Sum Rules and Physical Bounds in Electromagnetics

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Outline

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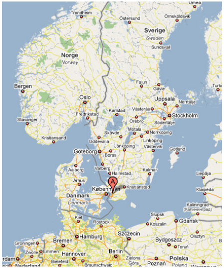


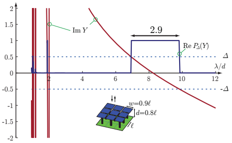
- Lund university was founded in 1666.
- Sweden's Largest University.
- Approximately 40 000 students.
- Department of Electrical and Information Technology. Broadband Communications, Circuits and Systems, Communication, Electromagnetic theory, Networking and Security, Signal Processing.

Outline

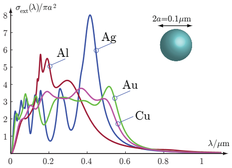
Sum rules and physical bounds

Construct identities and physical bounds using basic properties such as causality, linearity, passivity, and time invariance to, *e.g.*, analyze:

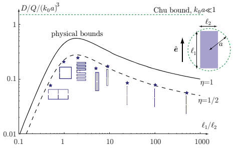
- Absorbers and High-impedance surfaces: How does the bandwidth depend on material and thickness?
- Extra ordinary transmission: Transmission through apertures?
- Scattering: How much can an object interact with an electromagnetic wave?
- Antennas: How does the performance depend on size, geometry, and material?
- Metamaterials: Bandwidth with $\epsilon(\omega) \approx -1$?
- Artificial magnets, cloaking, superluminal propagation, matching, filters....



High-impedance surface.

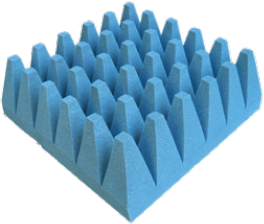


Cross sections of nano spheres.



A physical bound for absorbers

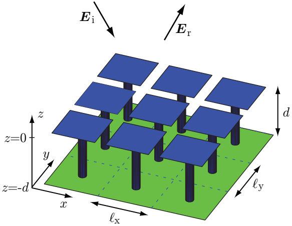
- A structure (above a ground plane) that absorbs incident EM waves.
- Pyramids, homogeneous, periodic, metamaterials,...
- Often desired to be thin and absorb power over large bandwidths.



Tradeoff between thickness d fractional bandwidth B and wavelength λ;

$$\lambda_2 - \lambda_1 = B\lambda_0 \leq \frac{2\pi^2 d\mu_{\mathrm{s}}}{\ln \mathit{\Gamma}_0^{-1}} \leq \frac{172 d\mu_{\mathrm{s}}}{|\mathit{\Gamma}_{0,\mathrm{dB}}|}$$

where $\mathit{\Gamma}_0 = \max_{\lambda_1\leq\lambda\leq\lambda_2} |\mathit{\Gamma}(\lambda)|$ and μ_{s} is the maximal static permeability of the absorber.



Rozanov, *Ultimate thickness to bandwidth ratio of radar absorbers*, IEEE-TAP, 2000.

Sum rules

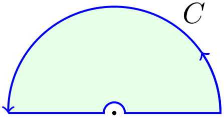
We use sum rules to derive physical bounds:
Kramers-Kronig relations (Hilbert Transform), *e.g.*, relative permittivity $\epsilon(\omega)$

$$\mathrm{Re}\,\epsilon(\omega) - 1 = \fint_{\mathbb{R}} \frac{\mathrm{Im}\,\epsilon(\omega')}{\omega' - \omega}\,\mathrm{d}\omega'$$

Evaluate for $\omega = 0$ (assume no (static) conductivity) to get the sum rule

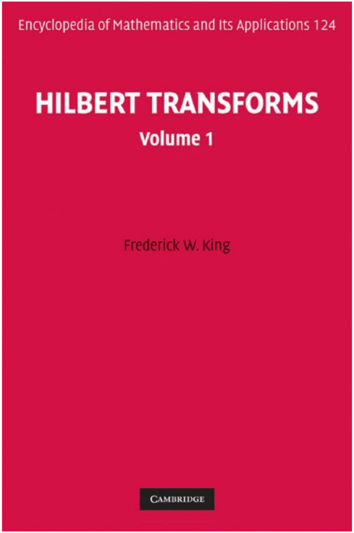
$$\epsilon(0) - 1 = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\mathrm{Im}\,\epsilon(\omega')}{\omega'}\,\mathrm{d}\omega' \geq 0.$$

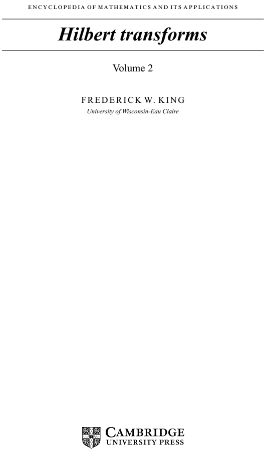
Complex analysis (Cauchy's theorem) can also be used to derive sum rules.



Passive systems: general approach used here.

Sum rules in Hilbert Transforms by King 2009, examples

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Hilbert transforms

Volume 2

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Sum rules in Hilbert Transforms by King 2009, examples

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Table 19.1. *Summary of sum rules for the dielectric constant*

Number	Sum rule	Reference
(1)	$\int_0^\infty \frac{\varepsilon_{\mathrm{i}}(\omega)\mathrm{d}\omega}{\omega} = \frac{\pi}{2}\{\varepsilon_{\mathrm{r}}(0) - \varepsilon_0\}$ (insulators)	Gorter and Kronig (1936)
(2)	$\int_0^\infty \frac{\{\varepsilon_{\mathrm{i}}(\omega) - \sigma(0)/\omega\}\mathrm{d}\omega}{\omega} = \frac{\pi}{2}\{\varepsilon_{\mathrm{r}}(0) - \varepsilon_0\}$	
(3)	$\int_0^\infty [\varepsilon_{\mathrm{r}}(\omega) - \varepsilon_0]\mathrm{d}\omega = 0$ (insulators)	Saslow (1970); Scaife (1972)
(4)	$\int_0^\infty \{\varepsilon_{\mathrm{r}}(\omega) - \varepsilon_0\}\mathrm{d}\omega = -\frac{\pi\sigma(0)}{2}$	Saslow (1970)
(5)	$\int_0^\infty \omega\varepsilon_{\mathrm{i}}(\omega)\mathrm{d}\omega = \frac{\pi\varepsilon_0\omega_{\mathrm{p}}^2}{2}$	Landau and Lifshitz (1960); Stern (1963)
(6)	$\int_0^\infty \{\varepsilon_{\mathrm{r}}(\omega) - \varepsilon_0\}\cos\omega t\,\mathrm{d}\omega = \int_0^\infty \varepsilon_{\mathrm{i}}(\omega)\sin\omega t\,\mathrm{d}\omega,\ t > 0$	Cole and Cole (1942); Scaife (1972); King (197
(7)	$\int_0^\infty \{\varepsilon_{\mathrm{r}}(\omega) - \varepsilon_0\}^2\,\mathrm{d}\omega = \int_0^\infty \varepsilon_{\mathrm{i}}(\omega)^2\,\mathrm{d}\omega$ (insulators)	
(8)	$\int_0^\infty \{\varepsilon_{\mathrm{r}}(\omega) - \varepsilon_0\}[\{\varepsilon_{\mathrm{r}}(\omega) - \varepsilon_0\}^2 - 3\varepsilon_{\mathrm{i}}(\omega)^2]\mathrm{d}\omega = 0$ (insulators)	
(9)	$\int_0^\infty \omega\varepsilon_{\mathrm{i}}(\omega)\{\varepsilon_{\mathrm{r}}(\omega) - \varepsilon_0\}\mathrm{d}\omega = 0$ (insulators)	Villani and Zimerman (1973b)

Sum rules in Hilbert Transforms by King 2009, examples

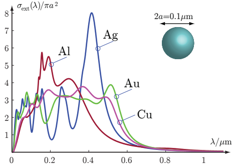
Table 19.2. *Summary of sum rules for the refractive index*

Number	Sum rule	Reference
(1)	$\int_0^\infty \{n(\omega)-1\}\mathrm{d}\omega = 0$	Saslow (1970); Altarelli *et al.* (1972); Smith (1985)
(2)	$\int_0^\infty \omega\kappa(\omega)\mathrm{d}\omega = \frac{\pi}{4}\omega_{\mathrm{p}}^2$	Kronig (1926)
(3)	$\int_0^\infty \frac{\kappa(\omega)\mathrm{d}\omega}{\omega} = \frac{\pi}{2}\{n(0)-1\}$ (insulators)	Moss (1961)
(4)	$\int_0^\infty \omega\kappa(\omega)n(\omega)\mathrm{d}\omega = \frac{\pi}{4}\omega_{\mathrm{p}}^2$	Villani and Zimerman (1973a)
(5)	$\int_0^\infty \{n(\omega)-1\}\cos\omega t\,\mathrm{d}\omega = \int_0^\infty \kappa(\omega)\sin\omega t\,\mathrm{d}\omega, \quad t>0$	
(6)	$\int_0^\infty \omega\kappa(\omega)[3n(\omega)^2-\kappa(\omega)^2]\mathrm{d}\omega = \frac{3\pi}{4}\omega_{\mathrm{p}}^2$	
(7)	$\int_0^\infty \omega\kappa(\omega)\{n(\omega)-1\}\mathrm{d}\omega = 0$	Stern (1963); Altarelli *et al.* (1972)
(8)	$\int_0^\infty \omega^m\kappa(\omega)[3\{n(\omega)-1\}^2-\kappa(\omega)^2]\mathrm{d}\omega = 0, \quad m=1,3$	Villani and Zimerman (1973b)
(9)	$\int_0^\infty \omega^m\{n(\omega)-1\}[\{n(\omega)-1\}^2-3\kappa(\omega)^2]\mathrm{d}\omega = 0,\ m=2,4$	Villani and Zimerman (1973b)

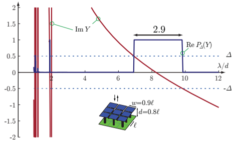
Outline

Sum rules and physical bounds on passive systems

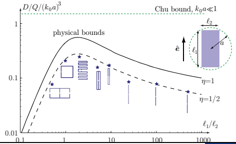
1. Identify a linear, time invariant, and passive (and causal) system.
2. Construct a Herglotz (or similar a positive real) function $h(z)$ that models the parameter of interest.
3. Investigate the asymptotic expansions of $h(z)$ as $z\hat{\to}0$ and $z\hat{\to}\infty$.
4. Use integral identities for Herglotz functions (moments or residue calculus) to relate the dynamic properties to the asymptotic expansions.
5. Bound the integral.



Cross sections of nano spheres (equal areas).



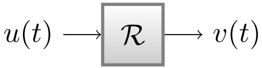
High-impedance surface $2.9 \leq \pi$.



Passive systems

Input-output system described by $\mathcal{R}$

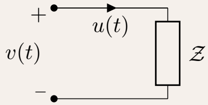
- input signal $u(t)$
- output signal $v(t) = \mathcal{R}\{u(\cdot)\}(t)$



Definition (Passivity)

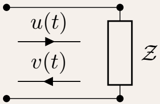
A system $(v = h * u)$ is admittance-passive if

$$\mathcal{W}_{\text{adm}}(T) = \operatorname{Re} \int_{-\infty}^{T} v^*(t) u(t) \, \mathrm{d}t \geq 0$$



and scatter-passive if

$$\mathcal{W}_{\text{scat}}(T) = \int_{-\infty}^{T} |u(t)|^2 - |v(t)|^2 \, \mathrm{d}t \geq 0,$$



for all $T \in \mathbb{R}$ and smooth functions of compact support u.

Zemanian, Distribution theory and transform analysis, 1965 [22]

Passive systems in EM

There are many passive systems (not more energy out than in) in electromagnetics (EM):

Admittance passive

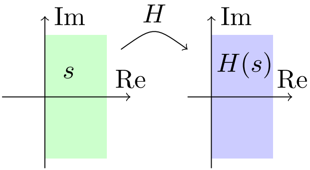
- Material models such as $P(s) = s\epsilon(s)$ and $h(\omega) = \omega\epsilon(\omega)$. Similarly for bi-anisotropic media.
- Antenna input impedance $P(s) = Z(s)$ and $h(\omega) = \mathrm{i}Z(\omega)$.
- Forward scattering of finite objects.

Scattering passive

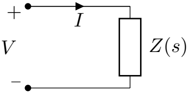
- Antenna and material reflection coefficients, $\mathit{\Gamma} = S_{11}$.
- Reflection and transmission coefficients of periodic structures.

Passive systems: transfer function $V(s) = H(s)U(s)$

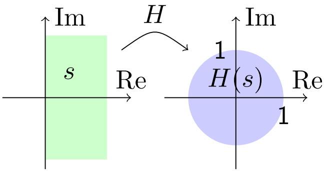
Admittance-passive: $H(s)$ analytic and $\operatorname{Re} H(s) \geq 0$ for $\operatorname{Re} s > 0$.



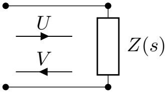
Example: Impedance $H(s) = Z(s)$ of a passive circuit, $V = ZI$.



Scatter-passive: $H(s)$ analytic and $|H(s)| \leq 1$ for $\operatorname{Re} s > 0$.



Example: Reflection coefficient $H(s) = \mathit{\Gamma}(s) = \frac{Z(s)-Z_0}{Z(s)+Z_0}$, $V = \mathit{\Gamma} U$.



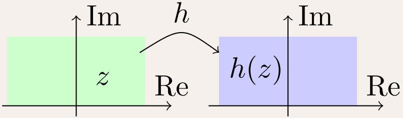
In both cases, $H(s)$ is holomorphic (analytic) for $\operatorname{Re} s > 0$, and can be related to a positive real (PR) (or Herglotz) function.

Youla *etal*(1959) [20], Zemanian (1963) [21], Wohlers and Beltrami (1965) [19], Zemanian (1965) [22]

Definition (Herglotz functions, $h(z)$)

A Herglotz (Nevanlinna, Pick, or R-) function $h(z)$ is holomorphic for $\operatorname{Im} z > 0$ and

$$\operatorname{Im} h(z) \geq 0 \quad \text{for } \operatorname{Im} z > 0$$



Representation for $\operatorname{Im} z > 0$, *cf.*, the Hilbert transform

$$h(z) = A_{\mathrm{h}} + Lz + \int_{-\infty}^{\infty} \frac{1}{\xi - z} - \frac{\xi}{1+\xi^2} \, \mathrm{d}\nu(\xi)$$

where $A_{\mathrm{h}} \in \mathbb{R}$, $L \geq 0$, and $\int_{\mathbb{R}} \frac{1}{1+\xi^2} \, \mathrm{d}\nu(\xi) < \infty$.

Gustav Herglotz
1881-1953

Rolf Nevanlinna
1895-1980

Georg Alexander Pick
1859-1942

Wilhelm Cauer
1900-1945

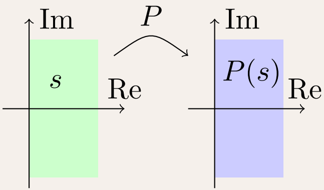
Positive real (PR) functions

Definition (Positive real (PR) functions)

A positive real function $P(s)$ is holomorphic for $\mathrm{Re}\, s > 0$ and

$$\mathrm{Re}\, P(s) \geq 0 \quad \text{for } \mathrm{Re}\, s > 0$$

with $P(s) = P^*(s^*)$.

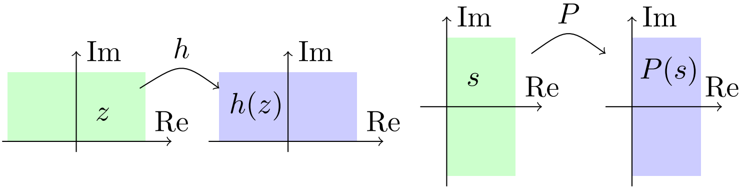


PR functions can be represented as

$$P(s) = Ls + \int_{-\infty}^{\infty} \frac{s}{s^2+\xi^2}\,\mathrm{d}\nu(\xi) = Ls + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{s\,\mathrm{Re}\{P(\mathrm{j}\xi)\}}{s^2+\xi^2}\,\mathrm{d}\xi$$

for $\mathrm{Re}\, s > 0$, where $L \geq 0$, $\int_{\mathbb{R}} \frac{1}{1+\xi^2}\,\mathrm{d}\nu(\xi) < \infty$, and we assume a sufficiently regular $P(\mathrm{j}\omega)$ in the final equality.

Herglotz functions and positive real functions



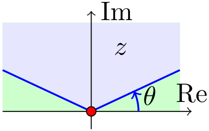
Note $z = \mathrm{i}s$, $h = \mathrm{i}P$. Here also with $h(z) = -h^*(-z^*)$ (real-valued in the time domain).

- Time conventions: $\mathrm{e}^{-\mathrm{i}\omega t}$ for Herglotz and $\mathrm{e}^{\mathrm{j}\omega t}$ for PR (Laplace parameter $s = \sigma + \mathrm{j}\omega$).
- Many contributors, Herglotz, Cauer, Nevanlinna, Pick, ...
- Also for maps between the unit circles.
- An impedance $Z(s)$ of a passive network is a typical PR function, see also ??.
- Applications: circuit synthesis, filter design, sum rules, operator theory, moment problem, ...

Integral identities for Herglotz functions

Herglotz functions with the symmetry $h(z) = -h^*(-z^*)$ (real-valued in the time domain) have asymptotic expansions ($N_0 \geq 0$ and $N_\infty \geq 0$)

$$\begin{cases} h(z) = \sum_{n=0}^{N_0} a_{2n-1} z^{2n-1} + o(z^{2N_0-1}) & \text{as } z \hat{\to} 0 \\ h(z) = \sum_{n=0}^{N_\infty} b_{1-2n} z^{1-2n} + o(z^{1-2N_\infty}) & \text{as } z \hat{\to} \infty \end{cases}$$



where $\hat{\to}$ denotes limits in the Stoltz domain $0 < \theta \leq \arg(z) \leq \pi - \theta$ 77. They satisfy the identities ($1 - N_\infty \leq n \leq N_0$)

$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_\varepsilon^{\frac{1}{\varepsilon}} \frac{\operatorname{Im} h(x + \mathrm{i}y)}{x^{2n}} \, \mathrm{d}x = a_{2n-1} - b_{2n-1} = \begin{cases} -b_{2n-1} & n < 0 \\ a_{-1} - b_{-1} & n = 0 \\ a_1 - b_1 & n = 1 \\ a_{2n-1} & n > 1 \end{cases}$$

Bernland, Luger, Gustafsson, *Sum rules and constraints on passive systems*, J. Phys. A: Math. Theor., 2011.

Integral identities for Herglotz functions

Common cases

Known low-frequency expansion $(a_1 \geq 0)$:

$$h(z) \sim \begin{cases} a_1 z & \text{as } z \hat{\to} 0 \\ b_1 z & \text{as } z \hat{\to} \infty \end{cases}$$

that gives the $n = 1$ identity (we drop the limits for simplicity)

$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{1/\varepsilon} \frac{\operatorname{Im} h(x + \mathrm{i}y)}{x^2} \, \mathrm{d}x \stackrel{\text{def}}{=} \frac{2}{\pi} \int_0^{\infty} \frac{\operatorname{Im} h(x)}{x^2} \, \mathrm{d}x = a_1 - b_1 \leq a_1$$

Known high-frequency expansion (short times) $(b_{-1} \leq 0)$:

$$h(z) \sim \begin{cases} a_{-1}/z & \text{as } z \hat{\to} 0 \\ b_{-1}/z & \text{as } z \hat{\to} \infty \end{cases}$$

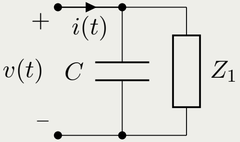
that gives the $n = 0$ identity

$$\frac{2}{\pi} \int_0^{\infty} \operatorname{Im} h(x) \, \mathrm{d}x = a_{-1} - b_{-1} \leq -b_{-1}.$$

Example (input impedance of circuit networks)

A classical sum rule for linear circuit networks is the *resistance-integral theorem* [2],[4],[18].

1. A circuit network composed of passive elements.
2. The impedance between two nodes $Z(s)$ is a PR function.
3. Consider the case with a shunt capacitor at the input terminal



$$Z(s) \sim \begin{cases} Z_1(0) & \text{as } s \hat{\to} 0 \\ \frac{1}{sC} & \text{as } s \hat{\to} \infty \end{cases}$$

where we assume that $Z_1(0)$ is finite.

4. Sum rule (integral identity with $n = 0$, $a_1 = 0$, $b_1 = 1/s$)

$$\frac{2}{\pi} \int_0^\infty R(\omega) \, \mathrm{d}\omega = \frac{1}{C}$$

Example (Temporally dispersive permittivity)

1. Linear passive material models with permittivity $\epsilon(\omega)$ ▸ ??.
2. $h_\epsilon(\omega) = \omega\epsilon(\omega)$ is a Herglotz function.
3. Consider the case without static conductivity

$$h_\epsilon(\omega) = \omega\epsilon(\omega) \sim \begin{cases} \omega\epsilon_{\mathrm{s}} = \omega\epsilon(0) & \text{as } \omega \hat{\to} 0 \\ \omega\epsilon_\infty = \omega\epsilon(\infty) & \text{as } \omega \hat{\to} \infty \end{cases}$$

4. Sum rule (integral identity with $n = 1$, $a_1 = \epsilon_{\mathrm{s}}$, $b_1 = \epsilon_\infty$)

$$\frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\{h_\epsilon(\omega)\}}{\omega^2}\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\{\epsilon(\omega)\}}{\omega}\,\mathrm{d}\omega = \epsilon_{\mathrm{s}} - \epsilon_\infty$$

Integrated losses are related to the difference $\epsilon_{\mathrm{s}} - \epsilon_\infty$, *cf.*, Landau-Lifshitz, *Electrodynamics of Continuous Media*[15] and Jackson, *Classical Electrodynamics*[12].

Example (Temporally dispersive permeability)

- Linear passive material models with permeability $\mu(\omega)$ satisfy the corresponding sum rule

$$\frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\{h_\mu(\omega)\}}{\omega^2}\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\{\mu(\omega)\}}{\omega}\,\mathrm{d}\omega = \mu_{\mathrm{s}} - \mu_\infty$$

  showing that $\mu_{\mathrm{s}} \geq \mu_\infty$, [15, 12].
- Sometimes considered a paradox for diamagnetic materials ($\mu_{\mathrm{s}} < 1$ and assuming $\mu_\infty = 1$). The paradox is resolved by considering the refractive index with $n_\infty \geq 1$ (due to special relativity) and hence

$$\frac{\epsilon_{\mathrm{s}} + \mu_{\mathrm{s}}}{2} \geq \sqrt{\epsilon_{\mathrm{s}}\mu_{\mathrm{s}}} = n_{\mathrm{s}} \geq n_\infty$$

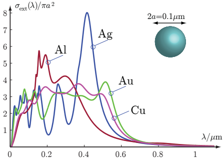
  showing that diamagnetic materials ($\mu_{\mathrm{s}} < 1$) have a static permittivity (and/or conductivity).

Sum rules and physical bounds on passive systems

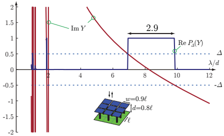
General simple approach

1. Identify a linear and passive system.
2. Construct a Herglotz (or similarly a positive real) function $h(z)$ that models the parameter of interest.
3. Investigate the asymptotic expansions of $h(z)$ as $z\hat{\to}0$ and $z\hat{\to}\infty$.
4. Use integral identities for Herglotz functions to relate the dynamic properties to the asymptotic expansions.
5. Bound the integral.

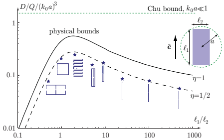
Examples: Matching networks [2, 3], Radar absorbers [16], Antennas [7, 8, 5], Scattering [17, 1], High-impedance surfaces [10], Metamaterials [6],Extraordinary transmission [9],Periodic structures [11]



Cross sections of nano spheres.



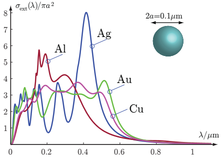
High-impedance surface.



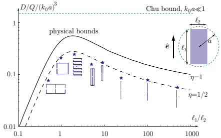
Antenna D/Q.

Sum rules and bounds in electromagnetic theory

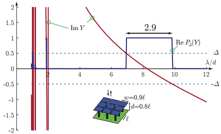
- Temporal dispersion, *e.g.*, from Kramers-Kronig relations (1926, 1927)
- Matching networks (Bode 1945, Fano 1950)
- Radar absorbers (Rozanov 2000)
- High impedance surfaces (Brewitt-Taylor 2007, Gustafsson+Sjöberg 2011)
- Extinction cross section (Sohl+etal 2007)
- Antennas (Gustafsson+etal 2007)
- Extraordinary transmission (Gustafsson 2009)
- Periodic arrays and FSS (Gustafsson+etal 2009, 2012)
- Antenna input impedance (Gustafsson 2010)
- Partial wave scattering (Bernland+etal 2010)
- Metamaterials (Gustafsson+Sjöberg 2010)
- Superluminal propagation (Gustafsson 2012)
- Array antennas (Doane, Sertel, Volakis 2013)



Cross sections of nano spheres.



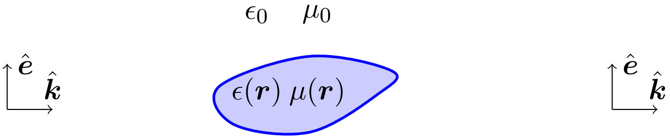
Antenna D/Q.



High-impedance surface.

Outline

Forward scattering sum rule: assumptions



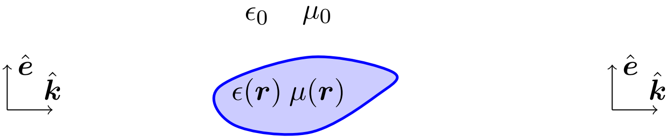
Assumptions:

- Finite scattering object composed of a linear, passive, and time translational invariant medium.
- Incident linearly polarized plane wave.

From physics:

- The propagation speed is limited by the speed of light.
- Optical theorem (energy conservation).
- Induced dipole moment in the static limit.

Forward scattering sum rule



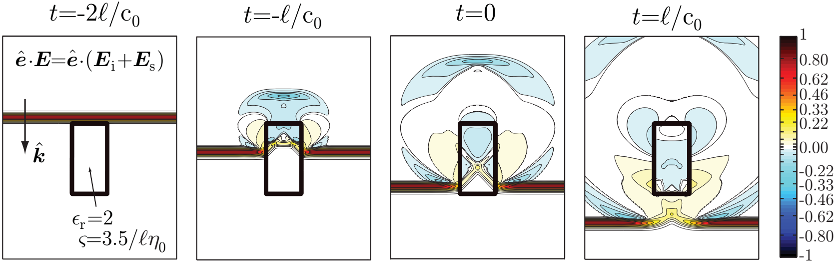
Use the $n = 1$ identity with
$a_1 = \gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$ and $b_1 = 0$, *i.e.*,

$$\frac{2}{\pi} \int_0^\infty \frac{\sigma_{\mathrm{ext}}(k)}{k^2} \, \mathrm{d}k = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$$

or written in the free-space wavelength $\lambda = 2\pi/k$

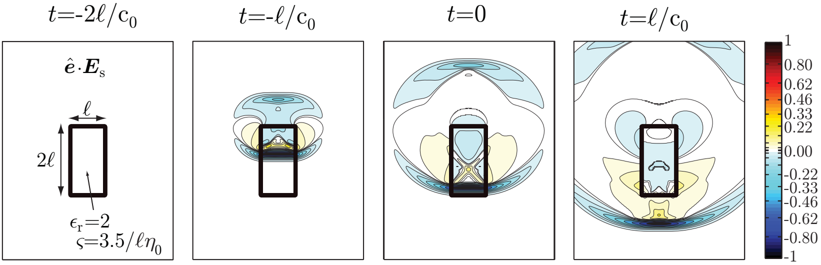
$$\frac{1}{\pi^2} \int_0^\infty \sigma_{\mathrm{ext}}(\lambda) \, \mathrm{d}\lambda = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$$

Propagation speed limited by the speed of light



- Causality in the sense that the scattered field cannot precede the incident field in the forward direction.
- Causal impulse response $h_{\rm t}(t)$.
- Analytic transfer function $h(k)$, (Fourier, Laplace transform of $h_{\rm t}(t)$, where $k = 2\pi f/{\rm c}_0$ denotes the free-space wavenumber.).

Propagation speed limited by the speed of light



- Causality in the sense that the scattered field cannot precede the incident field in the forward direction.
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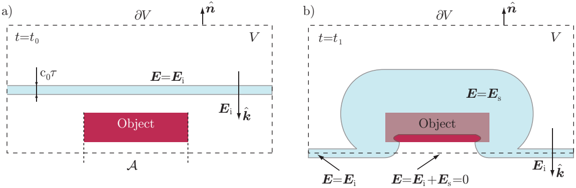
- Analytic transfer function $h(k)$, (Fourier, Laplace transform of $h_{\mathrm{t}}(t)$, where $k = 2\pi f/\mathrm{c}_0$ denotes the free-space wavenumber.).

Energy conservation (passivity)



$$W_{\mathrm{ext},\tau} = -\int_{\mathbb{R}}\int_{\partial V} \big(\boldsymbol{E}_\mathrm{i}(t,\boldsymbol{r})\times\boldsymbol{H}_\mathrm{s}(t,\boldsymbol{r})+\boldsymbol{E}_\mathrm{s}(t,\boldsymbol{r})\times\boldsymbol{H}_\mathrm{i}(t,\boldsymbol{r})\big)\cdot\hat{\boldsymbol{n}}(\boldsymbol{r})\,\mathrm{dS}\,\mathrm{d}t.$$

simplify to

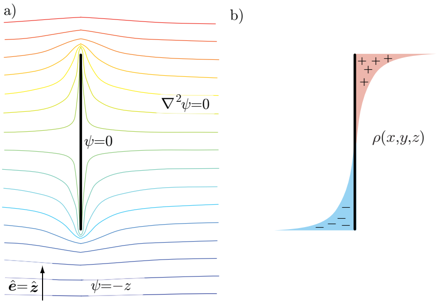
$$W_\mathrm{ext} = \int_{\mathbb{R}}\int_{\mathbb{R}} E(t)h_\mathrm{t}(\tau-t)E(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \geq 0$$

for all E implying

$$\operatorname{Im} h(k) = \sigma_\mathrm{ext}(k) \geq 0 \quad \text{for } \operatorname{Im} k \geq 0$$

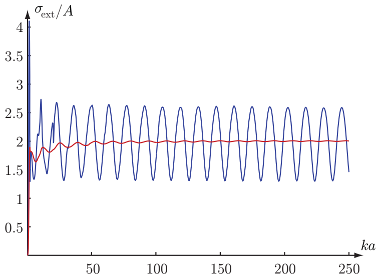
cf., the optical theorem.

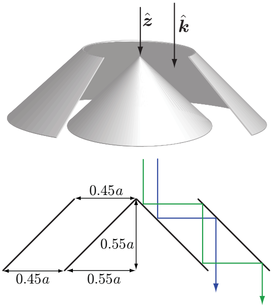
Low-frequency asymptotic expansion



- $h(k) = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \hat{\boldsymbol{e}} k + \mathcal{O}(k^2)$ as $k \to 0$ (Kleinman&Senior 1986).
- Polarizability dyadic $\boldsymbol{\gamma}_{\mathrm{e}}$.
- Induced dipole moment $\boldsymbol{p} = \epsilon_0 \boldsymbol{\gamma}_{\mathrm{e}} E_0 \hat{\boldsymbol{e}}$.
- Variational principles $\boldsymbol{\gamma}_{\mathrm{e}} \leq \boldsymbol{\gamma}_{\infty}$ (Jones 1985, Sjöberg 2009).
- High contrast polarizability dyadic $\boldsymbol{\gamma}_{\infty}$.

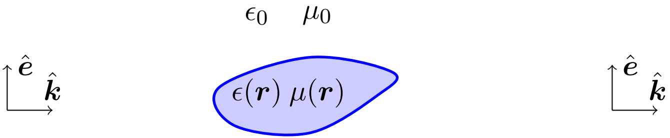
High-frequency asymptote





- Shadow scattering (Peierls 1979, Gustafsson *etal* 2008).
- $\operatorname{Im} h(k) = \sigma_{\mathrm{ext}}(k) \leq 2A$ on average as $k \to \infty$, *i.e.*, for $0 < \delta < \arg k < \pi - \delta$.
- the extinction paradox.

Forward scattering sum rule



Use the $n = 1$ identity with
$a_1 = \gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$ and $b_1 = 0$, *i.e.*,

$$\frac{2}{\pi}\int_0^\infty \frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$$

or written in the free-space wavelength $\lambda = 2\pi/k$

$$\frac{1}{\pi^2}\int_0^\infty \sigma_{\mathrm{ext}}(\lambda)\,\mathrm{d}\lambda = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})$$

Polarizability dyadic and induced dipole moment

The induced dipole moment can be written

$$\boldsymbol{p} = \epsilon_0 \boldsymbol{\gamma}_{\mathrm{e}} \cdot \boldsymbol{E}$$

where $\boldsymbol{\gamma}_{\mathrm{e}}$ is the polarizability dyadic.

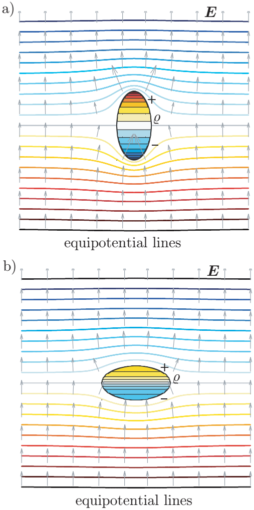
Example (Dielectric sphere)

A dielectric sphere with radius a and relative permittivity ϵ_{r} has the polarizability dyadic

$$\boldsymbol{\gamma}_{\mathrm{e}} = 4\pi a^3 \frac{\epsilon_{\mathrm{r}} - 1}{\epsilon_{\mathrm{r}} + 2} \mathbf{I} \to \boldsymbol{\gamma}_{\infty} = 4\pi a^3 \mathbf{I}$$

as $\epsilon_{\mathrm{r}} \to \infty$.

Analytic expressions for spheroids, elliptic discs, half spheres, hollow half spheres, touching spheres,...

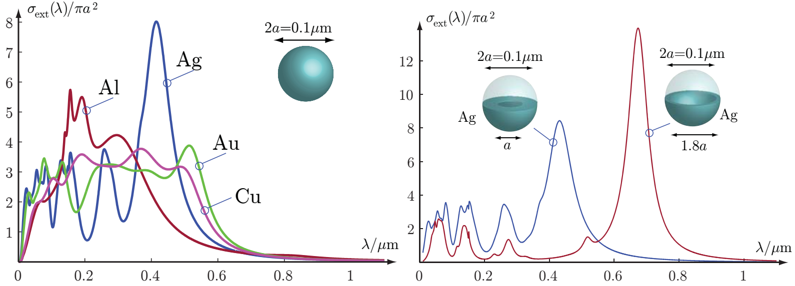


Extinction cross sections for $a = 50\,\mathrm{nm}$ spheres

$$\sigma_{\mathrm{ext}} = \sigma_{\mathrm{a}} + \sigma_{\mathrm{s}} = \frac{P_{\mathrm{a}} + P_{\mathrm{s}}}{|\boldsymbol{E}_{\mathrm{i}}|^2/2\eta_0}$$

Sum of the scattered and absorbed powers divided by the incident power flux. Integrate over the free-space wavelength $\lambda = 2\pi/k$

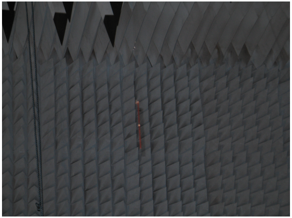
$$\frac{1}{\pi^2}\int_0^\infty \sigma_{\mathrm{ext}}(\lambda)\,\mathrm{d}\lambda = \hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}} = 4\pi a^3$$



The areas under the curves are identical.

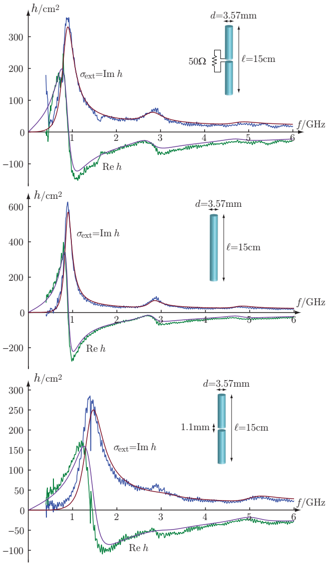
Time-domain approach to the forward scattering sum rule, Proc.R.Soc.A, 2010

Forward scattering of antennas



- Forward scattering measurement of a dipole antenna.
- Loaded, short, and open circuit.
- Length $15\,\mathrm{cm}$ and $0.5\,\mathrm{GHz}$ to $6\ \mathrm{GHz}$.

	in cm^3	loaded	short	open
sim:	γ	661	661	291
sim:	$\frac{2}{\pi}\int_0^{k_2} \frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k$	644	644	265
meas:	$\frac{2}{\pi}\int_{k_1}^{k_2} \frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k$	605	670	322



Forward scattering of loaded and unloaded antennas, IEEE-TAP, 2012.

Physical bounds on antennas

Given a geometry, *e.g.*, sphere, rectangle, spheroid, or cylinder. How does D/Q (directivity bandwidth product) depend on the geometry for optimal antennas?

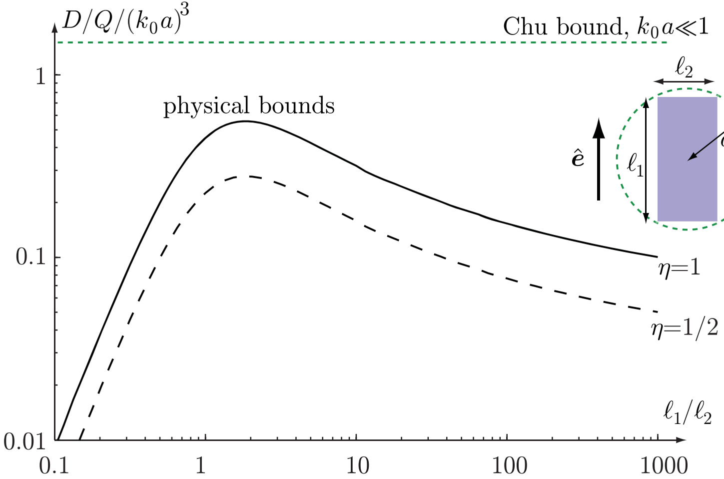
$$\frac{D}{Q} \leq \frac{\eta k_0^3}{2\pi}\big(\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}}+(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}})\cdot\boldsymbol{\gamma}_{\mathrm{m}}\cdot(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}})\big)$$

based on

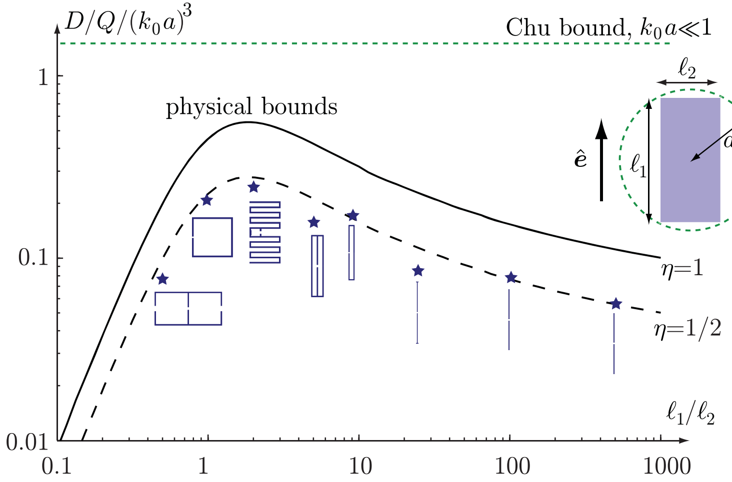
- Passive materials
- Antenna forward scattering
- Identities for Herglotz function

Physical limitations on antennas of arbitrary shape Proc R. Soc. A, 463. 2589-2607, 2007.
Illustrations of new physical bounds on linearly polarized antennas, IEEE Trans. Antennas Propagat., 2009.
Absorption Efficiency and Physical Bounds on Antennas, Int. J. of Antennas and Propagat., 946746, 2010
http://www.mathworks.com/matlabcentral/fileexchange/26806-antennaq

Circumscribing rectangles

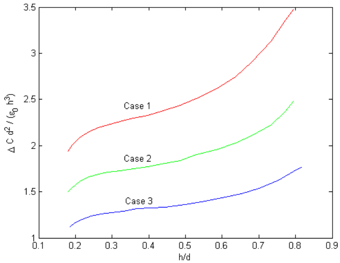


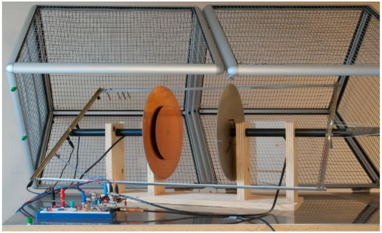
Circumscribing rectangles

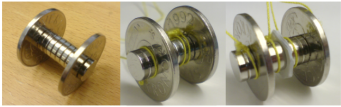


How can we measure the polarizability?

- Change of capacitance in a parallel plate capacitor.
- The polarizability in a parallel plate waveguide.
- The periodic polarizability for symmetric objects.







Case 1 Case 2 Case 3

Objects with increasing distance between the coins.
Large separation of charge give a large polarizability.

D. Lovrić, *Theoretical and experimental studies of polarizability dyadics* , 2011. Kristensson, *The polarizability and the capacitance change of a bounded object in a parallel plate capacitor*, Physica Scripta, 2012.

High-contrast polarizability dyadics: γ_∞

γ_∞ is determined from the induced normalized surface charge density, ρ, as

$$\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_\infty\cdot\hat{\boldsymbol{e}} = \frac{1}{E_0}\int_{\partial V}\hat{\boldsymbol{e}}\cdot\boldsymbol{r}\rho(\boldsymbol{r})\,\mathrm{dS}$$

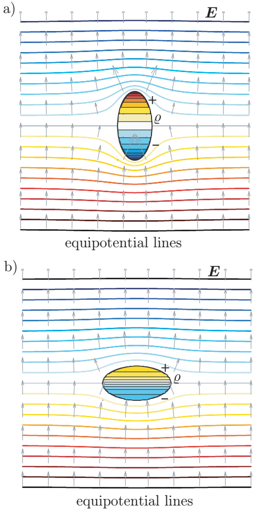
where ρ satisfies the integral equation

$$\int_{\partial V}\frac{\rho(\boldsymbol{r}')}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{dS}' = E_0\boldsymbol{r}\cdot\hat{\boldsymbol{e}} - V_n$$

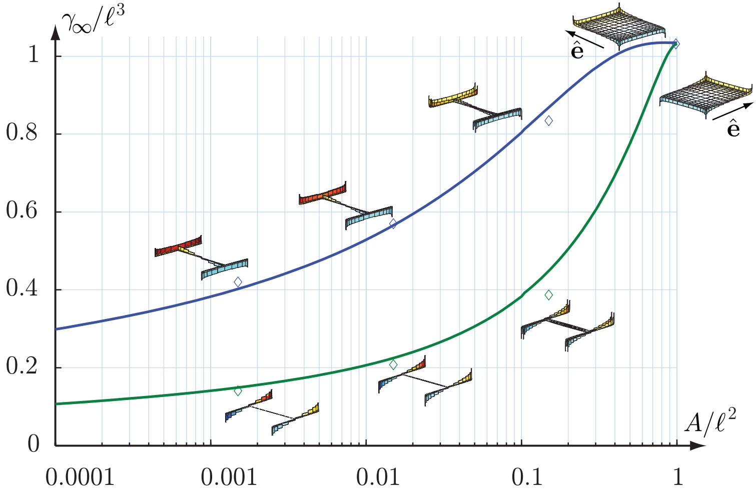
with the constraints of zero total charge

$$\int_{\partial V_n}\rho(\boldsymbol{r})\,\mathrm{dS} = 0$$

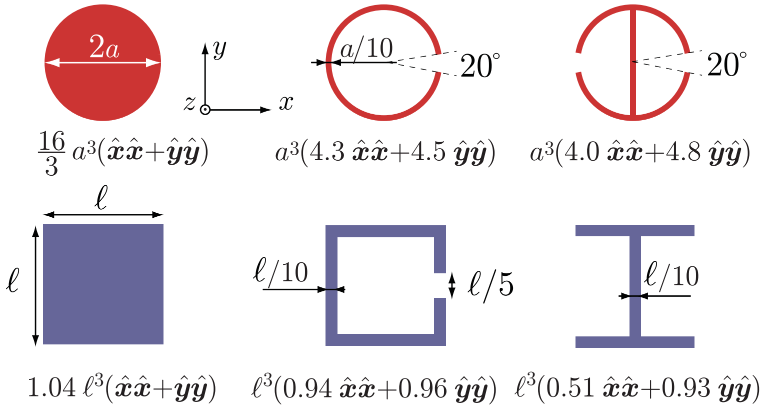
Can also use FEM (Laplace equation).



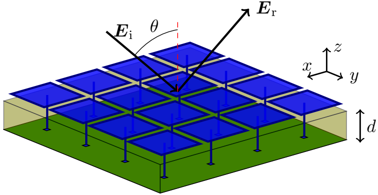
Removal of metal from a square plate

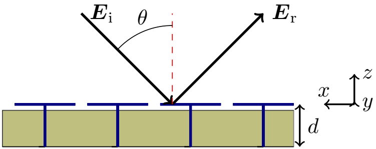


Removal of metal from a square plate and circular disk



Periodic structures

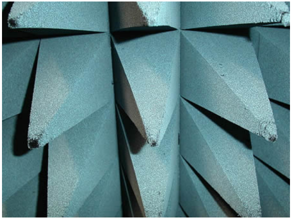


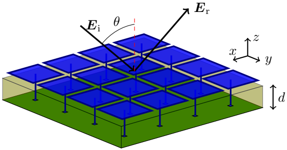


- Radar absorbers.
- Frequency selective surfaces (FSS).
- Aperture arrays for extraordinary optical transmission (EOT).
- Synthesize (meta) materials.
- Perfect lens (negative refractive index) to produce planar lenses and increased resolution.
- Cloaking to hide objects (negligible bi-static RCS).

Absorbers

- A (periodic) structure (above a ground plane) that absorbs incident EM waves.
- Often desired to be thin and absorb power over large bandwidths.
- The tradeoff between thickness and bandwidth (homogeneous multilayer structures) was first analyzed by Rozanov in 2000.
- Here, as an example also for the general periodic case.





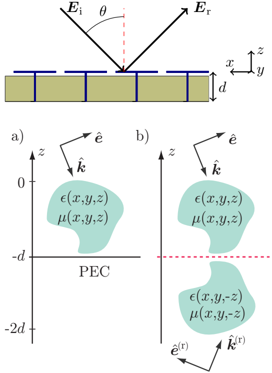
Rozanov, *Ultimate thickness to bandwidth ratio of radar absorbers*, IEEE-TAP, 2000.

Absorbers

1. Identify the reflection coefficient, $\varGamma$, as a passive system ($|\varGamma| \leq 1$).
2. Analyze the low- (and high) frequency behavior:

$$\varGamma(k) \sim -1 - \mathrm{i}k\big(2d\cos\theta + \gamma/A\big), \quad k \to 0$$

   where γ is the polarizability per unit cell. A well-defined static quantity that is easily determined.
3. Construct the Herglotz function $h = -\mathrm{i}\ln(\varGamma/B_{\mathrm{p}})$ and the sum rule



$$\frac{2}{\pi}\int_0^\infty \frac{1}{k^2}\ln\frac{1}{|\varGamma(k)|}\,\mathrm{d}k \leq 2d\cos\theta + \gamma/A \leq 2\mu_{\mathrm{s}} d$$

Absorbers and array antennas

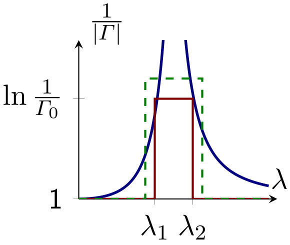
Rewrite in the wavelength $\lambda = 2\pi/k$ and estimate the integral, *e.g.*,

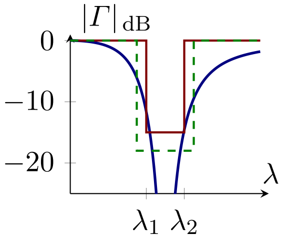
$$\frac{1}{\pi^2}(\lambda_2-\lambda_1)\ln\frac{1}{|\mathit{\Gamma}_0|}\,\mathrm{d}\lambda \leq \frac{1}{\pi^2}\int_{\lambda_1}^{\lambda_2}\ln\frac{1}{|\mathit{\Gamma}(\lambda)|}\,\mathrm{d}\lambda \leq \frac{1}{\pi^2}\int_0^{\infty}\ln\frac{1}{|\mathit{\Gamma}(\lambda)|}\,\mathrm{d}\lambda \leq 2\mu_{\mathrm{s}}d$$

with $\mathit{\Gamma}_0 = \max_{[\lambda_1,\lambda_2]}|\mathit{\Gamma}(\lambda)|$.

Bandwidth limited by the thickness d and (static) permeability μ_{s}.

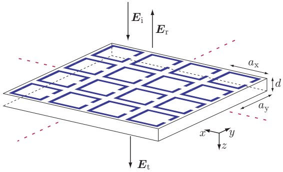
Extended to array antennas by Doane, Sertel, Volakis (IEEE-TAP 2013) & Jonsson, Kolitsidas, Hussain (IEEE-APWL 2013).



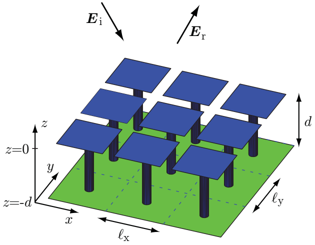


Identify passive systems

- The transmission coefficient, $T(k)$, represents a (time domain) passive system, *i.e.*, the transmitted energy cannot be greater than the incident energy for all times.
- The reflection coefficient, Γ, is passive if the reference plane is placed 'above' the structure.
- Note causality follows from (time domain) passivity.
- We can now construct many Herglotz functions (or PR) from T and Γ.



Screen with a periodic microstructure.

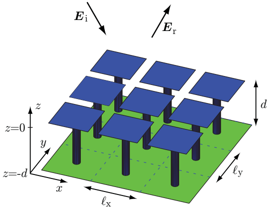


Array above a ground plane.

- Total cross section $h(k) = \mathrm{i}2\big(1 - T(k)\big)A$.
- Absorber $h = -\mathrm{i}\ln(\Gamma/B_{\mathrm{p}})$.
- High-impedance surfaces $Z = (1+\Gamma)/(1-\Gamma)$ and $h_{\mathrm{Y}} = \mathrm{i}Y$.

High-impedance (artificial magnetic) surfaces

- PEC surfaces have low impedance, *i.e.*, short circuit currents give $Z = 0$. They also have reflection coefficients $\Gamma = (Z - Z_0)/(Z + Z_0) = -1$.
- PMC surfaces have high impedance and $\Gamma = 1$ (no phase shift).
- Useful for low-profile antennas ,*i.e.*, planar antenna elements can be placed above a PMC.
- Also useful to stop surface waves, *cf.*, hard and soft surfaces.



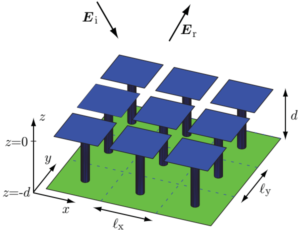
High-impedance surfaces are often composed by periodic structures above a PEC ground plane, here a mushroom structure.

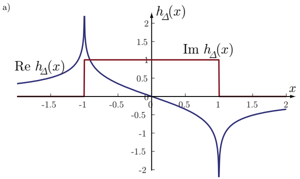
For what bandwidth can a periodic structure above a PEC plane have 'high' impedance (reflection coefficient $\Gamma \approx 1$)?

High-impedance surfaces

How does the bandwidth of high-impedance surfaces (artificial magnetic ground planes) depend on the thickness of the structure?

1. Passive system: Admittance $Y = 1/Z$ where $Z = (1+\Gamma)/(1-\Gamma)$ and Herglotz function $h_{\mathrm{Y}} = \mathrm{i}Y$.
2. Low-frequency expansion $h \sim -1/(k(d\cos\theta + \gamma/2A))$ as $k \to 0$.
3. Interested in the bandwidth with $|Y| \leq \Delta$. Compose with h_Δ, *i.e.*, $h_1 = h_\Delta(h_{\mathrm{Y}}) \sim 2k(d\cos\theta + \gamma/2A)\Delta/\pi$.
4. $n = 1$ sum rule.





High-impedance surfaces

Sum rule

$$\int_0^\infty \operatorname{Im} h_\Delta(h_{\mathrm{Y}}(\lambda))\,\mathrm{d}\lambda = 2\pi\left(d\cos\theta + \frac{\gamma}{2A}\right)\Delta,$$

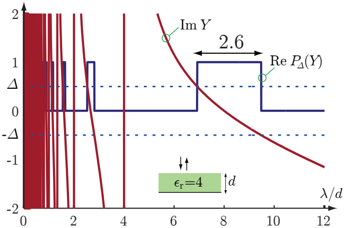
where $\lambda = 2\pi/k$.

Dielectric lossless slab

$$\frac{1}{d}\int_0^\infty \operatorname{Im} h_\Delta(h_{\mathrm{Y}}(\lambda))\,\mathrm{d}\lambda = 2\pi\Delta,$$

where

$$\operatorname{Im} h_\Delta(h_{\mathrm{Y}}(\lambda)) = \begin{cases} 0 & |Y| > \Delta \\ 1 & |Y| < \Delta \end{cases}$$



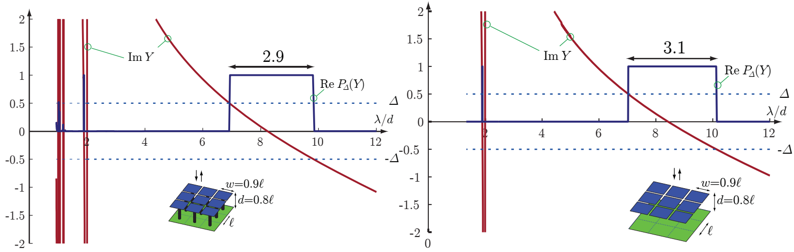
$\epsilon_{\mathrm{r}} = 4$ slab with $\Delta = 1/2$.
The area under the blue curve is $2\pi\Delta = \pi$.

High-impedance surfaces

Physical bound

$$\frac{B\lambda_0}{d} \leq 4\pi\mu_{\mathrm{s}}^{\max} \max_{\lambda\in\mathcal{B}} |Y(\lambda)| \begin{cases} 1 & \text{lossy} \\ 1/2 & \text{lossless,} \end{cases}$$

Non-magnetic and $\max|Y| \leq 1/2$ gives the normalized bandwidth $B\lambda_0/d \leq \pi$.



Physical bounds and sum rules for high-impedance surfaces, IEEE-TAP, 2011.

Additional sum rules for periodic structures

Cross section:

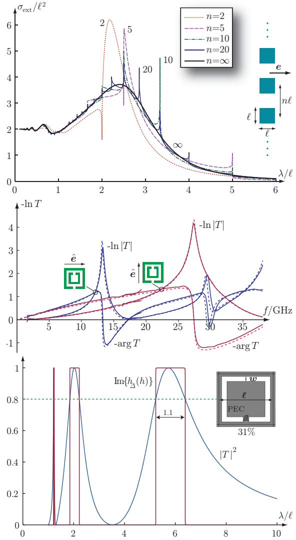
$$\frac{1}{\pi^2}\int_0^\infty \sigma_{\mathrm{ext}}(\lambda;\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})\,\mathrm{d}\lambda = \hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}}$$

Transmission blockage:

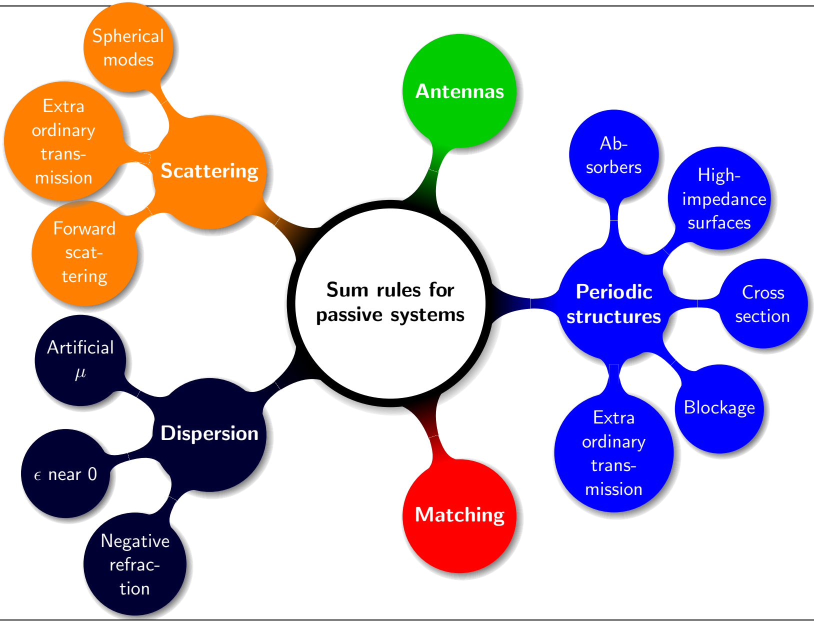
$$\frac{1}{\pi^2}\int_0^\infty \ln\frac{1}{|T(\lambda)|}\,\mathrm{d}\lambda \leq \hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}}$$

Extra ordinary transmission:

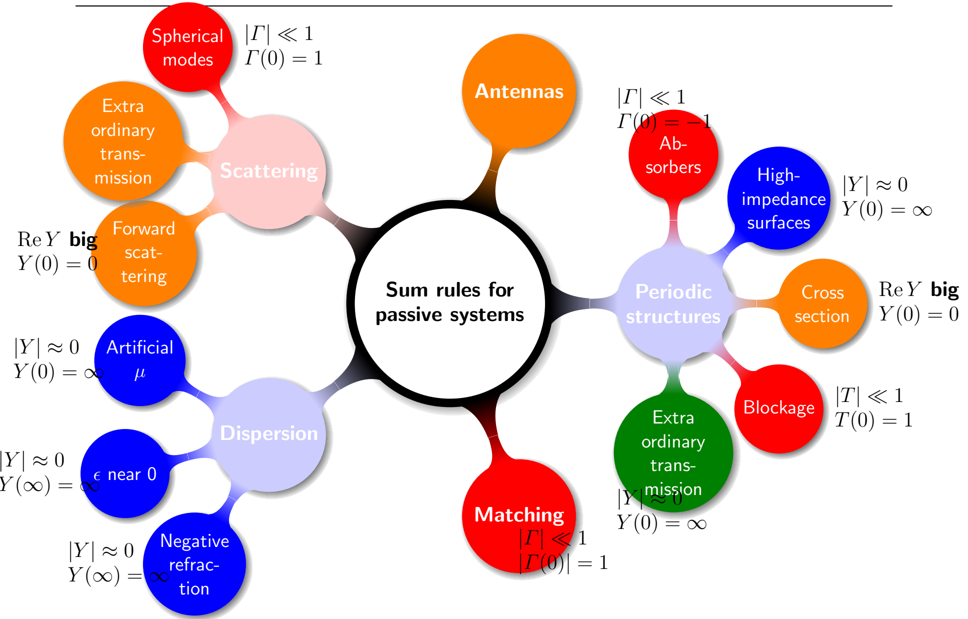
$$\int_0^\infty \mathrm{Im}\{h_\Delta(h(\lambda))\}\,\mathrm{d}\lambda = \frac{\gamma\Delta\pi}{A}$$



Sum rules for passive systems



Sum rules for passive systems



Basically four (or two) cases

admittance want
large $\operatorname{Re} Y(\omega_0)$ with $Y(0) = 0$.
forward scattering (cross section)
Use identity

S-parameter want
$|S(\omega_0)| \leq \delta$ with $|S(0)| = 1$.
absorber, matching, blockage, modes, ...
Use log+identity

admittance want
small $|Y(\omega_0)| \leq \delta$ with $Y(0) = \infty$.
high impedance surface, temporal dispersion.
Use pulse+identity

S-parameter want
$S(\omega_0) \approx 1$ with $S(0) = -1$.
high impedance surface, extra ordinary transmission
Use Cayley+pulse+identity

Passive constitutive relations

The linear, causal, time translational invariant, continuous, non-magnetic, and isotropic constitutive relations are

$$\boldsymbol{D}(t) = \epsilon_0\epsilon_\infty \boldsymbol{E}(t) + \epsilon_0 \int_{\mathbb{R}} \chi_{\mathrm{ee}}(t-t')\boldsymbol{E}(t')\,\mathrm{d}t'$$

where $\chi_{\mathrm{ee}}(t) = 0$ for $t < 0$, the dependence of the spatial coordinates is suppressed, and $\epsilon_\infty > 0$ is the instantaneous response. The material model is passive if

$$0 \leq \int_{-\infty}^{T} \boldsymbol{E}(t)\cdot\frac{\partial \boldsymbol{D}(t)}{\partial t}\,\mathrm{d}t = \epsilon_0 \int_{-\infty}^{T}\int_{\mathbb{R}} \boldsymbol{E}(t)\cdot\frac{\partial}{\partial t}\big(\epsilon_\infty\delta(t-t')+\chi_{\mathrm{ee}}(t-t')\big)\boldsymbol{E}(t')\,\mathrm{d}t'\,\mathrm{d}t$$

for all times T and fields $\boldsymbol{E}$.

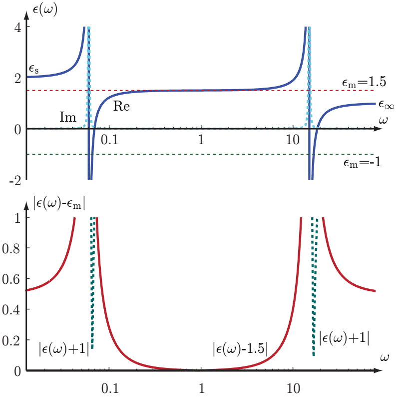
- Similarly for the magnetic fields.
- The presented results are also valid for the diagonal elements of general bi-anisotropic constitutive relations.
- Time-domain model, *e.g.*, used in FDTD.
- Fourier transform to get the frequency-domain model $\boldsymbol{D}(\omega) = \epsilon_0\epsilon(\omega)\boldsymbol{E}(\omega)$.

Temporal dispersion of passive metamaterials

Metamaterials are materials with unusual properties, *e.g.*, $\epsilon \approx \epsilon_\mathrm{m}$ where $\epsilon_\mathrm{m} = -1$ or $\epsilon_\mathrm{m} \approx 0$. Metamaterials are dispersive, *i.e.*, $\epsilon = \epsilon(\omega)$.

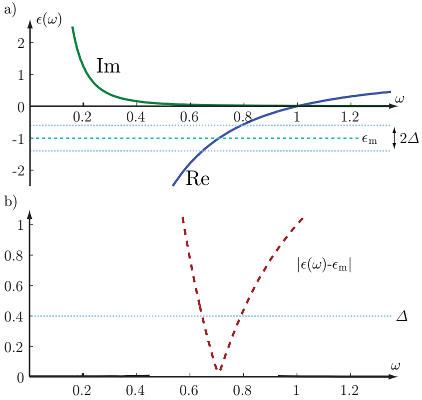
What is the minimum temporal dispersion of passive metamaterials over bandwidths $\mathcal{B} = [\omega_1, \omega_2]$?

- no limitation for $\epsilon_\infty \leq \epsilon_\mathrm{m} \leq \epsilon_\mathrm{s}$.
- limitations for $\epsilon_\mathrm{m} \leq \epsilon_\infty = \epsilon(\infty)$.
- limitations for $\epsilon_\mathrm{m} \geq \epsilon_\mathrm{s} = \epsilon(0)$.



$\epsilon(\omega)$ with 2 Lorentz terms: $\epsilon_\mathrm{s} = 2$: static value, $\epsilon_\infty = 1$: instantaneous value, and $\epsilon_\mathrm{m} = 1.5$ or $\epsilon_\mathrm{m} = -1$: target

Example: Drude model



The Drude model (common model for metals and metamaterials)

$$\epsilon(\omega) = 1 + \frac{1}{-\mathrm{i}\omega(0.01 - \mathrm{i}\omega)},$$

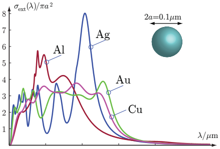
- Interested in the behavior of $\epsilon(\omega) \approx -1 = \epsilon_{\rm m}$
- $\epsilon(0.7) \approx -1 = \epsilon_{\rm m}$.
- Difference $|\epsilon(\omega) - \epsilon_{\rm m}| \leq \Delta = 0.4$ for approximately $|\omega - 0.7| < 0.1$.

Sum rules on passive systems

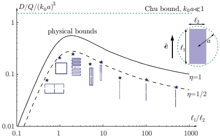
1. Identify a linear and passive (and causal) system (construct a Herglotz or positive real function). Here, *e.g.*, $h(\omega) = \omega\epsilon(\omega)$.
2. Determine the low- and high-frequency asymptotic expansions. Expressed in ϵ_{s} and ϵ_∞.
3. Use integral identities (for Herglotz functions) to relate the dynamic properties to the asymptotic expansion, *e.g.*,

$$\frac{2}{\pi}\int_0^\infty \frac{\operatorname{Im} h(\omega)}{\omega^2}\,\mathrm{d}\omega = a_1 - b_1 \leq a_1,$$

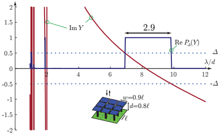
where $h(\omega) \sim a_1\omega$ as $\omega\hat{\to}0$ and $h(\omega) \sim b_1\omega$ as $\omega\hat{\to}\infty$.



Cross sections of nano spheres.



Antenna D/Q.



High-impedance surface.

Sum rules and constraints on passive systems J. Phys. A: Math. Theor. 44 145205, 2011

Kramers-Kronig relations

1. $\omega\epsilon(\omega)$ is a Herglotz function (passive material models).
2. without static conductivity

$$h_\epsilon(\omega) = \omega\epsilon(\omega) \sim \begin{cases} \omega\epsilon_\mathrm{s} = \omega\epsilon(0) & \text{as } \omega \hat{\to} 0 \\ \omega\epsilon_\infty = \omega\epsilon(\infty) & \text{as } \omega \hat{\to} \infty \end{cases}$$

3. Sum rule (integral identity with $n = 1$, $a_1 = \epsilon_\mathrm{s}$, $b_1 = \epsilon_\infty$)

$$\frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\{h_\epsilon(\omega)\}}{\omega^2}\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\{\epsilon(\omega)\}}{\omega}\,\mathrm{d}\omega = \epsilon_\mathrm{s} - \epsilon_\infty$$

Integrated losses are related to the difference $\epsilon_\mathrm{s} - \epsilon_\infty$, *cf.*, Landau-Lifshitz, *Electrodynamics of Continuous Media* and Jackson, *Classical Electrodynamics*.

How can we instead relate the temporal dispersion to the asymptotic values?

Composition of Herglotz functions

Construct a Herglotz function such that

- $|\epsilon - \epsilon_{\mathrm{m}}| < \Delta$ gives $\operatorname{Im} h \approx 1$.
- $|\epsilon - \epsilon_{\mathrm{m}}| > \Delta$ gives $\operatorname{Im} h \approx 0$.

Solution:

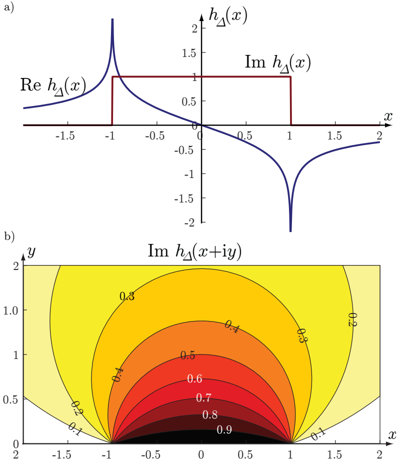
1. use the Herglotz function

$$h_1(\omega) = \frac{\omega}{\omega_0}\big(\epsilon(\omega) - \epsilon_{\mathrm{m}}\big)$$

   to map $\epsilon \approx \epsilon_{\mathrm{m}} \to 0$.

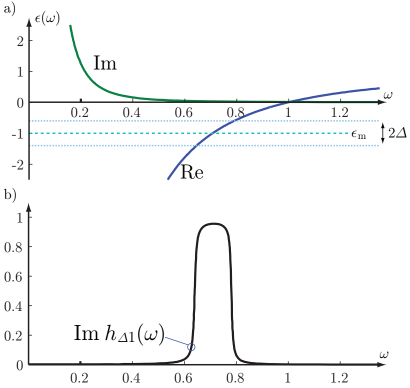
2. compose with a Herglotz function that has $\operatorname{Im} h_\Delta(z) \approx 1$ for $|z| < \Delta$ and $\operatorname{Im} h_\Delta(z) \approx 0$ for $|z| > \Delta$, *i.e.*,

$$h_\Delta(z) = \frac{1}{\pi}\int_{-\Delta}^{\Delta} \frac{1}{\xi - z}\,\mathrm{d}\xi = \frac{1}{\pi}\ln\frac{z-\Delta}{z+\Delta}$$



The Herglotz function $h_\Delta(z)$ with $\Delta = 1$. Note, it is not a rational function and it has logarithmic singularities at $\pm\Delta$.

Example: Drude model



$\mathrm{Im}\, h_{\Delta}(h_1(\omega))$ with $\Delta = 0.4$.

Sum rule

$$\int_0^{\infty} \mathrm{Im}\, h_{\Delta}(h_1(\omega))\, \mathrm{d}\omega = \frac{\omega_0 \Delta}{\epsilon_\infty - \epsilon_{\rm m}}$$

The Drude model

$$\epsilon(\omega) = 1 + \frac{1}{-\mathrm{i}\omega(0.01 - \mathrm{i}\omega)},$$

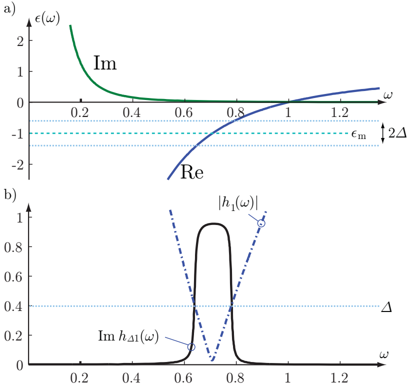
has $\epsilon(\omega) \approx -1 = \epsilon_{\rm m}$ for $\omega \approx 0.7$.

- The area under $\mathrm{Im}\, h_{\Delta}(h_1(\omega))$ is concentrated to the region where $|\epsilon(\omega) - \epsilon_{\rm m}| \leq \Delta$.
- This area is known

$$\frac{\omega_0 \Delta}{\epsilon_\infty - \epsilon_{\rm m}} \approx \frac{0.7 \cdot 0.4}{1 - (-1)} = 0.14$$

- area $\approx$ height $\times$ width gives the bandwidth, *i.e.*, bandwidth ≈ 0.14.

Example: Drude model



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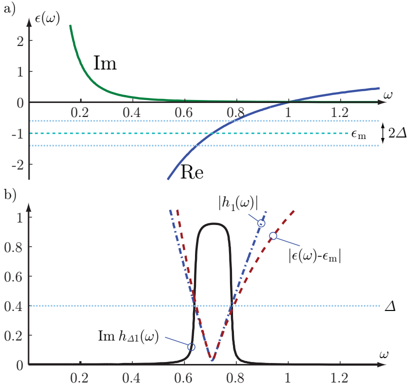

$\mathrm{Im}\, h_{\varDelta}(h_1(\omega))$ with $\varDelta = 0.4$.
Sum rule

$$\int_0^\infty \mathrm{Im}\, h_{\varDelta}(h_1(\omega))\,\mathrm{d}\omega = \frac{\omega_0 \varDelta}{\epsilon_\infty - \epsilon_{\rm m}}$$

The Drude model

$$\epsilon(\omega) = 1 + \frac{1}{-\mathrm{i}\omega(0.01 - \mathrm{i}\omega)},$$

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- area $\approx$ height $\times$ width gives the bandwidth, *i.e.*, bandwidth ≈ 0.14.

Temporal dispersion: constraints

Interval $\mathcal{B} = [\omega_1, \omega_2]$ with fractional bandwidth $B = (\omega_2 - \omega_1)/\omega_0$, $\omega_0 = (\omega_1 + \omega_2)/2$

ϵ_{s} =static, ϵ_∞ =instantaneous, ϵ_{m} =target values.

1. $\epsilon_{\mathrm{m}} < \epsilon_\infty$:

$$\max_{\omega\in\mathcal{B}} |\epsilon(\omega) - \epsilon_{\mathrm{m}}| \geq \frac{B}{1+B/2}(\epsilon_\infty - \epsilon_{\mathrm{m}}) \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case,} \end{cases}$$

2. without static conductivity

$$\max_{\omega\in\mathcal{B}} \frac{|\epsilon(\omega) - \epsilon_{\mathrm{m}}|}{|\epsilon(\omega) - \epsilon_\infty|} \geq \frac{B}{1+B/2} \frac{\epsilon_{\mathrm{s}} - \epsilon_{\mathrm{m}}}{\epsilon_{\mathrm{s}} - \epsilon_\infty} \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case,} \end{cases}$$

3. artificial magnetism $\mu_{\mathrm{m}} > \mu_{\mathrm{s}}$

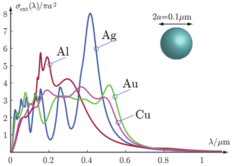
$$\max_{\omega\in\mathcal{B}} \frac{|\mu(\omega) - \mu_{\mathrm{m}}|}{|\mu(\omega) - \mu_\infty|} \geq \frac{B}{1+B/2} \frac{\mu_{\mathrm{m}} - \mu_{\mathrm{s}}}{\mu_{\mathrm{s}} - \mu_\infty} \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case,} \end{cases}$$

Sum rules and physical bounds on passive metamaterials, New Journal of Physics, Vol. 12, pp. 043046-, 2010.

Outline

Conclusions

- Sum rules derived from integral identities for Herglotz functions.
- Physical bounds.
- Extinction cross section, antennas, extra ordinary transmission, transmission blockage, high-impedance surfaces, radar absorbers, temporal dispersion, perfect lenses, artificial magnetism.



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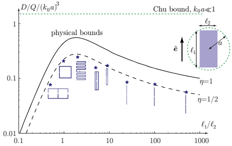
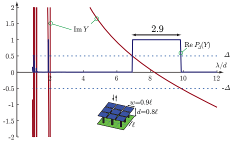


Antenna D/Q.



High-impedance surface.

Why physical bounds for passive systems?

- Realistic expectations. Possible/impossible.
- Possible design improvements. Is it worth it?
- Figure of merit for a design.
- Use of active and/or non-linear systems.

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Outline

7. **Definitions**
   - Distributions
   - Fourier- and Laplace transforms
   - Hilbert transform
   - Stoltz domain
8. **Polarizability dyadics**
9. **Extra ordinary transmission**
   - Extraordinary transmission
10. **Temporal dispersion**

Functions, distributions, and systems

Basic differences between functions, distributions, and systems:

Functions

$$\mathbb{R} \longrightarrow \boxed{u} \longrightarrow \mathbb{R}$$

Map numbers to numbers, *e.g.*, $\mathbb{R} \to \mathbb{R}$, $\mathbb{C} \to \mathbb{C}$, or matrix valued. Continuous, differentiable, or using equivalence classes such as integrable L^p.

Distributions

$$\mathcal{D} \longrightarrow \boxed{f} \longrightarrow \mathbb{R}$$

Map test functions to numbers, *e.g.*, $\mathcal{D} \to \mathbb{R}$, $\mathcal{D} \to \mathbb{C}$, $\mathcal{S} \to \mathbb{C}$.

Systems

$$\mathcal{D}' \longrightarrow \boxed{\mathcal{R}} \longrightarrow \mathcal{D}'$$

Many possibilities, *e.g.*, distributions to distributions $\mathcal{D}' \to \mathcal{D}'$ or functions to functions.

There are many similarities for LTI systems, $v = h * u$, where the impulse response h can be a function or a distribution.

Distributions

Definition (Test functions)

$\mathcal{D}$ is the space of smooth test functions with compact support.

Definition (Distribution)

The elements of the space $\mathcal{D}'$ of continuous linear functionals on $\mathcal{D}$ are distributions.

Linear functionals are often denoted $\langle f, \phi \rangle$. We can identify regular distributions (generated by functions) with the integral

$$\langle f, \phi \rangle = \int_{\mathbb{R}} f(t)\phi(t) \, \mathrm{d}t$$

We often suppress the difference between functions and distributions and use the same notation for distributions. In these cases it is important to realize the symbol $\int \cdot \cdot \mathrm{d} \cdot$ is just a notation for the corresponding linear functional $\langle \cdot, \cdot \rangle$.

Tempered distributions

Definition (Test functions of rapid descent)

The space $\mathcal{S}$ of smooth testing functions of rapid descent.

Definition (Tempered distribution)

The elements of the space $\mathcal{S}'$ of continuous linear functionals on $\mathcal{S}$ are tempered distributions (or distributions of slow growth).

- Subspace of $\mathcal{D}$.
- The Fourier transform of a tempered distribution is a tempered distribution.
- The Laplace transform of a casual tempered distribution is analytic for $\operatorname{Re} s > 0$.

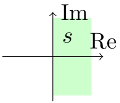
Causality

Definition (Causality)

A distribution u on $\mathcal{D}(\mathbb{R})$ is causal if $\langle u, \phi \rangle = 0$ for all test functions $\phi(t)$ such that $\phi(t) = 0$ for $t > 0$, *i.e.*, $\operatorname{supp} u \subset [0, \infty)$.??

The Laplace transform, $U(s) = \mathcal{L}\{u\}(s)$, of a causal tempered distribution is analytic for $\operatorname{Re} s > 0$. The limiting distribution at the frequency axis

$$\lim_{\sigma \to 0^+} \langle U(\sigma + \cdot), \phi \rangle = \lim_{\sigma \to 0^+} \int_{\mathbb{R}} U(\sigma + \mathrm{j}\omega)\phi(\omega)\, \mathrm{d}\omega$$



is a tempered distribution. Causality is hence not a very strong condition to restrict the class of distributions.

Example

Derivatives (and anti-derivatives) of the Dirac delta distribution are typical examples of causal distributions, *i.e.*,

$$u(t) = \delta^{(n)}(t) = \frac{\mathrm{d}^n \delta(t)}{\mathrm{d}t^n} \quad \text{with } U(s) = s^n$$

where we note that U is bounded for $n = 0$ and passive for $|n| \leq 1$.

Fourier- and Laplace transforms

The Fourier transform is usually defined for real-valued parameters (the frequency axis) but can also be considered for complex-valued parameters (*e.g.*, a half plane). There are also many common normalizations.

One particular illuminating case is the Fourier- and Laplace transforms of the unit step, *i.e.*,

$$\mathcal{F}\{\theta(t)\}(\omega) = \frac{1}{\mathrm{j}\omega} + \pi\delta(\omega) \quad \text{for } \omega \in \mathbb{R}$$

and

$$\mathcal{L}\{\theta(t)\}(s) = \frac{1}{s} \quad \text{for } \operatorname{Re} s > 0$$

where we note that $\mathcal{F}\{\theta\}$ is a distribution and $\mathcal{L}\{\theta\}$ is an analytic function in $s = \sigma + \mathrm{j}\omega$ for $\operatorname{Re} s > 0$. Moreover, $\mathcal{F}\{\theta\}$ is the limiting distribution of $\mathcal{L}\{\theta\}$ at the frequency axis, *i.e.*,

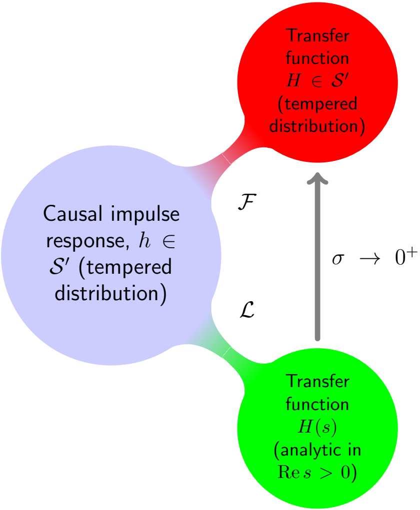
$$\lim_{\sigma \to 0^+} \langle \mathcal{L}\{\theta\}, \phi \rangle = \langle \mathcal{F}\{\theta\}, \phi \rangle$$

Fourier- and Laplace transforms

Consider a causal impulse response $h(t)$ in the form of a tempered distribution $h \in \mathcal{S}'$.

- The Laplace transform $\mathcal{L}\{h(t)\}(s)$ is analytic for $\sigma > 0$ with $s = \sigma + \mathrm{j}\omega$.
- The Fourier transform $\mathcal{F}\{h(t)\}(\omega)$ is a tempered distribution for $\omega \in \mathbb{R}$.
- They are related at the frequency axis $\sigma \to 0^+$.

For the sub class of passive impulse responses, we can use the representations for PR (or Herglotz) functions to restrict the spaces for the impulse response and transfer functions.



Hilbert transform

Definition (Hilbert transform)

The Hilbert transform is

$$\mathcal{H}\{u(\tau)\}(t) = \frac{1}{\pi} \int \frac{u(\tau)}{t-\tau} \, \mathrm{d}\tau$$

where a Cauchy principal value integral is used.

Properties

- Bounded in L^p for $1 < p < \infty$.
- Inverse $\mathcal{H}\{\mathcal{H}\{u\}\} = -u$.
- Convolution with the tempered distribution $h(t) = \mathrm{p.v.}\frac{1}{\pi t}$, $\mathcal{H}\{u\} = h * u$.
- Relates the real and imaginary parts of boundary functions in H^p for $1 < p < \infty$, ????

King, Hilbert Transforms I,II (2009) [13],[14].

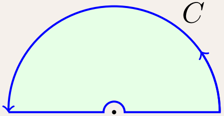
David Hilbert
1862-1943

SokhotskiPlemelj theorem

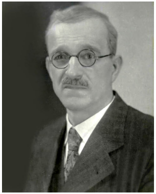
Theorem (SokhotskiPlemelj theorem)

The SokhotskiPlemelj theorem expresses the value of an analytic function as a Cauchy principal value integral over a (smooth) closed simple curve

$$f_{\pm}(z) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \pm \frac{f(z)}{2}$$



where $f_{\pm}(z)$ is the limit value from the interior/exterior of the curve C.



Josip Plemelj 1873-1967

Julian Karol Sochocki 1847-1927

Properties

- Interpreted as the Cauchy formula and half the residue of the pole.
- Similar to the Hilbert transform for half planes and sufficiently regular functions (decay at infinity).

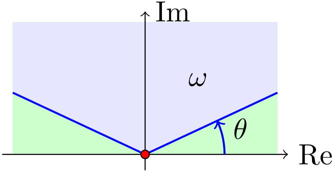
Stoltz domain

The symbol $\hat{\to}$ denotes limits in the Stoltz domain. For $\omega\hat{\to}0$ (upper) and $s\hat{\to}0$ (right) half planes, we use any $0 < \theta \leq \pi/2$ and

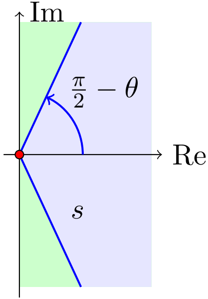
$$\theta \leq \arg\omega \leq \pi - \theta \quad \text{or} |\arg s| \leq \frac{\pi}{2} - \theta$$

and similarly for $\omega\hat{\to}\infty$ and $s\hat{\to}\infty$

upper half plane



right half plane



Example (time delay)

The time delay $\varGamma(s) = \mathrm{e}^{-s\tau}$ is scattering passive and imply the PR function

$$Y(s) = \frac{1-\varGamma}{1+\varGamma} = \frac{1-\mathrm{e}^{-s\tau}}{1+\mathrm{e}^{-s\tau}} = \tanh(\frac{s\tau}{2}) \to 1$$

as $s\hat{\to}\infty$ although the limit $s \to \infty$ for $s = \mathrm{j}\omega$ does not exist.

Outline

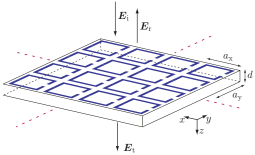
Optical theorem for periodic structures

- The optical theorem relates the forward scattering with the total cross section, $\sigma_{\mathrm{ext}} = \sigma_{\mathrm{a}} + \sigma_{\mathrm{s}}$.
- The incident power per unit cell is P_{i}.
- Transmitted power, $P_{\mathrm{t}} = |T|^2 P_{\mathrm{i}} + P_{\mathrm{t1}}$.
- The absorbed power, P_{a}

$$P_{\mathrm{a}} = P_{\mathrm{i}} - P_{\mathrm{r}} - P_{\mathrm{t}} = P_{\mathrm{i}} - P_{\mathrm{r}} - |T|^2 P_{\mathrm{i}} - P_{\mathrm{t1}}.$$

- The scattered power $P_{\mathrm{s}} = P_{\mathrm{r}} + |1-T|^2 P_{\mathrm{i}} + P_{\mathrm{t1}}$.
- The total absorbed and scattered power

$$\begin{aligned} P_{\mathrm{a}}+P_{\mathrm{s}} &= P_{\mathrm{i}}-P_{\mathrm{r}}-|T|^2P_{\mathrm{i}}-P_{\mathrm{t1}}+P_{\mathrm{r}}+|1-T|^2P_{\mathrm{i}}+P_{\mathrm{t1}} \\ &= 2\,\mathrm{Re}\{1-T\}P_{\mathrm{i}} \end{aligned}$$



Screen with a periodic microstructure.

(2) low- and high-frequency expansions

- The Herglotz function $h(k) = \mathrm{i}2\big(1 - T(k)\big)A$
- is bounded $|h(k)| \leq 4A$ so the high-frequency asymptotic is also bounded.
- The low-frequency asymptote of T is obtained by an expansion of the fields in powers of k giving

$$h(k) \sim k\gamma = k\big(\hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})\big)$$

  as $k \to 0$.
- The electric and magnetic static polarizabilities $\boldsymbol{\gamma}_{\mathrm{e}}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$ provide the induced electric and magnetic dipole moments per unit cell.
- Here, non-magnetic $\boldsymbol{\gamma}_{\mathrm{m}} = \mathbf{0}$.

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(3) Total cross section sum rule

How much can a low-pass FSS interact with EM fields?

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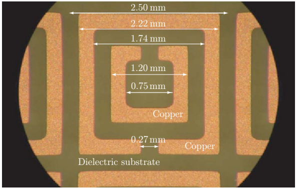
- The optical theorem

$$\sigma_{\text{ext}} = 2\operatorname{Re}\{1 - T\}A.$$

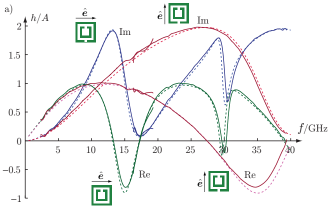
and the low-frequency expansion $h(k) \sim k\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\text{e}}\cdot\hat{\boldsymbol{e}}$ as $k \to 0$

- Sum rule $(n = 1)$

$$\frac{2}{\pi}\int_0^\infty \frac{\sigma_{\text{ext}}(k;\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{k^2}\,\mathrm{d}k = \hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\text{e}}\cdot\hat{\boldsymbol{e}}$$



Manufactured SRR structure.



Extinction cross sections for the SRR.

(3) Total cross section sum rule

- Using the wavelength $\lambda = 2\pi/k$

$$\frac{1}{\pi^2}\int_0^\infty \sigma_{\mathrm{ext}}(\lambda;\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})\,\mathrm{d}\lambda = \hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}}$$

- Physical bound (rectangle with the same area)

$$\frac{\lambda_2-\lambda_1}{\pi^2}\min_{\lambda\in[\lambda_1,\lambda_2]}\sigma_{\mathrm{ext}}(\lambda)\leq \hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}}$$

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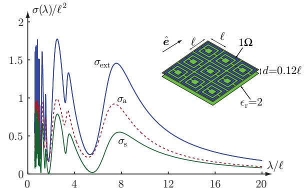
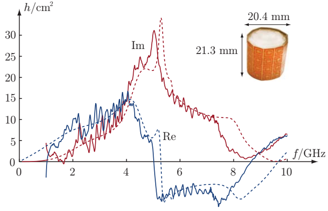
- Gives physical insight.

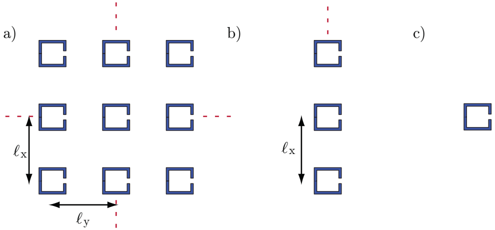


1 Ω SRR structure.



SRR cylinder.

Periodic in one direction and single object



Extend the sum rule to structures periodic in one direction and single objects

- Increase the (unit cell) distance ℓ_y between the objects to infinity to get periodic structures in one direction.
- Increase $\ell_\mathrm{x}, \ell_\mathrm{y}$ to infinity to get single objects.
- It turns out that they have identical sum rules

$$\frac{1}{\pi^2}\int_0^\infty \sigma_\mathrm{ext}(\lambda;\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})\,\mathrm{d}\lambda = \hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_\mathrm{e}\cdot\hat{\boldsymbol{e}}$$

Example 1D periodic

- Consider square patches with side lengths ℓ and that are repeated periodically in the $\hat{\boldsymbol{x}}$-direction with inter-element distances $\ell_{\mathrm{x}} = n\ell$, for $n = 2, 5, 10, 20$.
- Total cross section for a single square patch ($n = \infty$).
- (top) polarization $\hat{\boldsymbol{e}} = \hat{\boldsymbol{x}}$.
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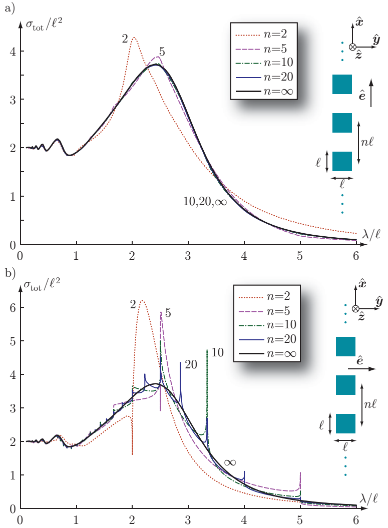
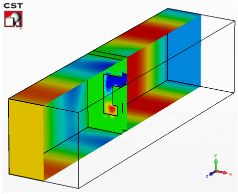
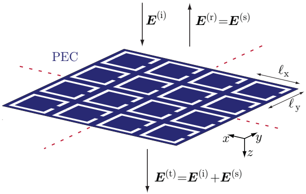


Outline

Extraordinary transmission

- Extraordinary optical transmission (EOT) is the phenomenon of greatly enhanced transmission of light through subwavelength apertures in an opaque metallic film.
- Ebbesen *et al.*, Extraordinary optical transmission through sub-wavelength hole arrays, Nature 1998.
- Localization of light, subwavelength optics, sensing, optoelectronics.
- Often in thin (*e.g.*, silver) films.
- Bandpass frequency selective surface.
- Here, we analyze apertures in PEC (perfectly conducting) sheets (zero thickness).

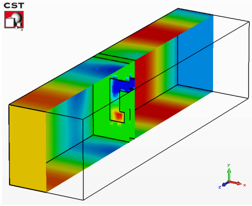


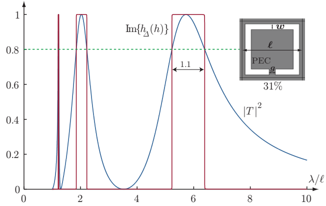


Array of apertures.

Extraordinary transmission through PEC sheets

Over what bandwidth can at least 80% of the power be transmitted?





- Construct a sum rule for $|T|^2 \geq 0.8$, *i.e.*, $\Delta = 0.5$ below

$$\int_0^\infty \mathrm{Im}\{h_\Delta(h(\lambda))\}\,\mathrm{d}\lambda = \frac{\gamma\Delta\pi}{A}$$

- Example with an aperture array of SRR in a PEC sheet.
- The area under $\mathrm{Im}\{h_\Delta(h(\lambda/\ell))\}$ is known: 1.56.
- Bandwidth with $|T|^2 \geq 0.8$ is ≈ 1.1 (bound 1.56).

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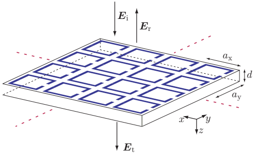

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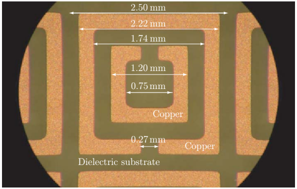

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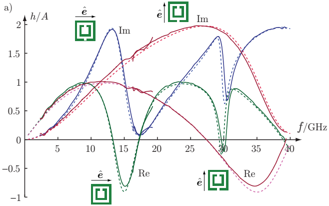

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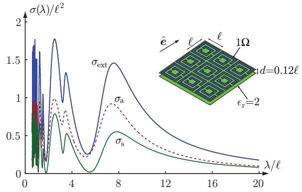

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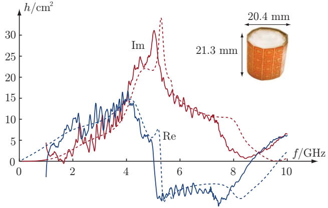

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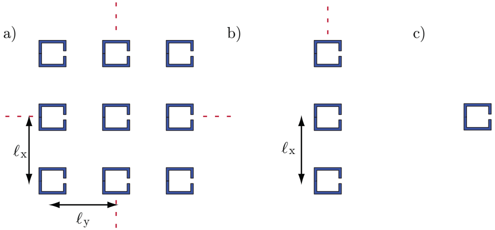

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