Convex Optimization for Optimal Design and Analysis of Small Antennas

Mats Gustafsson

Electrical and Information Technology, Lund University, Sweden
IEEE AP-S Distinguished Lecturer Program

Slides at www.eit.lth.se/staff/mats.gustafsson

Outline

Acknowledgments

- IEEE APS Distinguished Lecturer Program
- The Swedish Research Council
- Swedish Foundation for Strategic Research (SSF)
- Lund University

Collaboration with:

- Doruk Tayli, Lund University
- Marius Cismasu, Ericsson (was LU)
- Sven Nordebo, Linnæus University
- Lars Jonsson, KTH

VETENSKAPSRÅDET
THE SWEDISH RESEARCH COUNCIL

SWEDISH FOUNDATION *for* STRATEGIC RESEARCH

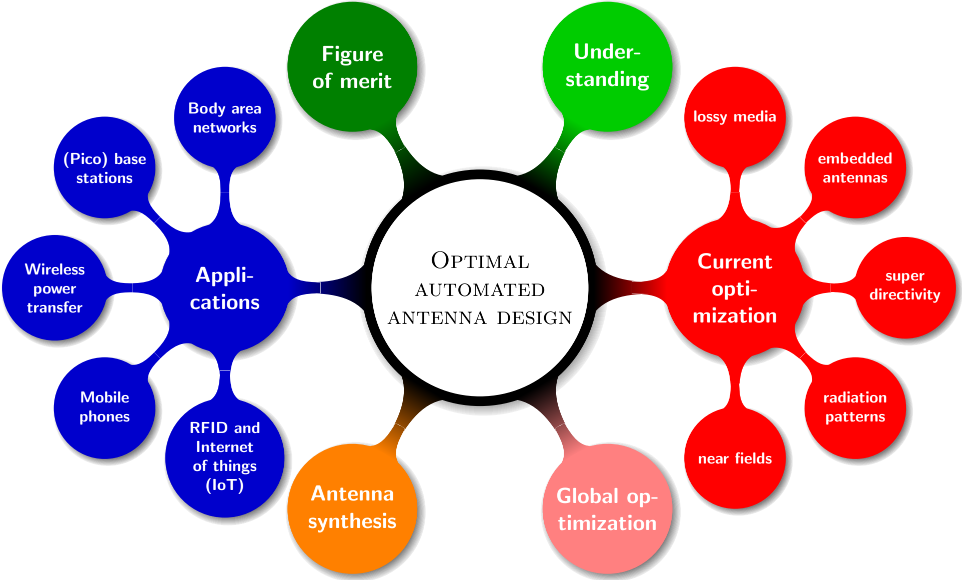


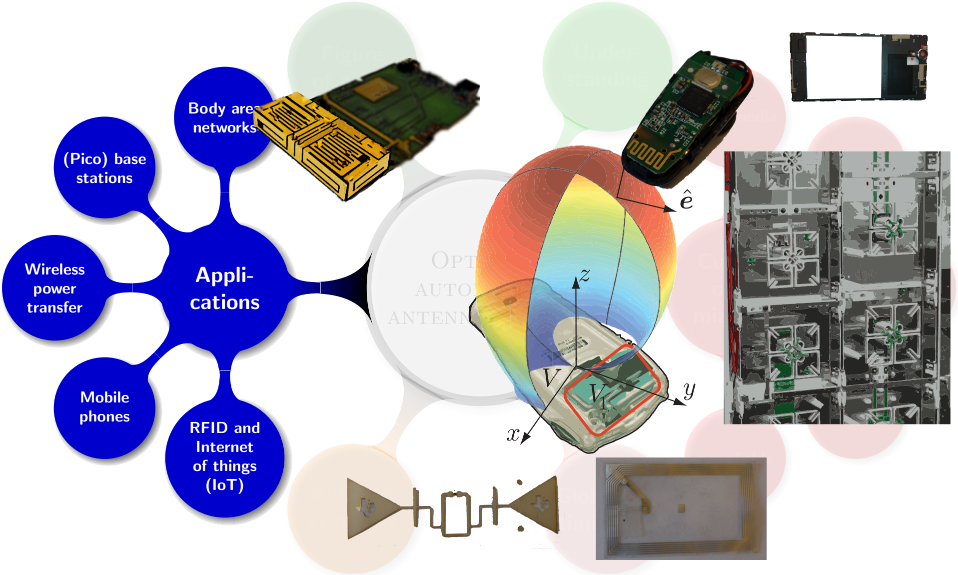


Lund University

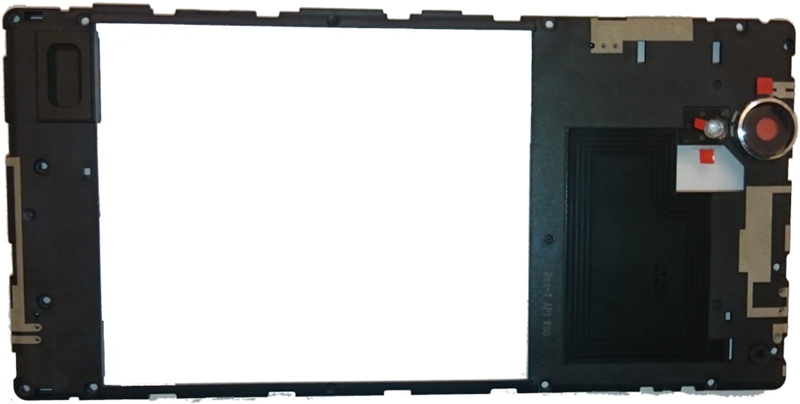
- Lund university was founded in 1666.
- Sweden's largest university.
- Approximately 40 000 students.
- Department of Electrical and Information Technology: Broadband Communications, Circuits and Systems, Communication, Electromagnetic theory, Networking and Security, Signal Processing.

Outline

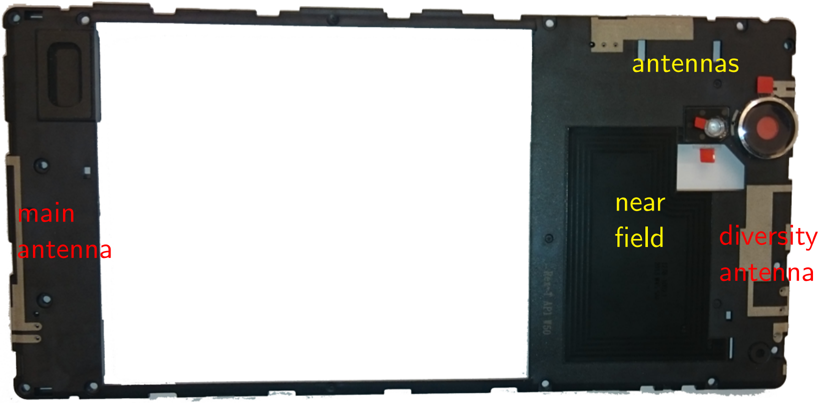

Optimal automated antenna design
Figure of merit
Under-standing
Appli-cations
Body area networks
(Pico) base stations
Wireless power transfer
Mobile phones
RFID and Internet of things (IoT)
Antenna synthesis
Global op-timization
Current opti-mization
lossy media
embedded antennas
super directivity
radiation patterns
near fields

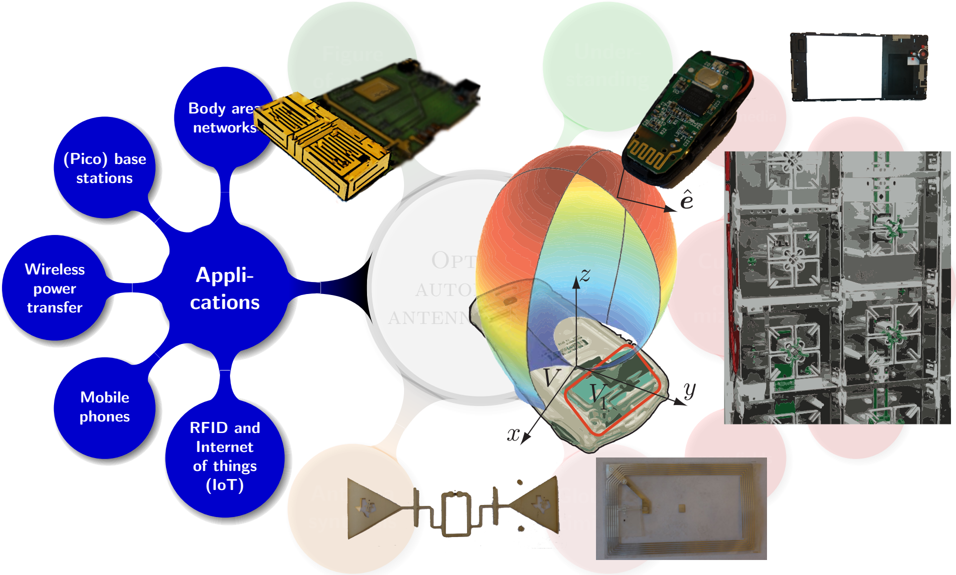

Appli-
cations
Body area
networks
(Pico) base
stations
Wireless
power
transfer
Mobile
phones
RFID and
Internet
of things
(IoT)
$\hat{e}$
z
y
x
V
V_1

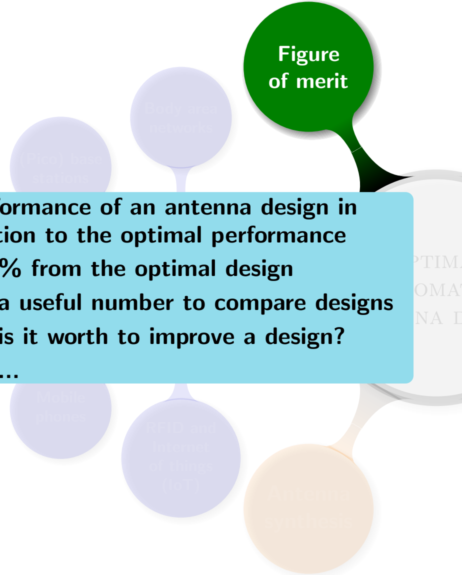
Frame integrated antennas (Sony Xperia)



Frame integrated antennas (Sony Xperia)

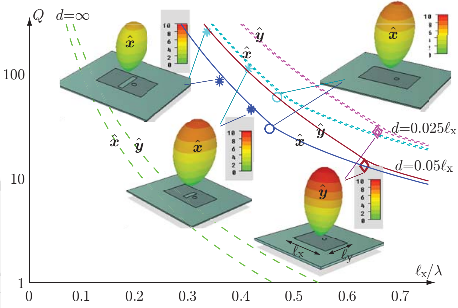


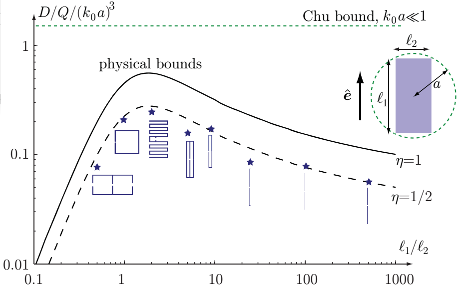

Body area networks
(Pico) base stations
Wireless power transfer
Appli-cations
Mobile phones
RFID and Internet of things (IoT)
$\hat{e}$
z
y
x
V
V_I



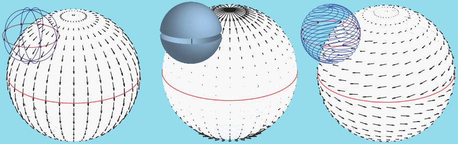
Performance of an antenna design in relation to the optimal performance

- % from the optimal design
- a useful number to compare designs
- is it worth to improve a design?
- ...

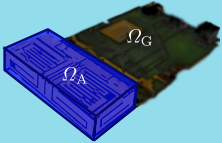




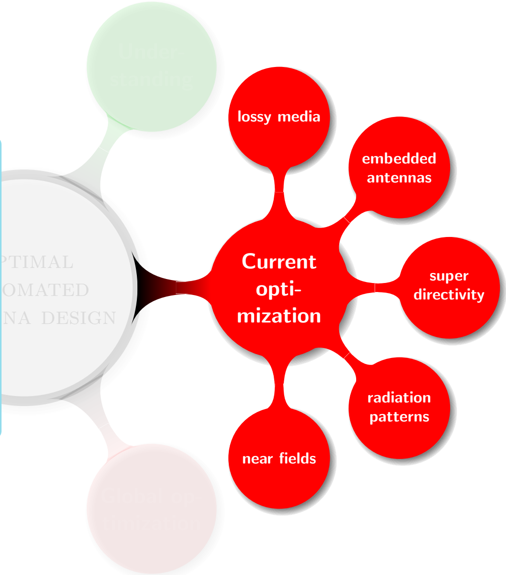
Under-standing

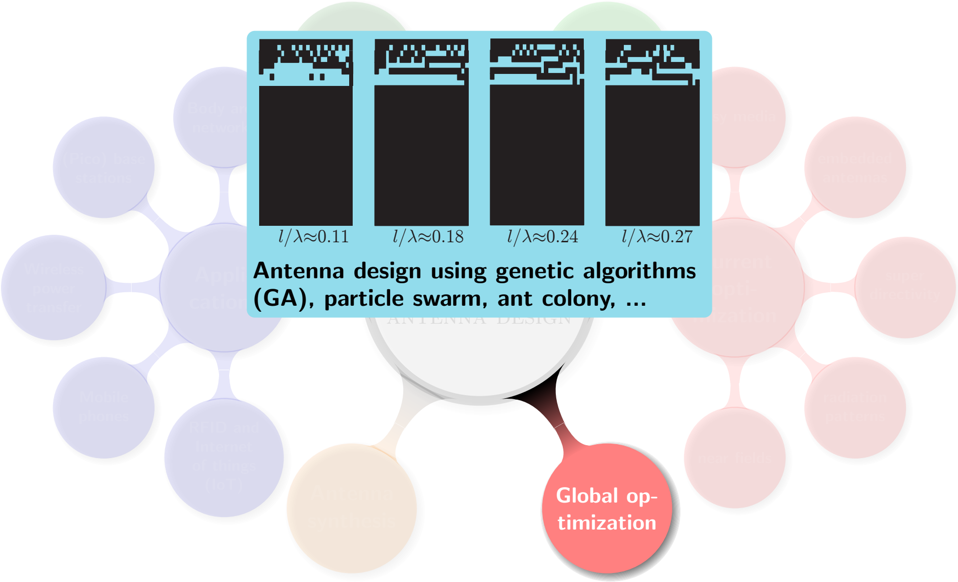


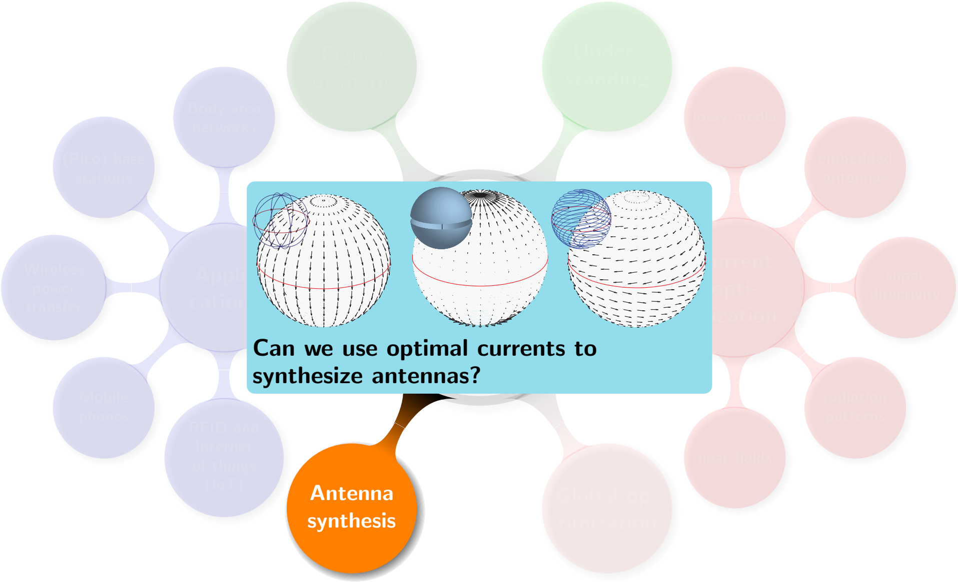
- **Physical bounds**
- **Current distributions**
- **Polarizability**

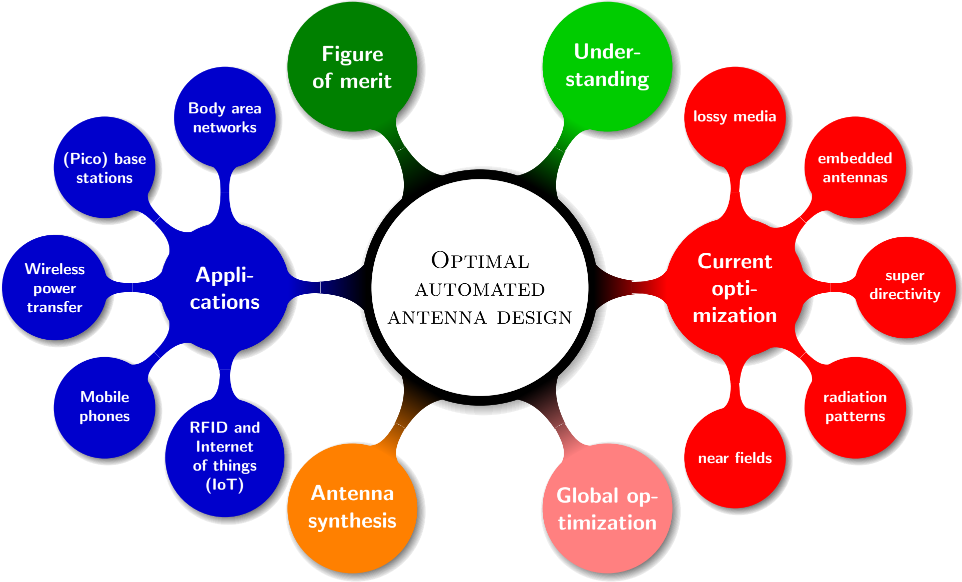


- **Optimal current distribution**
- **Physical bounds**
- **Convex optimization**
- **...**



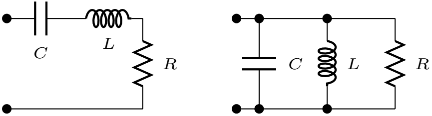

$l/\lambda \approx 0.11$
$l/\lambda \approx 0.18$
$l/\lambda \approx 0.24$
$l/\lambda \approx 0.27$
Antenna design using genetic algorithms (GA), particle swarm, ant colony, ...
Global optimization


Can we use optimal currents to synthesize antennas?
Antenna synthesis
Figure of merit
Under-standing
Body area networks
(Pico) base stations
Wireless power transfer
Mobile phones
RFID and Internet of things (IoT)
lossy media
embedded antennas
super directivity
radiation patterns
near fields


Optimal automated antenna design
Figure of merit
Under-standing
Appli-cations
Body area networks
(Pico) base stations
Wireless power transfer
Mobile phones
RFID and Internet of things (IoT)
Antenna synthesis
Global op-timization
Current opti-mization
lossy media
embedded antennas
super directivity
radiation patterns
near fields

Outline

Q-factor



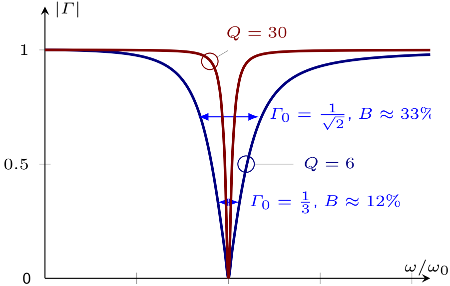
The Q-factor is defined as the ratio between the stored electric, W_{e}, and magnetic, W_{m}, energies and the dissipated power, *i.e.*,

$$W_{\mathrm{e}} = \frac{C|V|^2}{4} \quad \text{and} \quad W_{\mathrm{m}} = \frac{L|I|^2}{4}$$

$$Q = \frac{2\omega \max\{W_{\mathrm{e}}, W_{\mathrm{m}}\}}{P_{\mathrm{rad}} + P_{\mathrm{loss}}}.$$

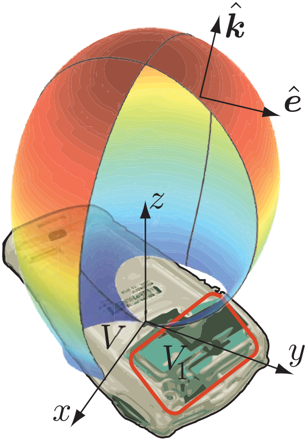
Fractional bandwidth for single resonances (RLC circuits)

$$B \approx \frac{2}{Q}\frac{\Gamma_0}{\sqrt{1-\Gamma_0^2}}$$



Physical bounds on antennas

- Tradeoff between performance and size.
- Performance, *e.g.*, in Q, (half-power fractional bandwidth $B \approx 2/Q$), directivity bandwidth product: D/Q, efficiency, capacity,....
- Properties of the best antenna confined to a given (arbitrary) geometry, *e.g.*, spheroid, cylinder, and rectangle.



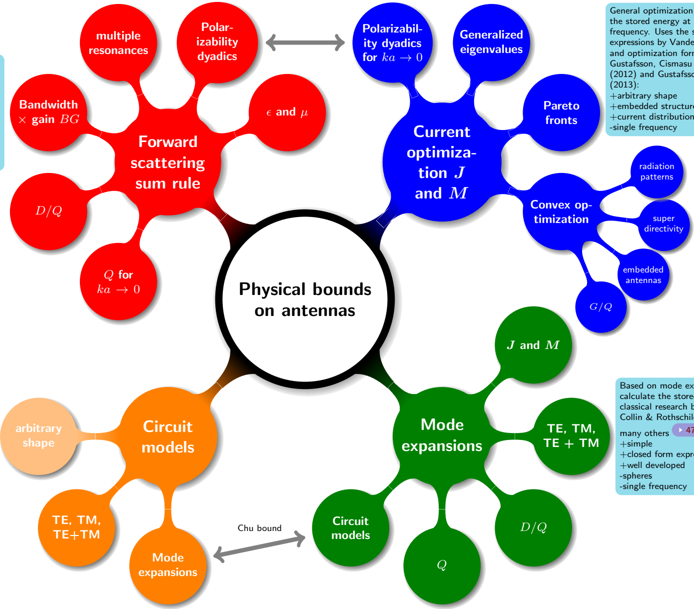
Physical bounds based on:

- Circuit models.
- Mode expansion (spheres).
- Forward scattering (arbitrary shape).
- Energy expressions in currents.

Gustafsson, Tayli, and Cismasu 2016

Physical bounds on antennas: methods

Uses the forward scattering sum rule to analyze receiving antennas, see Gustafsson, Sohl & Kristensson (2007,2009) ▸ 50:
+arbitrary shape and size
+bandwidth
+closed form expressions
+based on an identity
-entire volumes
-absorption efficiency



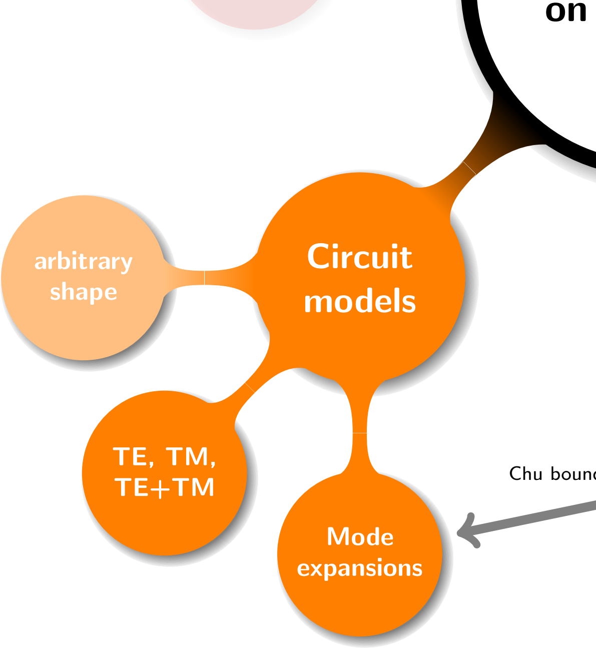
General optimization formulation for the stored energy at a fixed frequency. Uses the stored energy expressions by Vandenbosch (2010) and optimization formulations, see Gustafsson, Cismasu & Jonsson (2012) and Gustafsson & Nordebo (2013):
+arbitrary shape
+embedded structures
+current distributions
-single frequency

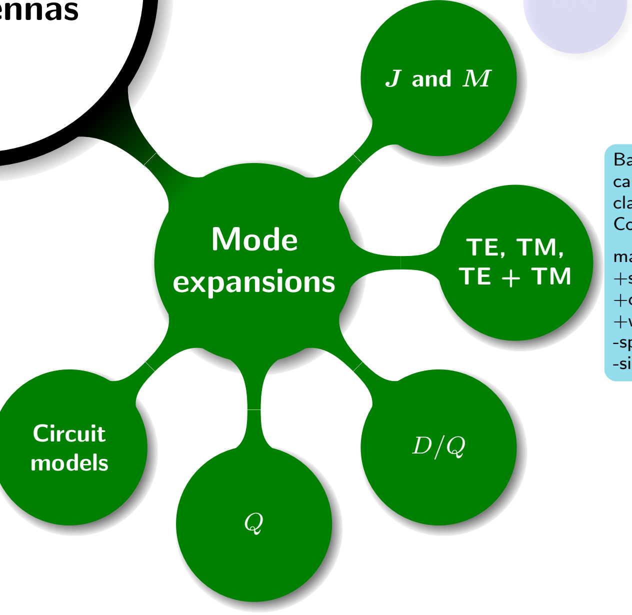
Uses circuit models for the antennas. Used originally by Wheeler (1947), Chu (1948) and later Thal (2006,2013)
+simple models
+physical intuition
± combined with mode expansions
+combined with matching
-approximate
-....

Based on mode expansions to calculate the stored energy, see the classical research by Chu (1948), Collin & Rothschild (1964), and many others ▸ 47:
+simple
+closed form expressions
+well developed
-spheres
-single frequency

Uses circuit models for the antennas.
Used originally by Wheeler (1947), Chu (1948) and later Thal (2006,2013)

+simple models
+physical intuition
± combined with mode expansions
+combined with matching
-approximate
-....

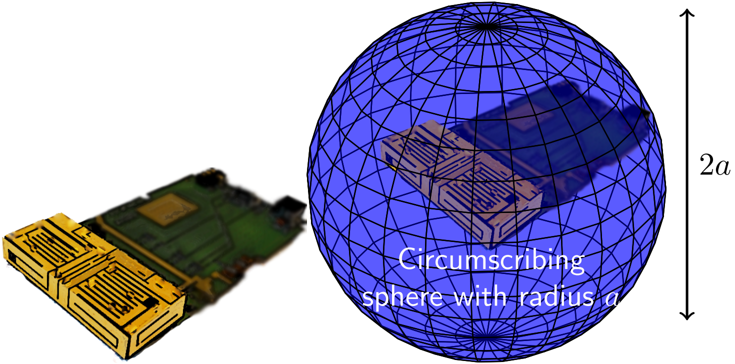




Based on mode expansions to calculate the stored energy, see the classical research by Chu (1948), Collin & Rothschild (1964), and many others 47:

+simple
+closed form expressions
+well developed
-spheres
-single frequency

ennas

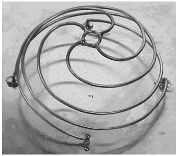


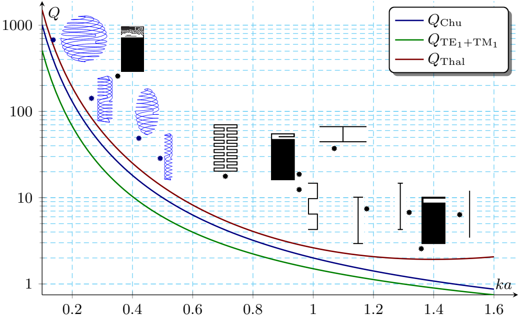
$$Q \geq Q_{\text{Chu}} = \frac{1}{(ka)^3} + \frac{1}{ka}$$ Chu (1948)

$$Q \geq Q_{\text{TE+TM}} = \frac{1}{2(ka)^3} + \frac{1}{ka}$$ Chu, McLean (1994), ...

$$Q \geq Q_{\text{Thal}} = \frac{1.5}{(ka)^3}, ka \ll 1$$ Thal (2006),...

ennas





Best 2004, Kim, Breinbjerg, and Yaghjian 2010, Sievenpiper et al. 2012

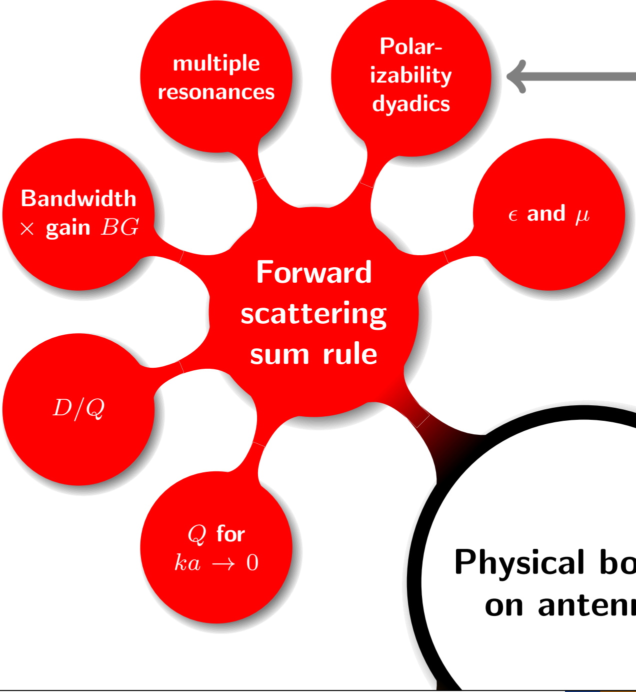
$$Q \geq Q_{\rm Chu} = \frac{1}{(ka)^3} + \frac{1}{ka} \qquad \text{Chu (1948)}$$

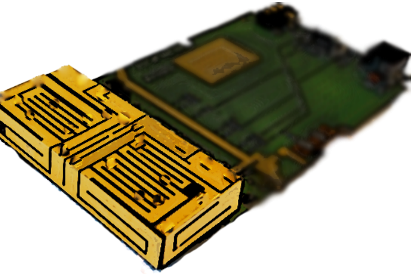
$$Q \geq Q_{\rm TE+TM} = \frac{1}{2(ka)^3} + \frac{1}{ka} \qquad \text{Chu, McLean (1994), ...}$$

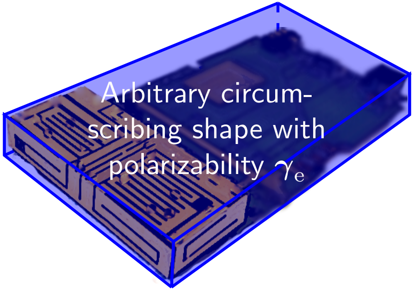
$$Q \geq Q_{\rm Thal} = \frac{1.5}{(ka)^3}, ka \ll 1 \qquad \text{Thal (2006),...}$$

Uses the forward scattering sum rule to analyze receiving antennas, see Gustafsson, Sohl & Kristensson (2007,2009) ▸ 50:

+arbitrary shape and size
+bandwidth
+closed form expressions
+based on an identity
-entire volumes
-absorption efficiency







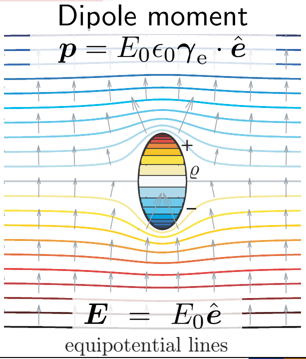
Forward scattering bound (2007)

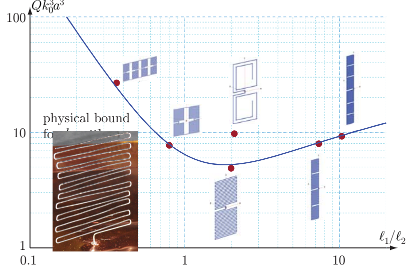
$$\frac{D}{Q} \leq \frac{\eta k^3}{2\pi} \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}}$$

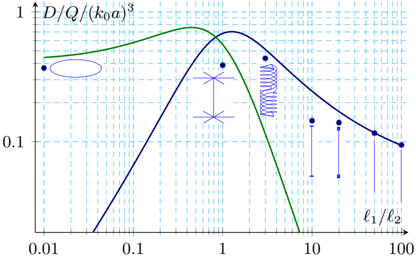
small electric dipoles ($D = 1.5$, $\eta = 0.5$)

$$Q \geq \frac{6\pi}{k^3 \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}}} \geq Q_{\mathrm{Thal}} = \frac{1.5}{(ka)^3}$$

$\boldsymbol{\gamma}_{\mathrm{e}}$ polarizability dyadic.







Best 2009; Best 2015; Gustafsson, Sohl, and Kristensson 2009

Forward scattering bound (2007)

$$\frac{D}{Q} \leq \frac{\eta k^3}{2\pi} \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}}$$

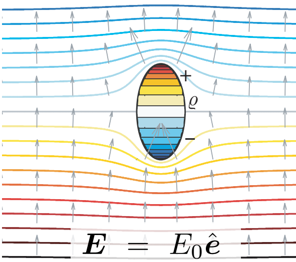
small electric dipoles ($D = 1.5$, $\eta = 0.5$)

$$Q \geq \frac{6\pi}{k^3 \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}}} \geq Q_{\mathrm{Thal}} = \frac{1.5}{(ka)^3}$$

$\boldsymbol{\gamma}_{\mathrm{e}}$ polarizability dyadic.

Dipole moment

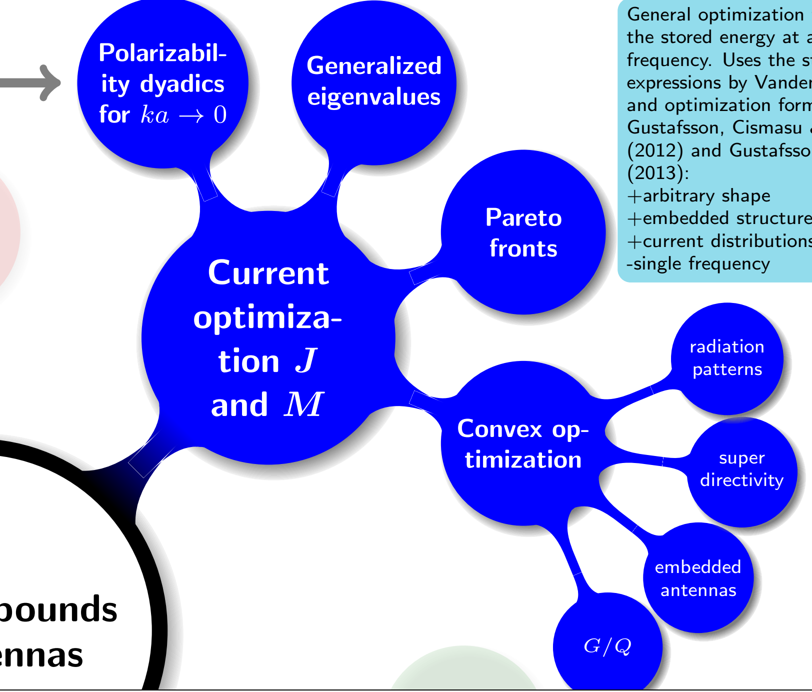
$$\boldsymbol{p} = E_0 \epsilon_0 \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}}$$

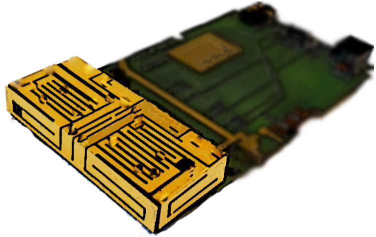


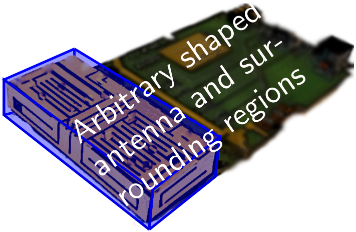
equipotential lines

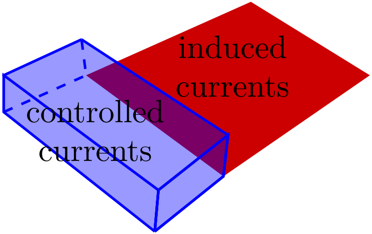
General optimization formulation for the stored energy at a fixed frequency. Uses the stored energy expressions by Vandenbosch (2010) and optimization formulations, see Gustafsson, Cismasu & Jonsson (2012) and Gustafsson & Nordebo (2013):

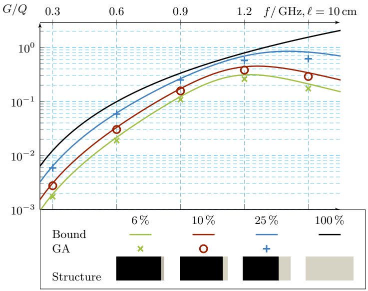
+arbitrary shape
+embedded structures
+current distributions
-single frequency











bounds

ennas

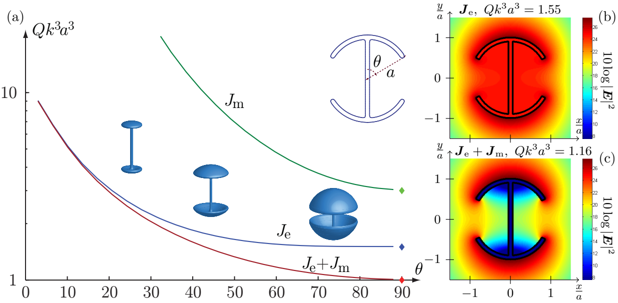
Small structures ($ka \ll 1$); electric currents

$$Q \geq \frac{6\pi}{k^3 \max \operatorname{eig}\{\boldsymbol{\gamma}_\mathrm{e}\}} \geq Q_\mathrm{Thal} = \frac{1.5}{(ka)^3}$$

electric and magnetic currents

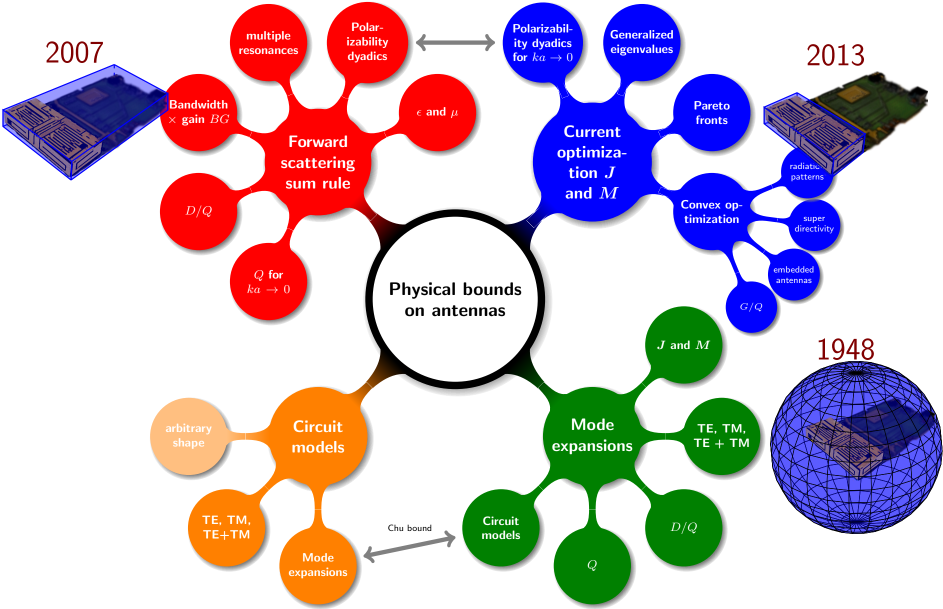
$$Q \geq \frac{6\pi}{k^3 \max \operatorname{eig}\{\boldsymbol{\gamma}_\mathrm{e} + \boldsymbol{\gamma}_\mathrm{m}\}} \geq Q_\mathrm{Chu} = \frac{1}{(ka)^3}$$

Yaghjian,Gustafsson,Jonsson (2013), Jonsson,Gustafsson (2015)



bounds
ennas

Physical bounds on antennas: methods



Outline

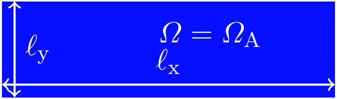
Antenna and antenna current optimization

Antenna design: produce the desired current distribution on the structure by shaping and choosing the materials.

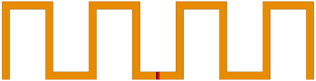
Have a given maximal size for the antenna structure.

- Antenna optimization: determine the shape and material properties for optimal performance.
- Antenna current optimization: determine an optimal current distribution from all possible currents in the available geometry.

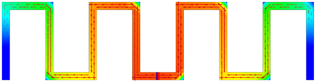
Maximal size of the antenna



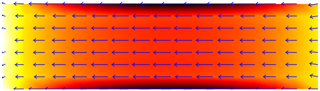
Antenna geometry with feed point



Current distribution on the antenna



Current distribution in the antenna antenna region

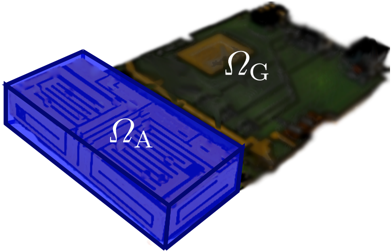


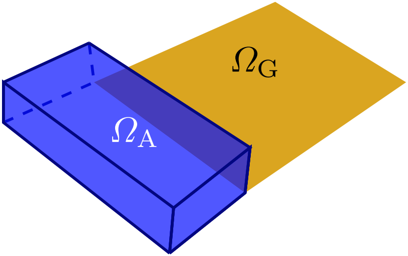
Antenna and antenna current optimization

Antenna design: produce the desired current distribution on the structure by shaping and choosing the materials.

Have a given maximal size for the antenna structure.

- Antenna optimization: determine the shape and material properties for optimal performance.
- Antenna current optimization: determine an optimal current distribution from all possible currents in the available geometry.





Optimization of antenna currents: examples

Gain over Q

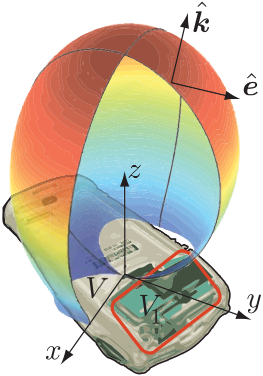
$$\begin{array}{ll} \text{minimize} & \text{Stored energy} \\ \text{subject to} & \text{Radiation intensity} = P_0 \end{array}$$

Q for superdirectivity $D \geq D_0$.

$$\begin{array}{ll} \text{minimize} & \text{Stored energy} \\ \text{subject to} & \text{Radiation intensity} = D_0 P_{\mathrm{rad}}/(4\pi) \\ & \text{Radiated power} \leq P_{\mathrm{rad}} \end{array}$$

Embedded structures

$$\begin{array}{ll} \text{minimize} & \text{Stored energy} \\ \text{subject to} & \text{Radiation intensity} = P_0 \\ & \text{Correct induced currents} \end{array}$$

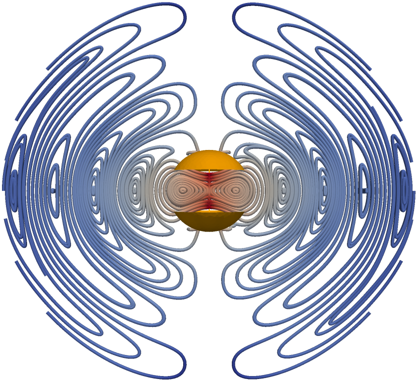


Need to:

1. Express the *stored energy* in the current density $\boldsymbol{J}$.
2. Solve the optimization problems.

Stored electromagnetic energy

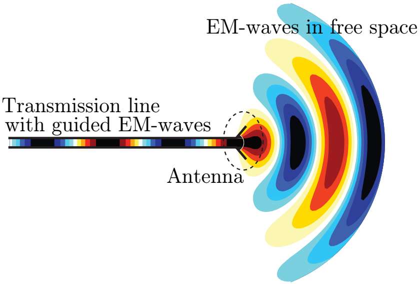
- Where is the energy stored?
  - Fields
  - Currents
  - Feed structure
- Stored according to what?
  - Input impedance
  - Material
  - Scatterer
- Why are we interested?
  - Basics physics
  - Antenna bandwidth
  - Physical bounds



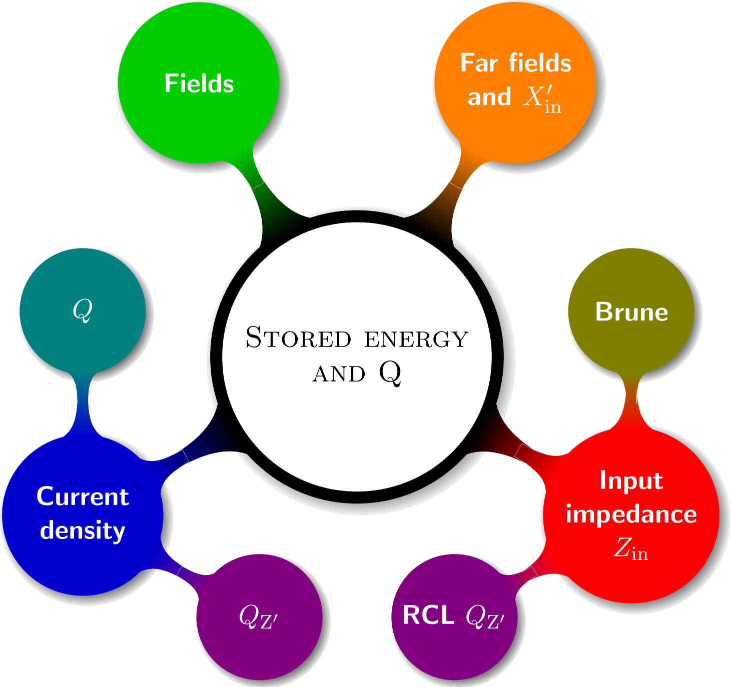
There are several proposals for the stored energy in the literature. They agree for many cases but differ for some. Differences often due to different interpretations, assumptions, and applications.

Stored electromagnetic energy

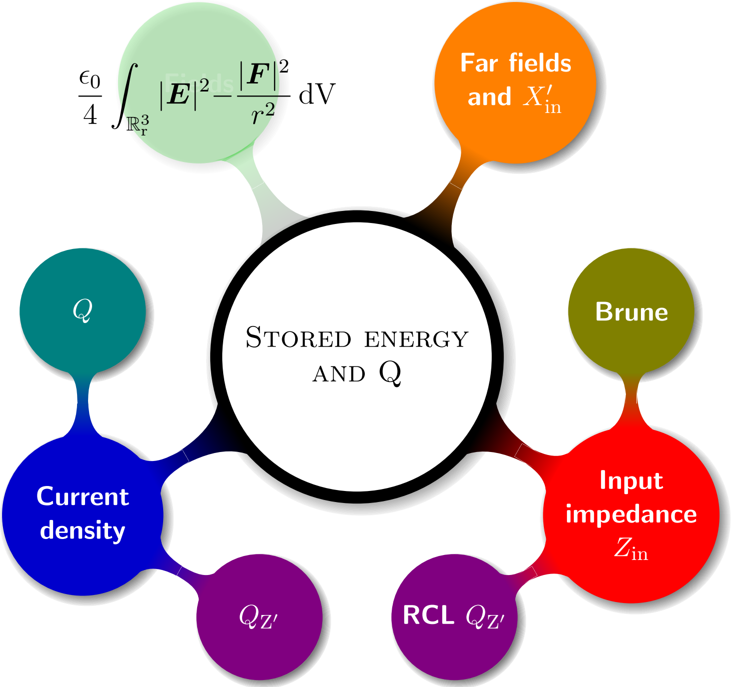
- Where is the energy stored?
  - Fields
  - Currents
  - Feed structure
- Stored according to what?
  - Input impedance
  - Material
  - Scatterer
- Why are we interested?
  - Basics physics
  - Antenna bandwidth
  - Physical bounds



There are several proposals for the stored energy in the literature. They agree for many cases but differ for some. Differences often due to different interpretations, assumptions, and applications.



Fields
Far fields and X'_{in}
Q
Stored energy and Q
Brune
Current density
Input impedance Z_{in}
$Q_{Z'}$
RCL $Q_{Z'}$


$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_\mathrm{r}} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
Far fields and X'_in
Stored energy and Q
Q
Brune
Current density
Input impedance Z_in
$Q_{\mathrm{Z}'}$
RCL $Q_{\mathrm{Z}'}$

$$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$$

Far fields and X'_{in}

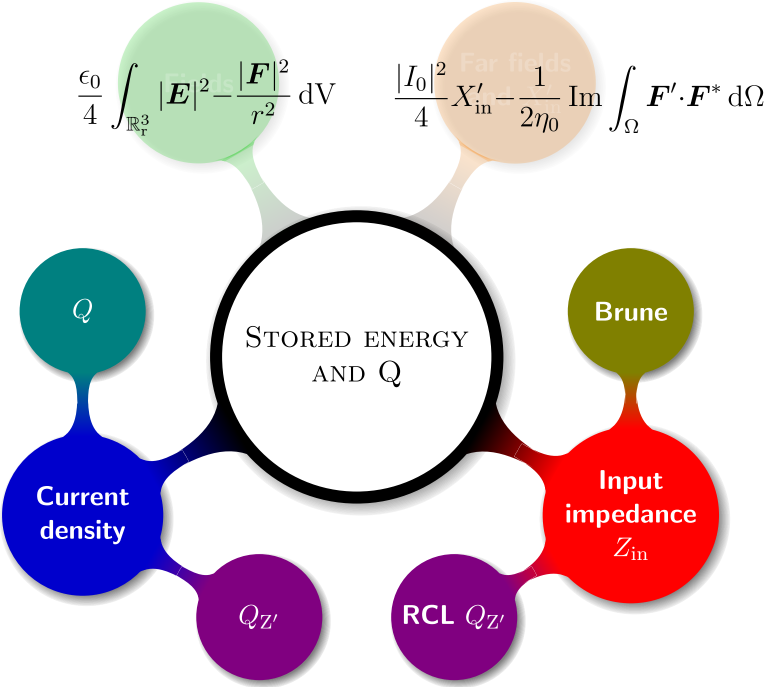
Subtract the 'radiated' energy density

- Power flow [1]
- Far field [2,3], ...
- Differ for standing waves
- Spherical modes
- Can be coordinate dependent [3]
- Inhomogeneous and lossy media?
- $\boldsymbol{F} = 0$ for a lossy background

1. Collin and Rothschild 1964
2. Fante 1969
3. Yaghjian and Best 2005

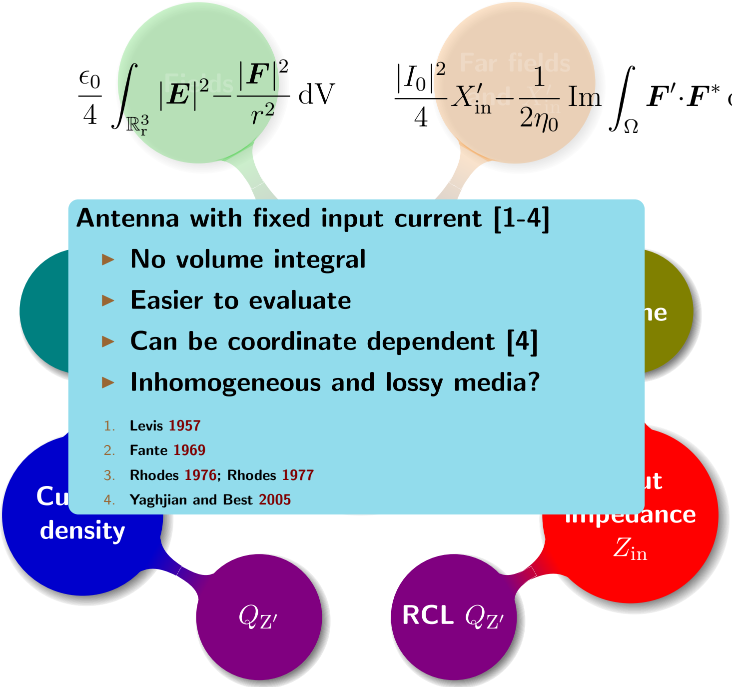
$Q_{\mathrm{Z}'}$

RCL $Q_{\mathrm{Z}'}$



$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
Far fields
$\frac{|I_0|^2}{4}X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega}\boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
Stored energy and Q
Q
Brune
Current density
Input impedance Z_{in}
$Q_{\mathrm{Z}'}$
RCL $Q_{\mathrm{Z}'}$

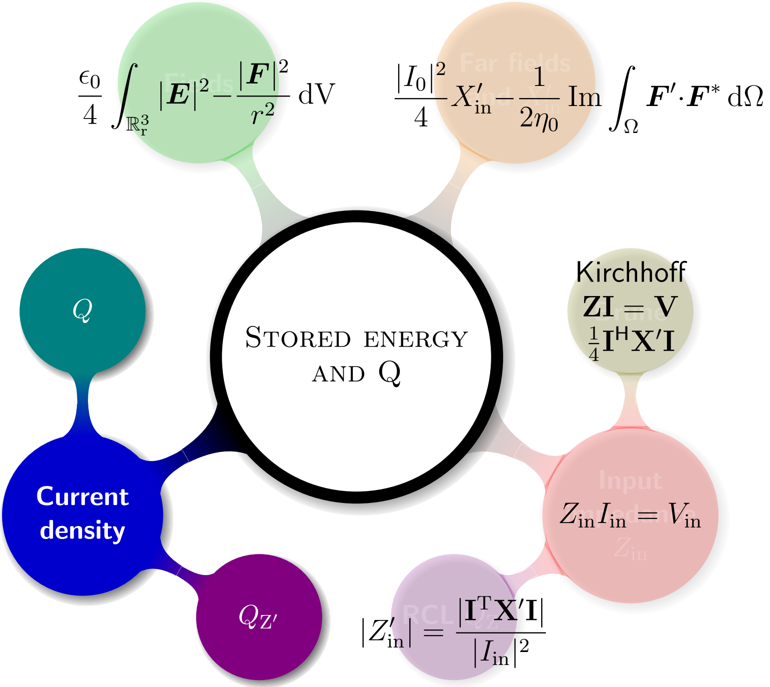
$$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV} \qquad \frac{|I_0|^2}{4}X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega} \boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$$



Antenna with fixed input current [1-4]

- No volume integral
- Easier to evaluate
- Can be coordinate dependent [4]
- Inhomogeneous and lossy media?

1. Levis 1957
2. Fante 1969
3. Rhodes 1976; Rhodes 1977
4. Yaghjian and Best 2005


Stored energy and Q
$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_r} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
Far fields
$\frac{|I_0|^2}{4}X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega}\boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
Kirchhoff
$\mathbf{ZI}=\mathbf{V}$
$\frac{1}{4}\mathbf{I}^{\mathrm{H}}\mathbf{X}'\mathbf{I}$
Q
Current density
Input
$Z_{\mathrm{in}}I_{\mathrm{in}} = V_{\mathrm{in}}$
$Q_{Z'}$
$|Z'_{\mathrm{in}}| = \frac{|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$

Q **from the input impedance:**

- **Synthesize a lumped circuit model [1,2]**
- **Stored energy in inductors and capacitors**
- **Differentiate the reactance matrix $\mathbf{X}'$ [3]**

$Q_{Z'}$ **from the input impedance:**

- **Differentiate Z_{in} to get $Q_{Z'} = \frac{\omega|Z'_{\mathrm{in}}|}{2R_{\mathrm{in}}}$ [4]**
- $Q \to Q_{Z'}$ **by $\mathbf{I}^{\mathsf{H}} \to \mathbf{I}^{\mathrm{T}}$ and $Q_{Z'} \leq Q$**
- $Q_{Z'} \approx 0$ **for multiple resonance cases [5,6]**

1. Brune 1931
2. Gustafsson and Jonsson 2015a
3. Gustafsson, Tayli, and Cismasu 2014
4. Yaghjian and Best 2005
5. Gustafsson and Nordebo 2006
6. Stuart, Best, and Yaghjian 2007

$$\frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega} \boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$$

Kirchhoff

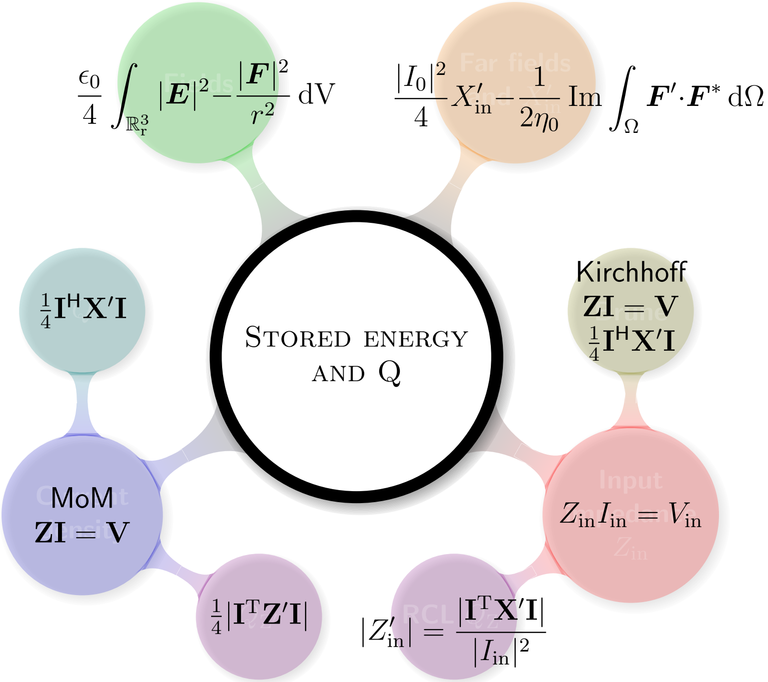
$$\mathbf{Z}\mathbf{I} = \mathbf{V}$$

$$\tfrac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$$

$$Z_{\mathrm{in}} I_{\mathrm{in}} = V_{\mathrm{in}}$$

$Q_{Z'}$

$$|Z'_{\mathrm{in}}| = \frac{|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$$



Stored energy and Q
$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
$\frac{|I_0|^2}{4}X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega}\boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
Kirchhoff
$\mathbf{ZI}=\mathbf{V}$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
MoM
$\mathbf{ZI}=\mathbf{V}$
$Z_{\mathrm{in}}I_{\mathrm{in}}=V_{\mathrm{in}}$
$\frac{1}{4}|\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}|$
$|Z'_{\mathrm{in}}| = \frac{|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$

$$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_r} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$$

$$\frac{|I_0|^2}{4}X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega}\boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$$

Stored energy expressed in the current density [1-5]

- **MoM implementation of the EFIE [3]**
- **Quadratic form in the current (antenna current optimization)**
- **Can be negative [4]**
- $Q_{\mathrm{Z}'}$ **expressed in the current [5,6]**

1. Vandenbosch 2010
2. Geyi 2003b
3. Harrington and Mautz 1972
4. Gustafsson, Cismasu, and Jonsson 2012
5. Capek et al. 2014
6. Gustafsson, Tayli, and Cismasu 2014

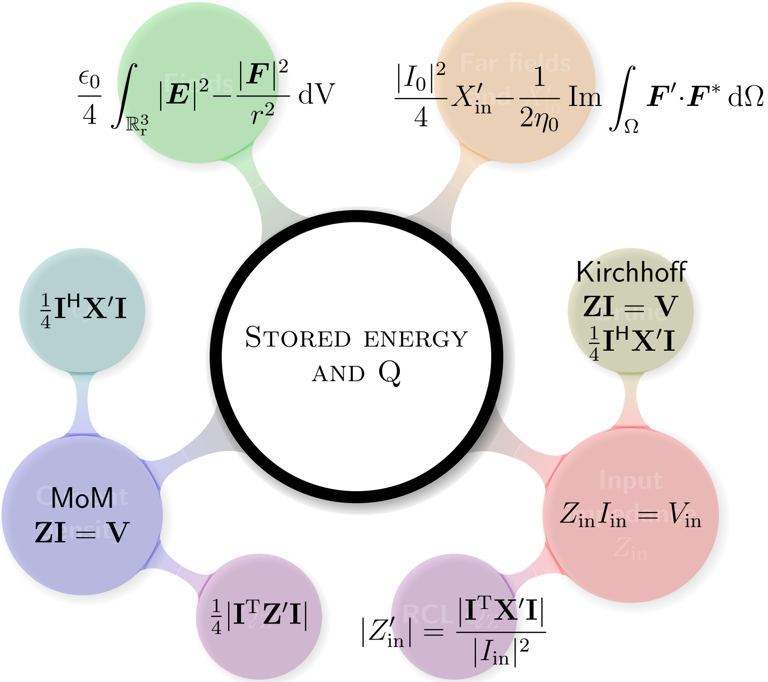
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$

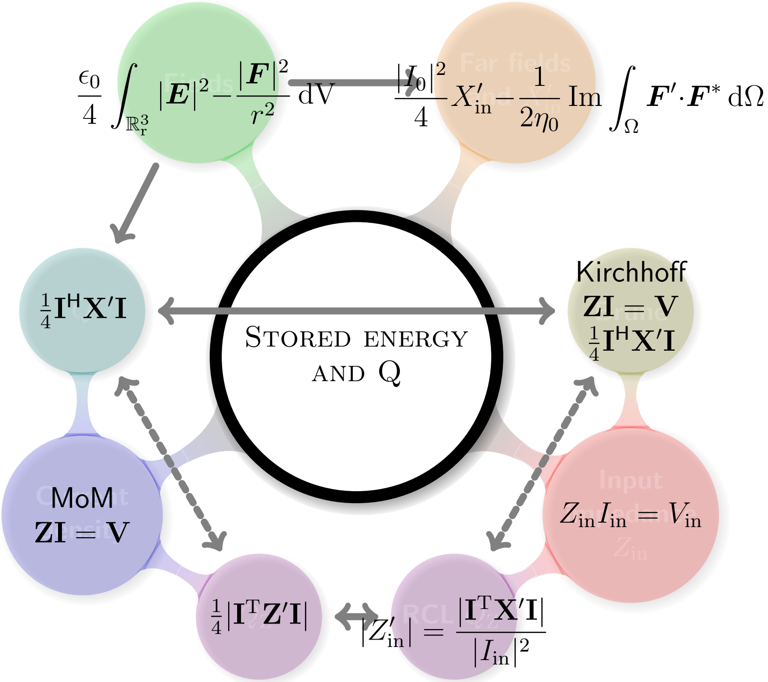
MoM

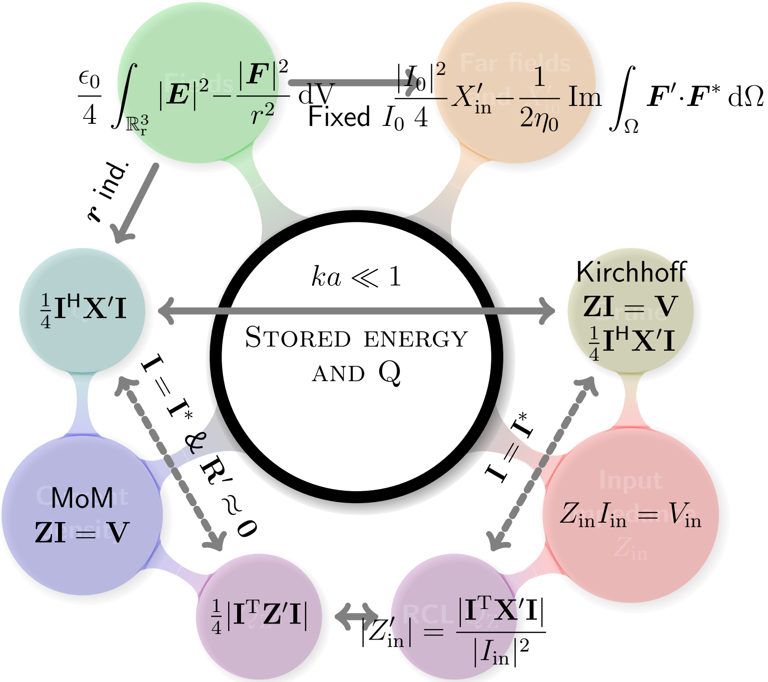
$\mathbf{ZI} = \mathbf{V}$

$\frac{1}{4}|\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}|$

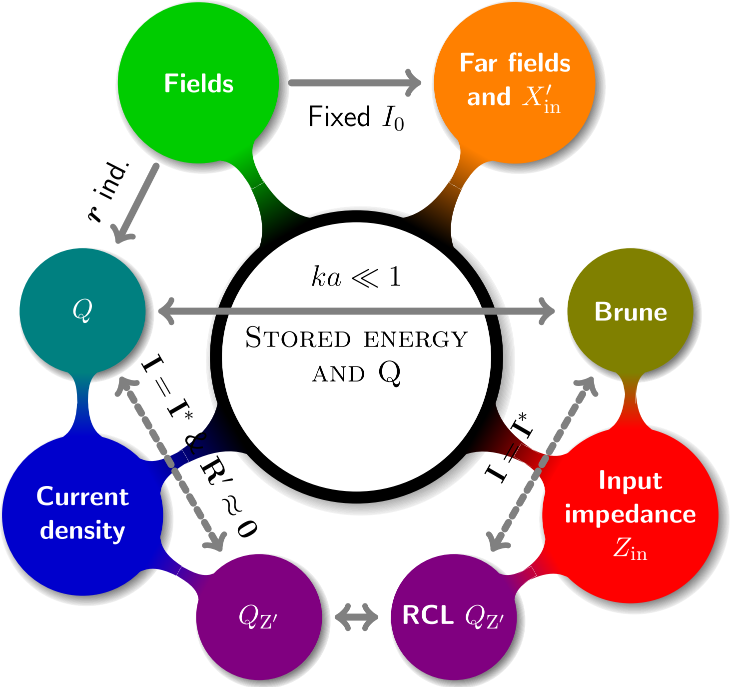
$$|Z'_{\mathrm{in}}| = \frac{|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$$


Stored energy and Q
$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
$\frac{|I_0|^2}{4}X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega}\boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
Kirchhoff
$\mathbf{ZI}=\mathbf{V}$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
MoM
$\mathbf{ZI}=\mathbf{V}$
$Z_{\mathrm{in}}I_{\mathrm{in}}=V_{\mathrm{in}}$
$\frac{1}{4}|\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}|$
$|Z'_{\mathrm{in}}| = \frac{|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$


$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_r} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
$\frac{|I_0|^2}{4}X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega}\boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
Kirchhoff
$\mathbf{ZI}=\mathbf{V}$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
Stored energy and Q
MoM
$\mathbf{ZI}=\mathbf{V}$
$Z_{\mathrm{in}}I_{\mathrm{in}}=V_{\mathrm{in}}$
$\frac{1}{4}|\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}|$
$|Z'_{\mathrm{in}}| = \frac{|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$



$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_r} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
Fixed I_0
$\frac{|I_0|^2}{4}X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_{\Omega}\boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
$\boldsymbol{r}$ ind.
$ka \ll 1$
Stored energy and Q
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
Kirchhoff
$\mathbf{ZI}=\mathbf{V}$
$\frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$
$\mathbf{I}=\mathbf{I}^*$ & $\mathbf{R}'\approx 0$
$\mathbf{I}=\mathbf{I}^*$
MoM
$\mathbf{ZI}=\mathbf{V}$
$Z_{\mathrm{in}}I_{\mathrm{in}}=V_{\mathrm{in}}$
$\frac{1}{4}|\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}|$
$|Z'_{\mathrm{in}}| = \frac{|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$


Fields
Far fields and X'_{in}
Fixed I_0
$\boldsymbol{r}$ ind.
$ka \ll 1$
Stored energy and Q
Q
Brune
$\mathbf{I} = \mathbf{I}^*$ & $\mathbf{R}' \gtrsim 0$
$\mathbf{I} = \mathbf{I}^*$
Current density
Input impedance Z_{in}
$Q_{Z'}$
RCL $Q_{Z'}$

From MoM to stored energy (I)

Use a method of moments (MoM) formulation of the electric field integral equation (EFIE). Impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$

$$\frac{Z_{mn}}{\eta} = \mathrm{j}\int_{\Omega}\int_{\Omega}\left(k\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2} - \frac{1}{k}\nabla_1\cdot\boldsymbol{\psi}_{m1}\,\nabla_2\cdot\boldsymbol{\psi}_{n2}\right)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}\,\mathrm{dS}_1\,\mathrm{dS}_2$$

where $\boldsymbol{\psi}_{n1} = \boldsymbol{\psi}_n(\boldsymbol{r}_1)$, $\boldsymbol{\psi}_{n2} = \boldsymbol{\psi}_n(\boldsymbol{r}_2)$, $m, n = 1, ..., N$, and $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. The current density is $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} I_n\boldsymbol{\psi}_n(\boldsymbol{r})$ with the expansion coefficients determined from $\mathbf{ZI} = \mathbf{V}$, where $\mathbf{V}$ is a column matrix with the excitation coefficients.

From MoM to stored energy (I)

Use a method of moments (MoM) formulation of the electric field integral equation (EFIE). Impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$

$$\frac{Z_{mn}}{\eta} = \mathrm{j}\int_{\Omega}\int_{\Omega} \Big(\overbrace{k\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2}}^{W_{\mathrm{m}}} - \overbrace{\frac{1}{k}\nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}}^{W_{\mathrm{e}}} \Big) \frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}\,\mathrm{dS}_1\,\mathrm{dS}_2$$

where $\boldsymbol{\psi}_{n1} = \boldsymbol{\psi}_n(\boldsymbol{r}_1)$, $\boldsymbol{\psi}_{n2} = \boldsymbol{\psi}_n(\boldsymbol{r}_2)$, $m, n = 1, ..., N$, and $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. The current density is $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} I_n\boldsymbol{\psi}_n(\boldsymbol{r})$ with the expansion coefficients determined from $\mathbf{ZI} = \mathbf{V}$, where $\mathbf{V}$ is a column matrix with the excitation coefficients.

From MoM to stored energy (I)

Use a method of moments (MoM) formulation of the electric field integral equation (EFIE). Impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$

$$\frac{Z_{mn}}{\eta} = \mathrm{j} \int_{\Omega}\int_{\Omega} \Big(\overbrace{k\boldsymbol{\psi}_{m1} \cdot \boldsymbol{\psi}_{n2}}^{W_\mathrm{m}} - \overbrace{\frac{1}{k}\nabla_1 \cdot \boldsymbol{\psi}_{m1} \nabla_2 \cdot \boldsymbol{\psi}_{n2}}^{W_\mathrm{e}} \Big) \frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}} \,\mathrm{dS}_1 \,\mathrm{dS}_2$$

where $\boldsymbol{\psi}_{n1} = \boldsymbol{\psi}_n(\boldsymbol{r}_1)$, $\boldsymbol{\psi}_{n2} = \boldsymbol{\psi}_n(\boldsymbol{r}_2)$, $m, n = 1, ..., N$, and $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. The current density is $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} I_n \boldsymbol{\psi}_n(\boldsymbol{r})$ with the expansion coefficients determined from $\mathbf{ZI} = \mathbf{V}$, where $\mathbf{V}$ is a column matrix with the excitation coefficients.

Differentiate the MoM impedance matrix

$$\frac{k\,\partial Z_{mn}}{\eta\,\partial k} = \int_{\Omega}\int_{\Omega} \mathrm{j} \left(k\boldsymbol{\psi}_{m1} \cdot \boldsymbol{\psi}_{n2} + \frac{1}{k}\nabla_1 \cdot \boldsymbol{\psi}_{m1} \nabla_2 \cdot \boldsymbol{\psi}_{n2} \right) \frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}$$
$$+ k \left(k\boldsymbol{\psi}_{m1} \cdot \boldsymbol{\psi}_{n2} - \frac{1}{k}\nabla_1 \cdot \boldsymbol{\psi}_{m1} \nabla_2 \cdot \boldsymbol{\psi}_{n2} \right) \frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi} \,\mathrm{dS}_1 \,\mathrm{dS}_2$$

From MoM to stored energy (I)

Use a method of moments (MoM) formulation of the electric field integral equation (EFIE). Impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$

$$\frac{Z_{mn}}{\eta} = \mathrm{j}\int_{\Omega}\int_{\Omega}\Big(\underbrace{k\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2}}_{W_\mathrm{m}} - \underbrace{\frac{1}{k}\nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}}_{W_\mathrm{e}}\Big)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}\,\mathrm{dS}_1\,\mathrm{dS}_2$$

where $\boldsymbol{\psi}_{n1} = \boldsymbol{\psi}_n(\boldsymbol{r}_1)$, $\boldsymbol{\psi}_{n2} = \boldsymbol{\psi}_n(\boldsymbol{r}_2)$, $m, n = 1, ..., N$, and $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. The current density is $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} I_n\boldsymbol{\psi}_n(\boldsymbol{r})$ with the expansion coefficients determined from $\mathbf{ZI} = \mathbf{V}$, where $\mathbf{V}$ is a column matrix with the excitation coefficients.

Differentiate the MoM impedance matrix

$$\frac{k\,\partial Z_{mn}}{\eta\,\partial k} = \int_{\Omega}\int_{\Omega}\mathrm{j}\Big(\underbrace{k\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2}}_{W_\mathrm{m}} + \underbrace{\frac{1}{k}\nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}}_{W_\mathrm{e}}\Big)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}} + k\Big(k\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2} - \frac{1}{k}\nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\Big)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi}\,\mathrm{dS}_1\,\mathrm{dS}_2$$

From MoM to stored energy (I)

Use a method of moments (MoM) formulation of the electric field integral equation (EFIE). Impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$

$$\frac{Z_{mn}}{\eta} = \mathrm{j}\int_{\Omega}\int_{\Omega}\Big(\underbrace{k\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2}}_{W_\mathrm{m}} - \underbrace{\frac{1}{k}\nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}}_{W_\mathrm{e}}\Big)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}\,\mathrm{dS}_1\,\mathrm{dS}_2$$

where $\boldsymbol{\psi}_{n1} = \boldsymbol{\psi}_n(\boldsymbol{r}_1)$, $\boldsymbol{\psi}_{n2} = \boldsymbol{\psi}_n(\boldsymbol{r}_2)$, $m,n = 1,...,N$, and $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. The current density is $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} I_n\boldsymbol{\psi}_n(\boldsymbol{r})$ with the expansion coefficients determined from $\mathbf{ZI} = \mathbf{V}$, where $\mathbf{V}$ is a column matrix with the excitation coefficients.

Differentiate the MoM impedance matrix

$$\frac{k\,\partial Z_{mn}}{\eta\,\partial k} = \int_{\Omega}\int_{\Omega}\mathrm{j}\Big(\underbrace{k\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2}}_{W_\mathrm{m}} + \underbrace{\frac{1}{k}\nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}}_{W_\mathrm{e}}\Big)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}$$

$$+\,k\Big(k\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2} - \frac{1}{k}\nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\Big)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi}\,\mathrm{dS}_1\,\mathrm{dS}_2$$

From MoM to stored energy (II)

Standard MoM implementations of the EFIE are easily modified to compute the stored energies. The sum and differences

$$W_{\mathrm{m}} + W_{\mathrm{e}} = \frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I} \quad \text{and } W_{\mathrm{m}} - W_{\mathrm{e}} = \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I}$$

gives the stored magnetic and electric energies

$$W_{\mathrm{m}} = \frac{1}{8}\mathbf{I}^{\mathsf{H}}\left(\frac{\partial \mathbf{X}}{\partial \omega} + \frac{\mathbf{X}}{\omega}\right)\mathbf{I} \quad \text{and } W_{\mathrm{e}} = \frac{1}{8}\mathbf{I}^{\mathsf{H}}\left(\frac{\partial \mathbf{X}}{\partial \omega} - \frac{\mathbf{X}}{\omega}\right)\mathbf{I},$$

respectively. Electric $\mathbf{X}_{\mathrm{e}}$, and magnetic $\mathbf{X}_{\mathrm{m}}$, reactance matrices

$$\mathbf{X}_{\mathrm{e}} = \frac{1}{2}\left(\omega\mathbf{X}' - \mathbf{X}\right) \quad \text{and } \mathbf{X}_{\mathrm{m}} = \frac{1}{2}\left(\omega\mathbf{X}' + \mathbf{X}\right)$$

Identical to the stored energy expression (free space) introduced by Vandenbosch 2010, see 69 and already considered by Harrington and Mautz 1972.

Matrix expressions for the stored EM energies

Method of Moments approximation (expand $\boldsymbol{J}$ in basis functions)

$$W_{\mathrm{e}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \quad \text{stored E-energy, } \mathbf{X}_{\mathrm{e}} \text{ electric reactance}$$

$$W_{\mathrm{m}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \quad \text{stored M-energy, } \mathbf{X}_{\mathrm{m}} \text{ magnetic reactance}$$

$$P_{\mathrm{rad}} \approx \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I} \quad \text{radiated power}$$

giving $\mathbf{Z} = \mathbf{R} + \mathrm{j}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})$. We also use

$$\hat{\boldsymbol{e}}^{*} \cdot \boldsymbol{F} \approx \mathbf{F}\mathbf{I} \quad \text{far field}$$

$$\boldsymbol{E} \approx \mathbf{N}\mathbf{I} \quad \text{near field}$$

$$\mathbf{I}_{\mathrm{G}} \approx \mathbf{C}\mathbf{I}_{\mathrm{A}} \quad \text{induced current on a PEC}$$

Matrix expressions for the stored EM energies

Method of Moments approximation (expand $\boldsymbol{J}$ in basis functions)

$$W_{\mathrm{e}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \quad \text{stored E-energy, } \mathbf{X}_{\mathrm{e}} \text{ electric reactance}$$

$$W_{\mathrm{m}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \quad \text{stored M-energy, } \mathbf{X}_{\mathrm{m}} \text{ magnetic reactance}$$

$$P_{\mathrm{rad}} \approx \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I} \quad \text{radiated power}$$

giving $\mathbf{Z} = \mathbf{R} + \mathrm{j}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})$. We also use

$$\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F} \approx \mathbf{F}\mathbf{I} \quad \text{far field}$$

$$\boldsymbol{E} \approx \mathbf{N}\mathbf{I} \quad \text{near field}$$

$$\mathbf{I}_{\mathrm{G}} \approx \mathbf{C}\mathbf{I}_{\mathrm{A}} \quad \text{induced current on a PEC}$$

Pre-computed matrices used in the optimization.

Optimization of the current distribution

Characteristic modes

Modes with small Rayleigh quotients

$$\frac{\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}} = \frac{\mathbf{I}^{\mathsf{H}}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

Eigenvalue problem

$$(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})\mathbf{I} = \nu\mathbf{R}\mathbf{I}$$

- Modes with low reactive power.
- Resonances $(\nu = 0)$
- Does not enforce low stored energy.

Stored energy

Minimize the energy Rayleigh quotient

$$\frac{\mathbf{I}^{\mathsf{H}}(\mathbf{X}_{\mathrm{m}} + \mathbf{X}_{\mathrm{e}})\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

Eigenvalue problem

$$(\mathbf{X}_{\mathrm{m}} + \mathbf{X}_{\mathrm{e}})\mathbf{I} = \nu\mathbf{R}\mathbf{I}$$

- Modes with low stored energy.
- Does not enforce resonance.

Q-factor

Minimize the Q-factor quotient

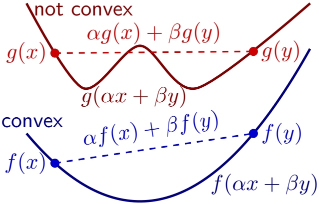
$$\frac{2\max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}\}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

- Currents with low Q-factors.
- Resonance by tuning.
- Need to solve these optimization problems $\Rightarrow$ convex optimization.

Chen and Wang 2015; Garbacz and Turpin 1971; Harrington and Mautz 1971, see also ▸ 77

Convex optimization

$$\begin{aligned} &\text{minimize} && f_0(\mathbf{x}) \\ &\text{subject to} && f_i(\mathbf{x}) \leq 0,\ i = 1, ..., N_1 \\ &&& \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$



where $f_i(x)$ are convex, *i.e.*, $f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$ for $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

Solved with efficient standard algorithms. No risk of getting trapped in a local minimum. A problem is 'solved' if formulated as a convex optimization problem.

Antenna performance expressed in the current density $\boldsymbol{J}$, *e.g.*,

- Radiated field $\boldsymbol{F}(\hat{\boldsymbol{k}}) = -\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} \times \int_\Omega \boldsymbol{J}(\boldsymbol{r}) \mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}} \,\mathrm{dV}$ is affine.
- Radiated power, stored electric and magnetic energies, and Ohmic losses are positive semi-definite quadratic forms in $\boldsymbol{J}$.

Convex optimization for antennas

Convex optimization offer many possibilities to analyze radiating structures. Common convex quantities:

- **linear forms** near fields $\mathbf{N}_{\mathrm{e}}\mathbf{I}$ and $\mathbf{N}_{\mathrm{m}}\mathbf{I}$, far field $\mathbf{F}\mathbf{I}$, and induced currents $\mathbf{C}\mathbf{I}$.
- **quadratic forms** radiated power $\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}$, absorbed power, stored electric energy $\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}$, stored magnetic energy $\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}$, ohmic losses $\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\Omega}\mathbf{I}$.
- **norms** field strengths $||\mathbf{N}\mathbf{I}||$, far-field levels $||\mathbf{F}\mathbf{I}||$
- **max** stored energy for tuned antennas $W = \max\{W_{\mathrm{e}}, W_{\mathrm{m}}\}$
- **logarithmic** capacity (can be).

in the current density. In convex optimization, we can

- minimize convex quantities.
- maximize concave quantities.

The linear (affine) quantities are both convex and concave.
Quadratic positive semidefinite forms are convex.

Currents for maximal G/Q

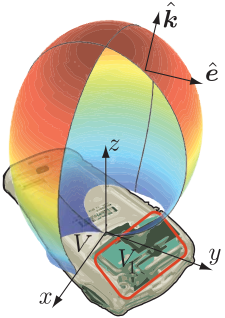
Determine a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V that maximizes the partial-gain Q-factor quotient $G(\hat{\boldsymbol{k}}, \hat{e})/Q$.

- Partial radiation intensity $P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})$

$$\frac{G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})}{\mathrm{c}_0 k \max\{W_{\mathrm{e}}, W_{\mathrm{m}}\}}.$$

- Scale $\boldsymbol{J}$ and reformulate $\max. P$ as $\max. \mathrm{Re}\{\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F}\}$.
- Convex optimization problem.

$$\begin{aligned} &\text{maximize} && \mathrm{Re}\{\mathbf{F}\mathbf{I}\} \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq 1 \\ & && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq 1 \end{aligned}$$



Determines a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V with maximal partial radiation intensity and unit stored EM energy.

Maximal $G(\hat{\boldsymbol{k}},\hat{\boldsymbol{x}})/Q$ for planar rectangles

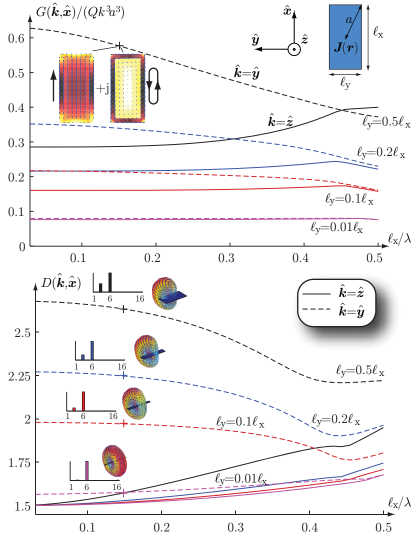
Solution of the convex optimization problem

$$\begin{aligned} \max. \quad & \mathrm{Re}\{\mathbf{FI}\} \\ \text{s.t.} \quad & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq 1 \\ & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq 1 \end{aligned}$$

or

$$\begin{aligned} \min. \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{s.t.} \quad & \mathbf{FI} = 1 \end{aligned}$$

for current densities confined to planar rectangles with side lengths ℓ_{x} and $\ell_{\mathrm{y}} = \{0.01, 0.1, 0.2, 0.5\}\ell_{\mathrm{x}}$.



Note $\ell_{\mathrm{x}}/\lambda = k\ell_{\mathrm{x}}/(2\pi)$, giving $\ell_{\mathrm{x}} = \lambda/2 \to k\ell_{\mathrm{x}} = \pi \to ka \geq \pi/2$.

D/Q (or G/Q) bounds

Typical (but not optimal) MATLAB code using CVX

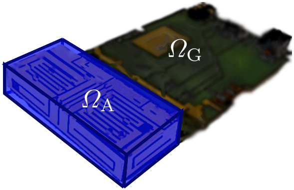
```
cvx_begin
   variable I(n) complex;        % current density
   maximize(real(F*I))           % far-field
   subject to
      quad_form(I,Xe) <= 1;      % stored E energy
      quad_form(I,Xm) <= 1;      % stored M energy
cvx_end
```

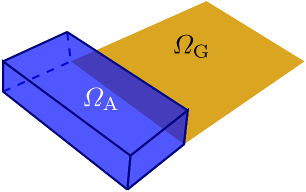
- Similar to the forward scattering bounds (2007) for TM.
- Can design 'optimal' electric dipole mode (TM) antennas.
- TE modes and TE+TM are not well understood.

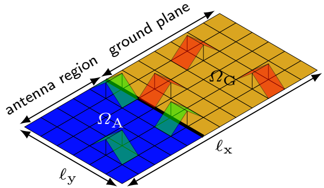
We now reformulate the complex optimization problem to analyze superdirectivity, antennas with a prescribed radiation pattern, losses, and antennas embedded in a PEC structure.

Optimal performance for embedded antennas

- Common with antennas embedded in metallic structures.
- The induced currents radiate but they are not arbitrary.
- Linear map from the antenna region adds a (convex) constraint.
- Here, we assume that the surrounding structure is PEC and add a constraint to account for the induced currents on the surrounding structure in the G/Q formulation.







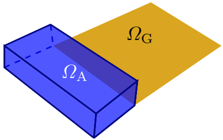
Currents for maximal G/Q for embedded antennas

Determine an optimal current density $\boldsymbol{J}_{\mathrm{A}}(\boldsymbol{r})$ in the region Ω_{A}. Assume that the ground plane $\Omega_{\mathrm{G}} = \Omega \setminus \Omega_{\mathrm{A}}$ is PEC.

Can minimize the stored energy for given radiated field

$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \mathbf{F}\mathbf{I} = 1 \\ & && \mathbf{I}_{\mathrm{G}} = \mathbf{C}\mathbf{I}_{\mathrm{A}} \end{aligned}$$

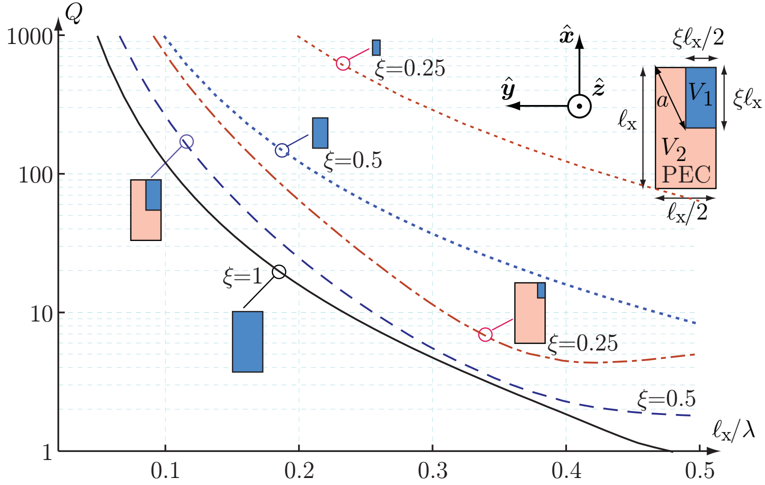
or maximize the radiated field for given stored energy



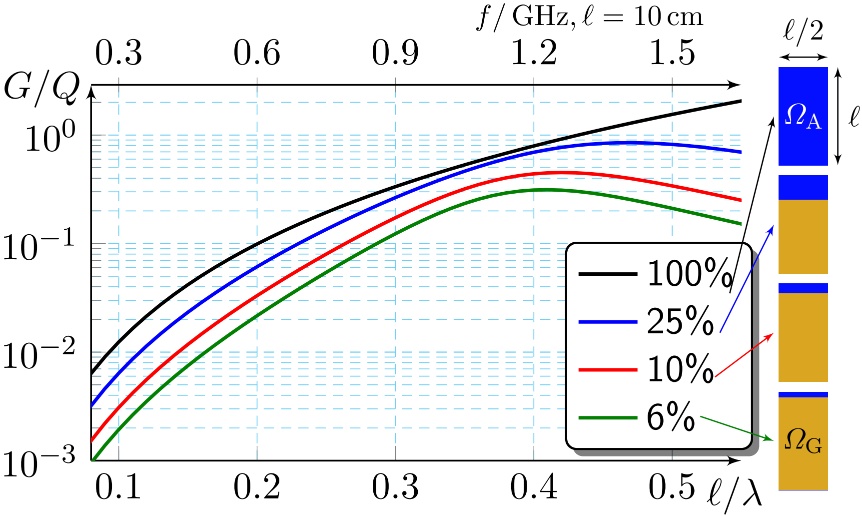
$$\begin{aligned} &\text{maximize} && \mathrm{Re}\{\mathbf{F}\mathbf{I}\} \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq 1 \\ & && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq 1 \\ & && \mathbf{I}_{\mathrm{G}} = \mathbf{C}\mathbf{I}_{\mathrm{A}} \end{aligned}$$

Can also eliminate $\mathbf{I}_{\mathrm{G}}$.

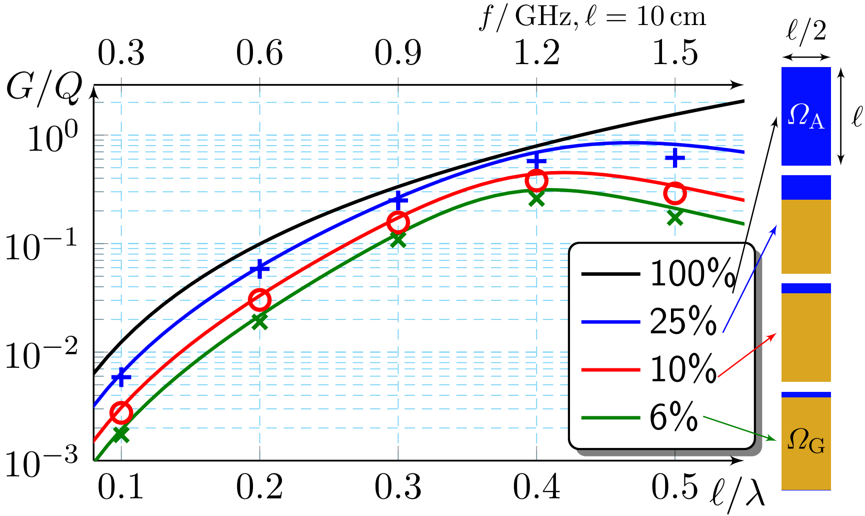
Embedded antennas in planar PEC rectangles



Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region



Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region



Why convex optimization?

Solved if formulated as a convex optimization problem.

Consider the G/Q problem

$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \mathbf{F}\mathbf{I} = 1 \end{aligned}$$

Many (optimization) algorithms can be used to solve this problem.

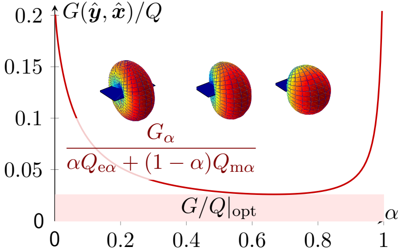
- Can *e.g.*, use any of the solvers included in CVX.
  - Very simple to use.
  - Good for small problems but less efficient for larger problems.
- A dedicated solver for quadratic programs.
  - More efficient for larger problems.
- Random search, eg genetic algorithms (GA), particle swarms,....
  - Very inefficient. Note you do not (should not) use (GA, ...) to solve *e.g.*, $\mathbf{A}\mathbf{x} = \mathbf{b}$ $(\min. ||\mathbf{A}\mathbf{x} - \mathbf{b}||)$.
- We also use a dual formulation
  - Computational efficient for large problems.
  - Illustrates dual problems and posteriori error estimates.

Why convex optimization: illustration

The upper bound on $G/Q|_{\mathrm{opt}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (red) curve

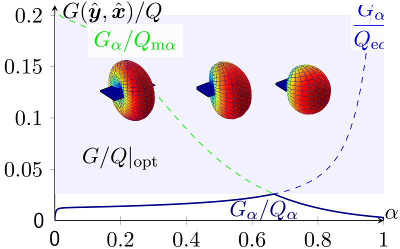
$$\frac{G}{Q}\Big|_{\mathrm{opt}} \leq \frac{G_\alpha}{\alpha Q_{\mathrm{e}\alpha} + (1-\alpha) Q_{\mathrm{m}\alpha}}$$

Efficiently solved with Newton iterations (cost $\mathbf{A}\mathbf{x} = \mathbf{b}$ per it).



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

Why convex optimization: illustration



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

We also compute the actual G/Q for the (dual) current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{\mathrm{e}\alpha},Q_{\mathrm{m}\alpha}\}} \leq \left.\frac{G}{Q}\right|_{\mathrm{opt}}$$

Why convex optimization: illustration

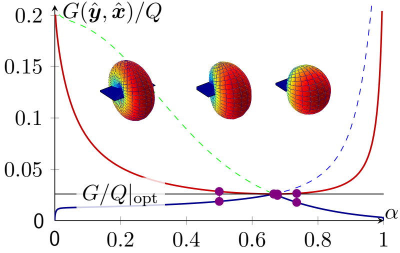
The upper bound on $G/Q|_{\mathrm{opt}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (red) curve

$$\frac{G}{Q}\bigg|_{\mathrm{opt}} \leq \frac{G_\alpha}{\alpha Q_{\mathrm{e}\alpha} + (1-\alpha) Q_{\mathrm{m}\alpha}}$$

Efficiently solved with Newton iterations (cost $\mathbf{Ax}=\mathbf{b}$ per it).

We also compute the actual G/Q for the (dual) current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{\mathrm{e}\alpha}, Q_{\mathrm{m}\alpha}\}} \leq \frac{G}{Q}\bigg|_{\mathrm{opt}}$$



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

The Newton iterations converge as $\alpha \approx 0.5$, 0.73536, 0.67677, 0.66629,0.66602, 0.66602 with the duality gap in G/Q approximately $10^{-\{2,2,3,4,8,16\}}$.

Why convex optimization: illustration

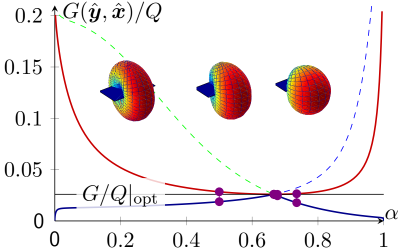
The upper bound on $G/Q|_{\mathrm{opt}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (red) curve

$$\frac{G}{Q}\bigg|_{\mathrm{opt}} \leq \frac{G_\alpha}{\alpha Q_{\mathrm{e}\alpha} + (1-\alpha)Q_{\mathrm{m}\alpha}}$$

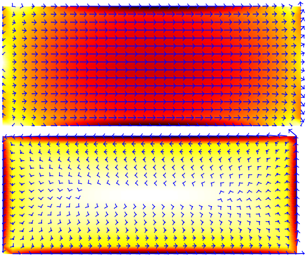
Efficiently solved with Newton iterations (cost $\mathbf{Ax}=\mathbf{b}$ per it).

We also compute the actual G/Q for the (dual) current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{\mathrm{e}\alpha}, Q_{\mathrm{m}\alpha}\}} \leq \frac{G}{Q}\bigg|_{\mathrm{opt}}$$



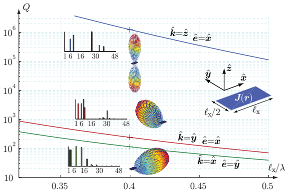
$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$



Why: simple optimization formulations

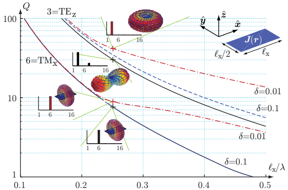
Super directivity:

$$\begin{array}{ll}\text{minimize} & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I},\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\}\\ \text{subject to} & \mathbf{F}\mathbf{I}=1\\ & \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}\leq 4\pi/(\eta_0 D_0)\end{array}$$



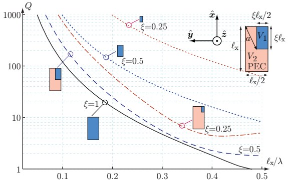
Prescribed far field:

$$\begin{array}{ll}\text{minimize} & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I},\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\}\\ \text{subject to} & \displaystyle\int_{\Omega}|\boldsymbol{F}(\hat{\boldsymbol{k}})-\boldsymbol{F}_0(\hat{\boldsymbol{k}})|^2\,\mathrm{d}\Omega_{\hat{\boldsymbol{k}}}<\delta\end{array}$$



Embedded antennas:

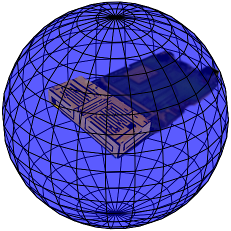
$$\begin{array}{ll}\text{minimize} & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I},\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\}\\ \text{subject to} & \mathbf{F}\mathbf{I}=1\\ & \mathbf{I}_{\mathrm{G}}=\mathbf{C}\mathbf{I}_{\mathrm{A}}\end{array}$$

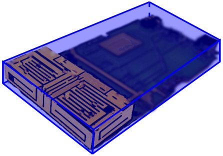


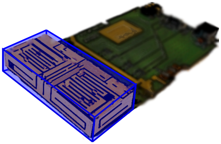
Outline

Summary

- Physical bounds from spheres (Chu 1948) and arbitrary shapes (Gustafsson *etal* 2007) to embedded antennas...
- Stored energy in the current density.
- Optimization of the antenna structure (global optimization) and the antenna currents (convex optimization).
- Convex optimization for bounds and optimal currents: G/Q, superdirective, embedded, ...
- Closed form solutions for small antennas.
- Non-Foster to overcome $B \sim 1/Q$ ...







Initial results for efficiency, more realistic geometries (phones), SAR, MIMO. Investigating dielectrics, volume currents, magnetic currents, ...

Slides: `http://www.eit.lth.se/staff/mats.gustafsson`

References

Antenna current optimization and physical bounds

- M. Gustafsson, M. Cismasu, B.L.G. Jonsson, *Physical bounds and optimal currents on antennas*, IEEE-TAP, 2012.
- M. Gustafsson, S. Nordebo, *Optimal antenna currents for Q, superdirectivity, and radiation patterns using convex optimization*, IEEE-TAP, 2013.
- M. Cismasu, M. Gustafsson, *Antenna Bandwidth Optimization with Single Frequency Simulation*, IEEE-TAP, 2014.
- M. Gustafsson *etal*, *Tutorial on antenna current optimization using MATLAB and CVX*, 2015.

Stored energy expressed in the current density

- G.A.E. Vandenbosch, *Reactive energies, impedance, and Q factor of radiating structures*, IEEE-TAP, 2010.
- M. Gustafsson, B.L.G. Jonsson, *Stored Electromagnetic Energy and Antenna Q*, arXiv:1211.5521, 2012.
- G.A.E. Vandenbosch, *Radiators in time domain, part I, II*, IEEE-TAP, 2013.
- M. Capek, L. Jelinek, P. Hazdra, and J. Eichler, *The measurable Q factor and observable energies of radiating structures*, IEEE-TAP, 2014.
- M. Gustafsson, D. Tayli, M. Cismasu, *Q factors for antennas in dispersive media*, arXiv:1408.6834, 2014.

Convex optimization

- S. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.
- M. Grant, S. Boyd, CVX, http://cvxr.com/cvx/

References I

Best, S. R. (2004). "The radiation properties of electrically small folded spherical helix antennas". *IEEE Trans. Antennas Propagat.* 52.4, pp. 953–960.

— (2009). "A Low Q Electrically Small Magnetic (TE Mode) Dipole". *Antennas and Wireless Propagation Letters, IEEE* 8, pp. 572–575.

— (2015). "Electrically Small Resonant Planar Antennas: Optimizing the quality factor and bandwidth." *IEEE Antennas and Propagation Magazine* 57.3, pp. 38–47.

Best, S. R. et al. (2008). "An impedance-matched 2-element superdirective array". *Antennas and Wireless Propagation Letters, IEEE* 7, pp. 302–305.

Boyd, S. P. and L. Vandenberghe (2004). *Convex Optimization*. Cambridge Univ. Pr.

Brune, O. (1931). "Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency". *MIT J. Math. Phys.* 10, pp. 191–236.

Capek, M., P. Hazdra, and J. Eichler (2012). "A method for the evaluation of radiation Q based on modal approach". *IEEE Trans. Antennas Propagat.* 60.10, pp. 4556–4567.

Capek, M. et al. (2014). "The Measurable Q Factor and Observable Energies of Radiating Structures". *IEEE Trans. Antennas Propagat.* 62.1, pp. 311–318.

Carpenter, C. J. (1989). "Electromagnetic energy and power in terms of charges and potentials instead of fields". *IEE Proc. A* 136.2, pp. 55–65.

Chalas, J., K. Sertel, and J. L. Volakis (2011). "Computation of the Q limits for arbitrary-shaped antennas using characteristic modes". In: *Antennas and Propagation (APSURSI), 2011 IEEE International Symposium on*. IEEE, pp. 772–774.

Chen, Y. and C.-F. Wang (2015). *Characteristic Modes: Theory and Applications in Antenna Engineering*. John Wiley & Sons.

Chu, L. J. (1948). "Physical Limitations of Omnidirectional Antennas". *J. Appl. Phys.* 19, pp. 1163–1175.

Cismasu, M. and M. Gustafsson (2014a). "Antenna Bandwidth Optimization with Single Frequency Simulation". *IEEE Trans. Antennas Propagat.* 62.3, pp. 1304–1311.

— (2014b). "Multiband Antenna Q Optimization using Stored Energy Expressions". *IEEE Antennas and Wireless Propagation Letters* 13.2014, pp. 646–649.

References II

Collin, R. E. and S. Rothschild (1964). "Evaluation of Antenna Q". *IEEE Trans. Antennas Propagat.* 12, pp. 23–27.

Derneryd, A. et al. (2009). "Application of gain-bandwidth bounds on loaded dipoles". *IET Microwaves, Antennas & Propagation* 3.6, pp. 959–966.

Fante, R. L. (1969). "Quality Factor of General Antennas". *IEEE Trans. Antennas Propagat.* 17.2, pp. 151–155.

Foltz, H. D. and J. S. McLean (1999). "Limits on the radiation Q of electrically small antennas restricted to oblong bounding regions". In: *IEEE Antennas and Propagation Society International Symposium*. Vol. 4. IEEE, pp. 2702–2705.

Garbacz, R. J. and R. H. Turpin (1971). "A generalized expansion for radiated and scattered fields". *IEEE Trans. Antennas Propagat.* 19.3, pp. 348–358.

Geyi, W. (2003a). "A method for the evaluation of small antenna Q". *IEEE Trans. Antennas Propagat.* 51.8, pp. 2124–2129.

— (2003b). "Physical limitations of antenna". *IEEE Trans. Antennas Propagat.* 51.8, pp. 2116–2123.

Gustafsson, M., M. Cismasu, and S. Nordebo (2010). "Absorption Efficiency and Physical Bounds on Antennas". *International Journal of Antennas and Propagation* 2010.Article ID 946746, pp. 1–7.

Gustafsson, M., J. Friden, and D. Colombi (2015). "Antenna Current Optimization for Lossy Media with Near Field Constraints". *Antennas and Wireless Propagation Letters, IEEE* 14, pp. 1538–1541.

Gustafsson, M. and B. L. G. Jonsson (2015a). "Antenna Q and stored energy expressed in the fields, currents, and input impedance". *IEEE Trans. Antennas Propagat.* 63.1, pp. 240–249.

— (2015b). "Stored Electromagnetic Energy and Antenna Q". *Progress In Electromagnetics Research (PIER)* 150, pp. 13–27.

Gustafsson, M. and S. Nordebo (2013). "Optimal Antenna Currents for Q, Superdirectivity, and Radiation Patterns Using Convex Optimization". *IEEE Trans. Antennas Propagat.* 61.3, pp. 1109–1118.

Gustafsson, M., C. Sohl, and G. Kristensson (2007). "Physical limitations on antennas of arbitrary shape". *Proc. R. Soc. A* 463, pp. 2589–2607.

— (2009). "Illustrations of New Physical Bounds on Linearly Polarized Antennas". *IEEE Trans. Antennas Propagat.* 57.5, pp. 1319–1327.

Gustafsson, M., M. Cismasu, and B. L. G. Jonsson (2012). "Physical bounds and optimal currents on antennas". *IEEE Trans. Antennas Propagat.* 60.6, pp. 2672–2681.

References III

Gustafsson, M. and B. L. G. Jonsson (2012). *Stored Electromagnetic Energy and Antenna Q*. Tech. rep. LUTEDX/(TEAT-7222)/1–25/(2012). Lund University.

Gustafsson, M. and S. Nordebo (2006). "Bandwidth, Q factor, and resonance models of antennas". *Progress in Electromagnetics Research* 62, pp. 1–20.

Gustafsson, M., D. Tayli, and M. Cismasu (2014). *Q factors for antennas in dispersive media*. Tech. rep. LUTEDX/(TEAT-7232)/1–24/(2014). Lund University.

— (2016). "Physical bounds of antennas". In: *Handbook of Antenna Technologies*. Ed. by Z. N. Chen. Springer-Verlag.

Hansen, R. C. and R. E. Collin (2009). "A new Chu formula for Q". *IEEE Antennas and Propagation Magazine* 51.5, pp. 38–41.

Hansen, T. V., O. S. Kim, and O. Breinbjerg (2012). "Stored Energy and Quality Factor of Spherical Wave Functions–in Relation to Spherical Antennas With Material Cores". *IEEE Trans. Antennas Propagat.* 60.3, pp. 1281–1290.

Harrington, R. F. and J. R. Mautz (1971). "Theory of characteristic modes for conducting bodies". *IEEE Trans. Antennas Propagat.* 19.5, pp. 622–628.

— (1972). "Control of radar scattering by reactive loading". *IEEE Trans. Antennas Propagat.* 20.4, pp. 446–454.

Jonsson, B. L. G. and M. Gustafsson (2015). "Stored energies in electric and magnetic current densities for small antennas". *Proc. R. Soc. A* 471.2176, p. 20140897.

Karlsson, A. (2004). "Physical limitations of antennas in a lossy medium". *IEEE Trans. Antennas Propagat.* 52, pp. 2027–2033.

Kim, O. (2012). "Minimum Q Electrically Small Antennas". *IEEE Trans. Antennas Propagat.* 60.8, pp. 3551–3558.

Kim, O., O. Breinbjerg, and A. Yaghjian (2010). "Electrically Small Magnetic Dipole Antennas With Quality Factors Approaching the Chu Lower Bound". *IEEE Trans. Antennas Propagat.* 58.6, pp. 1898–1906.

Levis, C. (1957). "A reactance theorem for antennas". *Proceedings of the IRE* 45.8, pp. 1128–1134.

McLean, J. S. (1996). "A Re-Examination of the Fundamental Limits on the Radiation Q of Electrically Small Antennas". *IEEE Trans. Antennas Propagat.* 44.5, pp. 672–676.

Rhodes, D. R. (1976). "Observable stored energies of electromagnetic systems". *Journal of the Franklin Institute* 302.3, pp. 225–237.

Rhodes, D. (1977). "A reactance theorem". *Proc. R. Soc. A* 353.1672, pp. 1–10.

References IV

Sievenpiper, D. F. et al. (2012). "Experimental Validation of Performance Limits and Design Guidelines for Small Antennas". *IEEE Trans. Antennas Propagat.* 60.1, pp. 8–19.

Sohl, C. and M. Gustafsson (2008). "A priori estimates on the partial realized gain of Ultra-Wideband (UWB) antennas". *Quart. J. Mech. Appl. Math.* 61.3, pp. 415–430.

Sohl, C., M. Gustafsson, and G. Kristensson (2007). "Physical limitations on broadband scattering by heterogeneous obstacles". *J. Phys. A: Math. Theor.* 40, pp. 11165–11182.

Sten, J. C.-E., P. K. Koivisto, and A. Hujanen (2001). "Limitations for the Radiation Q of a Small Antenna Enclosed in a Spheroidal Volume: Axial Polarisation". *AEÜ Int. J. Electron. Commun.* 55.3, pp. 198–204.

Sten, J.-E., A Hujanen, and P. Koivisto (2001). "Quality factor of an electrically small antenna radiating close to a conducting plane". *IEEE Trans. Antennas Propagat.* 49.5, pp. 829–837.

Stuart, H., S. Best, and A. Yaghjian (2007). "Limitations in Relating Quality Factor to Bandwidth in a Double Resonance Small Antenna". *Antennas and Wireless Propagation Letters* 6.

Thal, H. L. (2006). "New Radiation Q Limits for Spherical Wire Antennas". *IEEE Trans. Antennas Propagat.* 54.10, pp. 2757–2763.

— (2012). "Q Bounds for Arbitrary Small Antennas: A Circuit Approach". *IEEE Trans. Antennas Propagat.* 60.7, pp. 3120–3128.

Vandenbosch, G. A. E. (2010). "Reactive Energies, Impedance, and Q Factor of Radiating Structures". *IEEE Trans. Antennas Propagat.* 58.4, pp. 1112–1127.

— (2011). "Simple procedure to derive lower bounds for radiation Q of electrically small devices of arbitrary topology". *IEEE Trans. Antennas Propagat.* 59.6, pp. 2217–2225.

Vandenbosch, G. A. E. (2013a). "Radiators in time domain, part I: electric, magnetic, and radiated energies". *IEEE Trans. Antennas Propagat.* 61.8, pp. 3995–4003.

— (2013b). "Radiators in time domain, part II: finite pulses, sinusoidal regime and Q factor". *IEEE Trans. Antennas Propagat.* 61.8, pp. 4004–4012.

Volakis, J., C. C. Chen, and K. Fujimoto (2010). *Small Antennas: Miniaturization Techniques & Applications*. McGraw-Hill.

Wheeler, H. A. (1947). "Fundamental limitations of small antennas". *Proc. IRE* 35.12, pp. 1479–1484.

References V

Yaghjian, A. D., M. Gustafsson, and B. L. G Jonsson (2013). "Minimum Q for Lossy and Lossless Electrically Small Dipole Antennas". *Progress In Electromagnetics Research* 143, pp. 641–673.

Yaghjian, A. D. and H. R. Stuart (2010). "Lower Bounds on the Q of Electrically Small Dipole Antennas". *IEEE Trans. Antennas Propagat.* 58.10, pp. 3114–3121.

Yaghjian, A. D. and S. R. Best (2005). "Impedance, Bandwidth, and Q of Antennas". *IEEE Trans. Antennas Propagat.* 53.4, pp. 1298–1324.

Outline

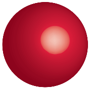
6. **Physical bounds**
   - Chu bound
   - Forward scattering
   - Polarizability dyadics

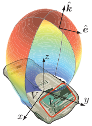
7. **Stored energy**

8. **Current optimization**

Background

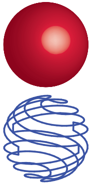
- 1947 Wheeler: *Bounds based on circuit models.*
- 1948 Chu: *Bounds on Q and D/Q for spheres.*
- 1964 Collin & Rothschild: *Closed form expressions of Q for arbitrary spherical modes, see also* Harrington, Collin, Fantes, Maclean, Gayi, Hansen, Hujanen, Sten, Best, Yaghjian, Kildal, Karlsson... *(most are based on Chu's approach using spherical modes.)*
- 1999 Foltz & McLean, 2001 Sten, Koivisto, and Hujanen: *Initial bounds for spheroidal volumes.*
- 2006 Thal: *Bounds on Q for small non-magnetic spherical antennas.*
- 2007 Gustafsson, Sohl & Kristensson: *Bounds on D/Q for arbitrary geometries (and Q for small antennas).*
- 2010 Yaghjian & Stuart: *Bounds on Q for dipole antennas in the limit* $ka \to 0$.
- 2011 Vandenbosch: *Bounds on Q for small (non-magnetic) antennas in the limit $ka \to 0$.*
- 2011 Chalas, Sertel & Volakis: *Bounds on Q using characteristic modes.*
- 2012 Gustafsson, Cismasu, & Jonsson: *Optimal charge and current distributions on antennas.*
- 2013 Gustafsson & Nordebo: *Optimal antenna Q, superdirectivity, and radiation patterns using convex optimization.*





Background

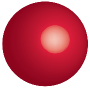
- 1948 Chu: *Bounds on Q and D/Q for spheres.*
- 1964 Collin & Rothschild: *Closed form expressions of Q for arbitrary spherical modes, see also* Harrington, Collin, Fantes, Maclean, Gayi, Hansen, Hujanen, Sten, Best, Yaghjian, Kildal, Karlsson... *(most are based on Chu's approach using spherical modes.)*



- 2006 Thal: *Bounds on Q for small non-magnetic spherical antennas.*

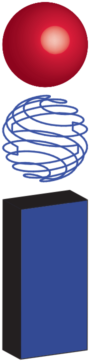
Background

- 1947 Wheeler: *Bounds based on circuit models.*
- 1948 Chu: *Bounds on Q and D/Q for spheres.*
- 1964 Collin & Rothschild: *Closed form expressions of Q for arbitrary spherical modes, see also* Harrington, Collin, Fantes, Maclean, Gayi, Hansen, Hujanen, Sten, Best, Yaghjian, Kildal, Karlsson... *(most are based on Chu's approach using spherical modes.)*
- 1999 Foltz & McLean, 2001 Sten, Koivisto, and Hujanen: *Initial bounds for spheroidal volumes.*
- 2006 Thal: *Bounds on Q for small non-magnetic spherical antennas.*



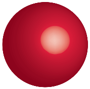
Background

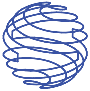
- 1947 Wheeler: *Bounds based on circuit models.*
- 1948 Chu: *Bounds on Q and D/Q for spheres.*
- 1964 Collin & Rothschild: *Closed form expressions of Q for arbitrary spherical modes, see also* Harrington, Collin, Fantes, Maclean, Gayi, Hansen, Hujanen, Sten, Best, Yaghjian, Kildal, Karlsson... *(most are based on Chu's approach using spherical modes.)*
- 1999 Foltz & McLean, 2001 Sten, Koivisto, and Hujanen: *Initial bounds for spheroidal volumes.*
- 2006 Thal: *Bounds on Q for small non-magnetic spherical antennas.*
- 2007 Gustafsson, Sohl & Kristensson: *Bounds on D/Q for arbitrary geometries (and Q for small antennas).*
- 2010 Yaghjian & Stuart: *Bounds on Q for dipole antennas in the limit $ka \to 0$.*
- 2011 Vandenbosch: *Bounds on Q for small (non-magnetic) antennas in the limit $ka \to 0$.*
- 2011 Chalas, Sertel & Volakis: *Bounds on Q using characteristic modes.*
- 2012 Gustafsson, Cismasu, & Jonsson: *Optimal charge and current distributions on antennas.*

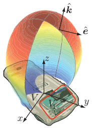


Background

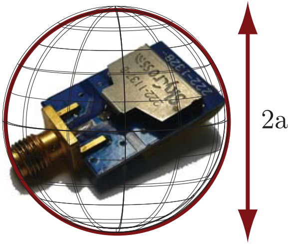
- 1947 Wheeler: *Bounds based on circuit models.*
- 1948 Chu: *Bounds on Q and D/Q for spheres.*
- 1964 Collin & Rothschild: *Closed form expressions of Q for arbitrary spherical modes, see also* Harrington, Collin, Fantes, Maclean, Gayi, Hansen, Hujanen, Sten, Best, Yaghjian, Kildal, Karlsson... *(most are based on Chu's approach using spherical modes.)*
- 1999 Foltz & McLean, 2001 Sten, Koivisto, and Hujanen: *Initial bounds for spheroidal volumes.*
- 2006 Thal: *Bounds on Q for small non-magnetic spherical antennas.*
- 2007 Gustafsson, Sohl & Kristensson: *Bounds on D/Q for arbitrary geometries (and Q for small antennas).*
- 2010 Yaghjian & Stuart: *Bounds on Q for dipole antennas in the limit* $ka \to 0$.
- 2011 Vandenbosch: *Bounds on Q for small (non-magnetic) antennas in the limit* $ka \to 0$.
- 2011 Chalas, Sertel & Volakis: *Bounds on Q using characteristic modes.*
- 2012 Gustafsson, Cismasu, & Jonsson: *Optimal charge and current distributions on antennas.*
- 2013 Gustafsson & Nordebo: *Optimal antenna Q, superdirectivity, and radiation patterns using convex optimization.*

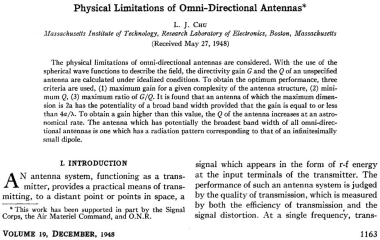






Chu 1948





Physical Limitations of Omni-Directional Antennas*

L. J. Chu
Massachusetts Institute of Technology, Research Laboratory of Electronics, Boston, Massachusetts
(Received May 27, 1948)

The physical limitations of omni-directional antennas are considered. With the use of the spherical wave functions to describe the field, the directivity gain G and the Q of an unspecified antenna are calculated under idealized conditions. To obtain the optimum performance, three criteria are used, (1) maximum gain for a given complexity of the antenna structure, (2) minimum Q, (3) maximum ratio of G/Q. It is found that an antenna of which the maximum dimension is 2a has the potentiality of a broad band width provided that the gain is equal to or less than $4a/\lambda$. To obtain a gain higher than this value, the Q of the antenna increases at an astronomical rate. The antenna which has potentially the broadest band width of all omni-directional antennas is one which has a radiation pattern corresponding to that of an infinitesimally small dipole.

I. INTRODUCTION

An antenna system, functioning as a transmitter, provides a practical means of transmitting, to a distant point or points in space, a signal which appears in the form of r-f energy at the input terminals of the transmitter. The performance of such an antenna system is judged by the quality of transmission, which is measured by both the efficiency of transmission and the signal distortion. At a single frequency, trans-

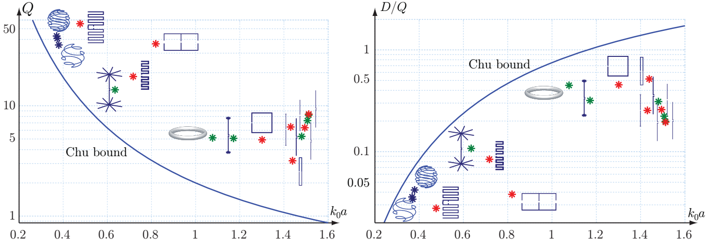
* This work has been supported in part by the Signal Corps, the Air Materiel Command, and O.N.R.

The stored energy and radiated power outside a sphere with radius a give the Chu-bounds for omni-directional antennas, *i.e.*,

$$Q \geq Q_{\mathrm{Chu}} = \frac{1}{(ka)^3} + \frac{1}{ka} \quad \text{and} \quad \frac{D}{Q} \leq \frac{3}{2Q_{\mathrm{Chu}}} \approx \frac{3}{2}(ka)^3$$

for $k_0 a \ll 1$, where $k = k_0$ is the resonance wavenumber $k = 2\pi/\lambda = 2\pi f/c_0$.

Chu 1948



The stored energy and radiated power outside a sphere with radius a give the Chu-bounds for omni-directional antennas, *i.e.*,

$$Q \geq Q_{\text{Chu}} = \frac{1}{(ka)^3} + \frac{1}{ka} \quad \text{and} \quad \frac{D}{Q} \leq \frac{3}{2Q_{\text{Chu}}} \approx \frac{3}{2}(ka)^3$$

for $k_0 a \ll 1$, where $k = k_0$ is the resonance wavenumber $k = 2\pi/\lambda = 2\pi f/c_0$.

see also Sievenpiper et al. 2012

Based on the approach of Chu 1948

Chu 1948 used mode expansions combined with circuit models to compute the stored energy. Fine for the dipole mode but technical for higher order modes. There have been a substantial amount of work following the approach by Chu, *e.g.*, (and many more...)

- Collin and Rothschild 1964: *EM fields for closed form expressions of Q for arbitrary spherical modes.*
- Fante 1969: *general TE+TM modes.*
- McLean 1996: *a re-examination of Q.*
- Foltz and McLean 1999; Sten, Koivisto, and Hujanen 2001: *extensions to spheroidal volumes.*
- Sten, Hujanen, and Koivisto 2001: *antennas close to a ground plane.*
- Geyi 2003b: *Q and G/Q for combined TE+TM.*
- Karlsson 2004: *lossy medium.*
- Thal 2006: *bounds on Q for small hollow spherical antennas.*
- Hansen, Kim, and Breinbjerg 2012: material core.
- Kim 2012: antennas that are close to the Chu limit.

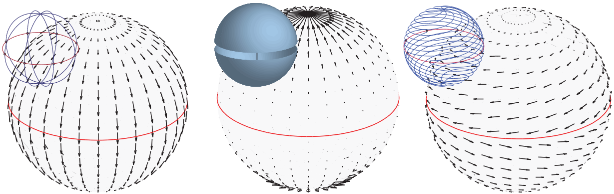
Thal 2006: non-magnetic spheres

The Chu bound is derived under the assumption of negligible stored energy in the interior of the sphere. Antennas without magnetic material (or magnetic currents) have internally stored energy.

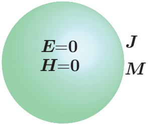
Thal 2006: *Bounds on Q for small non-magnetic spherical antennas.* Small electric dipole antennas

$$Q \geq \frac{1.5}{(ka)^3} = 1.5 Q_{\text{Chu}} \quad \text{for } ka \ll 1$$

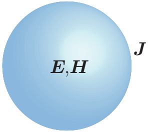
see also Gustafsson and Jonsson 2015b; Hansen and Collin 2009



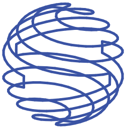
Illustrations of surface currents $\boldsymbol{J}$ for a dipole, capped dipole, and folded spherical helix. Gustafsson, Cismasu, and Jonsson 2012



Chu: $\boldsymbol{J}$, $\boldsymbol{M}$ currents.

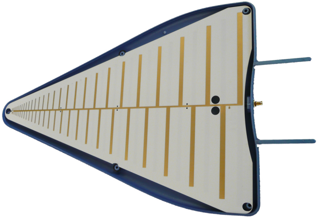


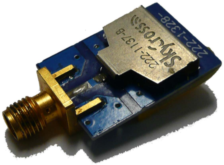
Thal: $\boldsymbol{J}$ currents.



Best 2004 Folded spherical helix $Q \approx 1.5 Q_{\text{Chu}}$.

Forward scattering bounds on antennas

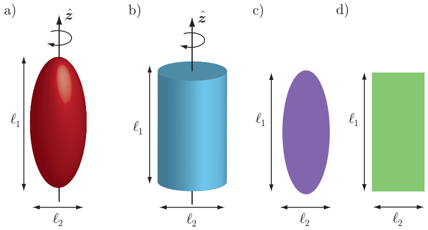




- Properties of the best antenna confined to a given (arbitrary) geometry, *e.g.*, spheroid, cylinder, elliptic disk, and rectangle.
- Performance in
  - Directivity bandwidth product: D/Q (half-power $B \approx 2/Q$).
  - Partial realized gain: $(1-|\Gamma|^2)G$ over a bandwidth.

Derneryd et al. 2009; Gustafsson, Sohl, and Kristensson 2007; Gustafsson, Sohl, and Kristensson 2009; Sohl and Gustafsson 2008

Forward scattering bounds on antennas



- Properties of the best antenna confined to a given (arbitrary) geometry, *e.g.*, spheroid, cylinder, elliptic disk, and rectangle.
- Performance in
  - Directivity bandwidth product: D/Q (half-power $B \approx 2/Q$).
  - Partial realized gain: $(1 - |\Gamma|^2)G$ over a bandwidth.

Derneryd et al. 2009; Gustafsson, Sohl, and Kristensson 2007; Gustafsson, Sohl, and Kristensson 2009; Sohl and Gustafsson 2008

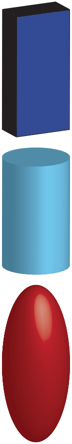
Forward scattering identity (2007)

The forward scattering identity (lossless, non-magnetic, linearly polarized ($\hat{e}$) antennas)

$$\int_0^\infty \frac{(1-|\Gamma(k)|^2)D(k;\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{k^4}\,\mathrm{d}k = \frac{\eta}{2}\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}}$$

gives a bound on D/Q (directivity bandwidth product) expressed in the high contrast polarizability dyadic $\boldsymbol{\gamma}_\infty \geq \boldsymbol{\gamma}_{\mathrm{e}}$:

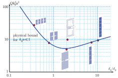
$$\frac{D}{Q} \leq \frac{\eta k_0^3}{2\pi}\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_\infty\cdot\hat{\boldsymbol{e}} \quad \text{and small E-dipoles } Q \geq \frac{6\pi}{k_0^3\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_\infty\cdot\hat{\boldsymbol{e}}}$$



Circumscribing geometries of arbitrary shape.

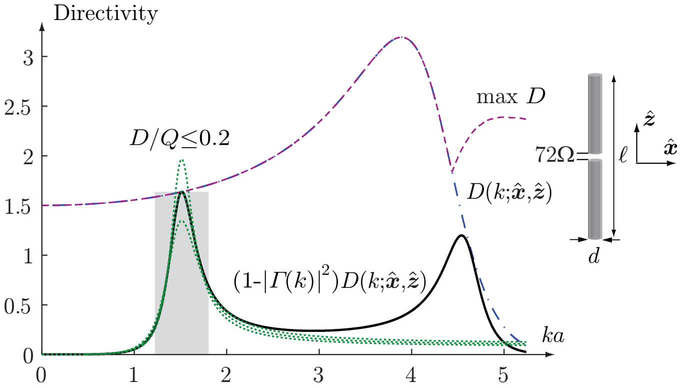
Performance proportional to the polarizability. ▸ 57

Identical to the Thal 2006 bound for spheres.



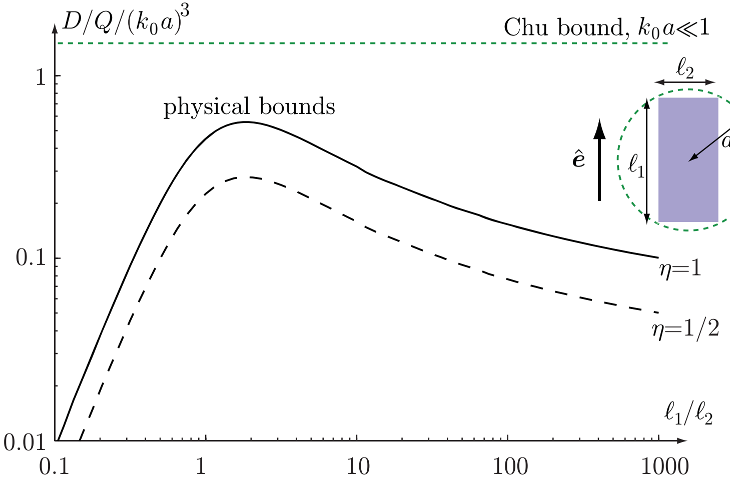
Derneryd et al. 2009; Gustafsson, Sohl, and Kristensson 2007; Gustafsson, Sohl, and Kristensson 2009; Sohl and Gustafsson 2008; Sohl, Gustafsson, and Kristensson 2007

Cylindrical dipole



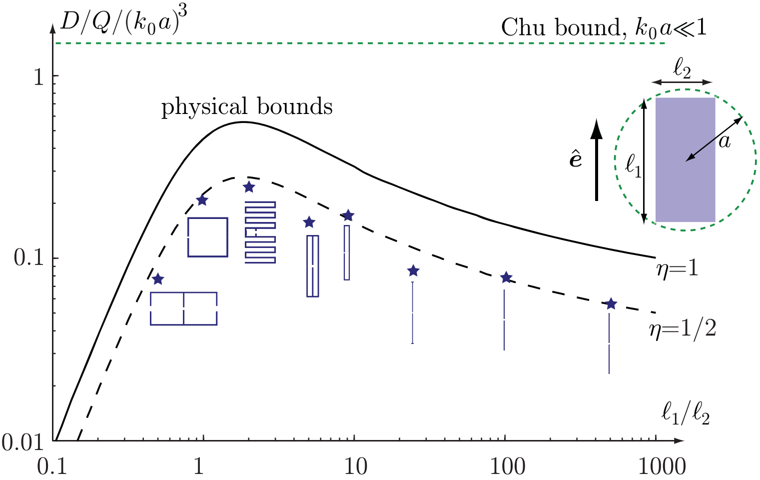
Lossless $\hat{\boldsymbol{z}}$-directed dipole, wire diameter $d = \ell/1000$, matched to $72\,\Omega$. Weighted area under the black curve (partial realized gain) is known. Note, half wavelength dipole for $ka = \pi/2 \approx 1.5$ with directivity $D \approx 1.64 \approx 2.15\,\mathrm{dB_i}$.

Circumscribing rectangles (2007)



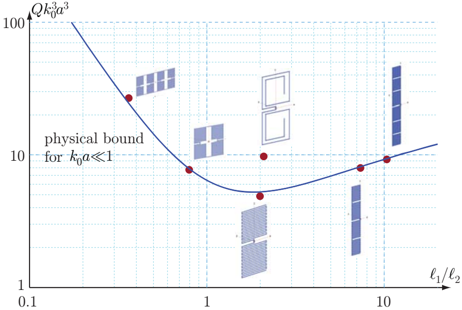
Note, $\eta \leq 1/2$ for small electric dipole antennas $k_0a \ll 1$.
Gustafsson, Sohl, and Kristensson 2009

Circumscribing rectangles (2007)



Note, $\eta \leq 1/2$ for small electric dipole antennas $k_0 a \ll 1$.
Gustafsson, Sohl, and Kristensson 2009

Small planar antennas

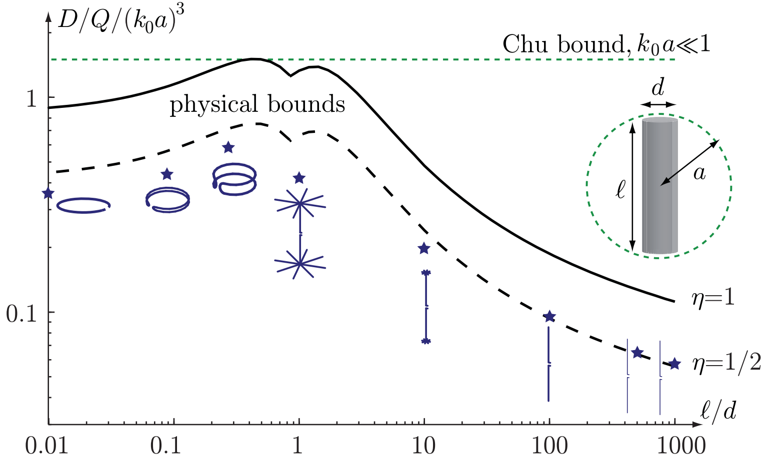


The dependence of $Qk_0^3a^3$ as a function of $\xi = \ell_1/\ell_2$.

- Multiplication of Q with $k_0^3a^3$ removes the dependence of the electrical size.
- A performance bound on $Qk_0^3a^3$ (for $k_0a \ll 1$) that only depends on the shape $\xi = \ell_1/\ell_2$
- Also explains the 'poor' performance of one of the antennas.

Best 2009

Circumscribing cylinders



Gustafsson, Sohl, and Kristensson 2009

Polarizability dyadic and induced dipole moment

The induced dipole moment can be written

$$\boldsymbol{p} = \epsilon_0 \boldsymbol{\gamma}_{\mathrm{e}} \cdot \boldsymbol{E}$$

where $\boldsymbol{\gamma}_{\mathrm{e}}$ is the polarizability dyadic.

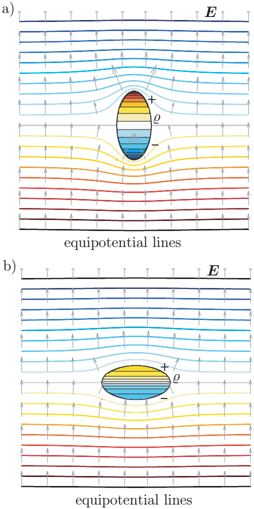
Example (Dielectric sphere)

A dielectric sphere with radius a and relative permittivity ϵ_{r} has the polarizability dyadic

$$\boldsymbol{\gamma}_{\mathrm{e}} = 4\pi a^3 \frac{\epsilon_{\mathrm{r}} - 1}{\epsilon_{\mathrm{r}} + 2} \mathbf{I} \to \boldsymbol{\gamma}_{\infty} = 4\pi a^3 \mathbf{I}$$

as $\epsilon_{\mathrm{r}} \to \infty$.

Analytic expressions for spheroids, elliptic discs, half spheres, hollow half spheres, touching spheres,...



High-contrast polarizability dyadics: γ_∞

γ_∞ is determined from the induced normalized surface charge density, ρ, as

$$\hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_\infty \cdot \hat{\boldsymbol{e}} = \frac{1}{E_0} \int_\Omega \hat{\boldsymbol{e}} \cdot \boldsymbol{r} \rho(\boldsymbol{r}) \, \mathrm{dS}$$

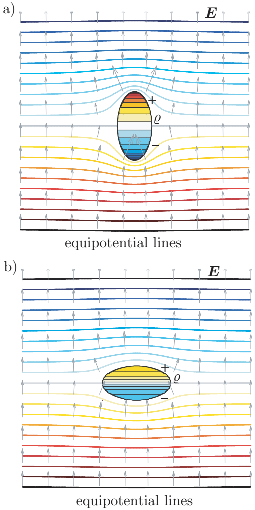
where ρ satisfies the integral equation

$$\int_\Omega \frac{\rho(\boldsymbol{r}')}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \, \mathrm{dS}' = E_0 \boldsymbol{r} \cdot \hat{\boldsymbol{e}} - V_n$$

with the constraints of zero total charge

$$\int_{\Omega_n} \rho(\boldsymbol{r}) \, \mathrm{dS} = 0$$

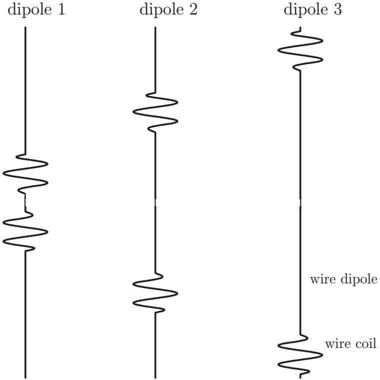
Can also use FEM (Laplace equation).



Polarizability $\gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}}$: Interpretation

Wire dipoles (length $\ell \approx 2a$) with coils

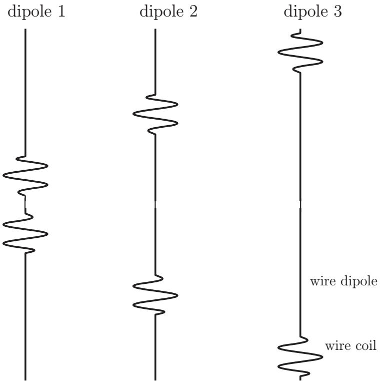
Geometries of the three wire dipoles



Polarizability $\gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}}$: Interpretation

Wire dipoles (length $\ell \approx 2a$) with coils

Geometries of the three wire dipoles

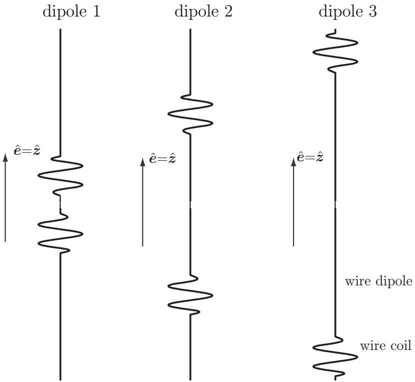


$\gamma/a^3 \approx 0.73$ $\gamma/a^3 \approx 0.95$ $\gamma/a^3 \approx 1.52$

Polarizability $\gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_\mathrm{e} \cdot \hat{\boldsymbol{e}}$: Interpretation

Wire dipoles (length $\ell \approx 2a$) with coils

External electrostatic field along the dipoles

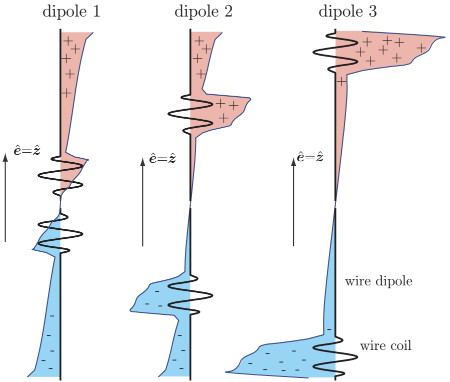


$\gamma/a^3 \approx 0.73$ $\quad\gamma/a^3 \approx 0.95$ $\quad\gamma/a^3 \approx 1.52$

Polarizability $\gamma = \hat{\boldsymbol{e}} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}}$: Interpretation

Wire dipoles (length $\ell \approx 2a$) with coils

Induced charge density on the wire

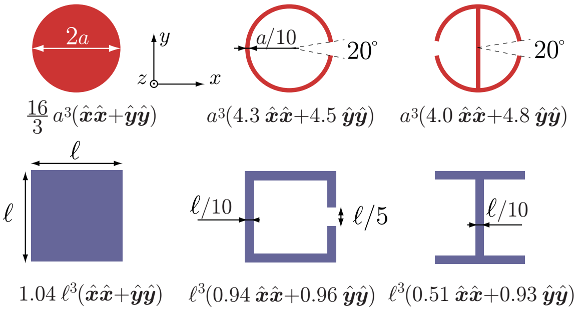


$\gamma/a^3 \approx 0.73$ $\gamma/a^3 \approx 0.95$ $\gamma/a^3 \approx 1.52$

Separation of charge for large polarizability.

Properties of the polarizability dyadics

Removal of metal from circular and square plates

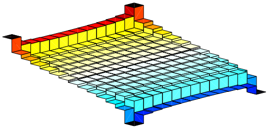


- The polarizability can not increase if you remove material.
- The metal in the center of the structure does not contribute much to the polarizability.
- Volume (and large area) is not necessary for a large polarizability.
- Important to be able to support a large separation of charge.

Numerical evaluation of γ_∞ (single object)

Expand the charge density in basis functions

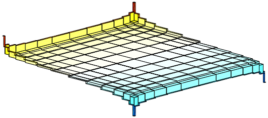
$$\rho(\boldsymbol{r}) = \sum_{n=1}^{N} \rho_n \psi_n(\boldsymbol{r}) = \boldsymbol{\psi}^{\mathsf{T}} \boldsymbol{\rho}$$



Equidistant mesh ($p = 1$)

and solve using Galerkin's method:

$$\begin{cases} \mathbf{W}_{\mathrm{e}}^{(0)} \boldsymbol{\rho} = E_0 \mathbf{f}_{\mathrm{e}} - \mathbf{n} V \\ \mathbf{f}_{\mathrm{e}}^{\mathsf{T}} \boldsymbol{\rho} = E_0/\gamma \\ \mathbf{n}^{\mathsf{T}} \boldsymbol{\rho} = 0 \end{cases} \qquad \begin{pmatrix} \mathbf{W}_{\mathrm{e}}^{(0)} & \mathbf{f}_{\mathrm{e}} & \mathbf{n} \\ \mathbf{f}_{\mathrm{e}}^{\mathsf{T}} & 0 & 0 \\ \mathbf{n}^{\mathsf{H}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\rho} \\ \gamma^{-1} \\ V \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -1 \\ 0 \end{pmatrix}$$



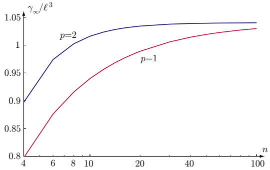
Constant charge ($p = 2$)

where $E_0 = -\gamma$ and ($N \times 1$ matrices)

$$\mathbf{f}_{\mathrm{e}} = \int_{\Omega} (\hat{\boldsymbol{e}} \cdot \boldsymbol{r}) \boldsymbol{\psi}(\boldsymbol{r}) \,\mathrm{dS}, \quad \mathbf{n} = \int_{\Omega} \boldsymbol{\psi}(\boldsymbol{r}) \,\mathrm{dS}$$

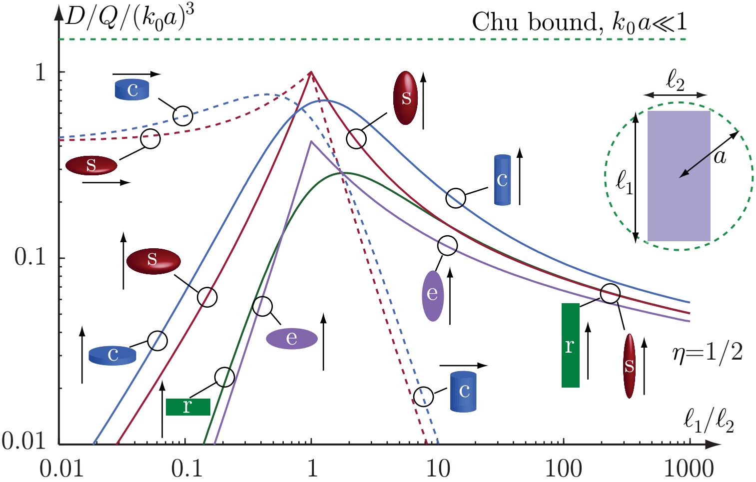
and the $N \times N$ matrix

$$\mathbf{W}_{\mathrm{e}}^{(0)} = \int_{\Omega} \int_{\Omega} \frac{\boldsymbol{\psi}(\boldsymbol{r}) \boldsymbol{\psi}^{\mathsf{T}}(\boldsymbol{r}')}{4\pi |\boldsymbol{r} - \boldsymbol{r}'|} \,\mathrm{dSdS}'.$$



Convergence

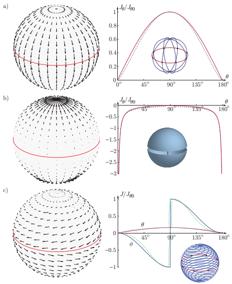
Rectangles, cylinders, elliptic disks, and spheroids (2007)

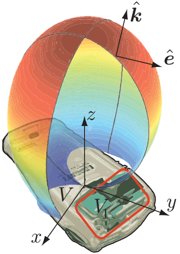


http://www.mathworks.com/matlabcentral/fileexchange/26806-antennaq

Bounds on D/Q (and Q for small antennas)

- Forward scattering (2007).
- Performance in the polarizability.
- Numerical simulations verify the results for electric dipole antennas.
- Similar results for small electric dipole antennas by Yaghjian & Stuart (2010), Vandenbosch (2011), Chalas, Sertel & Volakis (2011), and Gustafsson *etal*(2012).
- Many open questions for mixed modes (TE+TM) and magnetic materials.





What more can we do?

Antenna current optimization for

- embedded antennas (mobile phones).
- superdirectivity, efficiency, MIMO...
- current distribution for understanding.

Outline

Subtracted far field: negative W_e

The stored energy defined by subtraction of the far field

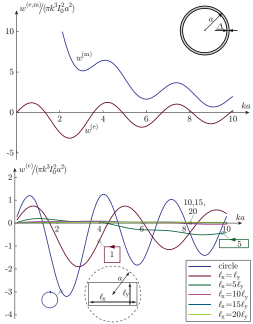
$$W_\mathrm{F}^{(\mathrm{E})} = \frac{\epsilon_0}{4}\int_{\mathbb{R}^3_\mathrm{r}} |\boldsymbol{E}(\boldsymbol{r})|^2 - \frac{|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2}{r^2}\,\mathrm{dV}$$

can produce negative values. Coordinate independent for symmetric radiation patterns and equals the energy expression in Vandenbosch 2010, see Gustafsson and Jonsson 2015b.

- Consider *e.g.*, the divergence free loop current $\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(\varrho - a)\delta(z)\hat{\boldsymbol{\phi}}$ in cylinder coordinates $\{\varrho, \phi, z\}$.
- Stored electric energy ($W_\mathrm{F}^{(\mathrm{M})} = \infty$)

$$W_\mathrm{F}^{(\mathrm{E})} = \frac{-\mu_0 k a^3 I_0^2}{16}\int_0^{2\pi} \sin\phi\sin(2ka\sin\frac{\phi}{2})\,\mathrm{d}\phi$$

- Can produce negative values for lager structures.



Gustafsson, Cismasu, Jonsson, *Physical Bounds and Optimal Currents on Antennas*, IEEE-TAP 2012.

Lumped circuits

Consider a voltage source and use the Kirchoffs' laws to construct the linear system $\mathbf{Z}\mathbf{I} = \mathbf{V}$, where the impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$ contains elements of the form

$$Z_{ij} = R_{ij} + \mathrm{j}X_{ij} = R_{ij} + \mathrm{j}\left(\omega L_{ij} - \frac{1}{\omega C_{ij}}\right)$$

The differentiated impedance matrix $\mathbf{Z}' = \mathrm{j}\mathbf{X}'$ is imaginary valued with the elements

$$X'_{ij} = \frac{\partial}{\partial\omega}\left(\omega L_{ij} - \frac{1}{\omega C_{ij}}\right) = L_{ij} + \frac{1}{\omega^2 C_{ij}}.$$

Differentiated input admittance and impedance

$$Y'_{\mathrm{in}} = -\mathrm{j}\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}/V_{\mathrm{in}}^2 \quad \text{and } Z'_{\mathrm{in}} = -Z_{\mathrm{in}}^2 Y'_{\mathrm{in}} = \mathrm{j}\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}/I_{\mathrm{in}}^2$$

Energy in lumped circuits

Time average stored energy in capacitors and in inductors

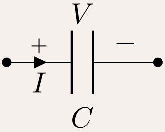
$$W_{\mathrm{e}} = \frac{C|V|^2}{4} = \frac{|I|^2}{4\omega^2 C} \quad \text{and } W_{\mathrm{m}} = \frac{L|I|^2}{4}$$

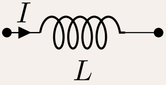
Contain absolute values $|I|^2$ and $|V|^2$, need to use Hermitian transpose. For a circuit network

$$W_{\mathrm{m}} - W_{\mathrm{e}} = \frac{\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I}}{4\omega} \quad \text{and } W_{\mathrm{e}} + W_{\mathrm{m}} = \frac{\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}}{4} \geq 0$$

reactance $\mathbf{X}$ for difference $W_{\mathrm{m}} - W_{\mathrm{e}}$ and differentiated reactance $\mathbf{X}'$ the sum $W_{\mathrm{m}} + W_{\mathrm{e}}$.

Energy in lumped circuits





Time average stored energy in capacitors and in inductors

$$W_\mathrm{e} = \frac{C|V|^2}{4} = \frac{|I|^2}{4\omega^2 C} \quad \text{and } W_\mathrm{m} = \frac{L|I|^2}{4}$$

Contain absolute values $|I|^2$ and $|V|^2$, need to use Hermitian transpose. For a circuit network

$$W_\mathrm{m} = \frac{1}{8}\mathbf{I}^\mathsf{H}\left(\frac{\partial \mathbf{X}}{\partial \omega} + \frac{\mathbf{X}}{\omega}\right)\mathbf{I} = \frac{1}{4}\sum_{i,j=1}^{N} I_i^* L_{ij} I_j \geq 0$$

$$W_\mathrm{e} = \frac{1}{8}\mathbf{I}^\mathsf{H}\left(\frac{\partial \mathbf{X}}{\partial \omega} - \frac{\mathbf{X}}{\omega}\right)\mathbf{I} = \frac{1}{4\omega^2}\sum_{i,j=1}^{N} I_i^* C_{ij}^{-1} I_j \geq 0,$$

Q and $Q_{Z'}$ for lumped circuits

Assume for simplicity a **self-resonant** circuit (antenna)

$$Q_{Z'} = \frac{\omega|Z'_{\mathrm{in}}|}{2R_{\mathrm{in}}} = \frac{\omega|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{2\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

and

$$Q = \frac{2\omega\max\{W_{\mathrm{e}}, W_{\mathrm{m}}\}}{P_{\mathrm{d}}} = \frac{\omega\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}}{2\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

Transpose for $Q_{Z'}$ and Hermitian transpose for Q

Also the inequality $Q \geq Q_{Z'}$ as ($\mathbf{X}' = \mathbf{U}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{U}$ real valued)

$$\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I} = (\mathbf{U}\mathbf{I})^{\mathsf{H}}\mathbf{\Lambda}\mathbf{U}\mathbf{I} \geq \left|(\mathbf{U}\mathbf{I})^{\mathrm{T}}\mathbf{\Lambda}\mathbf{U}\mathbf{I}\right| = \left|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}\right| \geq 0$$

with equality (to 0) for some current $\mathbf{I}$ (in the matrix case).

Electrostatic stored energy

- Consider the charge density $\rho(\boldsymbol{r})$ supported in $\Omega \subset \mathbb{R}^3$ in free space. Also assume that the total charge is zero, $\int \rho \,\mathrm{dV} = 0$.
- Have the alternative electric energy expressions

$$W_\mathrm{e} = \frac{1}{2}\int_{\mathbb{R}^3} \epsilon_0 |\boldsymbol{E}(\boldsymbol{r})|^2 \,\mathrm{dV} = \frac{1}{2}\int_{\Omega} \phi(\boldsymbol{r})\rho(\boldsymbol{r}) \,\mathrm{dV}$$
$$= \frac{1}{2\epsilon_0}\int_{\Omega}\int_{\Omega} \frac{\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \,\mathrm{dV}_1 \,\mathrm{dV}_2$$

where ϕ is the potential and ρ the charge density.

- Alternative interpretations: Energy in the fields or energy in the charges.
- Alternative computation: integral over $\mathbb{R}^3$ or over Ω.
- Positive definite quadratic form suitable for optimization.

Stored EM energy from current densities $\boldsymbol{J}$ in V

We use the expressions by Vandenbosch 2010 (and Carpenter 1989, Geyi 2003b for small antennas). Stored electric energy

$$W_{\mathrm{e}} = \frac{\eta_0}{4\omega}\int_{\Omega}\int_{\Omega}\nabla_1\cdot\boldsymbol{J}(\boldsymbol{r}_1)\nabla_2\cdot\boldsymbol{J}^*(\boldsymbol{r}_2)\frac{\cos(kr_{12})}{4\pi kr_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2 + W^{(2)}$$

Stored magnetic energy

$$W_{\mathrm{m}} = \frac{\eta_0}{4\omega}\int_{\Omega}\int_{\Omega}k^2\boldsymbol{J}(\boldsymbol{r}_1)\cdot\boldsymbol{J}^*(\boldsymbol{r}_2)\frac{\cos(kr_{12})}{4\pi kr_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2 + W^{(2)}$$

where $\mathrm{j}\omega\rho = -\nabla\cdot\boldsymbol{J}$, $\phi = \epsilon_0^{-1}g*\rho$, $\boldsymbol{A} = \mu_0 g*\boldsymbol{J}$, $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$

$$W^{(2)} = \frac{\eta_0}{8\omega}\int_{\Omega}\int_{\Omega}\left(\nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^* - k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^*\right)\frac{\sin(kr_{12})}{4\pi}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

Stored EM energy from current densities $\boldsymbol{J}$ in V

We use the expressions by Vandenbosch 2010 (and Carpenter 1989, Geyi 2003b for small antennas). Stored electric energy

$$W_{\mathrm{e}} = \frac{1}{4\epsilon_0}\,\mathrm{Re}\int_{\Omega}\int_{\Omega}\rho(\boldsymbol{r}_1)\rho^*(\boldsymbol{r}_2)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2 + W^{(2)}$$

Stored magnetic energy

$$W_{\mathrm{m}} = \frac{\mu_0}{4}\,\mathrm{Re}\int_{\Omega}\int_{\Omega}\boldsymbol{J}(\boldsymbol{r}_1)\cdot\boldsymbol{J}^*(\boldsymbol{r}_2)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2 + W^{(2)}$$

where $\mathrm{j}\omega\rho = -\nabla\cdot\boldsymbol{J}$, $\phi = \epsilon_0^{-1}g*\rho$, $\boldsymbol{A} = \mu_0 g*\boldsymbol{J}$, $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$

$$W^{(2)} = \frac{\eta_0}{8\omega}\int_{\Omega}\int_{\Omega}\left(\nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^* - k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^*\right)\frac{\sin(kr_{12})}{4\pi}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

Stored EM energy from current densities $\boldsymbol{J}$ in V

We use the expressions by Vandenbosch 2010 (and Carpenter 1989, Geyi 2003b for small antennas). Stored electric energy

$$W_{\mathrm{e}} = \frac{1}{4}\,\mathrm{Re}\int_{\Omega} \phi(\boldsymbol{r})\rho^{*}(\boldsymbol{r})\,\mathrm{dV} + W^{(2)}$$

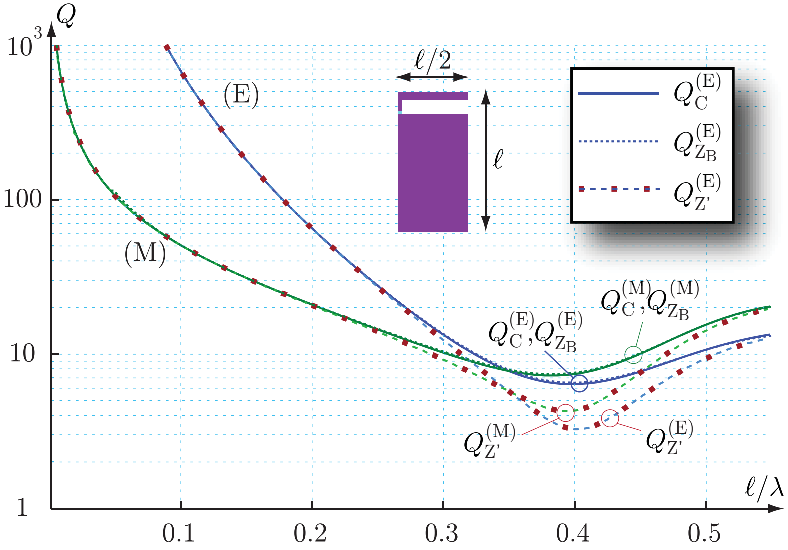
Stored magnetic energy

$$W_{\mathrm{m}} = \frac{1}{4}\,\mathrm{Re}\int_{\Omega} \boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{J}^{*}(\boldsymbol{r})\,\mathrm{dV} + W^{(2)}$$

where $\mathrm{j}\omega\rho = -\nabla\cdot\boldsymbol{J}$, $\phi = \epsilon_0^{-1} g * \rho$, $\boldsymbol{A} = \mu_0 g * \boldsymbol{J}$, $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$

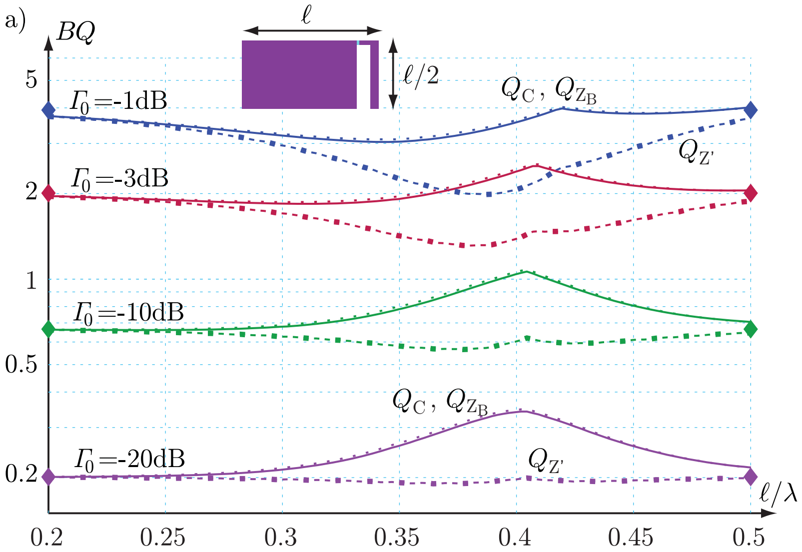
$$W^{(2)} = \frac{\eta_0}{8\omega}\int_{\Omega}\int_{\Omega}\left(\nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^{*} - k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^{*}\right)\frac{\sin(kr_{12})}{4\pi}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

Bandwidth



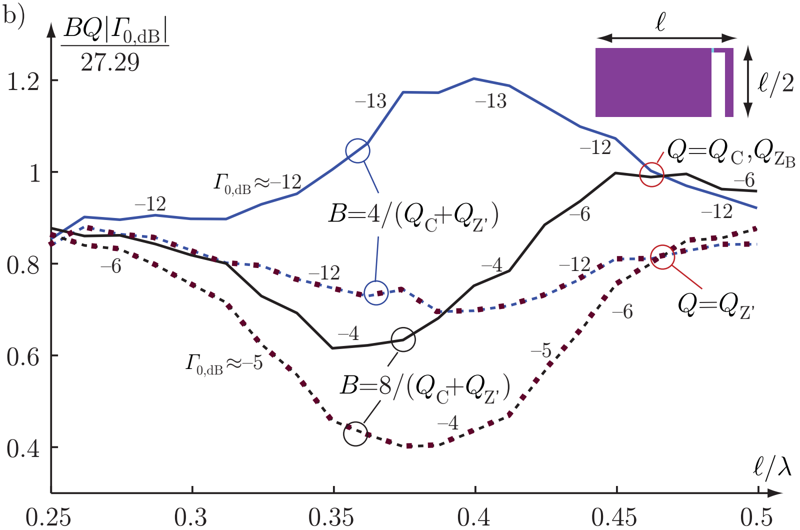
Q
10^3
100
10
1
0.1
0.2
0.3
0.4
0.5
ℓ/λ
(E)
(M)
$\ell/2$
ℓ
$Q_{\mathrm{C}}^{(\mathrm{E})}$
$Q_{\mathrm{Z_B}}^{(\mathrm{E})}$
$Q_{\mathrm{Z'}}^{(\mathrm{E})}$
$Q_{\mathrm{C}}^{(\mathrm{M})}, Q_{\mathrm{Z_B}}^{(\mathrm{M})}$
$Q_{\mathrm{C}}^{(\mathrm{E})}, Q_{\mathrm{Z_B}}^{(\mathrm{E})}$
$Q_{\mathrm{Z'}}^{(\mathrm{M})}$
$Q_{\mathrm{Z'}}^{(\mathrm{E})}$

Bandwidth



a)
BQ
ℓ
ℓ/2
5
2
1
0.5
0.2
Γ0 =-1dB
Γ0 =-3dB
Γ0 =-10dB
Γ0 =-20dB
QC , QZB
QZ'
QC , QZB
QZ'
ℓ/λ
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Bandwidth



b)
$\frac{BQ|\Gamma_{0,\mathrm{dB}}|}{27.29}$
1.2
1
0.8
0.6
0.4
0.25
0.3
0.35
0.4
0.45
0.5
ℓ/λ
ℓ
$\ell/2$
−13
−13
−12
−12
−12
−12
−12
−12
−6
−6
−6
−6
−4
−4
−4
−5
$\Gamma_{0,\mathrm{dB}}\approx-12$
$\Gamma_{0,\mathrm{dB}}\approx-5$
$B=4/(Q_{\mathrm{C}}+Q_{\mathrm{Z'}})$
$B=8/(Q_{\mathrm{C}}+Q_{\mathrm{Z'}})$
$Q=Q_{\mathrm{C}},Q_{\mathrm{Z_B}}$
$Q=Q_{\mathrm{Z'}}$

Dispersive media

The frequency derivative of the EFIE impedance matrix $\mathbf{Z}$ is

$$\omega\frac{\partial Z_{ij}}{\partial\omega} = k\frac{\partial(Z_{ij}/\eta)}{\partial k}\frac{\eta\omega}{k}\frac{\partial k}{\partial\omega} + \omega\frac{Z_{ij}}{\eta}\frac{\partial\eta}{\partial\omega}$$

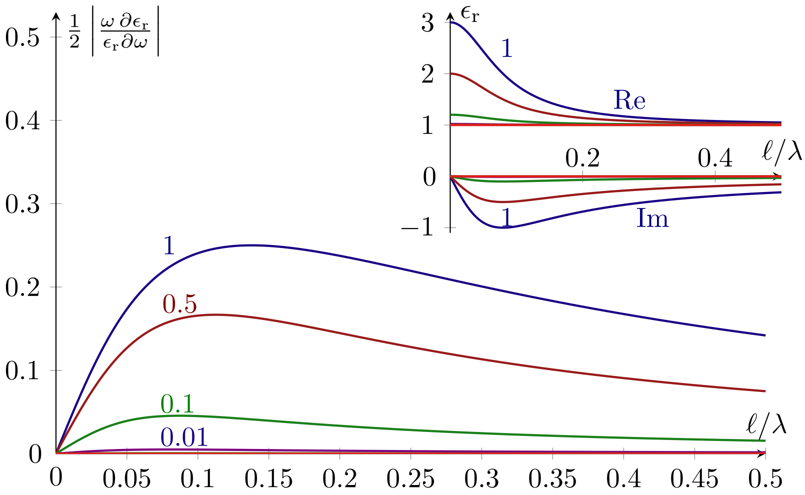
for a temporally dispersive background medium with $k = \omega\sqrt{\epsilon\mu}$ and $\eta = \sqrt{\mu/\epsilon}$. The derivative simplifies to

$$\omega\frac{\partial Z_{ij}}{\partial\omega} = k\frac{\partial(Z_{ij}/\eta)}{\partial k}\eta\left(\frac{\omega\partial\epsilon}{2\epsilon\partial\omega} + 1\right) - \frac{Z_{ij}}{2}\frac{\omega\partial\epsilon}{\epsilon\partial\omega}$$

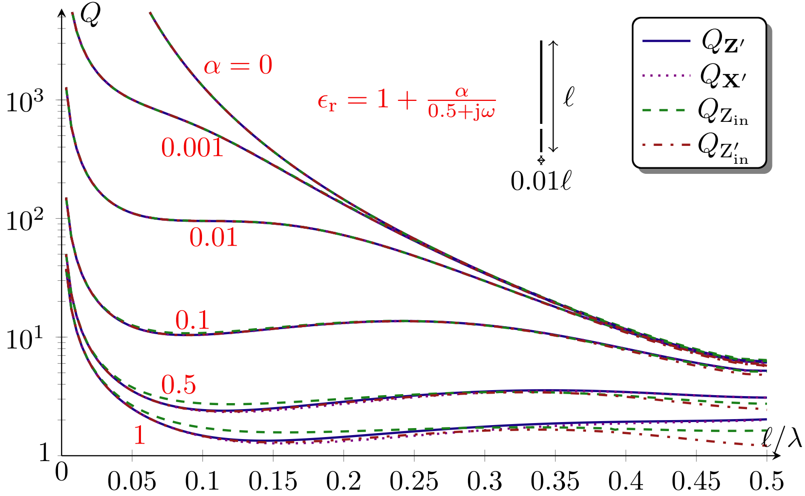
for the common case of a non-magnetic medium, $\mu_{\mathrm{r}} = 1$.

Multiplication of the previously calculated derivative (with respect to the wavenumber k in the medium) with a factor that only depends on the medium. The factor $\omega\epsilon' = (\omega\epsilon)' - \epsilon$ is similar to the classical approach used to define the energy density in dispersive media.

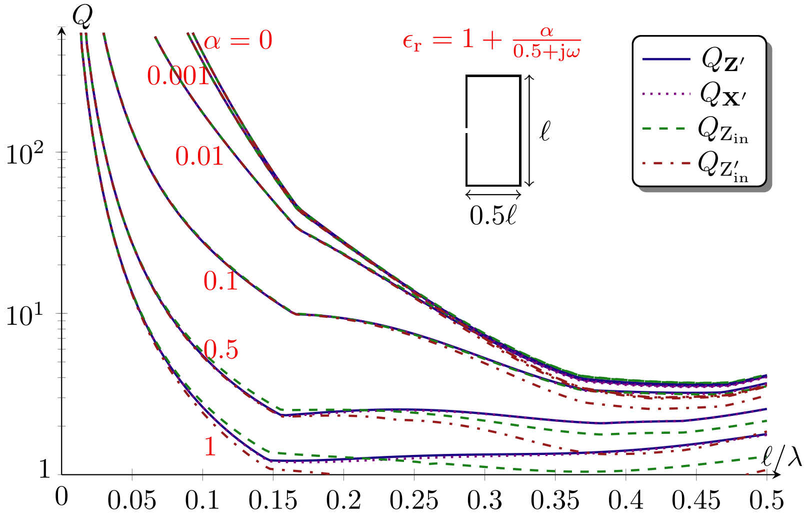
Numerical examples: Debye media



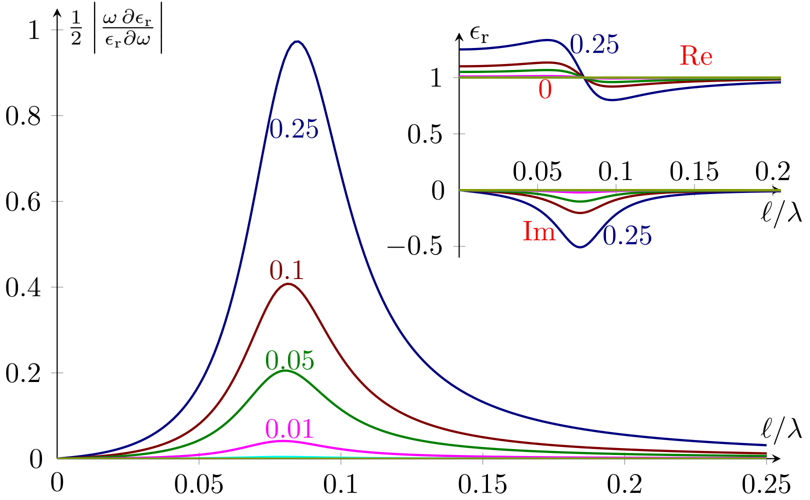
Numerical examples: Debye media



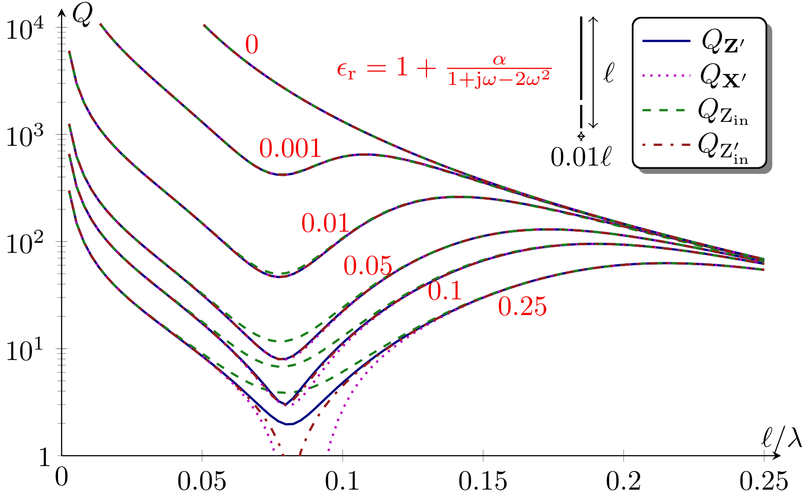
Numerical examples: Debye media



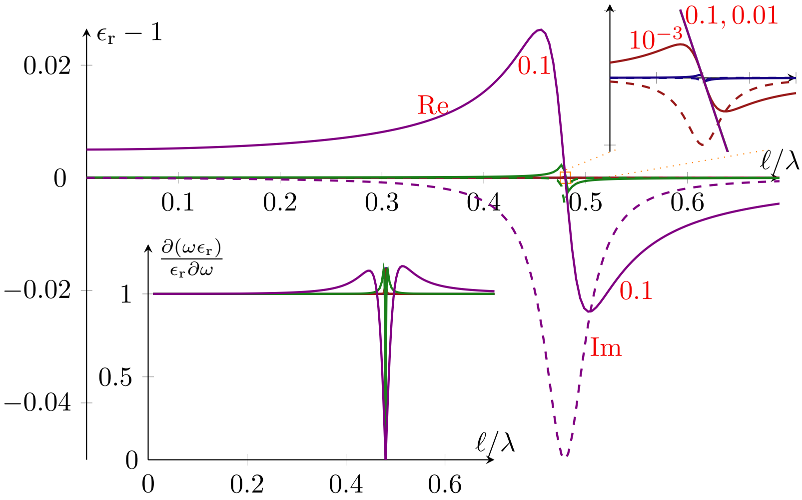
Numerical examples: Lorentz media



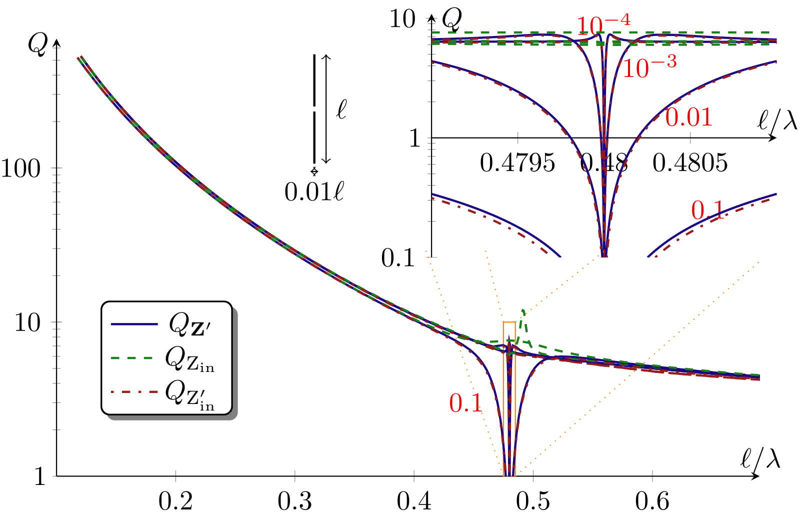
Numerical examples: Lorentz media



Numerical examples: Lorentz $\epsilon_r = \mu_r$ media



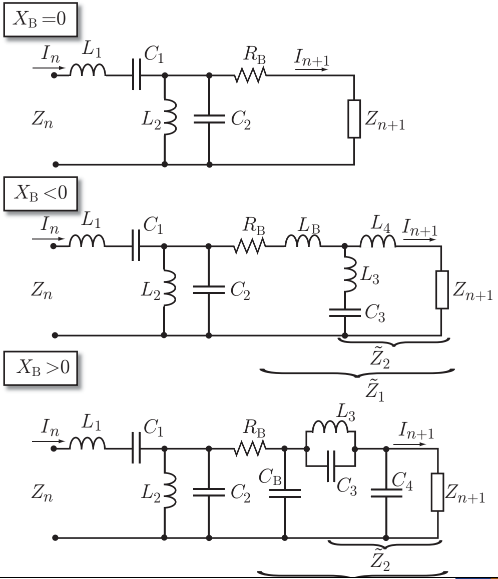
Numerical examples: Lorentz $\epsilon_r = \mu_r$ media



Brune synthesis

Iterative procedure to synthesize circuit models from PR (positive real rational functions) by Brune 1931.

1. Approximate the input impedance with a rational PR function (hard problem).
2. Apply Brune synthesis and compute the stored energy in the capacitors and inductors.

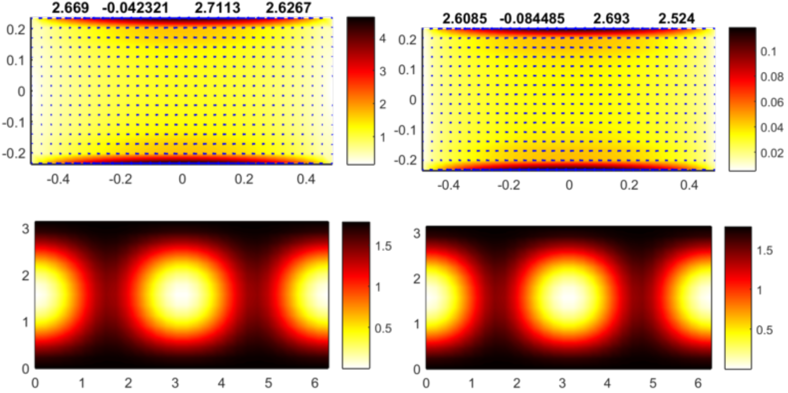


Outline

6. **Physical bounds**
   - Chu bound
   - Forward scattering
   - Polarizability dyadics
7. **Stored energy**
8. **Current optimization**

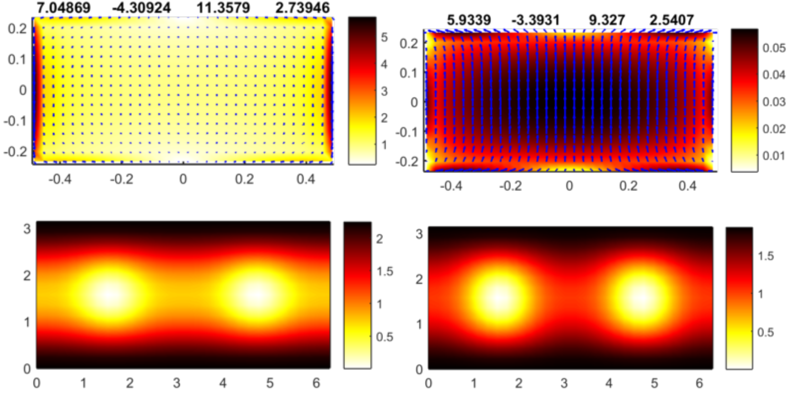
Characteristic modes and energy modes for a rectangle

$(Q_\mathrm{e}+Q_\mathrm{m})/2 \quad (Q_\mathrm{e}-Q_\mathrm{m})/2 \quad Q_\mathrm{e} \quad Q_\mathrm{m}$



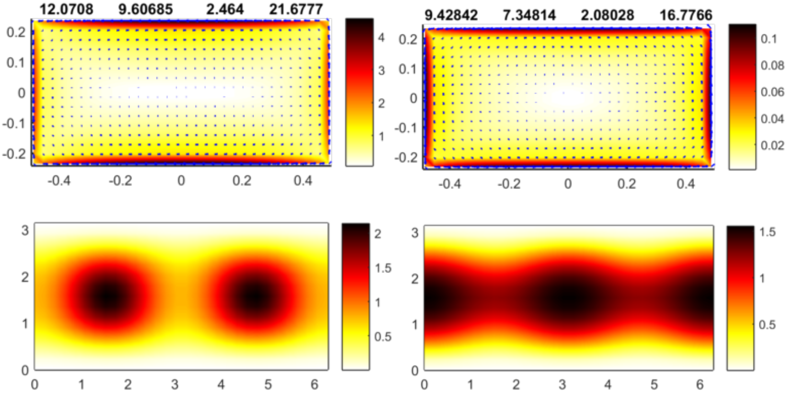
Characteristic modes and energy modes for a rectangle

$(Q_{\mathrm{e}}+Q_{\mathrm{m}})/2 \quad (Q_{\mathrm{e}}-Q_{\mathrm{m}})/2 \quad Q_{\mathrm{e}} \quad Q_{\mathrm{m}}$



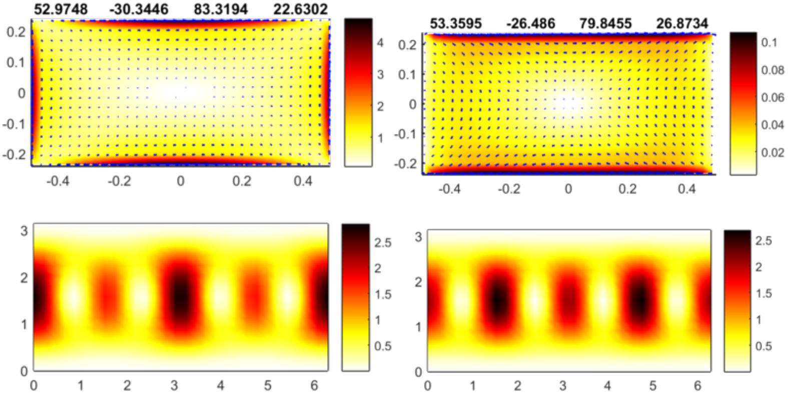
Characteristic modes and energy modes for a rectangle

$$(Q_\mathrm{e}+Q_\mathrm{m})/2 \quad (Q_\mathrm{e}-Q_\mathrm{m})/2 \quad Q_\mathrm{e} \quad Q_\mathrm{m}$$



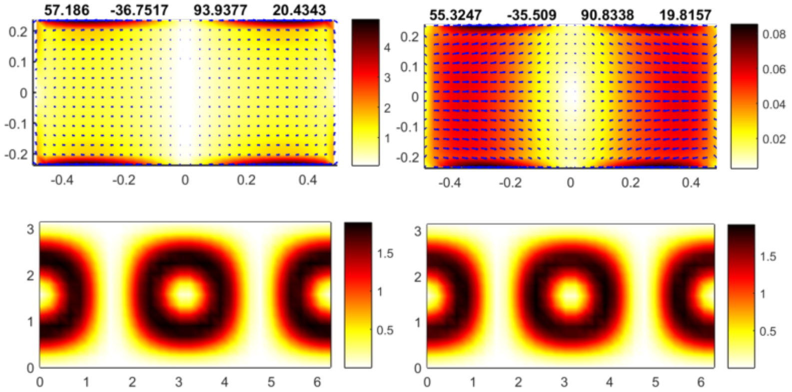
Characteristic modes and energy modes for a rectangle

$$(Q_\mathrm{e}+Q_\mathrm{m})/2 \quad (Q_\mathrm{e}-Q_\mathrm{m})/2 \quad Q_\mathrm{e} \quad Q_\mathrm{m}$$



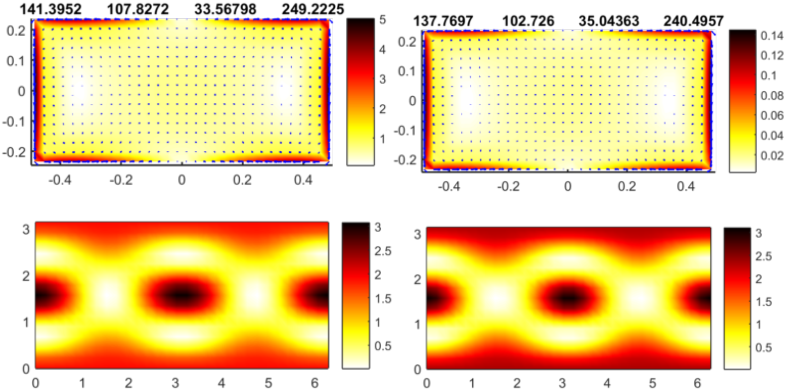
Characteristic modes and energy modes for a rectangle

$$(Q_{\mathrm{e}} + Q_{\mathrm{m}})/2 \quad (Q_{\mathrm{e}} - Q_{\mathrm{m}})/2 \quad Q_{\mathrm{e}} \quad Q_{\mathrm{m}}$$



Characteristic modes and energy modes for a rectangle

$$(Q_\mathrm{e}+Q_\mathrm{m})/2 \quad (Q_\mathrm{e}-Q_\mathrm{m})/2 \quad Q_\mathrm{e} \quad Q_\mathrm{m}$$



Minimization of Q

Compare maximization of G/Q with minimization of Q. Use the same inequality for $0 \leq \alpha \leq 1$

$$Q = \frac{\max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}} = \max\{Q_{\mathrm{e}}, Q_{\mathrm{m}}\}$$
$$\geq \alpha Q_{\mathrm{e}} + (1-\alpha)Q_{\mathrm{m}} = \frac{\mathbf{I}^{\mathsf{H}}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}})\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}}$$

The lower bound on Q, Q_{lb}, is a minimization problem for a Rayleigh quotient solved as a generalized eigenvalue problem

$$\min \mathrm{eig}(\alpha\mathbf{X}_{\mathrm{e}} + (1-\alpha)\mathbf{X}_{\mathrm{m}}, \mathbf{R}_{\mathrm{r}})$$

Let $Q_{\mathrm{e}\alpha}$ and $Q_{\mathrm{m}\alpha}$ denote the corresponding electric and magnetic Q-factors to get the estimate

$$\alpha Q_{\mathrm{e}\alpha} + (1-\alpha)Q_{\mathrm{m}\alpha} \leq Q_{\mathrm{lb}} \leq \max\{Q_{\mathrm{e}\alpha}, Q_{\mathrm{m}\alpha}\}$$

for the lower bound Q_{lb}.

Numerical illustration of $\min . Q$ and $\max . G/Q$

- The formulation for $\min . Q$ has a duality gap, *i.e.*, we have an interval for Q_{lb} here $88 \leq Q_{\text{lb}} \leq 106$.
- The optimization problem $\min . Q$ is not convex.
- The formulation for $\max . G/Q$ has no duality gap.
- This is common for many convex optimization problems.