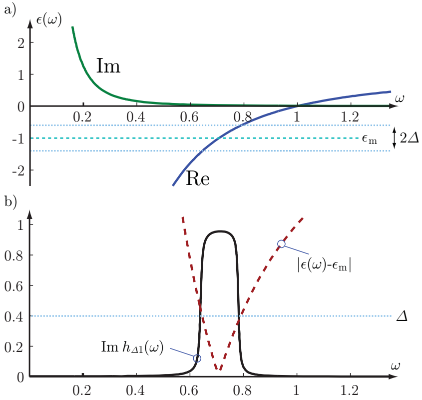

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Physical bounds

Consider an interval $\mathcal{B} = [\omega_1, \omega_2]$ and estimate the sum rule

$$(\omega_2 - \omega_1) \min_{\omega_1 \leq \omega \leq \omega_2} \operatorname{Im} h_\Delta(h(\omega)) \leq \int_{0^+}^{\infty} \operatorname{Im} h_\Delta(h(\omega)) \, \mathrm{d}\omega = \frac{\Delta}{b_1}$$

and use $\operatorname{Im} h_\Delta(z) \geq 1/2$ for $|z| \leq \Delta$ to get

$$\Delta = \max_{\omega_1 \leq \omega \leq \omega_2} |h(\omega)| \geq \frac{\omega_2 - \omega_1}{2} b_1$$

Reintroduce $h = \omega(\epsilon - \epsilon_\mathrm{m})/\omega_0$, $b_1 = (\epsilon_\infty - \epsilon_\mathrm{m})/\omega_0$ and use the fractional bandwidth $B = (\omega_2 - \omega_1)/\omega_0$, $\omega_0 = (\omega_1 + \omega_2)/2$, ϵ_∞ =instantaneous, and ϵ_m =target values to get

$$\max_{\omega \in \mathcal{B}} |\epsilon(\omega) - \epsilon_\mathrm{m}| \geq \frac{B}{1 + B/2} (\epsilon_\infty - \epsilon_\mathrm{m}) \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{cases},$$

where we also used that $\operatorname{Im} h_\Delta(z) = 1$ for the lossless case. Smoothly from the lossless case the lossy case as losses increases.

Similar bounds for other material cases

Interval $\mathcal{B} = [\omega_1, \omega_2]$ with fractional bandwidth $B = (\omega_2 - \omega_1)/\omega_0$, $\omega_0 = (\omega_1 + \omega_2)/2$

ϵ_{s} =static, ϵ_∞ =instantaneous, ϵ_{m} =target values.

1. $\epsilon_{\mathrm{m}} < \epsilon_\infty$:

$$\max_{\omega\in\mathcal{B}} |\epsilon(\omega) - \epsilon_{\mathrm{m}}| \geq \frac{B}{1+B/2}(\epsilon_\infty - \epsilon_{\mathrm{m}}) \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{cases},$$

2. without static conductivity

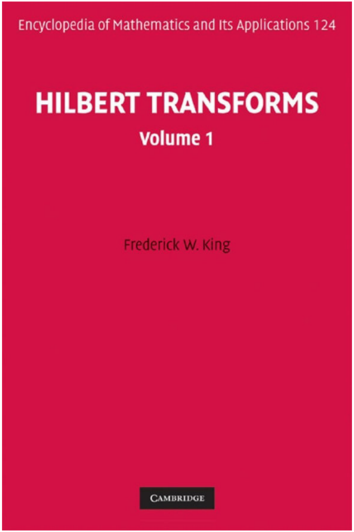
$$\max_{\omega\in\mathcal{B}} \frac{|\epsilon(\omega) - \epsilon_{\mathrm{m}}|}{|\epsilon(\omega) - \epsilon_\infty|} \geq \frac{B}{1+B/2} \frac{\epsilon_{\mathrm{s}} - \epsilon_{\mathrm{m}}}{\epsilon_{\mathrm{s}} - \epsilon_\infty} \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{cases},$$

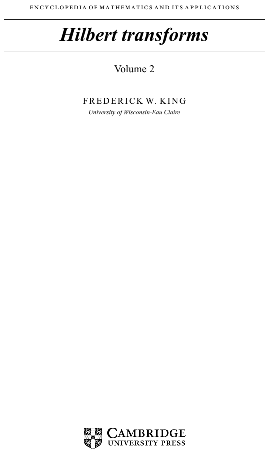
3. artificial magnetism $\mu_{\mathrm{m}} > \mu_{\mathrm{s}}$

$$\max_{\omega\in\mathcal{B}} \frac{|\mu(\omega) - \mu_{\mathrm{m}}|}{|\mu(\omega) - \mu_\infty|} \geq \frac{B}{1+B/2} \frac{\mu_{\mathrm{m}} - \mu_{\mathrm{s}}}{\mu_{\mathrm{s}} - \mu_\infty} \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{cases},$$

Sum rules and physical bounds on passive metamaterials, New Journal of Physics, Vol. 12, pp. 043046-, 2010.

Sum rules in Hilbert Transforms by King 2009, examples





ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Hilbert transforms

Volume 2

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CAMBRIDGE UNIVERSITY PRESS

Sum rules in Hilbert Transforms by King 2009, examples

240

Table 19.1. *Summary of sum rules for the dielectric constant*

Number	Sum rule	Reference
(1)	$\int_0^\infty \frac{\varepsilon_\mathrm{i}(\omega)\mathrm{d}\omega}{\omega} = \frac{\pi}{2}\{\varepsilon_\mathrm{r}(0) - \varepsilon_0\}$ (insulators)	Gorter and Kronig (1936)
(2)	$\int_0^\infty \frac{\{\varepsilon_\mathrm{i}(\omega) - \sigma(0)/\omega\}\mathrm{d}\omega}{\omega} = \frac{\pi}{2}\{\varepsilon_\mathrm{r}(0) - \varepsilon_0\}$	
(3)	$\int_0^\infty [\varepsilon_\mathrm{r}(\omega) - \varepsilon_0]\mathrm{d}\omega = 0$ (insulators)	Saslow (1970); Scaife (1972)
(4)	$\int_0^\infty \{\varepsilon_\mathrm{r}(\omega) - \varepsilon_0\}\mathrm{d}\omega = -\frac{\pi\sigma(0)}{2}$	Saslow (1970)
(5)	$\int_0^\infty \omega\varepsilon_\mathrm{i}(\omega)\mathrm{d}\omega = \frac{\pi\varepsilon_0\omega_\mathrm{p}^2}{2}$	Landau and Lifshitz (1960); Stern (1963)
(6)	$\int_0^\infty \{\varepsilon_\mathrm{r}(\omega) - \varepsilon_0\}\cos\omega t\,\mathrm{d}\omega = \int_0^\infty \varepsilon_\mathrm{i}(\omega)\sin\omega t\,\mathrm{d}\omega,\ t > 0$	Cole and Cole (1942); Scaife (1972); King (197
(7)	$\int_0^\infty \{\varepsilon_\mathrm{r}(\omega) - \varepsilon_0\}^2\,\mathrm{d}\omega = \int_0^\infty \varepsilon_\mathrm{i}(\omega)^2\,\mathrm{d}\omega$ (insulators)	
(8)	$\int_0^\infty \{\varepsilon_\mathrm{r}(\omega) - \varepsilon_0\}[\{\varepsilon_\mathrm{r}(\omega) - \varepsilon_0\}^2 - 3\varepsilon_\mathrm{i}(\omega)^2]\mathrm{d}\omega = 0$ (insulators)	
(9)	$\int_0^\infty \omega\varepsilon_\mathrm{i}(\omega)\{\varepsilon_\mathrm{r}(\omega) - \varepsilon_0\}\mathrm{d}\omega = 0$ (insulators)	Villani and Zimerman (1973b)

Sum rules in Hilbert Transforms by King 2009, examples

242

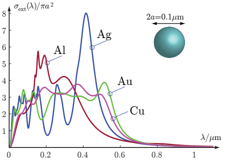
Table 19.2. *Summary of sum rules for the refractive index*

Number	Sum rule	Reference
(1)	$\int_0^\infty \{n(\omega)-1\}\mathrm{d}\omega = 0$	Saslow (1970); Altarelli *et al.* (1972); Smith (1985)
(2)	$\int_0^\infty \omega\kappa(\omega)\mathrm{d}\omega = \frac{\pi}{4}\omega_\mathrm{p}^2$	Kronig (1926)
(3)	$\int_0^\infty \frac{\kappa(\omega)\mathrm{d}\omega}{\omega} = \frac{\pi}{2}\{n(0)-1\}$ (insulators)	Moss (1961)
(4)	$\int_0^\infty \omega\kappa(\omega)n(\omega)\mathrm{d}\omega = \frac{\pi}{4}\omega_\mathrm{p}^2$	Villani and Zimerman (1973a)
(5)	$\int_0^\infty \{n(\omega)-1\}\cos\omega t\,\mathrm{d}\omega = \int_0^\infty \kappa(\omega)\sin\omega t\,\mathrm{d}\omega, \quad t>0$	
(6)	$\int_0^\infty \omega\kappa(\omega)[3n(\omega)^2-\kappa(\omega)^2]\mathrm{d}\omega = \frac{3\pi}{4}\omega_\mathrm{p}^2$	
(7)	$\int_0^\infty \omega\kappa(\omega)\{n(\omega)-1\}\mathrm{d}\omega = 0$	Stern (1963); Altarelli *et al.* (1972)
(8)	$\int_0^\infty \omega^m\kappa(\omega)[3\{n(\omega)-1\}^2-\kappa(\omega)^2]\mathrm{d}\omega = 0, \quad m=1,3$	Villani and Zimerman (1973b)
(9)	$\int_0^\infty \omega^m\{n(\omega)-1\}[\{n(\omega)-1\}^2-3\kappa(\omega)^2]\mathrm{d}\omega = 0,\ m=2,4$	Villani and Zimerman (1973b)

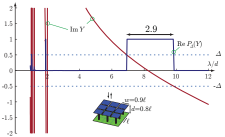
Sum rules and physical bounds on passive systems

General simple approach

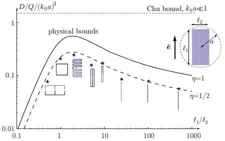
1. Identify a linear and passive system.
2. Construct a Herglotz (or similarly a positive real) function $h(z)$ that models the parameter of interest.
3. Investigate the asymptotic expansions of $h(z)$ as $z \hat{\to} 0$ and $z \hat{\to} \infty$.
4. Use integral identities for Herglotz functions to relate the dynamic properties to the asymptotic expansions.
5. Bound the integral.



Cross sections of nano spheres.



High-impedance surface.



Antenna D/Q.

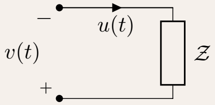
Examples: Matching networks (Bode 1945; Fano 1950), Radar absorbers (Rozanov 2000), Antennas (Gustafsson 2010a; Gustafsson, Sohl, and Kristensson 2007; Gustafsson, Sohl, and Kristensson 2009), Scattering (Bernland, Gustafsson, and Nordebo 2011; Sohl, Gustafsson, and Kristensson 2007), High-impedance surfaces (Gustafsson and Sjöberg 2011), Metamaterials (Gustafsson and Sjöberg 2010),Extraordinary transmission (Gustafsson 2009),Periodic structures (Gustafsson et al. 2012), ...

Passive systems

Definition (Passivity)

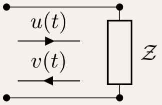
A system ($v = h * u$) is admittance-passive if

$$\mathcal{W}_{\mathrm{adm}}(T) = \mathrm{Re} \int_{-\infty}^{T} v^*(t)u(t)\,\mathrm{d}t \geq 0$$



and scatter-passive if

$$\mathcal{W}_{\mathrm{scat}}(T) = \int_{-\infty}^{T} |u(t)|^2 - |v(t)|^2\,\mathrm{d}t \geq 0,$$



for all $T \in \mathbb{R}$ and smooth functions of compact support u.

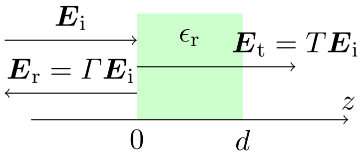
Passivity is a systems concept. Not sufficient with passive materials (devices). Need less energy in the output signal than in the input signal for all times and signals.

The transfer function, $H(s)$ is holomorphic (analytic) for $\mathrm{Re}\, s > 0$, and can be related to a positive real (PR) (or Herglotz) function.

(Wohlers and Beltrami 1965; Youla, Castriota, and Carlin 1959; Zemanian 1963; Zemanian 1965)

Passive systems: examples

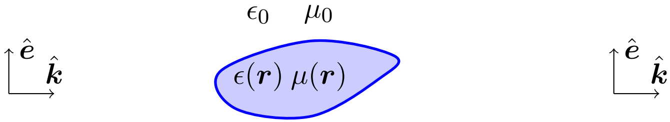
- Reflection and transmission of periodic slabs (scattering)



- Constitutive relations (admittance)

$$\boldsymbol{D}(t) = \epsilon_0\epsilon_\infty\boldsymbol{E}(t) + \epsilon_0\int_{\mathbb{R}} \chi_\mathrm{ee}(t-t')\boldsymbol{E}(t')\,\mathrm{d}t'$$

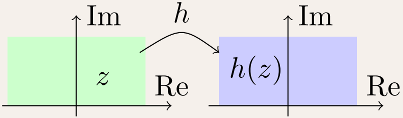
- Scattering (forward (admittance) and modes (scattering))



Definition (Herglotz functions, $h(z)$)

A Herglotz (Nevanlinna, Pick, or R-) function $h(z)$ is holomorphic for $\operatorname{Im} z > 0$ and

$$\operatorname{Im} h(z) \geq 0 \quad \text{for} \quad \operatorname{Im} z > 0$$



Representation for $\operatorname{Im} z > 0$, *cf.*, the Hilbert transform

$$h(z) = A_{\mathrm{h}} + Lz + \int_{-\infty}^{\infty} \frac{1}{\xi - z} - \frac{\xi}{1+\xi^2} \,\mathrm{d}\nu(\xi)$$

where $A_{\mathrm{h}} \in \mathbb{R}$, $L \geq 0$, and $\int_{\mathbb{R}} \frac{1}{1+\xi^2} \,\mathrm{d}\nu(\xi) < \infty$.

Gustav Herglotz
1881-1953

Rolf Nevanlinna
1895-1980

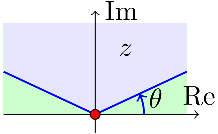
Georg Alexander
Pick 1859-1942

Wilhelm Cauer
1900-1945

Integral identities for Herglotz functions

Herglotz functions with the symmetry $h(z) = -h^*(-z^*)$ (real-valued in the time domain) have asymptotic expansions ($N_0 \geq 0$ and $N_\infty \geq 0$)

$$\begin{cases} h(z) = \sum_{n=0}^{N_0} a_{2n-1} z^{2n-1} + o(z^{2N_0-1}) & \text{as } z \hat{\to} 0 \\ h(z) = \sum_{n=0}^{N_\infty} b_{1-2n} z^{1-2n} + o(z^{1-2N_\infty}) & \text{as } z \hat{\to} \infty \end{cases}$$



where $\hat{\to}$ denotes limits in the Stoltz domain
$0 < \theta \leq \arg(z) \leq \pi - \theta$.They satisfy the identities ($1 - N_\infty \leq n \leq N_0$)

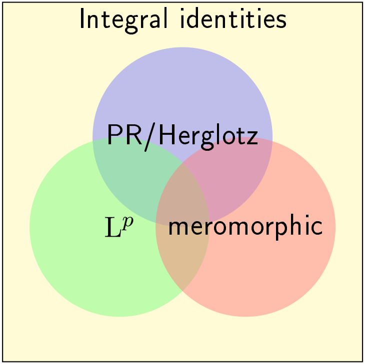
$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{\operatorname{Im} h(x+\mathrm{i}y)}{x^{2n}} \,\mathrm{d}x = a_{2n-1} - b_{2n-1} = \begin{cases} -b_{2n-1} & n < 0 \\ a_{-1} - b_{-1} & n = 0 \\ a_1 - b_1 & n = 1 \\ a_{2n-1} & n > 1 \end{cases}$$

Bernland, Luger, Gustafsson, *Sum rules and constraints on passive systems*, J. Phys. A: Math. Theor., 2011.

Derivation of the integral identities

Similar Integral identities can be derived under various assumptions.

- Passive system with PR (Herglotz) functions. Limits in the Stoltz domain.
- Holomorphic functions in a region that includes the frequency axis except for simple poles at the frequency axis. Limits in $\mathbb{C}_+$.
- L^p functions at the frequency axis with $1 < p < \infty$ and with limits along the frequency axis.



Easy to show passivity but difficult to show the last two properties.

Integral identities for Herglotz functions

Known low-frequency expansion ($a_1 \geq 0$):

$$h(z) \sim \begin{cases} a_1 z & \text{as } z \hat{\to} 0 \\ b_1 z & \text{as } z \hat{\to} \infty \end{cases}$$

which gives the $n = 1$ identity (we drop the limits for simplicity)

$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{1/\varepsilon} \frac{\operatorname{Im} h(x + \mathrm{i}y)}{x^2} \,\mathrm{d}x \overset{\text{def}}{=} \frac{2}{\pi} \int_0^{\infty} \frac{\operatorname{Im} h(x)}{x^2} \,\mathrm{d}x = a_1 - b_1 \leq a_1$$

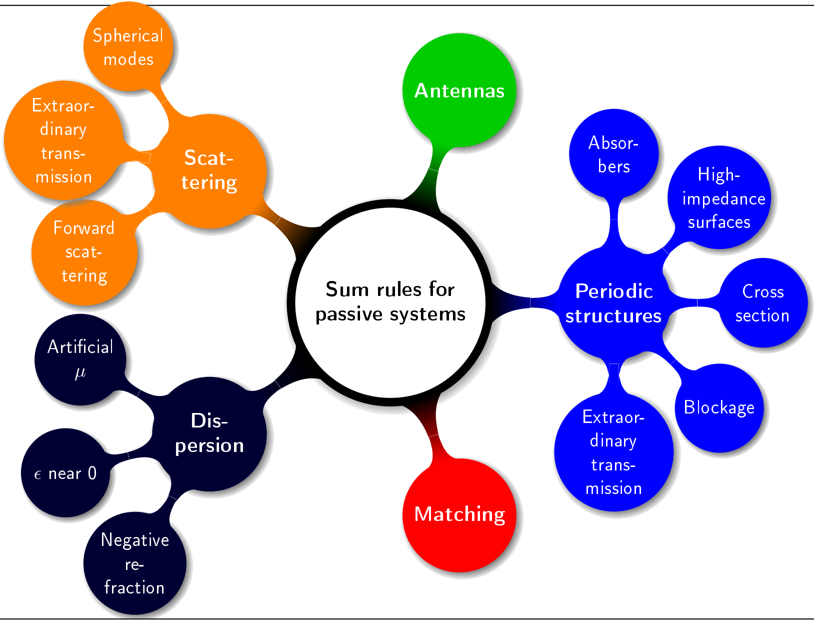
Known high-frequency expansion (short times) ($b_{-1} \leq 0$):

$$h(z) \sim \begin{cases} a_{-1}/z & \text{as } z \hat{\to} 0 \\ b_{-1}/z & \text{as } z \hat{\to} \infty \end{cases}$$

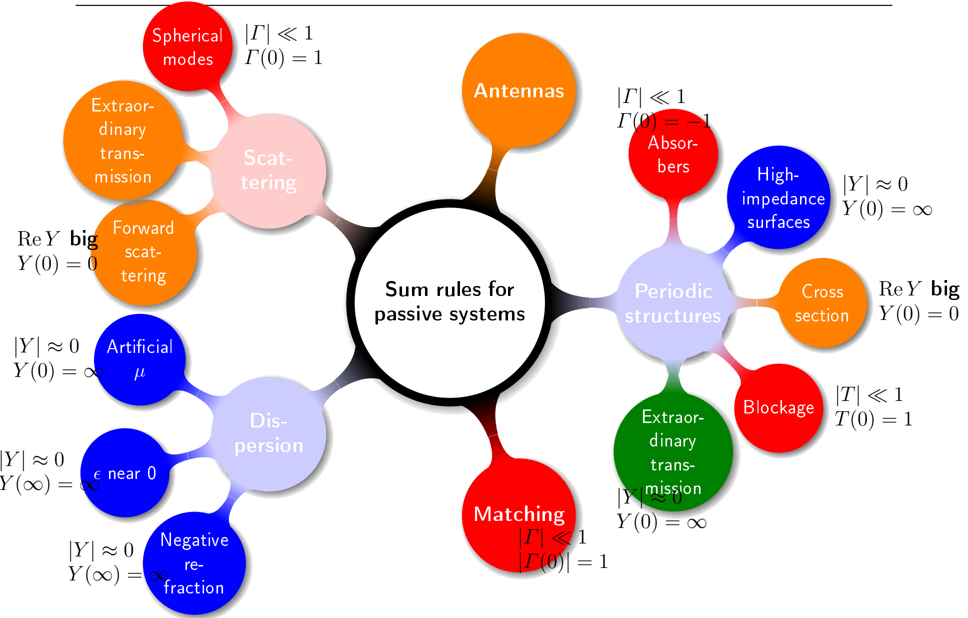
which gives the $n = 0$ identity

$$\frac{2}{\pi} \int_0^{\infty} \operatorname{Im} h(x) \,\mathrm{d}x = a_{-1} - b_{-1} \leq -b_{-1}.$$

Some sum rules for passive systems

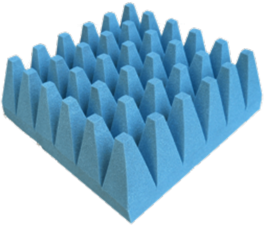


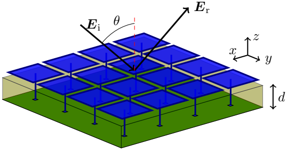
Some sum rules for passive systems



A physical bound for absorbers

- A structure (above a ground plane) that absorbs incident EM waves.
- Pyramids, homogeneous, periodic, metamaterials,...
- Often desired to be thin and absorb energy over large bandwidths.





Tradeoff between thickness d fractional bandwidth B and wavelength λ;

$$\lambda_2 - \lambda_1 = B\lambda_0 \leq \frac{2\pi^2 d\mu_{\mathrm{s}}}{\ln \varGamma_0^{-1}} \leq \frac{172 d\mu_{\mathrm{s}}}{|\varGamma_{0,\mathrm{dB}}|}$$

$\varGamma_0 = \max_{\lambda_1 \leq \lambda \leq \lambda_2} |\varGamma(\lambda)|$ and μ_{s} is the maximal static relative permeability of the absorber.

(Rozanov 2000)

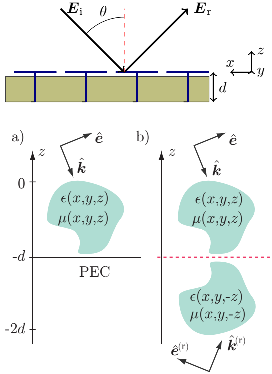
Absorbers

1. Identify the reflection coefficient, Γ, as a passive system ($|\Gamma| \leq 1$).
2. Analyze the low- (and high) frequency behavior:

$$\Gamma(k) \sim -1 - \mathrm{i}k\big(2d\cos\theta + \gamma/A\big), \quad k \to 0$$

where γ is the polarizability per unit cell. A well-defined static quantity which is easily determined.

3. Construct the Herglotz function $h = -\mathrm{i}\ln(\Gamma/\mathcal{B})$ and the sum rule



$$\frac{2}{\pi}\int_0^\infty \frac{1}{k^2}\ln\frac{1}{|\Gamma(k)|}\,\mathrm{d}k \leq 2d\cos\theta + \gamma/A \leq 2\mu_{\mathrm{s}}d$$

(Gustafsson and Sjöberg 2011; Rozanov 2000)

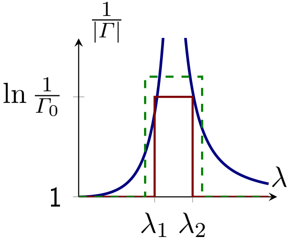
Rewrite in the wavelength $\lambda = 2\pi/k$ and estimate the integral, *e.g.*,

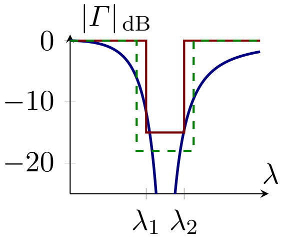
$$\frac{1}{\pi^2}(\lambda_2-\lambda_1)\ln\frac{1}{|\Gamma_0|}\,\mathrm{d}\lambda \leq \frac{1}{\pi^2}\int_{\lambda_1}^{\lambda_2}\ln\frac{1}{|\Gamma(\lambda)|}\,\mathrm{d}\lambda \leq \frac{1}{\pi^2}\int_0^{\infty}\ln\frac{1}{|\Gamma(\lambda)|}\,\mathrm{d}\lambda \leq 2\mu_{\mathrm{s}}d$$

with $\Gamma_0 = \max_{[\lambda_1,\lambda_2]}|\Gamma(\lambda)|$.

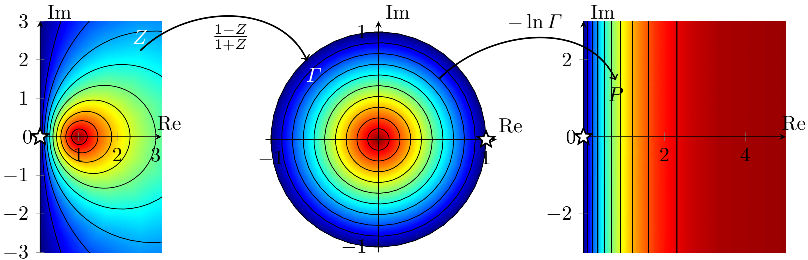
Bandwidth limited by the thickness d and (static) permeability μ_{s}.

Applied to array antennas in (Doane, Sertel, and Volakis 2013; Jonsson, Kolitsidas, and Hussain 2013).





Magnitude of scattering parameters

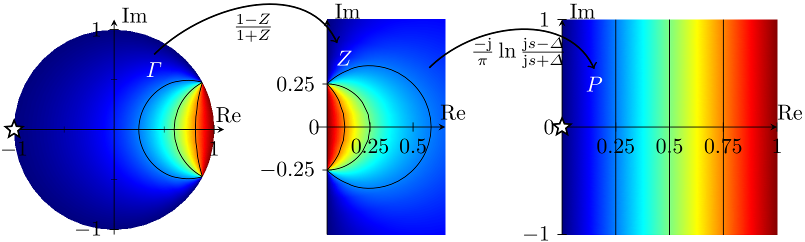


The absorber is an example with scattering parameters $|\Gamma(0)| = 1$ where it is desired to have $|\Gamma(\omega)| \leq \Gamma_0$.

Map to Herglotz/PR functions using $\ln(h/\mathcal{B})$ and use the $n = 1$ identity.

Similar approaches for matching (Bode 1945; Fano 1950), transmission blockage (Gustafsson et al. 2009; Sjöberg, Gustafsson, and Larsson 2010), mode scattering (Bernland 2012; Bernland, Gustafsson, and Nordebo 2011).

Magnitude of Herglotz/PR functions



- Scattering parameters $\Gamma(0) = \mp 1$ and desire $\Gamma(\mathrm{j}\omega) \approx \pm 1$
- Admittance $P(0) = \infty$ and desire $|P(\mathrm{j}\omega)| \leq P_0$.

Temporal dispersion (Gustafsson and Sjöberg 2010), high-impedance surfaces (Gustafsson and Sjöberg 2011), extraordinary transmission (Gustafsson, Sjöberg, and Vakili 2011), and superluminal transmission (Gustafsson 2012).

Basically four (or two) cases

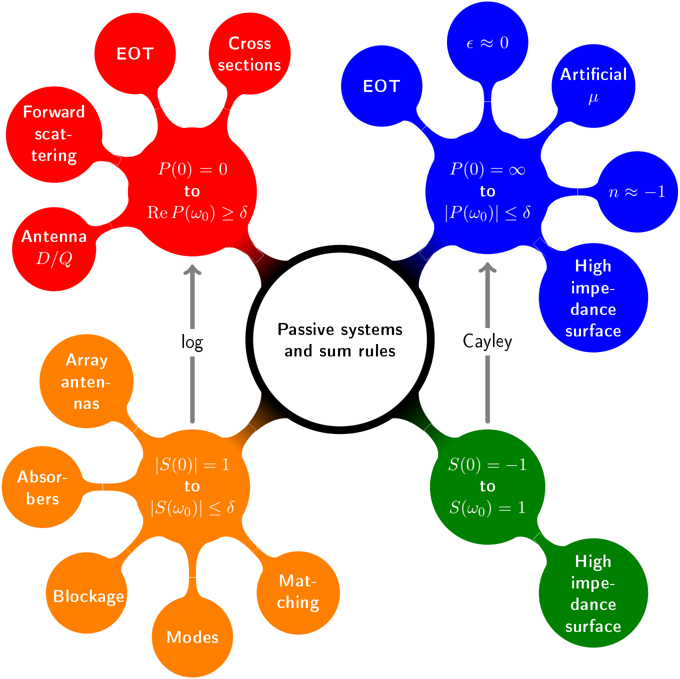
admittance want
large $\mathrm{Re}\, Y(\omega_0)$ with $Y(0) = 0$.
forward scattering (cross section)
Use identity

S-parameter want
$|S(\omega_0)| \leq \delta$ with $|S(0)| = 1$.
absorber, matching, blockage, modes, ...
Use log+identity

admittance want
small $|Y(\omega_0)| \leq \delta$ with $Y(0) = \infty$.
high impedance surface, temporal dispersion.
Use pulse+identity

S-parameter want
$S(\omega_0) \approx 1$ with $S(0) = -1$.
high impedance surface, extraordinary transmission
Use Cayley+pulse+identity

Many physical bounds based on sum rules can be formulated as these 4 (or 2) cases. Convex optimization can be used for some other cases.



Passive systems and sum rules
$P(0)=0$ to $\mathrm{Re}\,P(\omega_0)\geq\delta$
EOT
Cross sections
Forward scat- tering
Antenna D/Q
log
$P(0)=\infty$ to $|P(\omega_0)|\leq\delta$
$\epsilon\approx 0$
EOT
Artificial μ
$n\approx -1$
High impe- dance surface
Cayley
$|S(0)|=1$ to $|S(\omega_0)|\leq\delta$
Array anten- nas
Absor- bers
Blockage
Modes
Mat- ching
$S(0)=-1$ to $S(\omega_0)=1$
High impe- dance surface

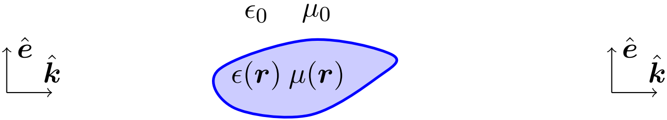
Challenges with the sum rules technique

Use of passive systems, Herglotz functions, and sum rules is a very powerful technique to derive bounds for EM design problems. Some challenges:

- Can easily change a problem to a physically equivalent problem where the approach does not work so well. For example: change of a PEC ground plane to Cu changes the expansion to $-d + \gamma\omega$ as $\omega \to 0$, where $d < 1$. Cannot use the integral identities.
- Multi-parameter cases, have often more than one parameter, *e.g.*, frequency and incident angle. Can the parameters be treated together?
- Have sometimes active systems or systems that are made of passive materials but not time-domain passive.

Some of these problems can be analyzed using convex optimization techniques.

How do we relate the antenna with Herglotz functions?



Assumptions:

- Finite scattering object composed of a linear, passive, and time translational invariant materials.
- Incident linearly polarized plane wave.

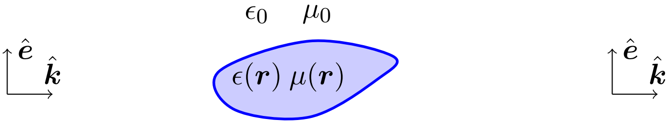
From physics:

- The propagation speed is limited by the speed of light.
- Optical theorem (energy conservation).
- Induced dipole moment in the static limit.

Passive system with $h(k) \sim \gamma k$ as $k \to 0$ and $\sigma_{\text{ext}} = \operatorname{Im} h$.

(Gustafsson 2010b; Purcell 1969; Sohl, Gustafsson, and Kristensson 2007)

Forward scattering sum rule



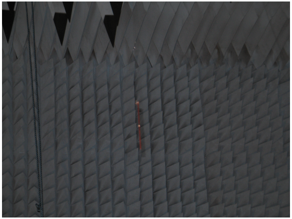
Use the $n = 1$ identity with
$a_1 = \gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$ and $b_1 = 0$, *i.e.*,

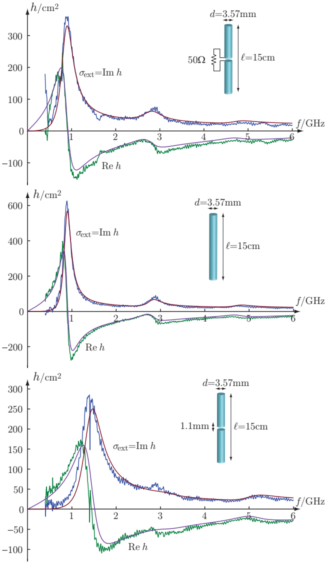
$$\frac{2}{\pi} \int_0^\infty \frac{\sigma_{\mathrm{ext}}(k)}{k^2} \, \mathrm{d}k = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$$

or written in the free-space wavelength $\lambda = 2\pi/k$

$$\frac{1}{\pi^2} \int_0^\infty \sigma_{\mathrm{ext}}(\lambda) \, \mathrm{d}\lambda = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$$

Forward scattering of antennas



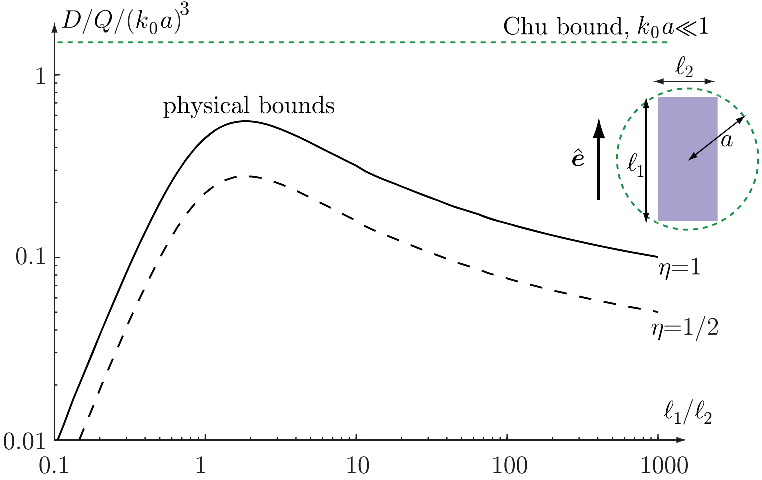


- Forward scattering measurement of a dipole antenna.
- Loaded, short, and open circuit.
- Length $15\,\mathrm{cm}$ and $0.5\,\mathrm{GHz}$ to $6\,\mathrm{GHz}$.

in cm^3		loaded	short	open
sim:	γ	661	661	291
sim:	$\frac{2}{\pi}\int_0^{k_2}\frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k$	644	644	265
meas:	$\frac{2}{\pi}\int_{k_1}^{k_2}\frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k$	605	670	322

Forward scattering of loaded and unloaded antennas, IEEE-TAP, 2012.

Antenna forward scattering bounds (rectangles)



(Gustafsson, Sohl, and Kristensson 2007; Gustafsson, Sohl, and Kristensson 2009)

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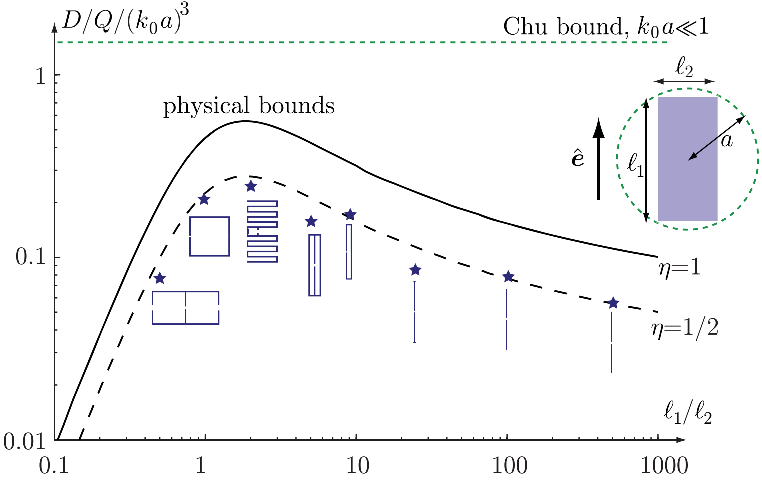
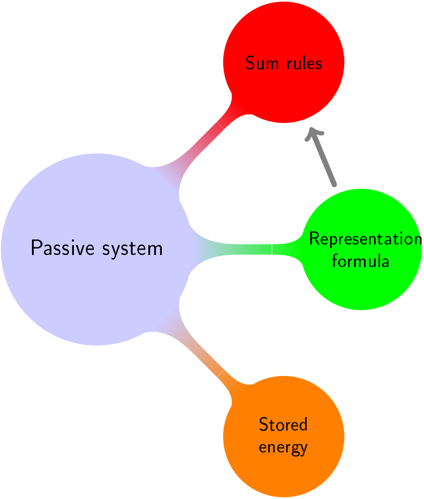


(Gustafsson, Sohl, and Kristensson 2007; Gustafsson, Sohl, and Kristensson 2009)

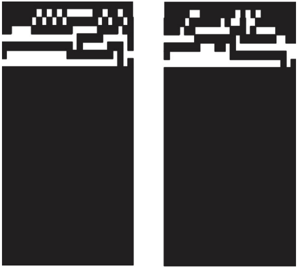
Sum rules, stored energy, and passive systems

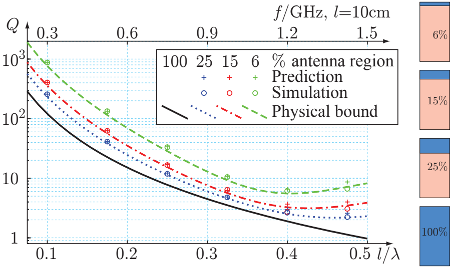
Sum rules (integral identities) and stored energy are commonly used to construct physical bounds. Both are related to passive systems

The stored energy is used to determine the Q-factor and hence an estimate of the bandwidth. This changes the perspective from optimization of the frequency behavior of the system to optimization of the states that models the system at a fixed frequency.



Stored energy and antenna current optimization





- Antenna current optimization and physical bounds [1].
- Single frequency antenna optimization, *e.g.*, minimize Q [2].

Express the stored energies as semi-positive quadratic forms in the current (density) $\mathbf{I}$:

$$W_\mathrm{e} \sim \mathbf{I}^\mathrm{H}\mathbf{X}_\mathrm{e}\mathbf{I} \geq 0 \quad \text{and} \quad W_\mathrm{m} \sim \mathbf{I}^\mathrm{H}\mathbf{X}_\mathrm{m}\mathbf{I} \geq 0$$

[1] Gustafsson and Nordebo, IEEE-TAP (2013), [2] Cismasu and Gustafsson, IEEE-TAP (2014)

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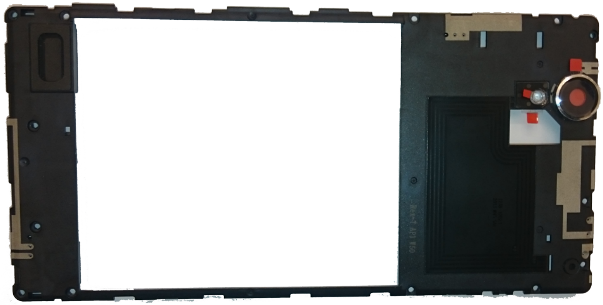

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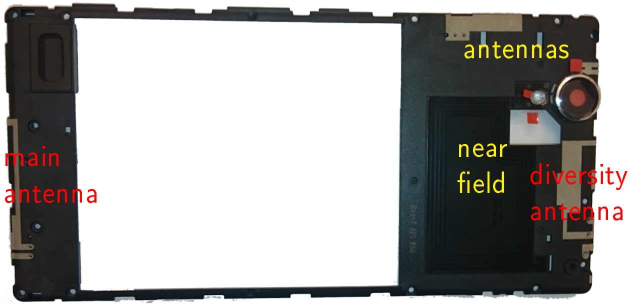

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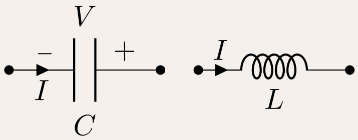
What is stored energy? (lumped circuits)

- Capacitor $i = C\frac{\mathrm{d}v}{\mathrm{d}t}$.
- Multiply with v to get the power $iv = \frac{\mathrm{d}}{\mathrm{d}t}\frac{C|v|^2}{2}$ and integrate

$$\int_{-\infty}^{T} iv \,\mathrm{d}t = \frac{C}{2}\left(|v(T)|^2 - |v(-\infty)|^2\right)$$

- Time harmonic case $v(t) = \mathrm{Re}\{V\mathrm{e}^{\mathrm{j}\omega t}\}$ gives the time average stored electric energy $W_\mathrm{e} = C|V|^2/4$

Lumped elements



Time average stored energy in capacitors

$$W_\mathrm{e} = \frac{C|V|^2}{4} = \frac{|I|^2}{4\omega^2 C}$$

and in inductors

$$W_\mathrm{m} = \frac{L|I|^2}{4}$$

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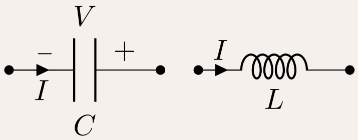
Stored energy from the $\frac{\mathrm{d}}{\mathrm{d}t}$-differentiated quadratic form. Frequency domain $\frac{\mathrm{d}}{\mathrm{d}t} \to s$ implies term proportional to the frequency, $s = \mathrm{j}\omega$.

Lumped elements



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Rational PR functions

Consider a rational PR function ($a_n, b_n \in \mathbb{R}$)

$$Z(s) = \frac{\sum_{n=0}^{N_b} b_n s^n}{\sum_{n=0}^{N_a} a_n s^n}$$

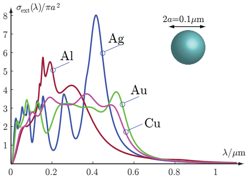
Often good for low frequencies but there are passive systems that are not well represented by rational functions.

Rewrite as a state-space model

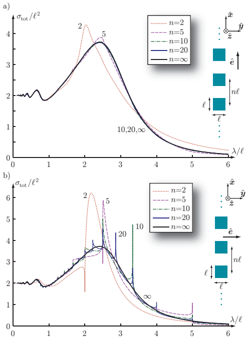
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} + Du \end{cases}$$

with

$$Z(s) = Y(s)/U = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} + D$$

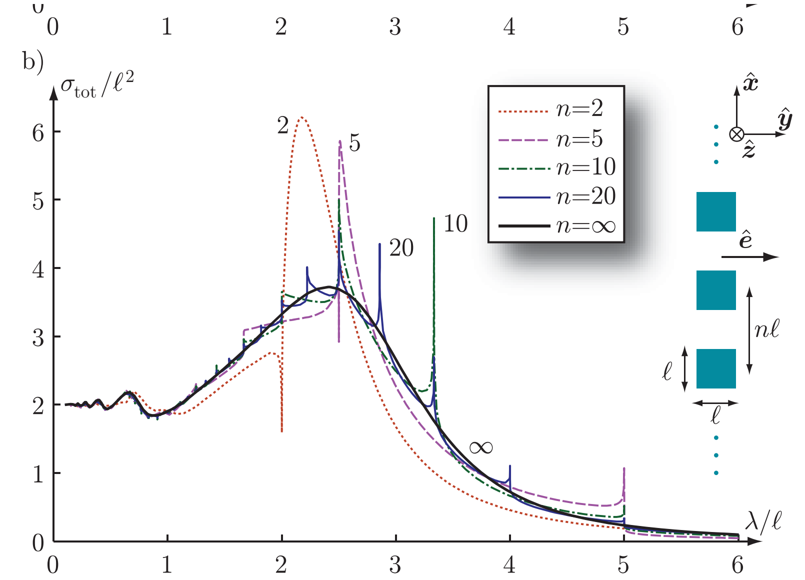


Forward scattering.



Periodic structure.

Rational PR functions



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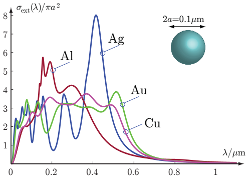
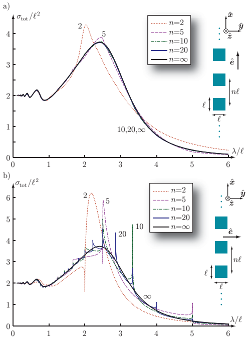


Forward scattering.



Periodic structure.

State-space model

Consider a state-space model with input u and output y

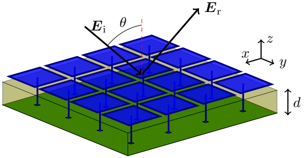
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} + Du \end{cases}$$

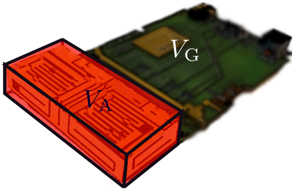
- Stored energy has been analyzed for state-space models.
- Need minimal representations and reciprocity (symmetry). Necessary but not sufficient.
- Still some open questions.
- What about for non-rational PR (or Herglotz) functions?
- Can we define a (useful) stored energy for a Herglotz (or PR) function?

J.C. Willems, Dissipative dynamical systems I,II (1972), ..., (2013)

Summary

- Herglotz/PR functions, physical bounds, and passive systems.
- Sufficient with passivity and low- and/or high-frequency expansion to derive sum rules and physical bounds.
- Antennas, scatterers, absorbers, high impedance surfaces, temporal dispersion,...
- The bounds are tight for many cases.
- Need active (non-foster), non-linear and/or time varying devices to overcome the bounds.
- Also convex optimization using the integral representation and minimization of stored energy.





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