An overview of stored electromagnetic energy

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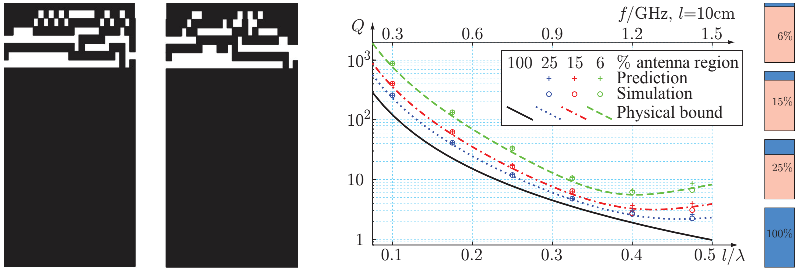
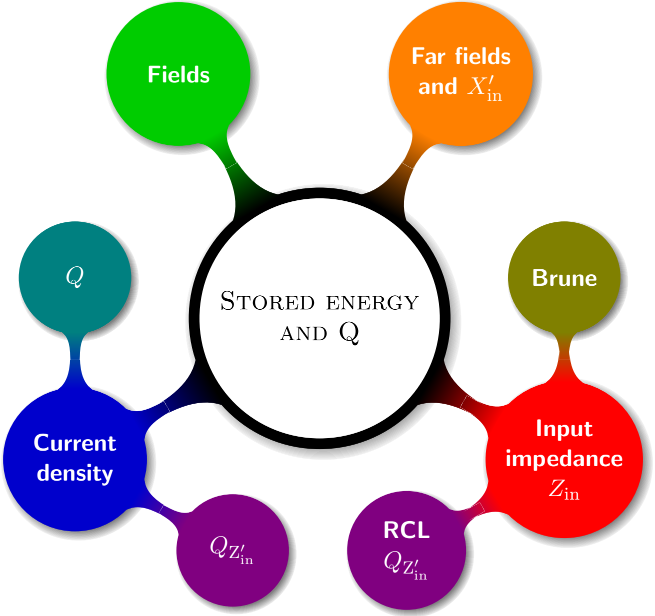
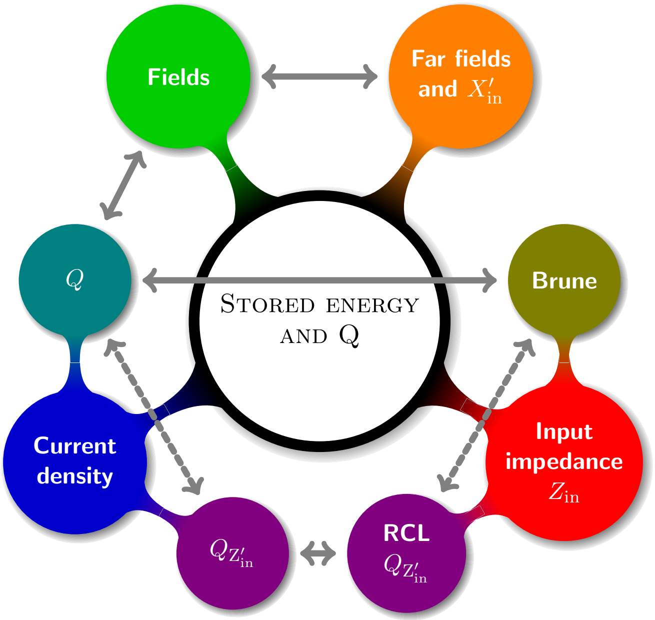
ICEAA, FIELDS AND WAVES, In Memoriam Professor Karl J. Langenberg, Aruba, August 6, 2014



- Single frequency antenna optimization, *e.g.*, minimize Q.
- Current optimization.
- Physical bounds.

Express the stored energy in the current density.


Fields
Far fields and X'_{in}
Q
Stored energy and Q
Brune
Current density
Input impedance Z_{in}
$Q_{Z'_{\mathrm{in}}}$
RCL $Q_{Z'_{\mathrm{in}}}$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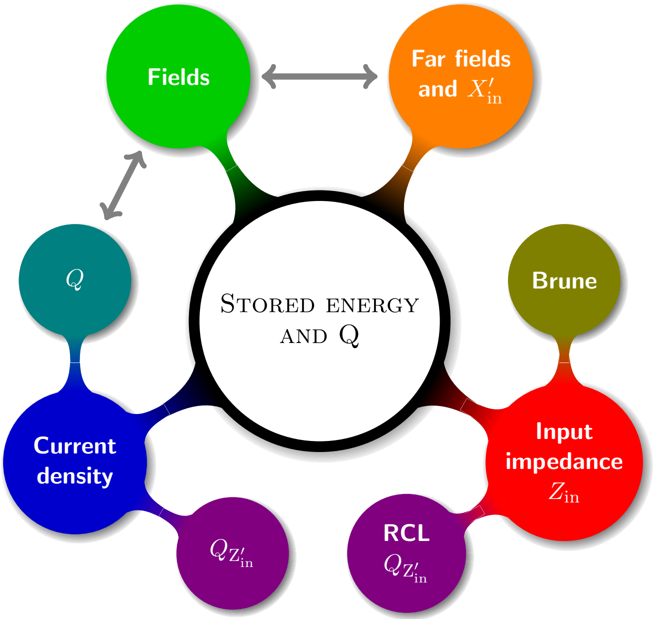
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Stored EM energy expressions (free space)

- Subtraction of the energy in the radiated field (far field $\boldsymbol{F}$) (Collin & Rothschild 1964, Yaghjian & Best 2005)

$$W_{\mathrm{F}}^{(\mathrm{E})} = \frac{\epsilon_0}{4} \int_{\mathbb{R}_{\mathrm{r}}^3} |\boldsymbol{E}(\boldsymbol{r})|^2 - \frac{|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2}{r^2} \,\mathrm{dV}$$

- Expressed in the frequency derivative of the reactance (Fante 1969, Yaghjian & Best 2005)

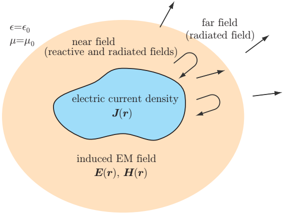
$$W_{\mathrm{F}}^{(\mathrm{E})} = \frac{|I_0|^2}{4} X_{\mathrm{in}}' - \frac{1}{2\eta_0} \operatorname{Im} \int_{\Omega} \boldsymbol{F}'(\hat{\boldsymbol{r}}) \cdot \boldsymbol{F}^*(\hat{\boldsymbol{r}}) \,\mathrm{d}\Omega$$

- In the current density (Vandenbosch 2010, see also Geyi 2003, Gustafsson & Jonsson 2012)

$$W_{\mathrm{C}}^{(\mathrm{E})} = \frac{\eta_0}{4\omega} \int_V \int_V \nabla_1 \cdot \boldsymbol{J}_1 \nabla_2 \cdot \boldsymbol{J}_2^* \frac{\cos(kr_{12})}{4\pi k r_{12}} - \left(k^2 \boldsymbol{J}_1 \cdot \boldsymbol{J}_2^* - \nabla_1 \cdot \boldsymbol{J}_1 \nabla_2 \cdot \boldsymbol{J}_2^*\right) \frac{\sin(kr_{12})}{8\pi} \,\mathrm{dV}_1 \,\mathrm{dV}_2$$

Subtracted far field approach

$$W_{\mathrm{F}}^{(\mathrm{E})} = \frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}(\boldsymbol{r})|^2 - \frac{|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2}{r^2}\,\mathrm{dV}$$



Have shown that $W_{\mathrm{F}}^{(\mathrm{E})} = W_{\mathrm{C}}^{(\mathrm{E})} + W_{\mathrm{c},\boldsymbol{0}}$:

$$W_{\mathrm{C}}^{(\mathrm{E})} = \frac{\eta_0}{4\omega}\int_V\int_V \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\frac{\cos(kr_{12})}{4\pi kr_{12}} - \left(k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^* - \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\right)\frac{\sin(kr_{12})}{8\pi}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

with $\boldsymbol{J}_n = \boldsymbol{J}(\boldsymbol{r}_n)$, $n=1,2$ and a **coordinate dependent** part

$$W_{\mathrm{c},\boldsymbol{0}} = \frac{\eta_0}{4\omega}\int_V\int_V \mathrm{Im}\left\{k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^* - \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\right\}\frac{r_1^2-r_2^2}{8\pi r_{12}}k\,\mathrm{j}_1(kr_{12})\,\mathrm{dV}_1\,\mathrm{dV}_2$$

where $\mathrm{j}_1(z) = (\sin(z) - z\cos(z))/z^2$ is a spherical Bessel function.

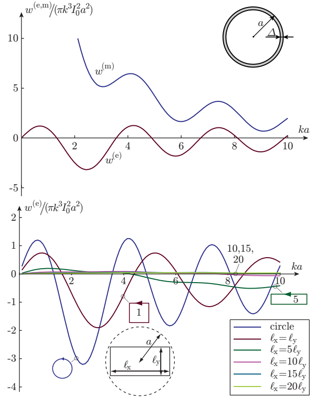
Gustafsson, Jonsson: Stored electromagnetic energy and antenna Q, 2012

Subtracted far field: comments

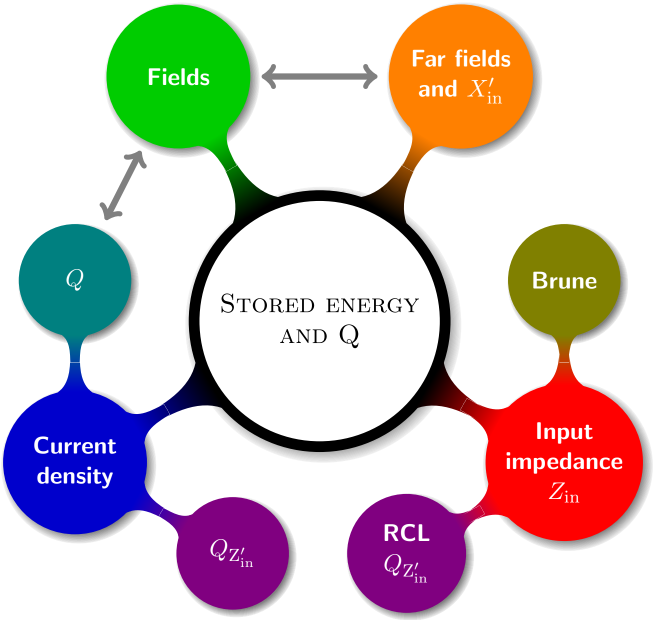
- Coordinate dependent for far-fields $\boldsymbol{F}$ with

$$W_{\mathrm{c},0} - W_{\mathrm{c},\boldsymbol{d}} = \frac{\epsilon_0}{4}\boldsymbol{d}\cdot\int_{\Omega}\hat{\boldsymbol{r}}|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2\,\mathrm{d}\Omega \neq 0$$

- Identical coordinate independent part as for the stored energy introduced by Vandenbosch 2010.
- Can produce negative values for lager structures.
- Difficult to generalize to antennas embedded in lossy media (no far field).



We now introduce an alternative approach to analyze antennas in lossy (dispersive) media.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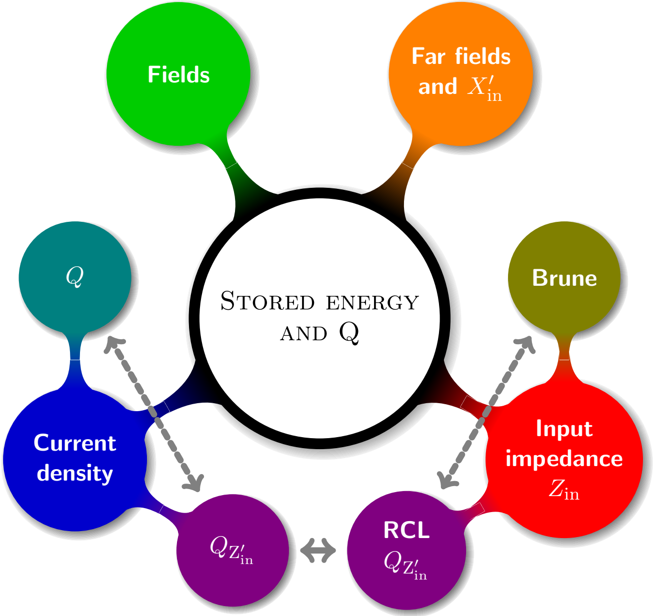

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Frequency derivatives of impedance/admittance matrices

The impedance and admittance matrices relate the voltages and currents

$$\mathbf{ZI} = \mathbf{V} \quad \text{or } \mathbf{I} = \mathbf{Z}^{-1}\mathbf{V} = \mathbf{YV}$$

The (angular) frequency derivative of the admittance matrix is

$$\mathbf{Y}' = \frac{\partial \mathbf{Y}}{\partial \omega} = \frac{\partial \mathbf{Z}^{-1}}{\partial \omega} = -\mathbf{Z}^{-1}\mathbf{Z}'\mathbf{Z}^{-1} = -\mathbf{YZ}'\mathbf{Y}$$

Note there are no complex conjugates. Hence, better to use quadratic forms with the transpose $\mathbf{V}^{\mathrm{T}}\mathbf{Y}'\mathbf{V}$ than Hermitian transpose $\mathbf{V}^{\mathsf{H}}\mathbf{Y}'\mathbf{V} = \mathbf{V}^{\mathrm{T}*}\mathbf{Y}'\mathbf{V}$.

For the case of a voltage source (frequency independent)

$$Y_{\mathrm{in}} = \frac{1}{Z_{\mathrm{in}}} = \frac{\mathbf{V}^{\mathrm{T}}\mathbf{YV}}{V_{\mathrm{in}}^2} \quad \text{and } V_{\mathrm{in}}^2 Y_{\mathrm{in}}' = \mathbf{V}^{\mathrm{T}}\mathbf{Y}'\mathbf{V} = -\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}.$$

Lumped circuits

Consider a voltage source and use the Kirchoffs' laws to construct the linear system $\mathbf{Z}\mathbf{I} = \mathbf{V}$, where the impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$ contains elements of the form

$$Z_{ij} = R_{ij} + \mathrm{j}X_{ij} = R_{ij} + \mathrm{j}\left(\omega L_{ij} - \frac{1}{\omega C_{ij}}\right)$$

The differentiated impedance matrix $\mathbf{Z}' = \mathrm{j}\mathbf{X}'$ is imaginary valued with the elements

$$X'_{ij} = \frac{\partial}{\partial \omega}\left(\omega L_{ij} - \frac{1}{\omega C_{ij}}\right) = L_{ij} + \frac{1}{\omega^2 C_{ij}}.$$

Differentiated input admittance and impedance

$$Y'_{\mathrm{in}} = -\mathrm{j}\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}/V_{\mathrm{in}}^2 \quad \text{and } Z'_{\mathrm{in}} = -Z_{\mathrm{in}}^2 Y'_{\mathrm{in}} = \mathrm{j}\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}/I_{\mathrm{in}}^2$$

Energy in lumped circuits

Time average stored energy in capacitors and in inductors

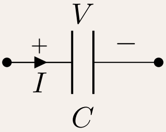
$$W^{(\mathrm{E})} = \frac{C|V|^2}{4} = \frac{|I|^2}{4\omega^2 C} \quad \text{and } W^{(\mathrm{M})} = \frac{L|I|^2}{4}$$

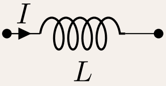
Contain absolute values $|I|^2$ and $|V|^2$, need to use Hermitian transpose. For a circuit network

$$W^{(\mathrm{M})} - W^{(\mathrm{E})} = \frac{\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I}}{4\omega} \quad \text{and } W^{(\mathrm{E})} + W^{(\mathrm{M})} = \frac{\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}}{4} \geq 0$$

reactance $\mathbf{X}$ for difference $W^{(\mathrm{M})} - W^{(\mathrm{E})}$ and differentiated reactance $\mathbf{X}'$ the sum $W^{(\mathrm{M})} + W^{(\mathrm{E})}$.

Energy in lumped circuits





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Contain absolute values $|I|^2$ and $|V|^2$, need to use Hermitian transpose. For a circuit network

$$W^{(\mathrm{M})} = \frac{1}{8}\mathbf{I}^{\mathsf{H}}\left(\frac{\partial \mathbf{X}}{\partial \omega} + \frac{\mathbf{X}}{\omega}\right)\mathbf{I} = \frac{1}{4}\sum_{i,j=1}^{N} I_i^* L_{ij} I_j \geq 0$$

$$W^{(\mathrm{E})} = \frac{1}{8}\mathbf{I}^{\mathsf{H}}\left(\frac{\partial \mathbf{X}}{\partial \omega} - \frac{\mathbf{X}}{\omega}\right)\mathbf{I} = \frac{1}{4\omega^2}\sum_{i,j=1}^{N} I_i^* C_{ij}^{-1} I_j \geq 0,$$

Q and $Q_{Z'_{\mathrm{in}}}$ for lumped circuits

Assume for simplicity a **self-resonant** circuit (antenna)

$$Q_{Z'_{\mathrm{in}}} = \frac{\omega |Z'_{\mathrm{in}}|}{2R_{\mathrm{in}}} = \frac{\omega |\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{2\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

and

$$Q = \frac{2\omega \max\{W^{(\mathrm{E})}, W^{(\mathrm{M})}\}}{P_{\mathrm{d}}} = \frac{\omega \mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}}{2\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

Transpose for $Q_{Z'_{\mathrm{in}}}$ and Hermitian transpose for Q

Also the inequality $Q \geq Q_{Z'_{\mathrm{in}}}$ as ($\mathbf{X}' = \mathbf{U}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{U}$ real valued)

$$\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I} = (\mathbf{U}\mathbf{I})^{\mathsf{H}}\mathbf{\Lambda}\mathbf{U}\mathbf{I} \geq \left|(\mathbf{U}\mathbf{I})^{\mathrm{T}}\mathbf{\Lambda}\mathbf{U}\mathbf{I}\right| = \left|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}\right| \geq 0$$

with equality (to 0) for some current $\mathbf{I}$ (in the matrix case).

Z_{in} for antennas using MoM

Use a method of moments (MoM) formulation of the electric field integral equation (EFIE). Impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$

$$\frac{Z_{ij}}{\eta} = \mathrm{j}\int_V\int_V \left(k^2\boldsymbol{\psi}_{i1}\cdot\boldsymbol{\psi}_{j2} - \nabla_1\cdot\boldsymbol{\psi}_{i1}\nabla_2\cdot\boldsymbol{\psi}_{j2}\right)\frac{\mathrm{e}^{-\mathrm{j}kR_{12}}}{4\pi kR_{12}}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

where $\boldsymbol{\psi}_i(\boldsymbol{r}_n)$ with $i = 1, ..., N$ and $n = 1, 2$ and $R_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$.
The current density is $\boldsymbol{J}(\boldsymbol{r}) = \sum_{i=1}^{N} I_i\boldsymbol{\psi}_i(\boldsymbol{r})$ with the expansion coefficients determined from

$$\mathbf{ZI} = \mathbf{V} \quad \text{or } \mathbf{I} = \mathbf{Z}^{-1}\mathbf{V} = \mathbf{YV}$$

where $\mathbf{V}$ is the column matrix with excitation coefficients and $\mathbf{Y} = \mathbf{G} + \mathrm{j}\mathbf{B}$ is the admittance matrix.
The input admittance, $Y_{\text{in}} = G_{\text{in}} + \mathrm{j}B_{\text{in}} = Z_{\text{in}}^{-1}$, is

$$Y_{\text{in}} = 1/Z_{\text{in}} = \mathbf{V}^{\mathrm{T}}\mathbf{YV}/V_{\text{in}}^2$$

where $Z_{\text{in}} = R_{\text{in}} + \mathrm{j}X_{\text{in}}$ is the input impedance.

$Q_{Z'_{\text{in}}}$ and Q for antennas (fields)

Differentiate the MoM impedance matrix

$$\frac{k\,\partial Z_{ij}}{\eta\,\partial k} = \int_V\int_V \mathrm{j}\left(k^2\boldsymbol{\psi}_{i1}\cdot\boldsymbol{\psi}_{j2} + \nabla_1\cdot\boldsymbol{\psi}_{i1}\nabla_2\cdot\boldsymbol{\psi}_{j2}\right)\frac{\mathrm{e}^{-\mathrm{j}kR_{12}}}{4\pi kR_{12}} + \left(k^2\boldsymbol{\psi}_{i1}\cdot\boldsymbol{\psi}_{j2} - \nabla_1\cdot\boldsymbol{\psi}_{i1}\nabla_2\cdot\boldsymbol{\psi}_{j2}\right)\frac{\mathrm{e}^{-\mathrm{j}kR_{12}}}{4\pi}\,\mathrm{dV}_1\,\mathrm{dV}_2$$

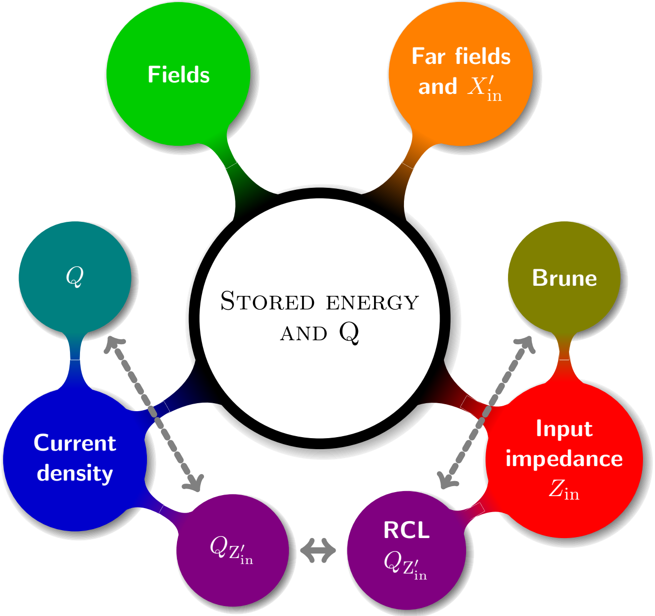
As for the lumped circuit case

$$V_{\text{in}}^2 Y'_{\text{in}} = (\mathbf{V}^{\mathrm{T}}\mathbf{Y}\mathbf{V})' = \mathbf{V}^{\mathrm{T}}\mathbf{Y}'\mathbf{V} = -\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}.$$

and the stored energy determined from $\mathbf{X}'$

$$W_{\mathrm{e}\mathbf{X}'} + W_{\mathrm{m}\mathbf{X}'} = \frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I}$$

is identical to the stored energy expressions introduced by Vandenbosch 2010.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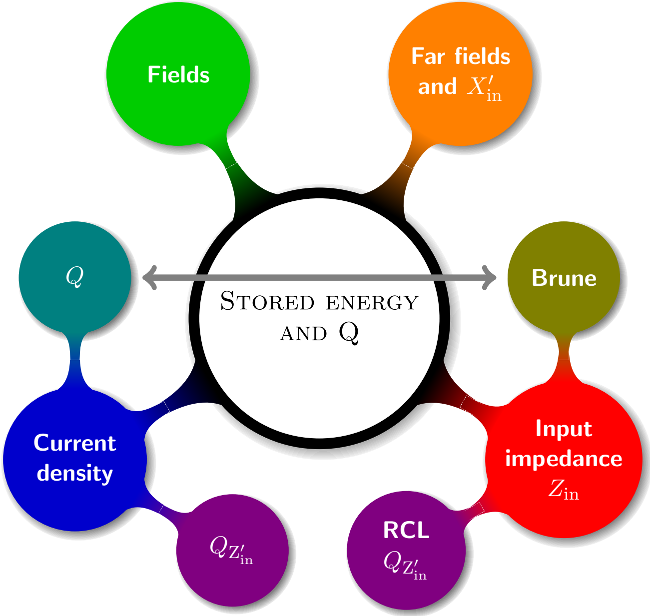
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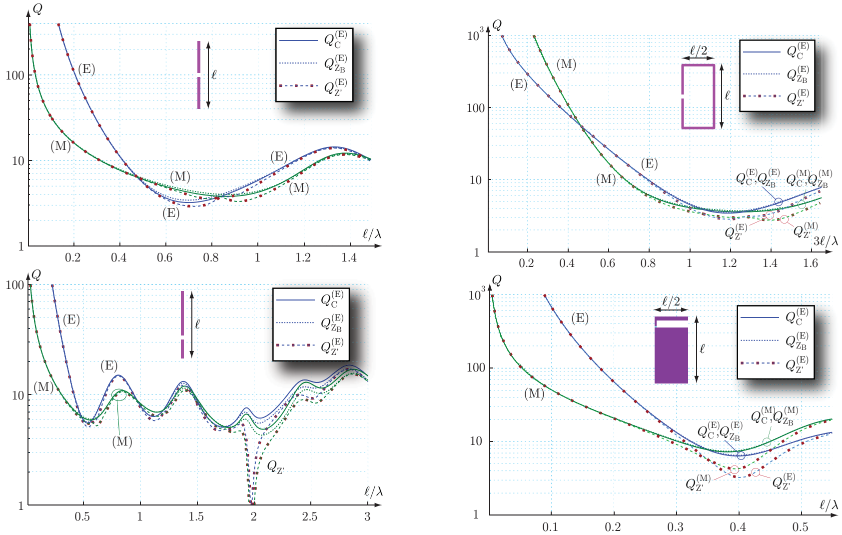
Antenna examples (free space)

Q from stored energy expressed in the current density Q_{C}, circuits $Q_{\mathrm{Z_B}}$, and differentiated impedance $Q_{\mathrm{Z'}}$



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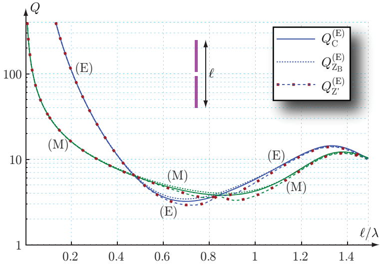
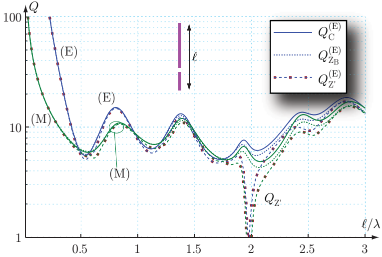


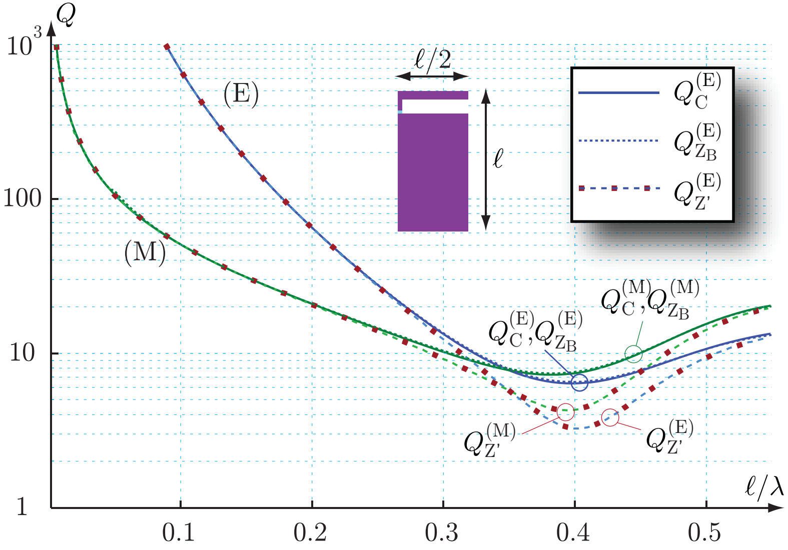


Q computed from

- the currents, Q_{C}.
- a circuit model synthesized from the input impedance using Brune synthesis (1931), $Q_{\mathrm{Z_B}}$.
- differentiation of the (tuned) input impedance,

$$Q_{\mathrm{Z}'_{\mathrm{in}}} = \frac{\omega_0 |Z'_{\mathrm{in}}|}{2R_{\mathrm{in}}} = \omega_0 |\Gamma'|.$$

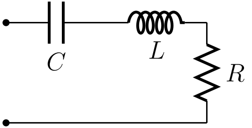
All agree for $Q \gg 1$ but the Q from the differentiated impedance ($Q_{\mathrm{Z}'_{\mathrm{in}}}$) is lower in some regions.
Which one is most accurate/best?



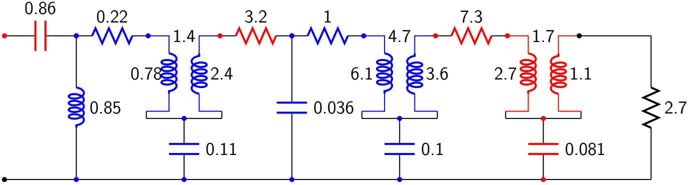
Gustafsson, Jonsson, 'Stored Electromagnetic Energy and Antenna Q', 2012.

Stored energy from circuit models

Resonance circuits Padé (local) approximation around the resonance frequency (also an all-pass filter), *cf.*, $Q_{\mathrm{Z}'} = \frac{\omega_0 |Z'|}{2R} = \omega_0 |\Gamma'|$



Brune synthesis Brune (1931) synthesized circuit from the input impedance. The negative quantities are replaced by ideal transformers. Here Q-factor $Q_{\mathrm{Z_B}}$



Dispersive media

The frequency derivative of the EFIE impedance matrix $\mathbf{Z}$ is

$$\omega\frac{\partial Z_{ij}}{\partial\omega} = k\frac{\partial(Z_{ij}/\eta)}{\partial k}\frac{\eta\omega}{k}\frac{\partial k}{\partial\omega} + \omega\frac{Z_{ij}}{\eta}\frac{\partial\eta}{\partial\omega}$$

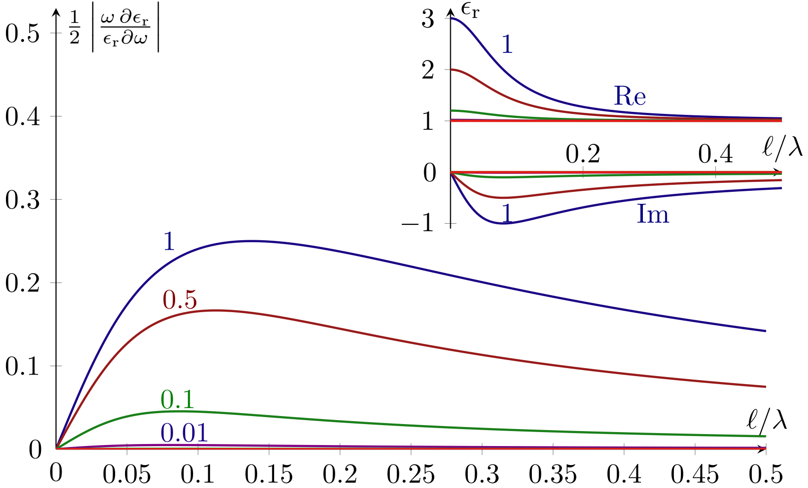
for a temporally dispersive background medium with $k = \omega\sqrt{\epsilon\mu}$ and $\eta = \sqrt{\mu/\epsilon}$. The derivative simplifies to

$$\omega\frac{\partial Z_{ij}}{\partial\omega} = k\frac{\partial(Z_{ij}/\eta)}{\partial k}\eta\left(\frac{\omega\partial\epsilon}{2\epsilon\partial\omega} + 1\right) - \frac{Z_{ij}}{2}\frac{\omega\partial\epsilon}{\epsilon\partial\omega}$$

for the common case of a non-magnetic medium, $\mu_{\mathrm{r}} = 1$.

Multiplication of the previously calculated derivative (with respect to the wavenumber k in the medium) with a factor that only depends on the medium. The factor $\omega\epsilon' = (\omega\epsilon)' - \epsilon$ is similar to the classical approach used to define the energy density in dispersive media.

Numerical examples: Debye media



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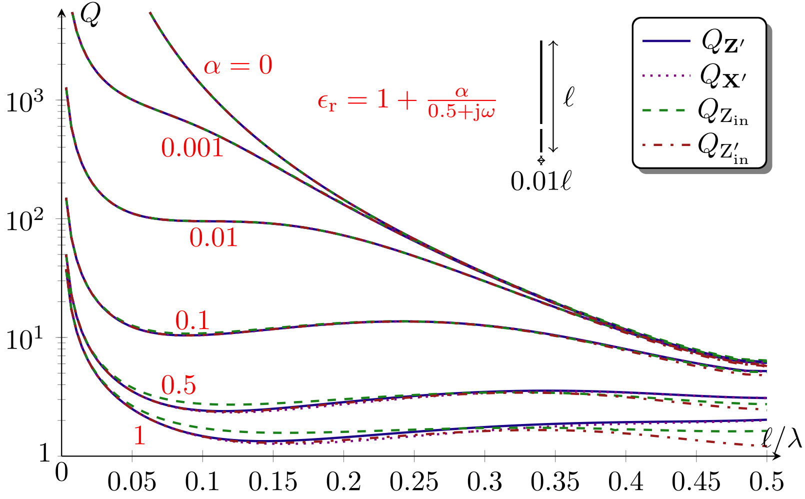

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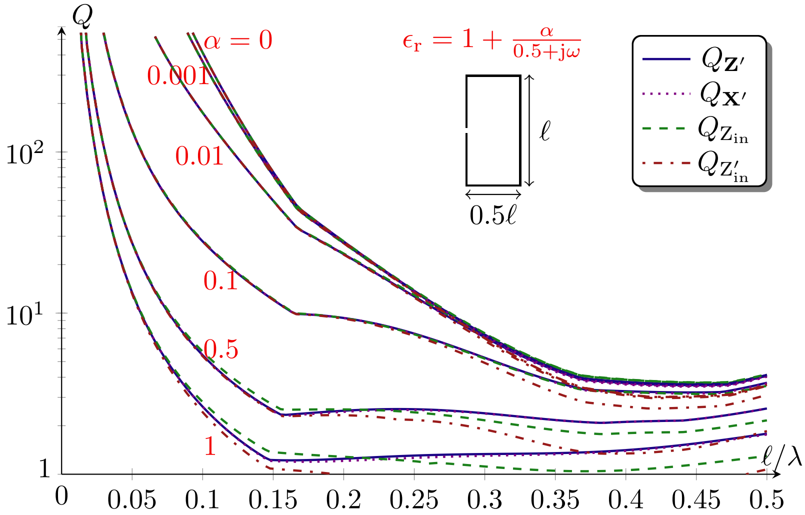
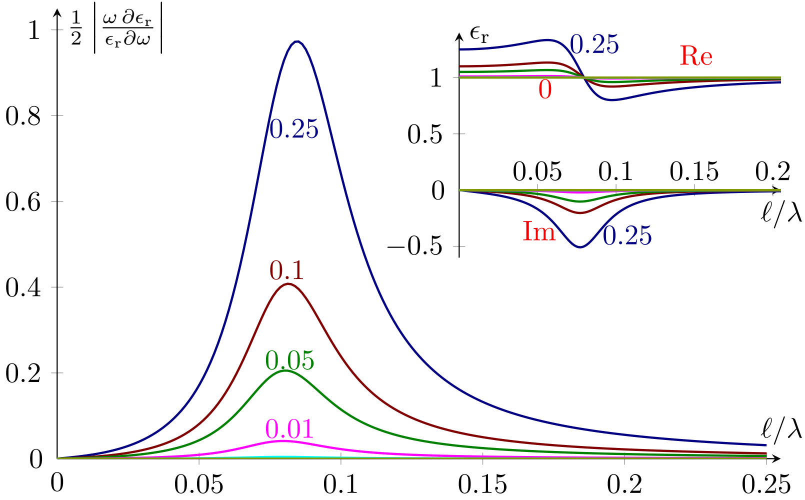


Numerical examples: Lorentz media



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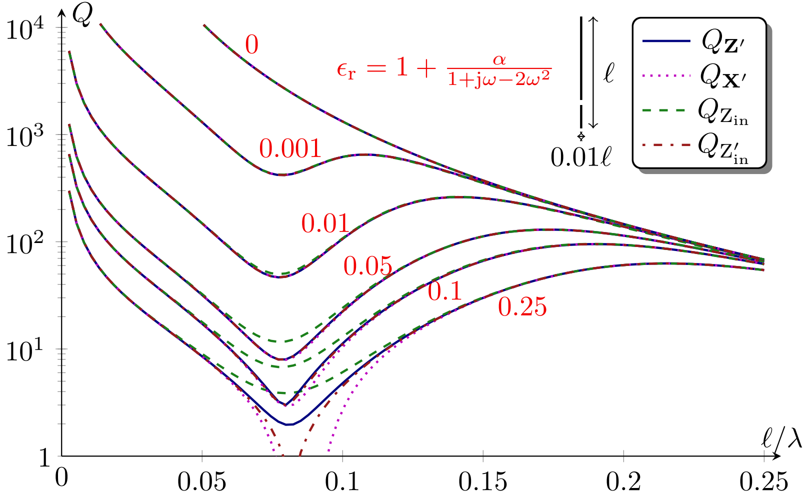
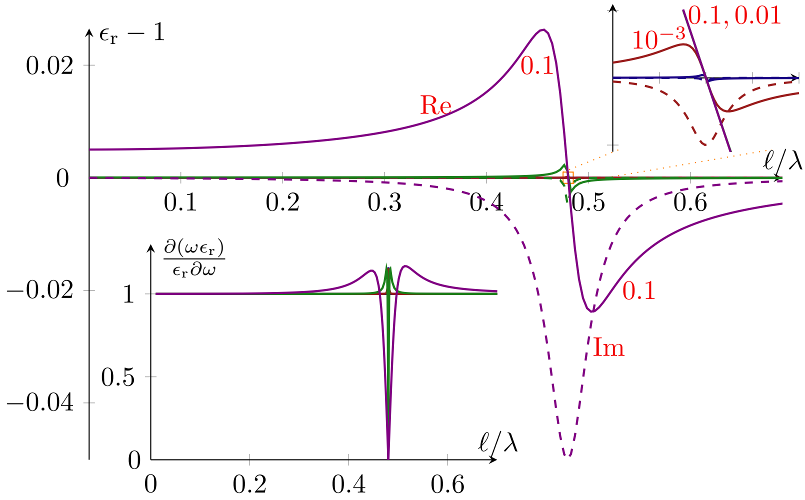


Numerical examples: Lorentz $\epsilon_r = \mu_r$ media



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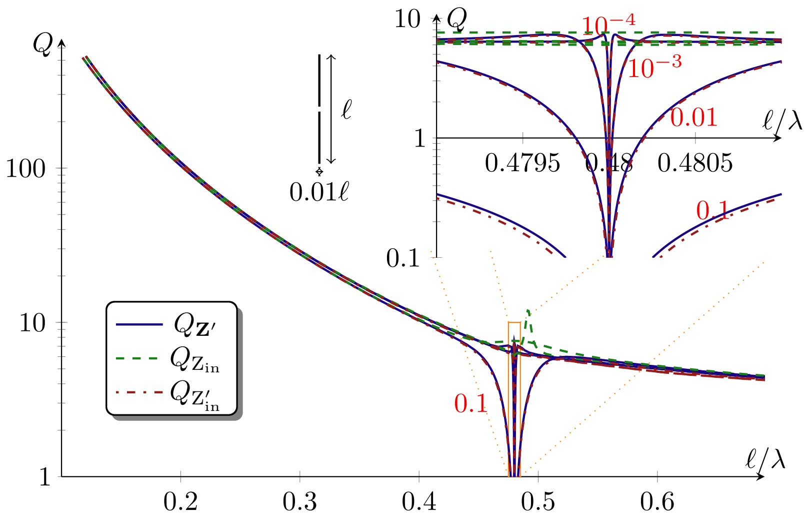

Summary: Stored EM energies

- Introduced by Vandenbosch in *Reactive energies, impedance, and Q factor of radiating structures*, IEEE-TAP 2010.
- In the limit $ka \to 0$ by Geyi, IEEE-TAP 2003 and also similar expressions by Carpenter 1989.
- Verification for wire antennas in Hazdra *etal*, IEEE-AWPL 2011.
- Some issues with 'negative stored energy' for large structures in Gustafsson *etal*, IEEE-TAP 2012. See also Gustafsson and Jonsson, *Stored Electromagnetic Energy and Antenna Q*, 2012.
- Time-domain version by Vandenbosch 2013.
- $Q_{Z'_{\mathrm{in}}}$ formulation by Capek *etal*, IEEE-TAP 2014.

One of the most powerful new tools in EM and antenna theory. Still many open questions and probably no consensus (yet).

- How do we interpret the stored energy? Subtracted far-field...
- How do we verify the expressions? Circuit models (Brune), unique,...
- Dialectics, losses, ... There are some suggestions and initial results..

Q-factor and stored energy

- The Q-factor for a tuned antenna is

$$Q = \max\{Q^{(\mathrm{E})}, Q^{(\mathrm{M})}\}, \quad Q^{(\mathrm{E})} = \frac{2\omega W^{(\mathrm{E})}}{P_\mathrm{r}}, \quad Q^{(\mathrm{M})} = \frac{2\omega W^{(\mathrm{M})}}{P_\mathrm{r}}$$

  and $W^{(\mathrm{E})}$ is the stored electric energy, $W^{(\mathrm{M})}$ the stored magnetic energy, and P_r the dissipated (radiated for a loss-less antenna) power.
- Fractional bandwidth for single resonance circuits

$$B = \frac{\omega_2 - \omega_1}{\omega_0} \approx \frac{2\Gamma_0}{Q\sqrt{1-\Gamma_0^2}},$$

  where $\omega_0 = (\omega_1 + \omega_2)/2$ and Γ_0 is the threshold of the reflection coefficient.
- The Fano limit for a single resonance circuit, $B \leq 27.29/(Q|\Gamma_{0,\mathrm{dB}}|)$, is an upper bound on the bandwidth after matching.

Brune synthesize

Iterative procedure to synthesize circuit models from PR (positive real rational functions) by Brune 1931.

1. Approximate the input impedance with a rational PR function (hard problem).
2. Apply Brune synthesis and compute the stored energy in the capacitors and inductors.