Mathematical modeling of causal signals and passive systems in electromagnetics

Mats Gustafsson

Department of Electrical and Information Technology

Lund University, Sweden

Outline

Motivation

LTI (linear time-translational and causal) systems are often used in modeling. These models are often too extensive and we use additional basic physical assumptions such as finite energy, causality, and passivity to restrict them. Here, we review and discuss

- what assumptions such as finite energy (L^2), causality, and passivity imply for systems and signals.
- when L^2 and causality are appropriate assumptions.
- when passivity is a good assumption.
- representation theorems for causal signals and passive systems.
- applications such as sum rules for passive systems.

Background

There is a considerable amount of literature on modeling of signals and systems. Some relevant references among many are

Circuit networks: Guillemin *Synthesis of passive networks* (1957) [9], *The Mathematics of Circuit Analysis* (1949) [8], Wing *Classical Circuit Theory* (2008) [27].

Analytic functions: Greene and Krantz *Function Theory of One Complex Variable* (2006) [7], Garnett *Bounded Analytic Functions* (2007) [6].

Physics: Nussenzveig *Causality and dispersion relations* (1972) [22], Jackson *Classical Electrodynamics* (1975) [17], Landau & Lifshitz *Electrodynamics of Continuous Media* (1984) [21].

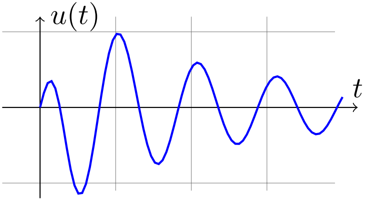
LTI systems: .

Functional analysis: Kreyszig *Introductory functional analysis with applications* (1978) [20].

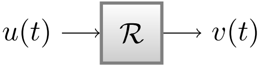
Transforms: Zemanian *Distribution Theory and Transform Analysis* (1987) [31], King *Hilbert Transforms, vol I,II* (2009) [18, 19], Widder *The Laplace Transform* (1946) [26].

Mathematical modeling of causal signals and passive systems

Signal



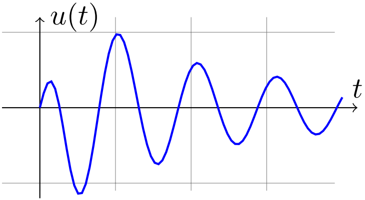
System



- Mathematical tools such as Fourier, Laplace, and Hilbert transforms and complex analysis are used to analyze and model signals and systems.
- Assumptions based on physical principles such as linearity, causality, stability, and passivity restrict the models.

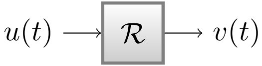
Here, we discuss and review the basic physical assumptions and the corresponding mathematical tools. We also discuss similarities and differences between signals and systems.

Signals and systems: assumptions and applications



Finite energy and causality are often used for signals. Finite energy gives $u \in \mathrm{L}^2$ and causality gives analyticity. **Applications:**

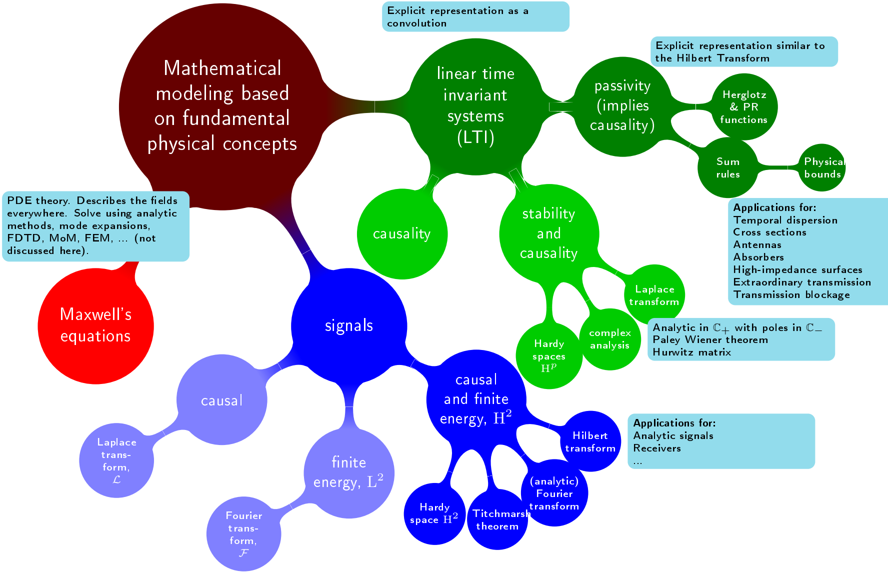
- analytic signals
- modulation
- receivers



Input output models of systems are often based on linearity, time (translational) invariance, and continuity (LTI). In addition we often use assumptions of causality, stability, and passivity. **Applications:**

- material modeling
- antenna input impedance
- reflection and transmission coefficients
- scattering parameters

Mathematical modeling of signals and LTI systems



Outline

Signals with finite energy

Definition (Finite energy (L^2))

A signal $u(t)$ has finite energy if it is square integrable

$$\mathcal{W} = \int_{-\infty}^{\infty} |u(t)|^2 \,\mathrm{d}t = \|u\|_2^2 < \infty$$

- Mathematically the equivalence class of L^2-functions $u \in \mathrm{L}^2(\mathbb{R})$.
- Fourier transform for $\omega \in \mathbb{R}$

$$U(\omega) = \mathcal{F}\{u(t)\}(\omega) = \lim_{T\to\infty} \int_{-T}^{T} \mathrm{e}^{\mathrm{i}\omega t} u(t) \,\mathrm{d}t$$

  There are many conventions for the Fourier transform, here we use the one corresponding to the time convention $\mathrm{e}^{-\mathrm{i}\omega t}$.
- Plancherel, Parseval's theorem, $\|u\|_2^2 = \frac{1}{2\pi}\|U\|_2^2$.

Causal signals with finite energy

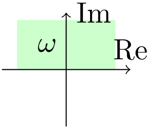
Definition (Causal signals)

A signal $u(t)$ is causal if $u(t) = 0$ for $t < 0$.

Properties ($u \in \mathrm{L}^2$ and $u(t) = 0$ for $t < 0$)

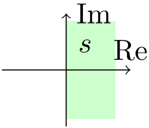
- Fourier transform analytic for $\operatorname{Im}\omega > 0$ and L^2 for $\operatorname{Im}\omega = 0$

$$U(\omega) = \mathcal{F}\{u(t)\}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega t} u(t)\,\mathrm{d}t$$



- Laplace transform (with slight abuse of notation, we use $U(s)$)

$$U(s) = \mathcal{L}\{u(t)\}(s) = \int_{0^-}^{\infty} \mathrm{e}^{-st} u(t)\,\mathrm{d}t$$



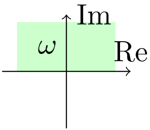
analytic for $\operatorname{Re} s > 0$. We use $s = \sigma + \mathrm{j}\omega$ with $\mathrm{j} = -\mathrm{i}$.

Upper, lower, and right half planes

The Laplace and (analytic) Fourier transform are in principle identical for causal signals. There are however some technical differences and there are several common versions of the Fourier transform, *e.g.*, different placements of 2π, signs $\pm\mathrm{i}$, and $\mathrm{j} = -\mathrm{i}$:

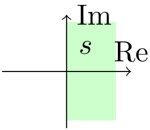
$$U(\omega) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega t} u(t)\,\mathrm{d}t$$

Fourier transform
analytic for $\operatorname{Im}\omega > 0$



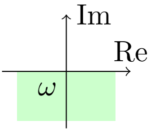
$$U(s) = \int_{0^-}^{\infty} \mathrm{e}^{-st} u(t)\,\mathrm{d}t$$

Laplace transform
analytic for $\operatorname{Re} s > 0$



$$U(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega t} u(t)\,\mathrm{d}t$$

Fourier transform
analytic for $\operatorname{Im}\omega < 0$



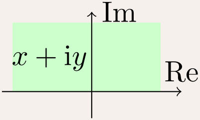
The same notation for the transformed function is used for simplicity. The variables ω and $s = \mathrm{j}\omega + \sigma$ differentiate them when necessary.

Hardy space H^2

Definition (Hardy space H^2)

The Hardy space H^2 is the space of holomorphic functions in the upper (or right) complex half plane with the norm

$$\|U\|_{\mathrm{H}^2} = \sup_{y>0} \left(\int_{\mathbb{R}} |U(x+\mathrm{i}y)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}$$



- Boundary values $U \in \mathrm{L}^2$ on the real axis.
- Hilbert transform 68 to relate the real and imaginary parts of $U(x)$.
- Also for the unit disk and other powers p, H^p, 67.

Godfrey Harold Hardy 1877-1947

Frigyes Riesz 1880-1956

Theorem (Titchmarsh's theorem)

If a square integrable function $U(\omega)$ *(*$U(\omega)\in\mathrm{L}^2$*) fulfills one of the conditions below it fulfills all of them:*

- *The inverse Fourier transform* $u(t)=\mathcal{F}^{-1}\{U(\omega)\}(t)=0$ *for* $t<0$*.*
- *The real and imaginary parts are related by the Hilbert transform (see also Sokhotski-Plemelj formulas)*

$$\mathrm{Re}\{U(\omega)\}=-\mathcal{H}\,\mathrm{Im}\{U\}(\omega)=-\frac{1}{\pi}\int_{\mathbb{R}}\frac{\mathrm{Im}\{U(\omega')\}}{\omega-\omega'}\,\mathrm{d}\omega'$$

$$\mathrm{Im}\{U(\omega)\}=\mathcal{H}\,\mathrm{Re}\{U\}(\omega)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{\mathrm{Re}\{U(\omega')\}}{\omega-\omega'}\,\mathrm{d}\omega'$$

- *The function* $U(\nu)$ *is holomorphic in* $\nu=\omega+\mathrm{i}\xi$ *for* $\xi>0$*. Furthermore, there is a constant* C *such that*

$$\int_{\mathbb{R}}|U(\omega+\mathrm{i}\xi)|^2\,\mathrm{d}\omega<C\quad\textit{for all}\ \ \xi>0$$

and $U(\omega)=\lim_{\xi\to 0^+}U(\omega+\mathrm{i}\xi)$ *for almost all* $\omega\in\mathbb{R}$

Edward Charles Titchmarsh
1899-1963

INTRODUCTION TO THE
THEORY OF
FOURIER INTEGRALS

E. C. TITCHMARSH

OXFORD
AT THE CLARENDON PRESS

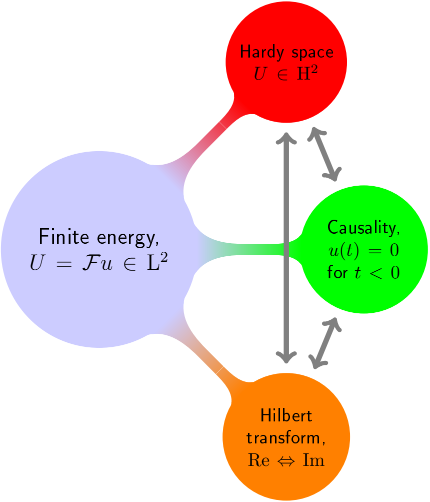
Titchmarsh (1948) [25], Nussenzveig (1972) [22], King (2009) [18]

Titchmarsh's theorem II

Titchmarsh's theorem shows that the conditions

- causality of $u(t)$
- Hilbert transform relations $\operatorname{Re} U = -\mathcal{H} \operatorname{Im} U$ and $\operatorname{Im} U = \mathcal{H} \operatorname{Re} U$.
- $U \in \mathrm{H}^2$ (Hardy space).

are equivalent (imply each other) for L^2 signals. There are some extensions to L^p spaces, $1 < p < \infty$, and distributions.



Titchmarsh's theorem: comments

Important (common) application

1. Have a (time domain) causal finite energy signal $u(t)$.
2. Unitary of the Fourier transform implies $U \in \mathrm{L}^2$.
3. Satisfies the first condition in Titchmarsh's theorem.
4. Can use either $\operatorname{Re} U$ or $\operatorname{Im} U$ to construct U, *e.g.*, $U = \operatorname{Re} U + \mathrm{i}\mathcal{H} \operatorname{Re} U$.

We note that L^2 is an essential and very natural assumption.

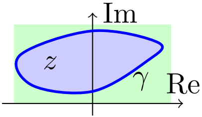
- Common signal such that $u(t) = \delta(t)$ (Dirac delta distributions) and $U(\omega) = 1/\omega$ (poles on the frequency axis) are not L^2 functions.
- For signals with infinite energy ($u(t) \notin \mathrm{L}^2$) it is sometimes possible to decompose the signal $u(t) = u_1(t) + u_2(t)$ where $u_2 \in \mathrm{L}^2$ and $u_1(t)$ is analyzed using other tools.
- There are partial generalizations to other L^p spaces and distributions.

Analytic (holomorphic) function

We have seen that causality is connected to analyticity in the Fourier (or Laplace) domain. Some important properties [7]:

- Holomorphic functions are defined in *open regions* (the domain of definition). This means that the frequency axis is in general not part of the domain of definition.
- Values on the boundary (closure) of the region can often but not always be defined as limits from the domain of definition.
- Cauchy's integral formula: for a simple closed curve γ in the region where $f(z)$ is analytic and with z in the interior of γ

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$



- Extensions of the Cauchy's integral formula to curves including the frequency axis are possible if the function is sufficiently regular at the frequency axis. The values should be interpreted as limits from the open interior domain.

Applications: analytic signals, IQ signals, modulation,...

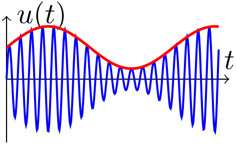
Analytic signals extend real valued signals, $u \in \mathrm{L}^2$ to complex valued signals

$$u_{\mathrm{a}}(t) = u(t) + \mathrm{i}\mathcal{H}\{u\}(t) = a(t)\mathrm{e}^{\mathrm{i}\phi(t)},$$

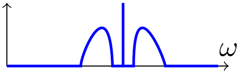
where $a(t)$, $\phi(t)$, and $\omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}$ are the envelope amplitude and instantaneous phase and angular frequency, respectively. Moreover, $\mathcal{F}\{u_{\mathrm{a}}(t)\}(\omega) = 2\mathcal{F}\{u(t)\}(\omega)$ for $\omega > 0$ and otherwise zero.

Single-sideband modulation (SSB) is a refinement of amplitude modulation (AM).

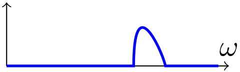
$$\begin{aligned} u_{\mathrm{ssb}}(t) &= \mathrm{Re}\{u_{\mathrm{a}}(t)\mathrm{e}^{\mathrm{i}\omega t}\} \\ &= u(t)\cos(\omega t) - \mathcal{H}\{u\}(t)\sin(\omega t) \end{aligned}$$



baseband signal

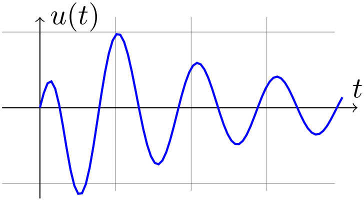


AM modulation



SSB modulation

Summary for signals



- The assumption of finite energy, L^2, is very good.
- Causality of $u(t)$ to get analyticity of $U(\omega)$ in a half plane and the Hilbert transform to relate the real and imaginary parts.
- Construct analytic time domain functions to remove negative frequencies components (causality in the frequency domain).
- There are cases where *e.g.*, bounded signals are useful $u \in \mathrm{L}^\infty$.

Note that L^2 functions are not point wise defined. Moreover, even if $u_1 \approx u_2$ in L^2 the differentiated signals u_1' and u_2' can be unbounded and very different.

Outline

Systems in convolution form (LTI systems)

Input-output system

- input signal $u(t)$
- output signal $v(t) = \mathcal{R}\{u(\cdot)\}(t)$

$$u(t) \longrightarrow \boxed{\mathcal{R}} \longrightarrow v(t)$$

where $\mathcal{R}$ is an operator.

Linear and time translational invariant (LTI) systems: the system is

Linear if $\mathcal{R}\{\alpha_1 u_1(t) + \alpha_2 u_2(t)\} = \alpha_1 \mathcal{R}\{u_1(t)\} + \alpha_2 \mathcal{R}\{u_2(t)\}$.

Time-translational invariant if

$$v(t) = \mathcal{R}\{u(\cdot)\} \Rightarrow v(t+\tau) = \mathcal{R}\{u(\cdot+\tau)\} \text{ for all } \tau.$$

Continuous if $u_n \to u \Rightarrow \mathcal{R}\, u_n \to \mathcal{R}\, u$.

Representation as convolutions

$$v(t) = h * u = \int_{\mathbb{R}} h(t-\tau) u(\tau)\,\mathrm{d}\tau$$

The class for the impulse response $h(t)$ includes distributions ▸ 62.

Causal and stable systems

Definition (Causal system)

A system on convolution form is causal if $h(t) = 0$ for $t < 0$.

The class for the kernel $h(t)$ includes all distributions such that $\operatorname{supp}\{h\} \subset [0, \infty)$.

Definition (Stable system)

There are many definitions of stability. One common version is bounded-input bounded-output (BIBO) stabillity that requires $h \in \mathrm{L}^1$ and hence $H \in \mathrm{H}^\infty$ (frequency axis in L^∞).

Computation

- in the time domain using convolution.
- by Laplace (Fourier) transformation, multiplication with the transfer function, and inverse transformation.

$$
\begin{tikzcd}
u(t) \arrow[r, "h*"] \arrow[d, "\mathcal{L}"'] & v(t) \\
U(s) \arrow[r, "H(s)"] & V(s) \arrow[u, "\mathcal{L}^{-1}"']
\end{tikzcd}
$$

Systems with signals in L^2 (finite energy)

Input and output signals in L^2 restrict the system. Start in the Fourier (Laplace) domain $V(\omega) = H(\omega)U(\omega)$ and assume functions

$$\|V\|_2^2 = \int_{\mathbb{R}} |H(\omega)U(\omega)|^2 \,\mathrm{d}\omega \leq \sup |H(\omega)|^2 \,\|U\|_2^2$$

or use the Hölder's inequality to show that $\|V\|_2 \leq \|H\|_\infty \|U\|_2$. Similar estimates for the impulse response $h(t)$.

We note that the assumptions of 'finite energy' are very different for signals and systems

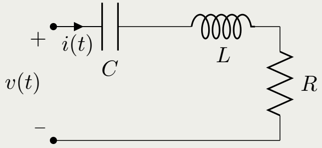
signals are square integrable $u \in \mathrm{L}^2$ and $U \in \mathrm{L}^2$.

systems integrable impulse response $h \in \mathrm{L}^1$ and bounded transfer function $H \in \mathrm{H}^\infty$.

But what does it mean that the signals are in L^2? Is it the 'appropriate' model?

Example (Resonance circuit)

Consider a series RCL resonance circuit:



The output signal (voltage)

$$v(t) = L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int_{-\infty}^{t} i(t')\,\mathrm{d}t' + Ri(t),\;\; V(s) = \big(sL + \frac{1}{sC} + R\big)I(s)$$

is unbounded in L^2, *i.e.*, there are input signals (currents) $i \in \mathrm{L}^2$ that do not produce output signals (voltages) $v(t)$ that are in L^2. However, the circuit is passive so it cannot produce energy. The problem is that although $\|v\|_2^2$ and $\|i\|_2^2$ are proportional to the energy for many cases the pertinent definition of the energy is $\int v(t)i(t)\,\mathrm{d}t$.

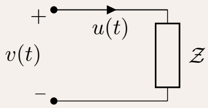
Passive systems

The concept of finite energy is very powerful to model signals. What is the corresponding property for systems? One possibility is passivity.

Definition (Passivity)

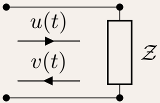
A system ($v = h * u$) is admittance-passive if

$$\mathcal{W}_{\text{adm}}(T) = \operatorname{Re} \int_{-\infty}^{T} v^*(t) u(t) \, \mathrm{d}t \geq 0$$



and scatter-passive if

$$\mathcal{W}_{\text{scat}}(T) = \int_{-\infty}^{T} |u(t)|^2 - |v(t)|^2 \, \mathrm{d}t \geq 0,$$

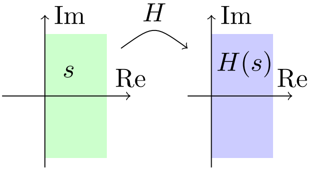


for all $T \in \mathbb{R}$ and smooth functions of compact support u.

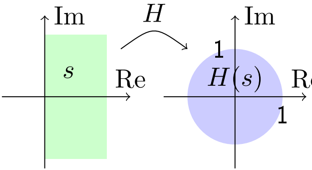
Zemanian, Distribution theory and transform analysis, 1965 [31]

Passive systems: transfer function $V(s) = H(s)U(s)$

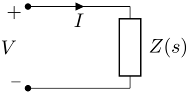
Admittance-passive: $H(s)$ analytic and $\operatorname{Re} H(s) \geq 0$ for $\operatorname{Re} s > 0$.



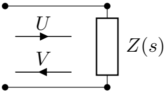
Scatter-passive: $H(s)$ analytic and $|H(s)| \leq 1$ for $\operatorname{Re} s > 0$.



Example: Impedance $H(s) = Z(s)$ of a passive circuit, $V = ZI$.



Example: Reflection coefficient $H(s) = \Gamma(s) = \frac{Z(s)-Z_0}{Z(s)+Z_0}$, $V = \Gamma U$.



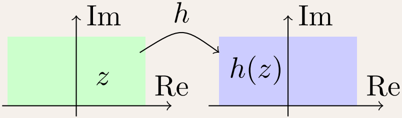
In both cases, $H(s)$ is holomorphic (analytic) for $\operatorname{Re} s > 0$, and can be related to a positive real (PR) (or Herglotz) function.

Youla *etal*(1959) [29], Zemanian (1963) [30], Wohlers and Beltrami (1965) [28], Zemanian (1965) [31]

Definition (Herglotz functions, $h(z)$)

A Herglotz (Nevanlinna, Pick, or R-) function $h(z)$ is holomorphic for $\operatorname{Im} z > 0$ and

$$\operatorname{Im} h(z) \geq 0 \quad \text{for} \quad \operatorname{Im} z > 0$$



Representation for $\operatorname{Im} z > 0$, *cf.*, the Hilbert transform

$$h(z) = A_{\mathrm{h}} + Lz + \int_{-\infty}^{\infty} \frac{1}{\xi - z} - \frac{\xi}{1+\xi^2} \, \mathrm{d}\nu(\xi)$$

where $A_{\mathrm{h}} \in \mathbb{R}$, $L \geq 0$, and $\int_{\mathbb{R}} \frac{1}{1+\xi^2} \, \mathrm{d}\nu(\xi) < \infty$.

The symbol $h = h(t)$ is also used for the impulse response in this presentation (very different).

Gustav Herglotz 1881-1953

Rolf Nevanlinna 1895-1980

Georg Alexander Pick 1859-1942

Wilhelm Cauer 1900-1945

Interpretation of the representation

The spectral function is

$$\nu(\xi) = \lim_{y\to 0^+} \frac{1}{\pi}\int_0^\xi \operatorname{Im}\{h(x+\mathrm{i}y)\}\,\mathrm{d}x \quad \text{and} \quad \mathrm{d}\nu(\xi) = \frac{1}{\pi}\operatorname{Im}\{h(\xi)\}\,\mathrm{d}\xi$$

for sufficiently regular cases. The 'convergence' term $\xi/(1+\xi^2)$ is odd and vanishes for symmetric integration intervals. Assume symmetry $\operatorname{Im}\{h(\xi)\} = \operatorname{Im}\{h(-\xi)\}$, then for $\operatorname{Im} z > 0$

$$\begin{aligned} h(z) &= A_{\mathrm{h}} + Lz + \int_{-\infty}^{\infty} \frac{1}{\xi - z} - \frac{\xi}{1+\xi^2}\,\mathrm{d}\nu(\xi) \\ &= Lz + \frac{1}{\pi}\lim_{R\to\infty}\int_{-R}^{R} \frac{\operatorname{Im}\{h(\xi)\}}{\xi - z}\,\mathrm{d}\xi = Lz + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{z\operatorname{Im}\{h(\xi)\}}{z^2-\xi^2}\,\mathrm{d}\xi \end{aligned}$$

where $\int_{\mathbb{R}} \frac{\operatorname{Im}\{h(\xi)\}}{1+\xi^2}\,\mathrm{d}\xi < \infty$, $L \geq 0$, and we assume a symmetric integration interval in the final equality.

- Reduces to the Hilbert transform ($\mathrm{Im} \to \mathrm{Re}$) and addition of Lz.
- Not necessary $\operatorname{Im}\{h(\xi)\} \in \mathrm{L}^2$ or asymptotic decay at ∞.
- Convergence term $\xi/(1+\xi^2)$ not needed in many cases.

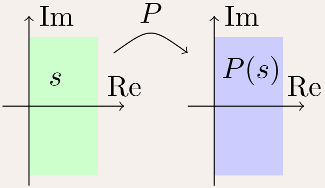
Positive real (PR) functions

Definition (Positive real (PR) functions)

A positive real function $P(s)$ is holomorphic for $\operatorname{Re} s > 0$ and

$$\operatorname{Re} P(s) \geq 0 \quad \text{for} \quad \operatorname{Re} s > 0$$

with $P(s) = P^*(s^*)$.

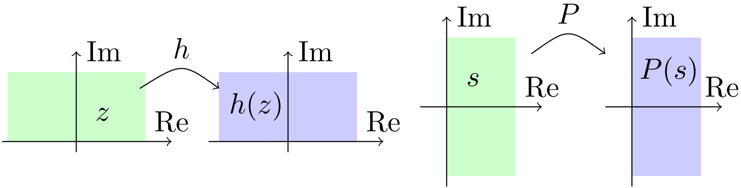


PR functions can be represented as

$$P(s) = Ls + \int_{-\infty}^{\infty} \frac{s}{s^2+\xi^2}\,\mathrm{d}\nu(\xi) = Ls + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{s\operatorname{Re}\{P(\mathrm{j}\xi)\}}{s^2+\xi^2}\,\mathrm{d}\xi$$

for $\operatorname{Re} s > 0$, where $L \geq 0$, $\int_{\mathbb{R}} \frac{1}{1+\xi^2}\,\mathrm{d}\nu(\xi) < \infty$, and we assume a sufficiently regular $P(\mathrm{j}\omega)$ in the final equality.

Herglotz functions and positive real functions



Note $z = \mathrm{i}s$, $h = \mathrm{i}P$. Here also with $h(z) = -h^*(-z^*)$ (real-valued in the time domain).

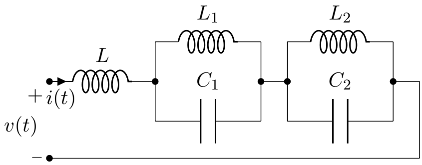
- Time conventions: $\mathrm{e}^{-\mathrm{i}\omega t}$ for Herglotz and $\mathrm{e}^{\mathrm{j}\omega t}$ for PR (Laplace parameter $s = \sigma + \mathrm{j}\omega$).
- Many contributors, Herglotz, Cauer, Nevanlinna, Pick, ...
- Also for maps between the unit circles.
- An impedance $Z(s)$ of a passive network is a typical PR function, see also 75.
- Applications: circuit synthesis, filter design, sum rules, operator theory, moment problem, ...

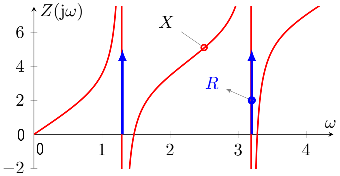
Point measures and Dirac delta distributions

The representation theorems admit point measures (can be interpreted as Dirac delta distributions) that give PR functions of the form (remember $\operatorname{Re} s > 0$)

$$P(s) = Ls + \sum_n \frac{\alpha_n s}{s^2 + \xi_n^2} = sL + \sum_n \frac{1}{sC_n + \frac{1}{sL_n}}$$

where $L \geq 0$, $\alpha_n = C_n^{-1} > 0$, $L_n = 1/(C_n \xi_n^2)$, and we identify the PR function with the input impedance of a series of parallel LC resonators, *i.e.*, $P(s) = Z(s)$ and *e.g.*,





An arbitrary PR function P can be decomposed into a sum of two PR functions $P = P_{\mathrm{r}} + P_{\mathrm{s}}$, where P_{s} only has point measures.

Applications of PR and Herglotz functions

Circuit synthesis: We can synthesize networks with lumped circuit elements from rational PR functions [27], *e.g.*, Brune and Darlington synthesis. The basic procedure is to to reduce the order of the PR function by iterative subtraction of simple PR functions. Iterate subtraction of: point measures of $Z, Y = Z^{-1}$, minimal resistance $R_0 = \min \mathrm{Re}\, Z$, negative sL or $1/(sC)$ (not PR). Transform the negative elements to ideal transformers.

Filters: Synthesis of filters from the amplitude of the transfer function.

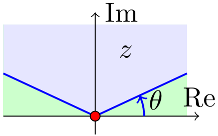
Sum rules: Integral identities for Herglotz and PR functions are instrumental for the general procedure to construct sum rules for passive systems [3].

Mathematics: Herglotz (and PR) functions appear in a variety of mathematical proplems, *e.g.*, the moment problems [1], ...

Integral identities for Herglotz functions

Herglotz functions with the symmetry $h(z) = -h^*(-z^*)$ (real-valued in the time domain) have asymptotic expansions ($N_0 \geq 0$ and $N_\infty \geq 0$)

$$\begin{cases} h(z) = \displaystyle\sum_{n=0}^{N_0} a_{2n-1} z^{2n-1} + o(z^{2N_0-1}) & \text{as } z \hat{\to} 0 \\ h(z) = \displaystyle\sum_{n=0}^{N_\infty} b_{1-2n} z^{1-2n} + o(z^{1-2N_\infty}) & \text{as } z \hat{\to} \infty \end{cases}$$



where $\hat{\to}$ denotes limits in the Stoltz domain $0 < \theta \leq \arg(z) \leq \pi - \theta$ 70. They satisfy the identities ($1 - N_\infty \leq n \leq N_0$)

$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{\operatorname{Im} h(x + \mathrm{i}y)}{x^{2n}} \, \mathrm{d}x = a_{2n-1} - b_{2n-1} = \begin{cases} -b_{2n-1} & n < 0 \\ a_{-1} - b_{-1} & n = 0 \\ a_1 - b_1 & n = 1 \\ a_{2n-1} & n > 1 \end{cases}$$

Bernland, Luger, Gustafsson, *Sum rules and constraints on passive systems*, J. Phys. A: Math. Theor., 2011.

Known low-frequency expansion ($a_1 \geq 0$):

$$h(z) \sim \begin{cases} a_1 z & \text{as} \quad z \hat{\to} 0 \\ b_1 z & \text{as} \quad z \hat{\to} \infty \end{cases}$$

that gives the $n = 1$ identity (we drop the limits for simplicity)

$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{1/\varepsilon} \frac{\operatorname{Im} h(x + \mathrm{i}y)}{x^2} \, \mathrm{d}x \overset{\text{def}}{=} \frac{2}{\pi} \int_0^{\infty} \frac{\operatorname{Im} h(x)}{x^2} \, \mathrm{d}x = a_1 - b_1 \leq a_1$$

Known high-frequency expansion (short times) ($b_{-1} \leq 0$):

$$h(z) \sim \begin{cases} a_{-1}/z & \text{as} \quad z \hat{\to} 0 \\ b_{-1}/z & \text{as} \quad z \hat{\to} \infty \end{cases}$$

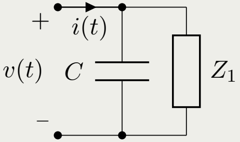
that gives the $n = 0$ identity

$$\frac{2}{\pi} \int_0^{\infty} \operatorname{Im} h(x) \, \mathrm{d}x = a_{-1} - b_{-1} \leq -b_{-1}.$$

Example (input impedance of circuit networks)

A classical sum rule for linear circuit networks is the *resistance-integral theorem* [4],[9],[27].

1. A circuit network composed of passive elements.
2. The impedance between two nodes $Z(s)$ is a PR function.
3. Consider the case with a shunt capacitor at the input terminal



$$Z(s) \sim \begin{cases} Z_1(0) & \text{as } s \hat{\to} 0 \\ \frac{1}{sC} & \text{as } s \hat{\to} \infty \end{cases}$$

   where we assume that $Z_1(0)$ is finite.
4. Sum rule (integral identity with $n = 0$, $a_1 = 0$, $b_1 = 1/s$)

$$\frac{2}{\pi} \int_0^{\infty} R(\omega)\, \mathrm{d}\omega = \frac{1}{C}$$

Example (Temporally dispersive permittivity)

1. Linear passive material models with permittivity $\epsilon(\omega)$ 56.
2. $h_\epsilon(\omega) = \omega\epsilon(\omega)$ is a Herglotz function.
3. Consider the case without static conductivity

$$h_\epsilon(\omega) = \omega\epsilon(\omega) \sim \begin{cases} \omega\epsilon_{\mathrm{s}} = \omega\epsilon(0) & \text{as } \omega \hat{\to} 0 \\ \omega\epsilon_\infty = \omega\epsilon(\infty) & \text{as } \omega \hat{\to} \infty \end{cases}$$

4. Sum rule (integral identity with $n = 1$, $a_1 = \epsilon_{\mathrm{s}}$, $b_1 = \epsilon_\infty$)

$$\frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\{h_\epsilon(\omega)\}}{\omega^2}\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\{\epsilon(\omega)\}}{\omega}\,\mathrm{d}\omega = \epsilon_{\mathrm{s}} - \epsilon_\infty$$

Integrated losses are related to the difference $\epsilon_{\mathrm{s}} - \epsilon_\infty$, *cf.*, Landau-Lifshitz, *Electrodynamics of Continuous Media*[21] and Jackson, *Classical Electrodynamics*[17].

Example (Temporally dispersive permeability)

1. Linear passive material models with permeability $\mu(\omega)$ satisfy the corresponding sum rule

$$\frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\{h_\mu(\omega)\}}{\omega^2}\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\{\mu(\omega)\}}{\omega}\,\mathrm{d}\omega = \mu_\mathrm{s} - \mu_\infty$$

showing that $\mu_\mathrm{s} \geq \mu_\infty$, [21, 17].

2. Sometimes considered a paradox for diamagnetic materials ($\mu_\mathrm{s} < 1$ and assuming $\mu_\infty = 1$). The paradox is resolved by considering the refractive index with $n_\infty \geq 1$ (due to special relativity) and hence

$$\frac{\epsilon_\mathrm{s} + \mu_\mathrm{s}}{2} \geq \sqrt{\epsilon_\mathrm{s}\mu_\mathrm{s}} = n_\mathrm{s} \geq n_\infty$$

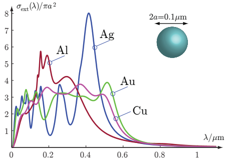
showing that diamagnetic materials ($\mu_\mathrm{s} < 1$) have a static permittivity (and/or conductivity).

Sum rules and physical bounds on passive systems

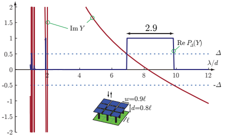
General simple approach

1. Identify a linear and passive system.
2. Construct a Herglotz (or similarly a positive real) function $h(z)$ that models the parameter of interest.
3. Investigate the asymptotic expansions of $h(z)$ as $z \hat{\to} 0$ and $z \hat{\to} \infty$.
4. Use integral identities for Herglotz functions to relate the dynamic properties to the asymptotic expansions.
5. Bound the integral.

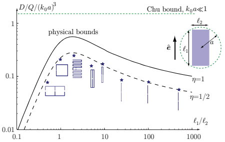
Examples: Matching networks [4, 5], Radar absorbers [23], Antennas [12, 13, 10], Scattering [24, 2], High-impedance surfaces [15], Metamaterials [11], Extraordinary transmission [14], Periodic structures [16]



Cross sections of nano spheres.



High-impedance surface.



Antenna D/Q.

Systems for fixed frequency

We are in most cases interested in the system (transfer function) for fixed (real-valued) frequencies. How do we define, measure, and use $U(\omega)$ for $\omega \in \mathbb{R}$?

Definition $U(\omega) = \lim_{\xi\to 0^+} U(\omega + \mathrm{i}\xi)$ if it exists. Analytic functions are defined in open regions, *e.g.*, $\operatorname{Re} s > 0$ or $\operatorname{Im} \omega > 0$.

Measure using a finite time-domain pulse, *i.e.*, as $U(\omega) = \lim_{\xi\to 0^+} U(\omega + \mathrm{i}\xi)$. (VNA???)

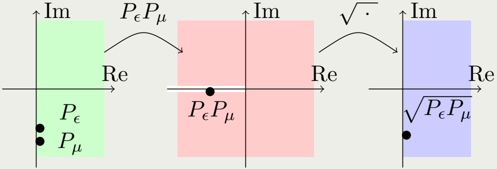
Use in many mathematical proofs of well posedness of the Maxwell equations, we use the uniqueness for negligible losses (or equivalently a complex frequency $\omega + \mathrm{i}\xi$ and send $\xi \to 0^+$, *i.e.*, $U(\omega) = \lim_{\xi\to 0^+} U(\omega + \mathrm{i}\xi)$. The sum rules (integral identities) are defined as the limit $\xi \to 0^+$.

In many cases we consider the frequency domain value as the limit from the complex valued frequency (*i.e.*, from the open half plane).

Example (Negative refraction, or how to interpret $\sqrt{-1\cdot -1} = -1$?)

Consider a permittivity $\epsilon(s)$ and permeability $\mu(s)$. Passivity imply that $P_\epsilon = s\epsilon(s)$ and $P_\mu = s\mu(s)$ are PR functions. The refractive index $n(s)$ can be determined from the PR function $P_\mathrm{n}(s) = sn(s)$, *i.e.*, $n(s) = P_\mathrm{n}(s)/s$, where we use the square root with branch cut at the negative real axis

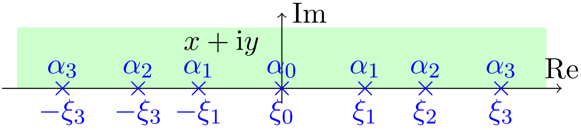
$$sn(s) = P_\mathrm{n}(s) = \sqrt{P_\epsilon(s)P_\mu(s)} = \sqrt{s\epsilon(s)s\mu(s)}$$



The case $\epsilon \approx -1$, $\mu \approx -0.75$, and $\omega \approx 1$ is depicted in the figure. We have

- $P_\epsilon \approx -0.75\mathrm{j}$ and $P_\mu \approx -\mathrm{j}$ giving $P_\epsilon P_\mu \approx -0.75$, $P_\mathrm{n} \approx -0.87\mathrm{i}$, and $n \approx -0.87$.
- The values are limits from an open region (*e.g.*, the half plane, $\mathrm{Re}\, s > 0$).
- Note, the corresponding case without PR functions (the multiplications with s) requires a square root operator with branch cut at the positive real axis. Additional modifications of $\sqrt{\cdot}$ for analyticity in the upper half plane (Herglotz functions).

Simple poles at the frequency axis



Passive transfer functions with simple poles at the frequency axis are common. The following interpretations are similar:

Point measures with amplitude $\alpha_n \geq 0$ at ξ_n for $0 = 1, .., N$ in the representation theorem of PR (or Herglotz functions) giving

$$P(s) = sL + \sum_{n=-N}^{N} \frac{\alpha_{|n|} s}{s^2 + \xi_{|n|}^2} \quad \text{for} \quad \mathrm{Re}\, s > 0.$$

Dirac delta distributions in the resistance *i.e.*,

$R(\omega) = \mathrm{Re}\, P(\mathrm{j}\omega) = \frac{\pi}{2} \sum \alpha_{|n|} \delta(\omega - \mathrm{sign}(n)\xi_{|n|})$. (neq freq?)

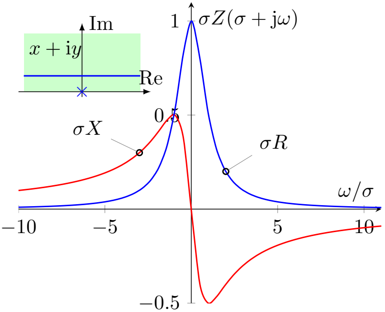
Fourier transform of the unit step is $\frac{1}{\mathrm{j}\omega} + \pi\delta(\omega)$ that corresponds to the limit of $1/s$ as $\mathrm{Re}\, s \to 0^+$.

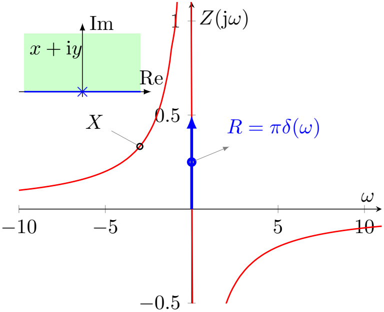
Simple poles at the frequency axis II

Consider for simplicity the simple pole $-1/z$ with $z = x + \mathrm{i}y$, *i.e.*,

$$\frac{-1}{z} = \frac{-1}{x + \mathrm{i}y} = \frac{-x + \mathrm{i}y}{x^2 + y^2} \quad \text{with} \quad \lim_{y \to 0^+} \int \phi(x) \frac{y}{x^2 + y^2} \, \mathrm{d}x = \pi \phi(0)$$

for smooth functions of compact support $\phi(x)$.





Some disadvantages with distributions for transfer functions

The impulse response is usually a tempered distribution and the corresponding output is given by temporal convolutions. The situation is very different for the transfer function where the output is given by multiplications.

The limiting 'functions' at the frequency axis of passive transfer functions (analytic in the open half plane) can contain Dirac delta distributions. The case of causal transfer functions contain more general distributions. Although, distributions are (one of) the most general representations of a linear functional, there are also some drawbacks

Multiplication: It is in general very difficult to multiply distributions (see the wave front set), *e.g.*, what is $\epsilon \boldsymbol{E}$ if ϵ is a distribution? Also $s\delta(s) = 0$ so the algebra for $1/s$ is non-trivial for $\operatorname{Re} s = 0$.

Numbers: In the end we usually want numbers (not functionals).

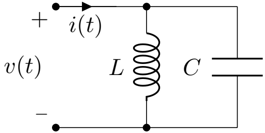
Too large class: We want a function class that covers all relevant cases but not more.

We avoid the problems with distributions for the transfer function in calculations and analysis by considering the frequency domain value as the limit from the complex valued frequency (*i.e.*, from the open half plane).

Are lossless one-ports lossless?

The archetype of lossless one-ports are networks composed of ideal capacitors and inductors. Obviously there is no dissipation of power in ideal capacitors and inductors but what happens at resonances?

Consider the LC-circuit

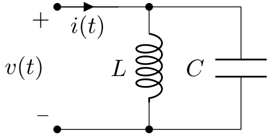


$$Z(s) = \frac{V}{I} = \frac{sL}{1 + s^2 LC}$$

The impedance is lossless away from the resonance frequency, *i.e.*, $R(\omega) = \mathrm{Re}\{Z(\mathrm{j}\omega)\} = 0$ for $\omega \neq \omega_0 = 1/\sqrt{LC}$. We investigate the resistance at the resonance frequency using

- absorbed energy in the time domain using the causal input signal $u(t) = \sin(\omega t)$, $t > 0$ and $u(t) = 0$, $t < 0$, see 86.
- sum rules that relate the all spectrum integral of R to the low- and high-frequency asymptotic expansions.
- limiting value from the open half plane that can be interpreted as a point measure (Dirac delta distribution) in the resistance.

Are lossless one-ports lossless II?



$$Z(s) = \frac{V}{I} = \frac{sL}{1+s^2LC} = \sqrt{\frac{L}{C}}\frac{s/\omega_0}{1+s^2/\omega_0^2}$$

Time domain: The causal input current $i(t) = I_0 \sin(\omega t)$ for $t > 0$ and $i(t) = 0$ for $t < 0$ gives the output voltage $\frac{\omega I_0 Z_0}{2}\sin(\omega t)t$ and an absorbed energy $\sim t^2$ for $\omega = \omega_0 = 1/\sqrt{LC}$, see 86.

Sum rule: The resistance $\mathrm{Re}\, Z(\mathrm{j}\omega) = 0$ for $\omega \neq \pm\omega_0$ and satisfies [3]

$$\lim_{\varepsilon\to 0^+}\lim_{\sigma\to 0^+}\frac{2}{\pi}\int_{\varepsilon}^{\varepsilon^{-1}}\frac{\mathrm{Re}\{Z(\mathrm{j}\omega+\sigma)\}}{\omega^{2p}}\,\mathrm{d}\omega = \sqrt{\frac{L}{C}}\omega_0^{-2p+1} \quad \text{for } \; p = 0, \pm 1, \ldots$$

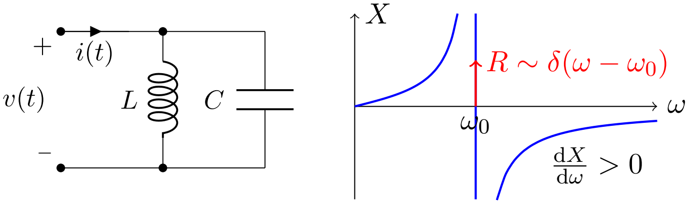
that shows that the singularity at $\omega = \omega_0$ contributes to the resistance. We can model the contribution as from point sources at $\pm\omega_0$ that is similar to the delta distributions $R(\mathrm{j}\omega) = \frac{\pi}{2C}\big(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\big)$.

Limit: The limits from the poles at $\omega = \pm\omega_0$ imply a point measure in the resistance.

Foster's reactance theorem

Theorem (Foster's reactance theorem)

The reactance $X(\omega)$ of a lossless one-port monotonically increases with frequency, i.e., $\frac{\mathrm{d}X}{\mathrm{d}\omega} > 0$ for $\omega \in \Omega = (\omega_1, \omega_2)$ if $Z = \mathrm{j}X$ for $\omega \in \Omega$.



The reactance increases in the interval away from the singularity at $\omega = \omega_0 = 1/\sqrt{LC}$. Note that the assumption of no losses is essential for the Foster's reactance theorem. The reactance decreases rapidly (or is undefined) if the resistance is singular.

Foster's reactance theorem

Theorem (Foster's reactance theorem)

Foster's reactance theorem states that the reactance of a passive, lossless one-port monotonically increases with frequency, i.e., $\frac{\mathrm{d}X}{\mathrm{d}\omega} > 0$ *for* $\omega \in \Omega = (\omega_1, \omega_2)$ *if* $Z = \mathrm{j}X$ *for* $\omega \in \Omega$.

Proof.

Use the representation

$$P(s) = sL + \int_{-\infty}^{\infty} \frac{s}{s^2 + \xi^2}\,\mathrm{d}\nu(\xi)$$

in an open interval $\Omega = (\omega_1, \omega_2)$ with $\mathrm{d}\nu(\xi) = 0$ to get

$$X(\omega) = \operatorname{Im} P(\mathrm{j}\omega) = \omega L + \int_{\mathbb{R}\setminus\Omega} \frac{\omega}{\xi^2 - \omega^2}\,\mathrm{d}\nu(\xi) \quad \text{for} \quad \omega \in \Omega$$

We note that $X(\omega)$ is a smooth function and that we can differentiate $X(\omega)$ with respect to ω for $\omega \in \Omega = (\omega_1, \omega_2)$ giving

$$\frac{\mathrm{d}X(\omega)}{\mathrm{d}\omega} = L + \int_{\mathbb{R}\setminus\Omega} \frac{\xi^2 + \omega^2}{(\xi^2 - \omega^2)^2}\,\mathrm{d}\nu(\xi) \geq 0 \quad \text{for} \quad \omega \in \Omega$$

□

Foster's reactance theorem for constitutive relations

A result similar to the Foster's reactance theorem is often used for the permittivity $\epsilon_{\mathrm{r}}(\omega)$, see [21, 11]. Using that $s\epsilon_r(s)$ is a PR function and $\omega\epsilon_{\mathrm{r}}(\omega)$ is a Herglotz function. Their corresponding representations give ($\operatorname{Im}\epsilon(\omega)=0$ for $\omega\in\Omega$)

$$\frac{\mathrm{d}h_\epsilon}{\mathrm{d}\omega}=\frac{\mathrm{d}(\omega\epsilon_{\mathrm{r}}(\omega))}{\mathrm{d}\omega}=\epsilon_\infty+\int_{\mathbb{R}\setminus\Omega}\frac{\xi^2+\omega^2}{(\xi^2-\omega^2)^2}\,\mathrm{d}\nu(\xi)\geq\epsilon_\infty\quad\text{for}\ \ \omega\in\Omega$$

This pointwise bound on the derivative is not true when losses are present, even if the loss (*i.e.*, the imaginary part $\operatorname{Im}h(\omega)$) is arbitrarily small. Consider, *e.g.*, the Lorentz model

$$h_\epsilon(\omega)=\epsilon_\infty\omega+\frac{\omega\nu^{3/2}}{1-\omega^2-\mathrm{i}\nu\omega}\quad\text{with}\ \ h(1)=\epsilon_\infty+\mathrm{i}\nu^{1/2}\approx\epsilon_\infty$$

where $\epsilon_\infty,\nu>0$ and $\nu\ll 1$. However

$$\frac{\mathrm{d}h_\epsilon}{\mathrm{d}\omega}(1)=\epsilon_\infty+\mathrm{i}\nu^{1/2}-\frac{2+\mathrm{i}\nu}{\nu^{1/2}}\approx\epsilon_\infty-\frac{2}{\nu^{1/2}}\to-\infty\quad\text{as}\ \ \nu\to 0.$$

This simple example shows that it is very difficult to bound the derivative of Herglotz functions (and hence ϵ and μ) pointwise at the frequency axis.

Passive systems in EM

There are many passive systems (not more energy out than in) in electromagnetics (EM):

Admittance passive

- Material models such as $P(s) = s\epsilon(s)$ and $h(\omega) = \omega\epsilon(\omega)$. Similar for bi-anisotropic media.
- Antenna input impedance $P(s) = Z(s)$ and $h(\omega) = \mathrm{i}Z(\omega)$.
- Forward scattering of finite objects.

Scattering passive

- Antenna and material reflection coefficients, $\mathit{\Gamma} = S_{11}$.
- Reflection and transmission coefficients of periodic structures.

Example (Passive systems: material modeling (Laplace))

The Maxwell equations in the Laplace domain are

$$s\boldsymbol{D} = \nabla \times \boldsymbol{H} - \boldsymbol{J} \quad \text{and} \quad s\boldsymbol{B} = -\nabla \times \boldsymbol{E}$$

where $\boldsymbol{J}$ is the current density. Modeling of the interaction between the EM fields and material is done expressing the electric flux density $\boldsymbol{D}$ and/or the current density $\boldsymbol{J}$ in the electric field intensity $\boldsymbol{E}$. More general bi-anisotropic models are treated similarly. It is also customary to decompose the current density into one part that is proportional to $\boldsymbol{E}$ and one part that is forced or controlled (here we suppress this part).

The linear, passive, time translational invariant, continuous, non-magnetic, and isotropic constitutive relations are:

$\boldsymbol{E} \to \boldsymbol{D}$: $\boldsymbol{D}(s) = \epsilon_0\epsilon_{\mathrm{r}}(s)\boldsymbol{E}(s)$ where $s\epsilon_{\mathrm{r}}(s)$ is a PR function.

$\boldsymbol{E} \to \boldsymbol{J}$: $\boldsymbol{J}(s) = \sigma(s)\boldsymbol{E}(s)$ where $\sigma(s)$ is a PR function

The two models are (basically) equivalent and

- $\sigma(s) = s\epsilon_0(\epsilon_{\mathrm{r}}(s) - 1)$, note that $\sigma(s)$ assumes a corresponding high-frequency response $\epsilon_\infty \geq 1$ in the $\epsilon(s)$ case.
- Similar models for the general bi-anisotropic case using matrix PR (Herglotz) functions.

Examples (Passive systems: material modeling (time domain))

The linear, causal, time translational invariant, continuous, non-magnetic, and isotropic constitutive relations are

$$\boldsymbol{D}(t) = \epsilon_0\epsilon_\infty \boldsymbol{E}(t) + \epsilon_0 \int_{\mathbb{R}} \chi(t-t')\boldsymbol{E}(t')\,\mathrm{d}t' \qquad \boldsymbol{E}(t) \longrightarrow \boxed{\varepsilon} \longrightarrow \boldsymbol{D}(t)$$

where $\chi(t) = 0$ for $t < 0$ and $\epsilon_\infty > 0$ is the instantaneous response. The material model is passive if

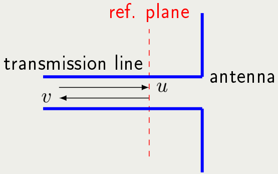
$$0 \leq \int_{-\infty}^{T} \boldsymbol{E}(t)\cdot\frac{\partial \boldsymbol{D}(t)}{\partial t}\,\mathrm{d}t = \epsilon_0 \int_{-\infty}^{T}\int_{\mathbb{R}} \boldsymbol{E}(t)\cdot\frac{\partial}{\partial t}\big(\epsilon_\infty\delta(t-\tau)+\chi(t-\tau)\big)\boldsymbol{E}(\tau)\,\mathrm{d}\tau\,\mathrm{d}t$$

for all times T and smooth compactly supported fields $\boldsymbol{E}$.

- Similarly for the magnetic fields.
- The presented results are also valid for the diagonal elements of general bi-anisotropic constitutive relations.
- Fourier transform to get the frequency-domain model $\boldsymbol{D}(\omega) = \epsilon_0\epsilon_{\mathrm{r}}(\omega)\boldsymbol{E}(\omega)$, where $\omega\epsilon_{\mathrm{r}}(\omega)$ is a Herglotz function 79.

Examples (Passive systems: antenna reflection coefficient)

The reflection coefficient of a structure composed of passive materials is passive if it is causal, *i.e.*, the reference plane is placed 'in front' of the structure.

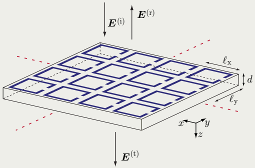


S-parameter measurements of an antenna with a VNA.

- Reflection coefficients (or more general scattering parameters) are defined in transmission lines. We expand the fields in the transmission line in modes and define the reflection coefficient as the scattering parameter of the lowest order mode.
- Although the scattering parameters are defined for all frequencies we are usually only interested in the results where the transmission line has a single propagating mode.

Examples (Passive systems: periodic structures)

Consider a linear polarized incident plane wave $\boldsymbol{E}^{(\mathrm{i})} = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}kz}$ on a periodic structure. Decompose the transmitted field, $\boldsymbol{E}^{(\mathrm{t})}$, in Floquet modes outside the structure ($z > 0$)



Periodic structure.

$$\boldsymbol{E}^{(\mathrm{t})}(k;\boldsymbol{r}) = \sum_{m,n=-\infty}^{\infty} \boldsymbol{E}_{mn}^{(\mathrm{t})}(k) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_{mn}\cdot\boldsymbol{\rho}} \mathrm{e}^{\mathrm{i}k_{z,mn}z}$$

where $\boldsymbol{k}_{mn} = m2\pi/\ell_{\mathrm{x}}\hat{\boldsymbol{x}} + n2\pi/\ell_{\mathrm{y}}\hat{\boldsymbol{y}}$, and $k_{z,mn} = \sqrt{k^2 - |\boldsymbol{k}_{mn}|^2}$ is the wavenumber in the z direction for the mn mode.

The transmission coefficient for the co-polarized lowest order mode is $T(k) = \boldsymbol{E}_0^* \cdot \boldsymbol{E}_{00}^{(\mathrm{t})}(k)/|\boldsymbol{E}_0^2|$. The transmission coefficient is passive if the periodic structure does not increase the wavefront speed (such as structures in free space) and is composed of passive material constituents.

Note, the wavefront speed can be increased for structures embedded in high-permittivity media and for the corresponding acoustic case.

Time-domain representation of passive systems

The impulse response of a passive system has the representation

$$h(t) = L\delta'(t) + \theta(t)\int_{\mathbb{R}} \cos(\xi t)\,\mathrm{d}\nu(\xi)$$

where $L \geq 0$ and $\int_{\mathbb{R}} \frac{1}{1+\xi^2}\,\mathrm{d}\nu(\xi) < \infty$ [31]. Also note that $\mathcal{L}\{\cos(\xi t)\theta(t)\} = \frac{s}{s^2+\xi^2}$.

Example

The impulse response and transfer function

$$h(t) = L\delta'(t) + R_1\delta(t) + R_2\delta(t-\tau), \quad H(s) = sL + R_1 + R_2\mathrm{e}^{-s\tau}$$

are passive if $L, \tau \geq 0$ and $R_1 > |R_2|$.

Time-domain representation of passive systems II

The admittance passivity is (h is a distribution)

$$\mathcal{W}(T) = \mathrm{Re}\int_{-\infty}^{T} v^*(t)u(t)\,\mathrm{d}t = \mathrm{Re}\int_{-\infty}^{T}\int_{\mathbb{R}} u^*(t)h(t-\tau)u(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \geq 0$$

for all $u \in \mathcal{D}$. Here, we note that passivity is related to positive semidefiniteness of the kernel $h(t-\tau)$.
The impulse response and transfer function
Bochner's theorem, positive-definite function, Toeplitz kernels??

Examples (Passive systems: time-domain material modeling)

The linear, causal, time translational invariant, continuous, non-magnetic, and isotropic constitutive relations are

$$\boldsymbol{D}(t) = \epsilon_0\epsilon_\infty \boldsymbol{E}(t) + \epsilon_0 \int_{\mathbb{R}} \chi(t-t')\boldsymbol{E}(t')\,\mathrm{d}t' \qquad \boldsymbol{E}(t) \longrightarrow \boxed{\varepsilon} \longrightarrow \boldsymbol{D}(t)$$

where $\chi(t) = 0$ for $t < 0$ and $\epsilon_\infty > 0$ is the instantaneous response. The material model is passive if

$$0 \leq \int_{-\infty}^{T} \boldsymbol{E}(t)\cdot\frac{\partial \boldsymbol{D}(t)}{\partial t}\,\mathrm{d}t = \epsilon_0 \int_{-\infty}^{T}\int_{\mathbb{R}} \boldsymbol{E}(t)\cdot\frac{\partial}{\partial t}\big(\epsilon_\infty\delta(t-\tau)+\chi(t-\tau)\big)\boldsymbol{E}(\tau)\,\mathrm{d}\tau\,\mathrm{d}t$$

for all times T and smooth compactly supported fields $\boldsymbol{E}$. Comparing with the general passive system, we get the relation

$$\frac{\partial}{\partial t}\big(\epsilon_\infty\delta(t)+\chi(t)\big) = L\delta'(t) + \theta(t)\int_{\mathbb{R}} \cos(\xi t)\,\mathrm{d}\nu(\xi)$$

that corresponding to the PR-function $s\epsilon(s) = P(s)$. We have $L = \epsilon_\infty$ and

$$\chi(t) = \int_{0^-}^{t}\int_{\mathbb{R}} \cos(\xi\tau)\,\mathrm{d}\nu(\xi)\,\mathrm{d}\tau$$

Examples (Passive systems: time-domain material modeling)

The linear, causal, time translational invariant, continuous, non-magnetic, and isotropic constitutive relations are

$$\boldsymbol{D}(t) = \epsilon_0\epsilon_\infty \boldsymbol{E}(t) + \epsilon_0 \int_{\mathbb{R}} \chi(t-t')\boldsymbol{E}(t')\,\mathrm{d}t' \qquad \boldsymbol{E}(t) \longrightarrow \boxed{\varepsilon} \longrightarrow \boldsymbol{D}(t)$$

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for all times T and smooth compactly supported fields $\boldsymbol{E}$.

- Similarly for the magnetic fields.
- The presented results are also valid for the diagonal elements of general bi-anisotropic constitutive relations.
- Fourier transform to get the frequency-domain model $\boldsymbol{D}(\omega) = \epsilon_0\epsilon(\omega)\boldsymbol{E}(\omega)$, where $\omega\epsilon(\omega)$ is a Herglotz function 79.

Outline

Why passive systems?

- We can often show that we have a passive system from simple energy arguments and causality.
- Note that causality is a necessary conditions for passivity.
- Passivity offers rich and powerful mathematics.
- Composition of two Herglotz functions is a new Herglotz function.

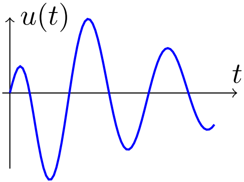
also the alternatives to passivity can be difficult to show or insufficient for analysis

- Causality is in general not sufficient to restrict the transfer function to something useful. We also need some assumptions of stability and/or restriction to some L^p space.
- Often 'very hard' to show that a transfer function of an EM system belongs to *e.g.*, L^2 or L^p spaces or is stable.

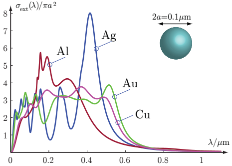
Conclusions

- Finite energy (L^2) and causality are natural assumptions for signals. Titchmarsh's theorem connects causality with analyticity in a half plane.
- Passivity and causality for systems.
- Herglotz and PR functions.
- Representation theorem.
- Frequency domain values as limits from the interior of the half plane (similar to time harmonic signals that need $t \to \infty$).
- Sum rules.

A prior knowledge (assumption) of passivity is often very easy to deduce and is very useful as it offers many powerful mathematical tools.



$$u(t) \longrightarrow \boxed{\mathcal{R}} \longrightarrow v(t)$$



Outline

5. **Appendix**
   - Distributions
   - Fourier- and Laplace transforms
   - Hardy space
   - Hilbert transform
   - Stoltz domain
   - Herglotz and PR functions
   - Integral identities
   - Lossless one-ports
6. References

Functions, distributions, and systems

Basic differences between functions, distributions, and systems:

Functions

$$\mathbb{R} \longrightarrow \boxed{u} \longrightarrow \mathbb{R}$$

Map numbers to numbers, *e.g.*, $\mathbb{R} \to \mathbb{R}$, $\mathbb{C} \to \mathbb{C}$, or matrix valued. Continuous, differentiable, or using equivalence classes such as integrable L^p.

Distributions

$$\mathcal{D} \longrightarrow \boxed{f} \longrightarrow \mathbb{R}$$

Map test functions to numbers, *e.g.*, $\mathcal{D} \to \mathbb{R}$, $\mathcal{D} \to \mathbb{C}$, $\mathcal{S} \to \mathbb{C}$.

Systems

$$\mathcal{D}' \longrightarrow \boxed{\mathcal{R}} \longrightarrow \mathcal{D}'$$

Many possibilities, *e.g.*, distributions to distributions $\mathcal{D}' \to \mathcal{D}'$ or functions to functions.

There are many similarities for LTI systems, $v = h * u$, where the impulse response h can be a function or a distribution.

Distributions

Definition (Test functions)

$\mathcal{D}$ is the space of smooth test functions with compact support.

Definition (Distribution)

The elements of the space $\mathcal{D}'$ of continuous linear functionals on $\mathcal{D}$ are distributions.

Linear functionals are often denoted $\langle f, \phi \rangle$. We can identify regular distributions (generated by functions) with the integral

$$\langle f, \phi \rangle = \int_{\mathbb{R}} f(t)\phi(t)\,\mathrm{d}t$$

We often suppress the difference between functions and distributions and use the same notation for distributions. In these cases it is important to realize the symbol $\int \cdot \cdot \mathrm{d}\cdot$ is just a notation for the corresponding linear functional $\langle \cdot, \cdot \rangle$.

Tempered distributions

Definition (Test functions of rapid descent)

The space $\mathcal{S}$ of smooth testing functions of rapid descent.

Definition (Tempered distribution)

The elements of the space $\mathcal{S}'$ of continuous linear functionals on $\mathcal{S}$ are tempered distributions (or distributions of slow growth).

- Subspace of $\mathcal{D}$.
- The Fourier transform of a tempered distribution is a tempered distribution.
- The Laplace transform of a casual tempered distribution is analytic for $\operatorname{Re} s > 0$.

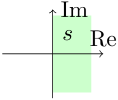
Causality

Definition (Causality)

A distribution u on $\mathcal{D}(\mathbb{R})$ is causal if $\langle u, \phi \rangle = 0$ for all test functions $\phi(t)$ such that $\phi(t) = 0$ for $t > 0$, *i.e.*, $\operatorname{supp} u \subset [0, \infty)$.??

The Laplace transform, $U(s) = \mathcal{L}\{u\}(s)$, of a causal tempered distribution is analytic for $\operatorname{Re} s > 0$. The limiting distribution at the frequency axis

$$\lim_{\sigma \to 0^+} \langle U(\sigma + \cdot), \phi \rangle = \lim_{\sigma \to 0^+} \int_{\mathbb{R}} U(\sigma + \mathrm{j}\omega)\phi(\omega)\, \mathrm{d}\omega$$



is a tempered distribution. Causality is hence not a very strong condition to restrict the class of distributions.

Example

Derivatives (and anti-derivatives) of the Dirac delta distribution are typical examples of causal distributions, *i.e.*,

$$u(t) = \delta^{(n)}(t) = \frac{\mathrm{d}^n \delta(t)}{\mathrm{d}t^n} \quad \text{with} \quad U(s) = s^n$$

where we note that U is bounded for $n = 0$ and passive for $|n| \leq 1$.

Fourier- and Laplace transforms

The Fourier transform is usually defined for real-valued parameters (the frequency axis) but can also be considered for complex-valued parameters (*e.g.*, a half plane). There are also many common normalizations.

One particular illuminating case is the Fourier- and Laplace transforms of the unit step, *i.e.*,

$$\mathcal{F}\{\theta(t)\}(\omega) = \frac{1}{\mathrm{j}\omega} + \pi\delta(\omega) \quad \text{for} \quad \omega \in \mathbb{R}$$

and

$$\mathcal{L}\{\theta(t)\}(s) = \frac{1}{s} \quad \text{for} \quad \mathrm{Re}\, s > 0$$

where we note that $\mathcal{F}\{\theta\}$ is a distribution and $\mathcal{L}\{\theta\}$ is an analytic function in $s = \sigma + \mathrm{j}\omega$ for $\mathrm{Re}\, s > 0$. Moreover, $\mathcal{F}\{\theta\}$ is the limiting distribution of $\mathcal{L}\{\theta\}$ at the frequency axis, *i.e.*,

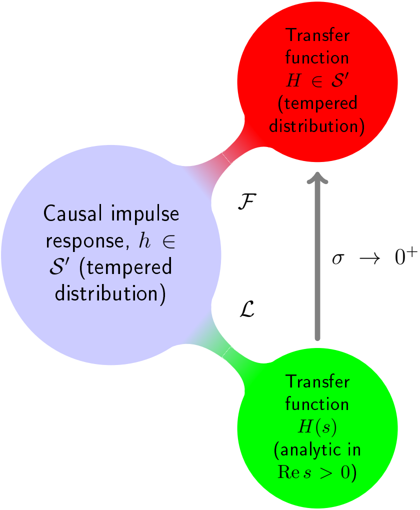
$$\lim_{\sigma\to 0^+} \langle \mathcal{L}\{\theta\}, \phi\rangle = \langle \mathcal{F}\{\theta\}, \phi\rangle$$

Fourier- and Laplace transforms

Consider a causal impulse response $h(t)$ in the form of a tempered distribution $h \in \mathcal{S}'$.

- The Laplace transform $\mathcal{L}\{h(t)\}(s)$ is analytic for $\sigma > 0$ with $s = \sigma + \mathrm{j}\omega$.
- The Fourier transform $\mathcal{F}\{h(t)\}(\omega)$ is a tempered distribution for $\omega \in \mathbb{R}$.
- They are related at the frequency axis $\sigma \to 0^+$.

For the sub class of passive impulse responses, we can use the representations for PR (or Herglotz) functions to restrict the spaces for the impulse response and transfer functions.

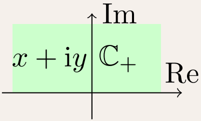


Hardy space H^p

Definition (Hardy space H^p)

The Hardy space H^p is the space of holomorphic functions in the upper (or right) half plane with the norm

$$\|u\|_{\mathrm{H}^p} = \sup_{y>0} \left(\int_{\mathbb{R}} |u(x+\mathrm{i}y)|^p \,\mathrm{d}x \right)^{\frac{1}{p}}$$



and the bounded analytic functions

$$\|u\|_{\mathrm{H}^\infty} = \sup_{z\in\mathbb{C}_+} |u(z)| \quad \text{where } \mathbb{C}_+ = \{x+\mathrm{i}y : x\in\mathbb{R}, y>0\}$$

- Boundary values $u \in \mathrm{L}^p$ on the real axis.
- Also for the unit disk.

Godfrey Harold Hardy 1877-1947

Frigyes Riesz 1880-1956

Definition (Hilbert transform)

The Hilbert transform is

$$\mathcal{H}\{u(\tau)\}(t) = \frac{1}{\pi}\int \frac{u(\tau)}{t-\tau}\,\mathrm{d}\tau$$

where a Cauchy principal value integral is used.

Properties

- Bounded in L^p for $1 < p < \infty$.
- Inverse $\mathcal{H}\{\mathcal{H}\{u\}\} = -u$.
- Convolution with the tempered distribution $h(t) = \mathrm{p.v.}\frac{1}{\pi t}$, $\mathcal{H}\{u\} = h * u$.
- Relates the real and imaginary parts of boundary functions in H^p for $1 < p < \infty$, ????

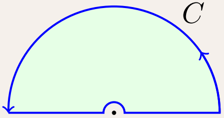
King, Hilbert Transforms I,II (2009) [18],[19].

David Hilbert
1862-1943

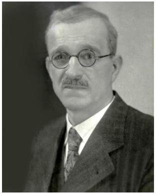
Theorem (Sokhotski–Plemelj theorem)

The Sokhotski–Plemelj theorem expresses the value of an analytic function as a Cauchy principal value integral over a (smooth) closed simple curve

$$f_{\pm}(z) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \pm \frac{f(z)}{2}$$



where $f_{\pm}(z)$ is the limit value from the interior/exterior of the curve C.



Josip Plemelj
1873-1967

Julian Karol Sochocki
1847-1927

Properties

- Interpreted as the Cauchy formula and half the residue of the pole.
- Similar to the Hilbert transform for half planes and sufficiently regular functions (decay at infinity).

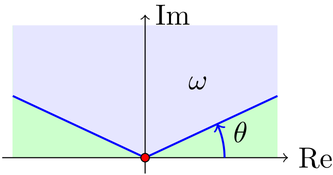
Stoltz domain

The symbol $\hat{\to}$ denotes limits in the Stoltz domain. For $\omega\hat{\to}0$ (upper) and $s\hat{\to}0$ (right) half planes, we use any $0 < \theta \leq \pi/2$ and

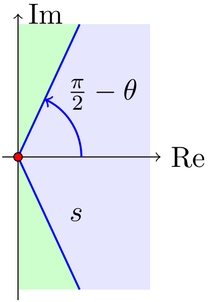
$$\theta \leq \arg\omega \leq \pi - \theta \quad \text{or } |\arg s| \leq \frac{\pi}{2} - \theta$$

and similarly for $\omega\hat{\to}\infty$ and $s\hat{\to}\infty$

upper half plane



right half plane



Example (time delay)

The time delay $\mathit{\Gamma}(s) = \mathrm{e}^{-s\tau}$ is scattering passive and imply the PR function

$$Y(s) = \frac{1-\mathit{\Gamma}}{1+\mathit{\Gamma}} = \frac{1-\mathrm{e}^{-s\tau}}{1+\mathrm{e}^{-s\tau}} = \tanh(\frac{s\tau}{2}) \to 1$$

as $s\hat{\to}\infty$ although the limit $s \to \infty$ for $s = \mathrm{j}\omega$ does not exist.

Hurwitz polynomials

Definition (Hurwitz polynomial)

A Hurwitz polynomial satisfies

1. $P(s) = P^*(s^*)$ (real valued coefficients)
2. The roots have real parts that are non-positive.

All coefficients have the same sign (usually chosen positive). Divide $P(s)$ into its even $m(s)$ and odd $n(s)$ parts. A necessary and sufficient condition that $P(s) = m(s) + n(s)$ is Hurwitz is that the continued fraction expansion

$$\frac{m(s)}{n(s)} = C_1 s + \cfrac{1}{C_2 s + \cfrac{1}{C_3 s + \cfrac{1}{... + \cfrac{1}{C_p s}}}}$$

has $C_1, C_2, C_3, ..., C_p > 0$.

Adolf Hurwitz
1859-1919

PR and Hurwitz polynomials

PR functions are sometimes restricted to be rational functions [8, 9]. This is common in network analysis and directly applicable to the input impedance for lumped circuit networks. Decompose the polynomials of a rational PR function in even, m_1, m_2, and odd, n_1, n_2, parts

$$P(s) = \frac{m_1(s) + n_1(s)}{m_2(s) + n_2(s)}$$

we have the Hurwitz polynomials $m_1 + n_1, m_1 + n_2, m_2 + n_1, m_2 + n_2$, and

$$\frac{m_1(s)}{n_1(s)}, \ \frac{m_1(s)}{n_2(s)}, \ \frac{m_2(s)}{n_1(s)}, \ \frac{m_2(s)}{n_2(s)}$$

are the impedance of LC ladder networks.

Positive Real lemma

Consider the (controllable) state-space model

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

with the transfer function

$$\mathbf{H}(s) = \mathbf{D} + \mathbf{C}(s\,\mathrm{Im} - \mathbf{A})^{-1}\mathbf{B}$$

Theorem ... $\mathbf{H}(s)$ is passive if and only of there exists a symmetric positive definite ($\mathbf{K} = \mathbf{K}^{\mathrm{T}}\ \mathbf{K} \geq 0$) such that

$$\begin{pmatrix} -\mathbf{A}^{\mathrm{T}}\mathbf{K} - \mathbf{K}\mathbf{A} & -\mathbf{K}\mathbf{B} + \mathbf{C}^{\mathrm{T}} \\ -\mathbf{B}^{\mathrm{T}}\mathbf{K} + \mathbf{C} & \mathbf{D} + \mathbf{D}^{\mathrm{T}} \end{pmatrix} \geq 0$$

Herglotz...

There are several alternative to the representation of Herglotz functions from $\operatorname{Im} z > 0 \to \operatorname{Im} z > 0$.

unit circle to right half-plane

$$f(z) = \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} \, \mathrm{d}\mu(\theta)$$

Herglotz and PR functions: Simple examples

Elementary Herglotz functions, $h(z)$, are

$$z, \quad \frac{-1}{z}, \quad \mathrm{i}\sqrt{-\mathrm{i}z}, \quad \tan(z)$$

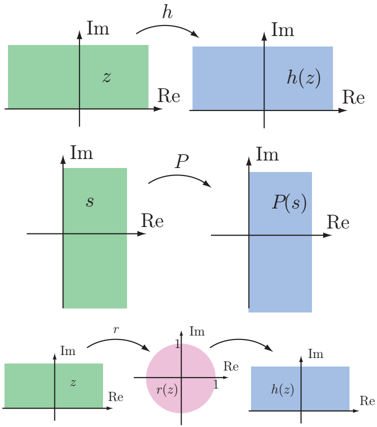
with the related PR-functions, $P(s)$ ($z = \mathrm{i}s$ and $h = \mathrm{i}P$)

$$s, \quad \frac{1}{s}, \quad \sqrt{s}, \quad \tanh(s)$$

Also with the Cayley transform

$$h(z) = \mathrm{i}\frac{1 + r(z)}{1 - r(z)},$$

where $|r(z)| \leq 1$ and holomorphic for $\mathrm{Im}\, z > 0$ (a passive reflection coefficient).



Herglotz, PR, and reflection coefficients.

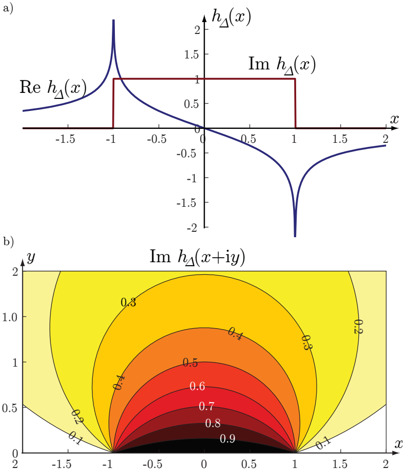
Herglotz functions: pulse function

We can also use the representation (Hilbert transform) to construct Herglotz functions, *e.g.*, the pulse function

$$h_\Delta(z) = \frac{1}{\pi}\int_{|\xi|\leq\Delta} \frac{1}{\xi - z}\,\mathrm{d}\xi = \frac{1}{\pi}\ln\frac{z-\Delta}{z+\Delta}$$

Composition of Herglotz function offers additional possibilities, *e.g.*,

$$h_\Delta(\tan(z)), \quad \text{and} \quad h_\Delta(\tan(-1/z))$$



The Herglotz function $h_\Delta(z)$ with $\Delta = 1$.

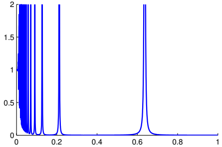
Simple examples of integral identities I

- $\tan(z)$ and $\tan(-1/z)$ are Herglotz functions.
- Asymptotic expansions

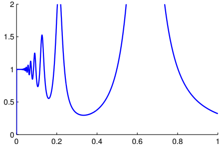
$$\tan(-1/z) \sim \begin{cases} \mathrm{i}, & \text{as} \quad z \hat{\to} 0 \\ -1/z, & \text{as} \quad z \hat{\to} \infty \end{cases}$$

- Note, the limit $z \hat{\to} 0$ is for $0 < \theta \leq \arg(z) \leq \pi - \theta$. The limit for $z = x \to 0$ is not well defined.
- Integral identities for $n = \ldots, 0$, *e.g.*, $n = 0$

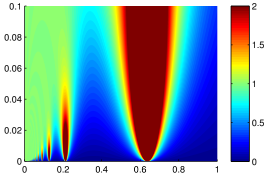
$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{1/\varepsilon} \mathrm{Im}\{\tan \frac{-1}{x + \mathrm{i}y}\} \, \mathrm{d}x = 1$$



$\mathrm{Im}\{\tan(-1/z)\}$, $z = x + \mathrm{i}0.0001$.



$\mathrm{Im}\{\tan(-1/z)\}$, $z = x(1 + \mathrm{i}0.1)$.

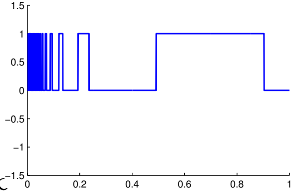


$\mathrm{Im}\{\tan(-1/z)\}$ with $z = x + \mathrm{i}y$.

Simple examples of integral identities II

The Herglotz function $-\cot(-1/z)$ has

$$-\cot(-1/z) = -1/\tan(-1/z) \sim \begin{cases} \mathrm{i}, & \text{as } z\hat{\to}0 \\ z, & \text{as } z\hat{\to}\infty \end{cases}$$



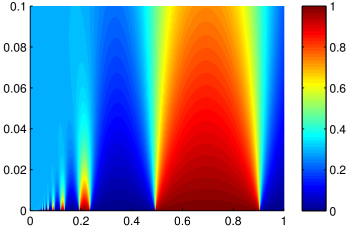
$\mathrm{Im}\{h_1(x)\}$ for $\Delta = 1/2$ in blue with the area (under the blue curve) Δ.

Compose with the pulse function

$$h_1(z) = h_\Delta(-\cot(\frac{-1}{z})) \sim \begin{cases} h_\Delta(\mathrm{i}) = \frac{\mathrm{i}}{2}, & \text{as } z\hat{\to}0 \\ \frac{-2\Delta}{\pi z}, & \text{as } z\hat{\to}\infty \end{cases}$$

Integral identity for $n = 0$

$$\int_0^\infty \mathrm{Im}\{h_1(x))\}\,\mathrm{d}x = \Delta$$



$\mathrm{Im}\{h_1(x+\mathrm{i}y)\}$ for $\Delta = 1/2$, $0 \le x \le 1$, and $0 \le y \le 0.1$.

Simple examples of integral identities II

The Herglotz function $-\cot(-1/z)$ has

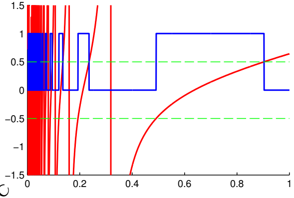
$$-\cot(-1/z) = -1/\tan(-1/z) \sim \begin{cases} \mathrm{i}, & \text{as} \quad z\hat{\to}0 \\ z, & \text{as} \quad z\hat{\to}\infty \end{cases}$$

Compose with the pulse function

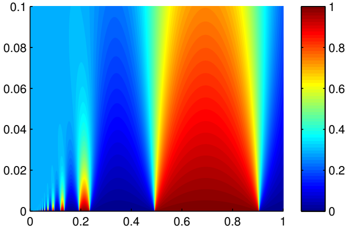
$$h_1(z) = h_\varDelta(-\cot(\frac{-1}{z})) \sim \begin{cases} h_\varDelta(\mathrm{i}) = \frac{\mathrm{i}}{2}, & \text{as } z\hat{\to}0 \\ \frac{-2\varDelta}{\pi z}, & \text{as } z\hat{\to}\infty \end{cases}$$

Integral identity for $n = 0$

$$\int_0^\infty \mathrm{Im}\{h_1(x))\}\,\mathrm{d}x = \varDelta$$



$\mathrm{Im}\{h_1(x)\}$ for $\varDelta = 1/2$ in blue with the area (under the blue curve) $\varDelta$. $\cot(1/z)$ in red.



$\mathrm{Im}\{h_1(x+\mathrm{i}y)\}$ for $\varDelta = 1/2$, $0 \le x \le 1$, and $0 \le y \le 0.1$.

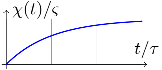
Drude model for the permittivity I

The Drude model is passive (and causal) and can be interpreted as the difference between conductivity and Debye terms

$$\epsilon_{\mathrm{r}}(s) = 1 + \frac{\varsigma}{s(1+s\tau)} = 1 + \frac{\varsigma}{s} - \frac{\varsigma\tau}{1+s\tau} \quad \text{for} \quad \operatorname{Re} s > 0$$

where $\varsigma, \tau \geq 0$. The corresponding time-domain representation is

$$\mathcal{L}^{-1}\{\epsilon_{\mathrm{r}}(s)\}(t) = \delta(t) + \chi(t) = \delta(t) + \varsigma(1 - \mathrm{e}^{-t/\tau})\theta(t)$$



where $\theta(t)$ is the unit step $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$.

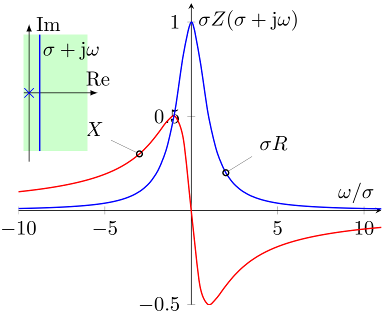
We note that $\epsilon_{\mathrm{r}}(s)$ can be continued to the frequency axis $s = \mathrm{j}\omega$ except for $s = 0$, where we have a simple pole. We can identify the contribution at $s = 0$ as from a point measure with amplitude $\pi\varsigma$ (or as a delta distribution $\varsigma\pi\delta(s)$), see ▸ 80.

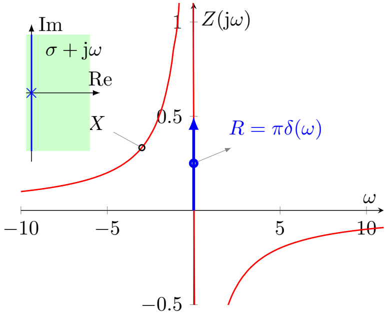
Drude model II

Consider for simplicity the simple pole $Z = 1/s$ with $s = \sigma + \mathrm{j}\omega$

$$\frac{1}{s} = \frac{1}{\sigma + \mathrm{j}\omega} = \frac{\sigma - \mathrm{j}\omega}{\sigma^2 + \omega^2} \quad \text{with} \quad \lim_{\sigma \to 0^+} \int \phi(\omega) \frac{\sigma}{\sigma^2 + \omega^2} \, \mathrm{d}\omega = \pi \phi(0)$$

for smooth functions of compact support $\phi(\omega)$.





Drude model III

An alternative interpretation of the Dirac delta distributions is offered by comparing the Fourier and Laplace transforms 65. The time-domain representation

$$\delta(t) + \chi(t) = \delta(t) + \varsigma(1 - \mathrm{e}^{-t/\tau})\theta(t)$$

has the Laplace transform

$$\epsilon_{\mathrm{r}}(s) = 1 + \frac{\varsigma}{s(1+s\tau)} = 1 + \frac{\varsigma}{s} - \frac{\varsigma\tau}{1+s\tau} \quad \text{for} \quad \operatorname{Re} s > 0 \qquad \text{(L)}$$

and the Fourier transform

$$\epsilon_{\mathrm{r}}(\omega) = 1 + \frac{\varsigma}{\mathrm{j}\omega} - \frac{\varsigma\tau}{1+\mathrm{j}\omega\tau} + \varsigma\pi\delta(\omega) \quad \text{for} \quad \omega \in \mathbb{R} \qquad \text{(F)}$$

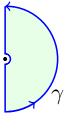
Drude model for the permittivity IV

(L) (Best to directly use the inverse Laplace transform but we can also use) Cauchy's integral theorem can be used for the Laplace (or analytic Fourier) transformed permittivity. We need to use a curve in the open half plane $\mathrm{Re}\, s > 0$ giving

$$\int_\gamma \mathrm{e}^{st} \epsilon_{\mathrm{r}}(s) \,\mathrm{d}s = 0$$

Consider $t < 0$ and a curve γ consisting of the line $s = \sigma$ and a large semi-circle with radius R to get

$$0 = \lim_{R\to\infty} \int_\gamma \mathrm{e}^{st} \epsilon_{\mathrm{r}}(s) \,\mathrm{d}s = -\mathrm{j} \int_{\mathbb{R}} \mathrm{e}^{(\sigma+\mathrm{j}\omega)t} \epsilon_{\mathrm{r}}(\sigma + \mathrm{j}\omega) \,\mathrm{d}\omega$$



The right hand side is recognized as the inverse Laplace transform and shows that the Drude model is causal. We can subsequently let $\sigma \to 0$ (and keep a small semicircle at $s = 0$) to obtain the frequency domain values.

(F) We can also use the inverse Fourier transform (on tempered distributions)

$$\left\langle \mathcal{F}^{-1}\{\epsilon_{\mathrm{r}}(\omega)\}, \phi \right\rangle = \langle \delta(t) + \chi(t), \phi \rangle$$

where it is necessary to consider distributions in both the frequency and time domains.

Drude model for the permittivity IIV

The Drude model

$$\epsilon_{\mathrm{r}}(s) = 1 + \frac{\varsigma}{s(1+s\tau)} = 1 + \frac{\varsigma}{s} - \frac{\varsigma\tau}{1+s\tau}$$

defines the PR function ($\varsigma, \tau \geq 0$)

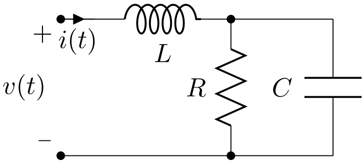
$$P_\epsilon(s) = s\epsilon_{\mathrm{r}}(s) = s + \frac{\varsigma}{1+s\tau}$$

The corresponding impulse response is

$$h(t) = \delta'(t) + \frac{\varsigma}{\tau}\mathrm{e}^{-t/\tau}\theta(t)$$

where $\theta(t)$ is the unit step $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$.

We also note that the PR function is identical to the input impedance of the network, with $L = 1$, $R = \varsigma$, and $C = \tau/\varsigma$.



Integral identities for PR functions

PR functions have asymptotic expansions ($N_0 \geq 0$ and $N_\infty \geq 0$)

$$\begin{cases} P(s) = \sum_{n=0}^{N_0} a_{2n-1} s^{2n-1} + o(s^{2N_0-1}) & \text{as} \quad s \hat{\to} 0 \\ P(s) = \sum_{n=0}^{N_\infty} b_{1-2n} s^{1-2n} + o(s^{1-2N_\infty}) & \text{as} \quad s \hat{\to} \infty \end{cases}$$

They satisfy the identities ($1 - N_\infty \leq n \leq N_0$)

$$\lim_{\varepsilon \to 0^+} \lim_{\sigma \to 0^+} \frac{2}{\pi} \int_\varepsilon^{1/\varepsilon} \frac{\mathrm{Re}\, P(\sigma + \mathrm{j}\omega)}{\omega^{2n}} \, \mathrm{d}\omega = \begin{cases} (-1)^n b_{2n-1} & n < 0 \\ b_{-1} - a_{-1} & n = 0 \\ a_1 - b_1 & n = 1 \\ (-1)^{n+1} a_{2n-1} & n > 1 \end{cases}$$

For notational simplicity the limits are (often) omitted.

Bernland, Luger, Gustafsson, *Sum rules and constraints on passive systems*, J.Phys.A: Math.Theor.,2011.

Integral identities for PR functions: Common cases

Known low-frequency expansion ($a_1 \geq 0$):

$$P(s) \sim \begin{cases} a_1 s & \text{as} \quad s \hat{\to} 0 \\ b_1 s & \text{as} \quad s \hat{\to} \infty \end{cases}$$

that gives the $n = 1$ identity (we drop the limits for simplicity)

$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{\operatorname{Re} P(\sigma + \mathrm{j}\omega)}{\omega^2} \, \mathrm{d}\omega \stackrel{\text{def}}{=} \frac{2}{\pi} \int_0^{\infty} \frac{\operatorname{Re} P(\mathrm{j}\omega)}{\omega^2} \, \mathrm{d}\omega = a_1 - b_1 \leq a_1$$

Known high-frequency expansion ($b_{-1} \geq 0$):

$$P(s) \sim \begin{cases} a_{-1}/s & \text{as} \quad s \hat{\to} 0 \\ b_{-1}/s & \text{as} \quad s \hat{\to} \infty \end{cases}$$

that gives the $n = 0$ identity

$$\frac{2}{\pi} \int_0^{\infty} \operatorname{Re} P(\mathrm{j}\omega) \, \mathrm{d}\omega = b_{-1} - a_{-1} \leq b_{-1}.$$

Are lossless one-ports lossless? (details time domain)

Consider the causal input current $i(t) = I_0 \sin(\omega t)$ for $t > 0$ and $i(t) = 0$ for $t < 0$. We have the ODE

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \omega_0^2 v = \frac{1}{C}\frac{\mathrm{d}i}{\mathrm{d}t}$$

with the impulse response

$$h(t) = \omega_0^{-1}\sin(\omega_0 t) \quad \text{for} \quad t > 0 \quad \text{and} \quad h(t) = 0 \quad \text{for} \quad t < 0$$

and the solution for $\omega = \omega_0 = 1/\sqrt{LC}$ and using $Z_0 = \sqrt{L/C}$

$$v(t) = Z_0 \int_{-\infty}^{t} \sin(\omega(t-\tau))\frac{\mathrm{d}i}{\mathrm{d}\tau}(\tau)\,\mathrm{d}\tau = \omega I_0 Z_0 \int_0^t \sin(\omega(t-\tau))\cos(\omega\tau)\,\mathrm{d}\tau = \frac{\omega I_0 Z_0}{2}\int_0^t (\sin(\omega t) + \sin(\omega t - 2\omega\tau))\,\mathrm{d}\tau$$

$$= \frac{\omega I_0 Z_0}{2}\left[\sin(\omega t)\tau - \frac{\cos(\omega t - 2\omega\tau)}{2\omega}\right]_0^t = \frac{\omega I_0 Z_0}{2}\left(\sin(\omega t)t - \frac{\cos(\omega t)}{2\omega} + \frac{\cos(\omega t)}{2\omega}\right) = \frac{\omega I_0 Z_0}{2}\sin(\omega t)t,$$

where it is noted that the amplitude of $v(t)$ increases with t which is consistent with the pole at ω_0.
The power $i(t)v(t) = \frac{\omega I_0^2 Z_0}{2}\sin^2(\omega t)t > 0$ and the energy

$$\mathcal{W}(T) = \frac{\omega I_0^2 Z_0}{2}\int_0^T \sin^2(\omega t)t\,\mathrm{d}t = \frac{\omega I_0^2 Z_0}{4}\int_0^T t - t\cos(2\omega t)\,\mathrm{d}t = \frac{\omega I_0^2 Z_0}{4}\left(\left[\frac{t^2}{2} - \frac{t\sin(2\omega t)}{2\omega}\right]_0^T - \int_0^T \frac{\sin(2\omega t)}{2\omega}\,\mathrm{d}t\right)$$

$$= \frac{\omega I_0^2 Z_0}{4}\left(\frac{T^2}{2} - \frac{T\sin(2\omega T)}{2\omega} + \left[\frac{\cos(2\omega t)}{4\omega^2}\right]_0^T\right) = \frac{I_0^2 L}{8}\left(\omega^2 T^2 - T\omega\sin(2\omega T) + \frac{1}{2} - \frac{\cos(2\omega T)}{2}\right)$$

The corresponding stored energies in the capacitor and inductor are ($\omega Z_0 C = Z_0/(\omega L) = 1$)

$$\mathcal{W}_{\mathrm{C}}(T) = \frac{Cv^2(T)}{2} = \frac{I_0^2 L}{8}\omega^2 T^2\sin^2(\omega T),\ \mathcal{W}_{\mathrm{L}}(T) = \frac{Li_{\mathrm{L}}^2(T)}{2} = \frac{\omega^2 Z_0^2 I_0^2}{8L}\Big(\int_0^T \sin(\omega t)t\,\mathrm{d}t\Big)^2 = \frac{LI_0^2}{8}(\sin(\omega T) - \omega T\cos(\omega T))^2$$

that verifies that the energy is stored in the capacitor and inductor, *i.e.*,

$$\mathcal{W}_{\mathrm{C}}(T) + \mathcal{W}_{\mathrm{L}}(T) = \frac{LI_0^2}{8}\left(\omega^2 T^2 - \omega T 2\sin(\omega T)\cos(\omega T) + \sin^2(\omega T)\right) = \mathcal{W}(T)$$

Outline

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