Stored Energy and Antenna Current Optimization

Mats Gustafsson

Electrical and Information Technology, Lund University, Sweden

Slides at www.eit.lth.se/staff/mats.gustafsson

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- Lars Jonsson, KTH





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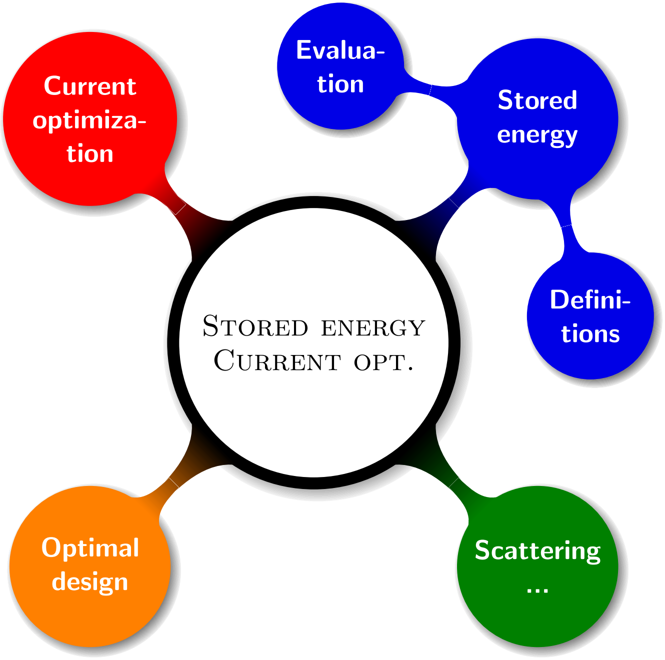

Stored energy
Current opt.
Current optimiza-tion
Evalua-tion
Stored energy
Defini-tions
Optimal design
Scattering
...

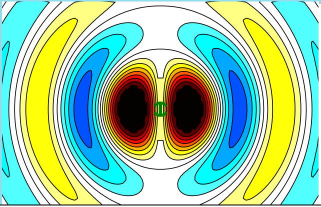
Evalua-
tion

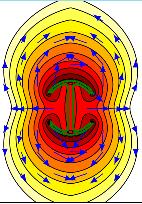
Stored
energy

Defini-
tions

Stored energy for radiating structures (antennas)

- What is it? What is it used for?
- How is it evaluated?
- Here, time harmonic fields
- Different definitions. Consensus for
  - Small structures (sub wavelength)
  - Free space (vacuum)
- What about:
  - Larger structures
  - Inhomogeneous materials
  - Temporal dispersion





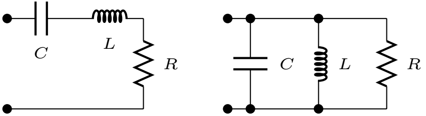
Q-factor and stored energy

The Q-factor is defined as the ratio between the stored electric, W_e, and magnetic, W_m, energies and the dissipated power, *i.e.*,

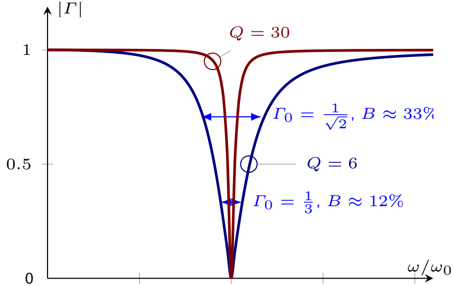
$$Q = \frac{2\omega \max\{W_\mathrm{e}, W_\mathrm{m}\}}{P_\mathrm{rad} + P_\mathrm{loss}}.$$

Fractional bandwidth for single resonances (RLC circuits) Yaghjian and Best 2005

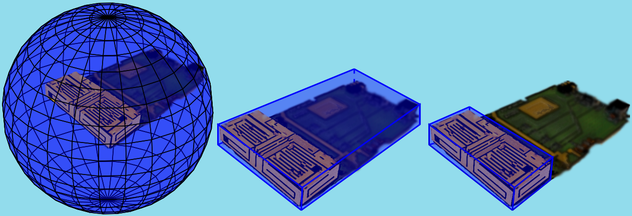
$$B \approx \frac{2}{Q}\frac{\Gamma_0}{\sqrt{1-\Gamma_0^2}}$$



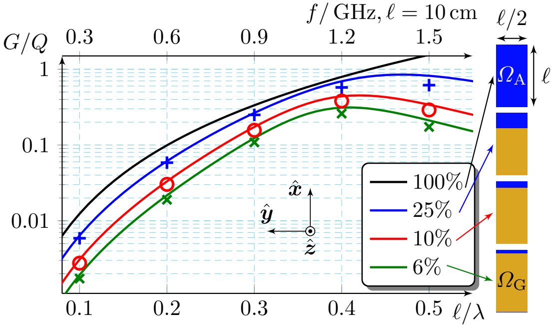
$$W_\mathrm{e} = \frac{C|V|^2}{4} \quad \text{and } W_\mathrm{m} = \frac{L|I|^2}{4}$$

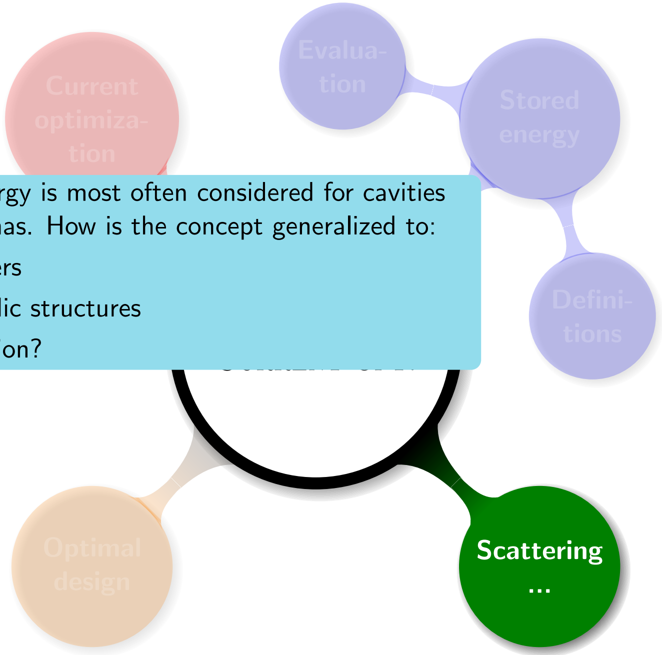


- Optimize over the current density
  - Maximum G/Q (Gain/Q-factor).
  - Minimum Q for prescribed radiated field.
  - Minimum Q for superdirectivity.
  - Efficiency
- Physical bounds for given size and geometry.
- Consensus for small antennas in free space.
- What about larger structures, embedded antennas, ...?



- Optimal design of antennas in a given geometry.
  - Optimization of the currents for optimal performance.
  - Convex optimization problems.
- Optimization of the device
  - Heuristic optimization algorithms (GA,...)
  - How do we use the currents?

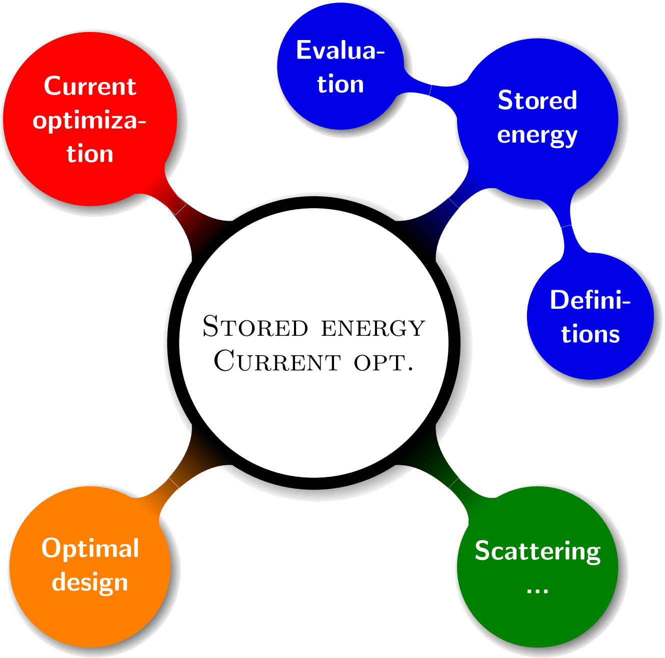




Stored energy is most often considered for cavities and antennas. How is the concept generalized to:

- Scatters
- Periodic structures

Interpretation?


Stored energy
Current opt.
Current optimiza-tion
Evalua-tion
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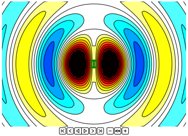
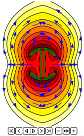
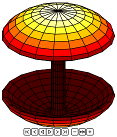
Defini-tions
Optimal design
Scattering
...

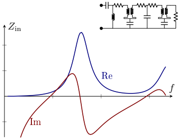
Stored electromagnetic energy

- Where is the energy stored?
  - Fields
  - Currents
  - Feed structure
- Stored according to what?
  - Input impedance
  - Material
  - Scatterer
- Why are we interested?
  - Physics, EM-theory
  - Antenna bandwidth
  - Physical bounds



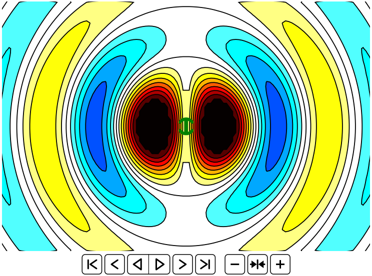


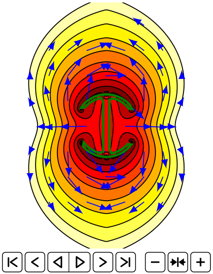


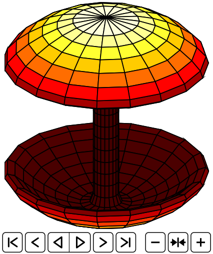


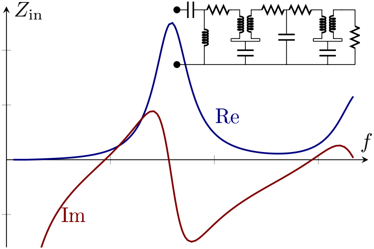
There are several proposals for the stored energy. They agree for many cases but differ for some. Differences often due to different interpretations, assumptions, and applications.

Stored energy expressed in fields, currents, and circuits

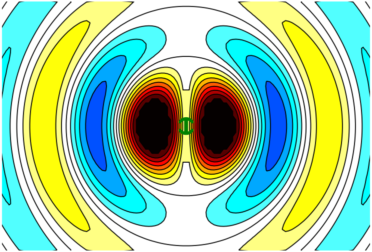








Stored energy expressed in fields, currents, and circuits



$$\int_{\mathbb{R}^3} \frac{\epsilon_0}{4} |\boldsymbol{E}(\boldsymbol{r})|^2 \,\mathrm{d}V = \infty$$

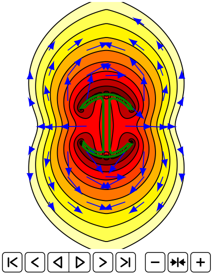
For time harmonic fields.

- Everything known...
- Time average electric energy density $\epsilon_0 |\boldsymbol{E}|^2/4$. Also known for temporally dispersive media (Loudon 1970; Ruppin 2002; Tretyakov 2005).
- Unbounded total energy (integration of the energy density over $\mathbb{R}^3$). Need to subtract something to get the stored energy.

Stored energy expressed in **fields**, currents, and circuits

$$W_{\mathrm{F}}^{(\mathrm{E})} = \frac{\epsilon_0}{4} \int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}(\boldsymbol{r})|^2 - \frac{|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2}{r^2} \,\mathrm{dV}$$

also expressed in the input reactance and far field. Alternatively with subtraction of the power flow.



- Many results for spherical modes Collin and Rothschild 1964; Fante 1969, ...
- Coordinate dependent for non-symmetric far-fields $\boldsymbol{F}$

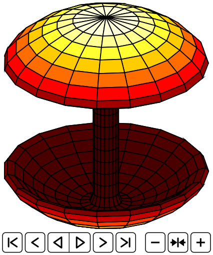
$$\frac{\epsilon_0}{4}\boldsymbol{d} \cdot \int_{\Omega} \hat{\boldsymbol{r}} |\boldsymbol{F}(\hat{\boldsymbol{r}})|^2 \,\mathrm{d}\Omega \neq 0$$

  (Gustafsson and Jonsson 2015b; Yaghjian and Best 2005)
- Difficult to generalize to antennas embedded in lossy media (vanishing far field $\boldsymbol{F} = \boldsymbol{0}$).

Stored energy expressed in fields, **currents**, and circuits

Stored electric energy by Vandenbosch 2010 (Geyi 2003b, $ka \to 0$).

$$W_{\mathrm{e}} = \frac{\eta_0}{4\omega} \int_{\Omega} \int_{\Omega} \nabla_1 \cdot \boldsymbol{J}(\boldsymbol{r}_1) \nabla_2 \cdot \boldsymbol{J}^*(\boldsymbol{r}_2) \frac{\cos(k|\boldsymbol{r}_1 - \boldsymbol{r}_2|)}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|} - \frac{k}{2}\big(k^2 \boldsymbol{J}(\boldsymbol{r}_1)\cdot\boldsymbol{J}^*(\boldsymbol{r}_2) - \nabla_1\cdot\boldsymbol{J}(\boldsymbol{r}_1)\nabla_2\cdot\boldsymbol{J}^*(\boldsymbol{r}_2)\big) \sin(k|\boldsymbol{r}_1 - \boldsymbol{r}_2|) \,\mathrm{dV}_1 \,\mathrm{dV}_2$$



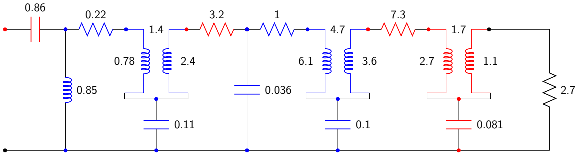
- Derived from the subtracted far-field energy (Gustafsson and Jonsson 2015b; Vandenbosch 2010).
- Negative values (Gustafsson, Cismasu, and Jonsson 2012).
- Need only the current density.
- Can be used in convex optimization (Gustafsson and Nordebo 2013).

Stored energy expressed in fields, currents, and **circuits**

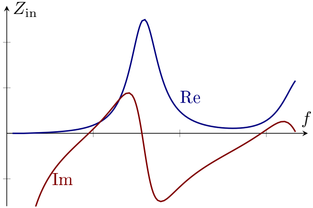
Use the input impedance $Z_{\mathrm{in}}(\omega)$ to estimate the stored energy.

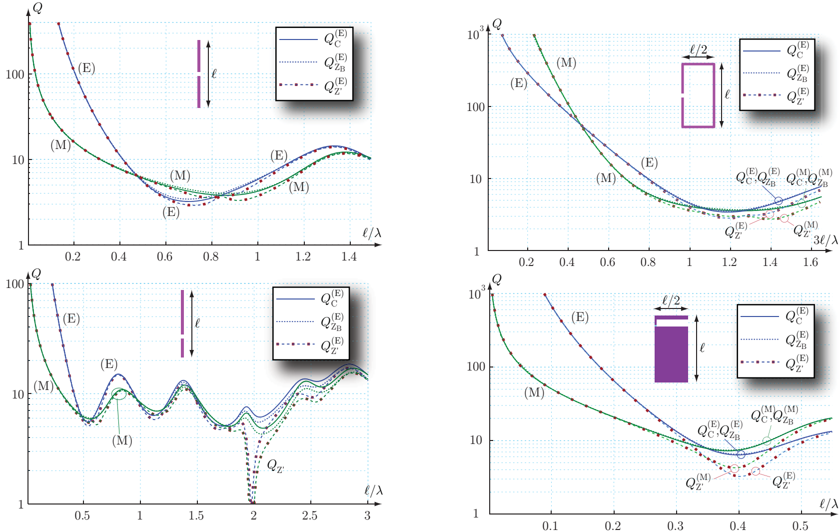
Differentiation $Q_{\mathrm{Z'}} = \omega|Z'_{\mathrm{in}}|/(2R_{\mathrm{in}}) = \omega|\Gamma'|$ (Yaghjian and Best 2005).

Brune synthesis (Brune 1931) synthesized circuit (Gustafsson and Jonsson 2015a). Here Q-factor $Q_{\mathrm{Z_B}}$

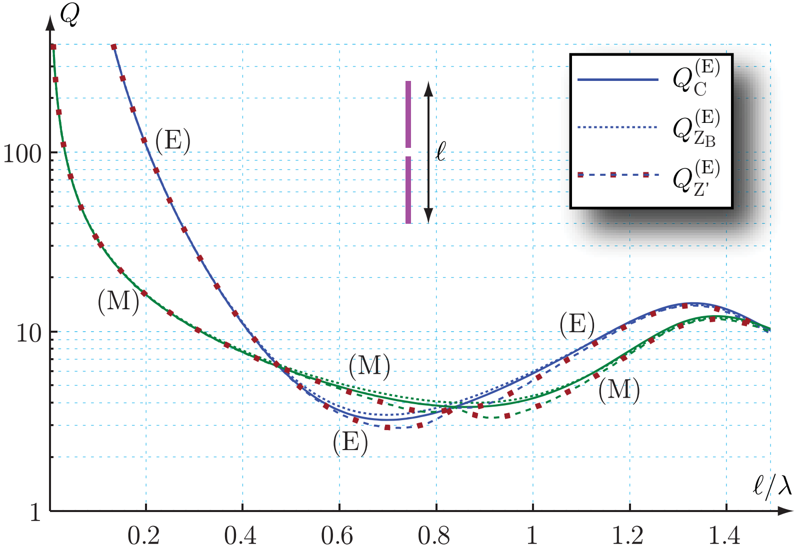


- Q-factors related to Z_{in} and hence to the bandwidth.
- Constructed from the antenna (geometry and feed).
- Difficult to use in optimization.

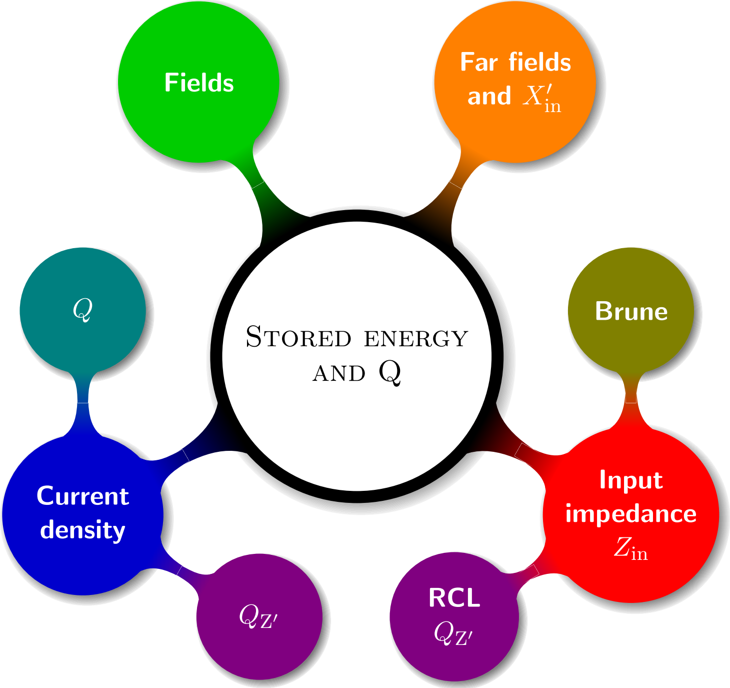


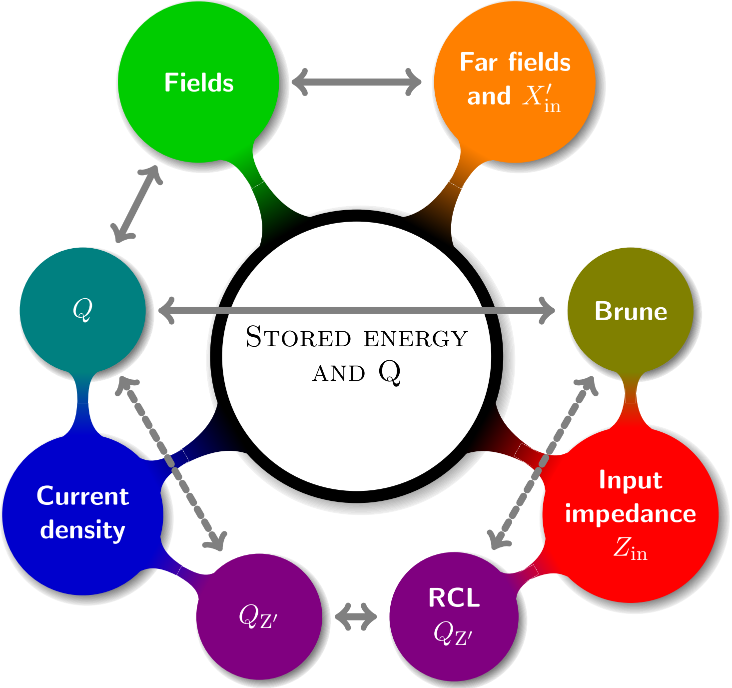


Agree well except for cases with multiple resonances.



Gustafsson and Jonsson 2015a


Stored energy and Q
Fields
Far fields and X'_{in}
Q
Brune
Current density
Input impedance Z_{in}
$Q_{Z'}$
RCL $Q_{Z'}$


Fields
Far fields and X'_{in}
Stored energy and Q
Q
Brune
Current density
Input impedance Z_{in}
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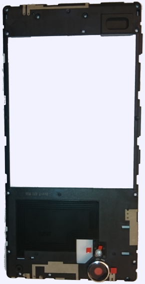
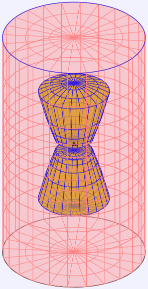
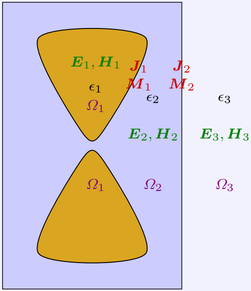
- Good understanding for small antennas in free space.
- The different approaches agree well and the differences are understood.
- Differentiated method-of-moments (MoM) matrices in Harrington and Mautz 1972.
- Time-domain derivation in Vandenbosch 2013b.
- Magnetic current densities in Jonsson and Gustafsson 2015; Jonsson and Gustafsson 2016; Kim 2016.
- Other approaches to define stored energy, see Capek, Jelinek, and Vandenbosch 2016; Carpenter 1989; Kaiser 2011; Mikki and Antar 2011.

- Small antennas in free space is a very important case but there are many other cases.
- The antenna region can be small but the radiating structure large.
- What about antennas embedded in an inhomogeneous dispersive media?
- How do we define/evaluate the stored energy?



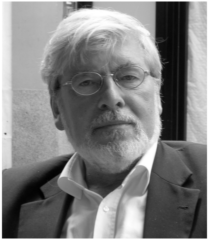




A system approach to stored energy

Can one determine the stored energy from the in- and output signals?

Jan C. Willems, Dissipative Dynamical Systems II. Linear Systems with Quadratic Supply Rates, Archive Rat. Mech. Anal. 1972



Jan C. Willems (1939-2013)

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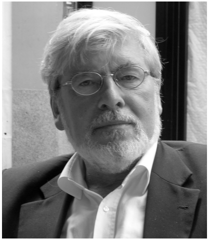
- State-space models
- Minimal (observable and controllable)
- Reciprocity



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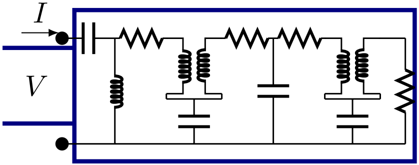
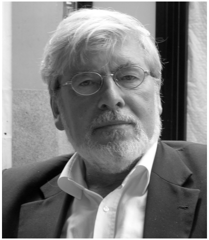




Jan C. Willems (1939-2013)

- State-space models
- Minimal (observable and controllable)
- Reciprocity

Brune circuit synthesis is one approach.

Brune circuit

The circuit network with input impedance $Z_{\rm in} = V_{\rm in}/I_{\rm in}$ can be written ($s = {\rm j}\omega$)

$$\mathbf{Z}\mathbf{I} = (\mathbf{R} + s\mathbf{L} + \frac{1}{s}\mathbf{C}_{\rm i})\mathbf{I} = \mathbf{V} = \mathbf{B}V_{\rm in}$$

and

$$I_{\rm in} = \mathbf{B}^{\rm T}\mathbf{I}$$

where the impedance matrix is decomposed in its resistance $\mathbf{R}$, inductance $\mathbf{L}$, and $\mathbf{C} = \mathbf{C}_{\rm i}^{-1}$ capacitance matrices. As a first order model Willems 1972b

$$s\begin{pmatrix}\mathbf{L} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}\end{pmatrix}\begin{pmatrix}\mathbf{I}\\ \mathbf{U}\end{pmatrix} + \begin{pmatrix}\mathbf{R} & \mathbf{1}\\ -\mathbf{1} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{I}\\ \mathbf{U}\end{pmatrix} = \begin{pmatrix}\mathbf{V}\\ \mathbf{0}\end{pmatrix}$$

where the voltage state $\mathbf{U} = \frac{1}{s}\mathbf{C}_{\rm i}\mathbf{I}$ is introduced.
How do we determine the stored energy?

Stored energy

Time domain ($s \to \frac{\partial}{\partial t}$)

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{L} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}\end{pmatrix}\begin{pmatrix}\mathbf{I}\\ \mathbf{U}\end{pmatrix}+\begin{pmatrix}\mathbf{R} & \mathbf{1}\\ -\mathbf{1} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{I}\\ \mathbf{U}\end{pmatrix}\begin{pmatrix}\mathbf{V}\\ \mathbf{0}\end{pmatrix}$$

Multiply with the states and integrate

$$\left[\frac{\mathbf{I}^{\mathrm{T}}\mathbf{L}\mathbf{I}+\mathbf{U}^{\mathrm{T}}\mathbf{C}\mathbf{U}}{2}\right]_{t_1}^{t_2}+\int_{t_1}^{t_2}\mathbf{I}^{\mathrm{T}}\mathbf{R}\mathbf{I}\,\mathrm{d}t=\int_{t_1}^{t_2}\mathbf{I}^{\mathrm{T}}\mathbf{V}\,\mathrm{d}t$$

Stored energy from the quadratic form obtained form the differentiated term (term proportional to s).

1. Construct a 'symmetric' state-space model
2. Differentiate the system matrix with respect to s
3. Stored energy from the quadratic form

Stored energy for electric currents in free space I

A MoM implementation of the EFIE determines the impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$ can be written

$$\mathbf{Z} = s\mu\mathbf{L} + \frac{1}{s\epsilon}\mathbf{C}_{\mathrm{i}}$$

where the matrix $\mathbf{L}$ has the elements

$$L_{mn} = \int_{\Omega}\int_{\Omega} \boldsymbol{\psi}_m(\boldsymbol{r}_1) \cdot \boldsymbol{\psi}_n(\boldsymbol{r}_2) \frac{\mathrm{e}^{-\kappa|\boldsymbol{r}_1 - \boldsymbol{r}_2|}}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \,\mathrm{dS}_1 \,\mathrm{dS}_2,$$

with $\kappa = \sqrt{s^2\epsilon\mu}$ and the matrix $\mathbf{C}_{\mathrm{i}}$ with the elements

$$C_{\mathrm{i}mn} = \int_{\Omega}\int_{\Omega} \nabla_1 \cdot \boldsymbol{\psi}_m(\boldsymbol{r}_1) \nabla_2 \cdot \boldsymbol{\psi}_n(\boldsymbol{r}_2) \frac{\mathrm{e}^{-\kappa|\boldsymbol{r}_1 - \boldsymbol{r}_2|}}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \,\mathrm{dS}_1 \,\mathrm{dS}_2.$$

Stored energy for electric currents in free space II

Introduce a voltage state $\mathbf{U} = \frac{1}{s\epsilon}\mathbf{C}_{\mathrm{i}}\mathbf{I}$ to get

$$\mathbf{Z}\mathbf{I} = (s\mu\mathbf{L} + \frac{1}{s\epsilon}\mathbf{C}_{\mathrm{i}})\mathbf{I} = s\mu\mathbf{L}\mathbf{I} + \mathbf{U} = \mathbf{V}$$

and the *state-space model* (Willems 1972a) note $s \to \frac{\partial}{\partial t}$

$$s\begin{pmatrix}\mu\mathbf{L} & \mathbf{0}\\ \mathbf{0} & \epsilon\mathbf{C}\end{pmatrix}\begin{pmatrix}\mathbf{I}\\ \mathbf{U}\end{pmatrix} = \begin{pmatrix}\mathbf{0} & -\mathbf{1}\\ \mathbf{1} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{I}\\ \mathbf{U}\end{pmatrix} + \begin{pmatrix}\mathbf{V}\\ \mathbf{0}\end{pmatrix}$$

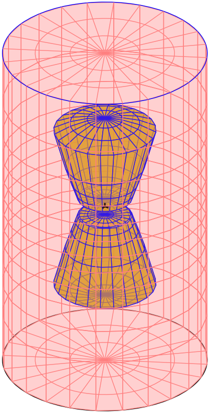
Use differentiation with respect to s of the state-space model to determine the term that is proportional to s and hence the time average stored energy

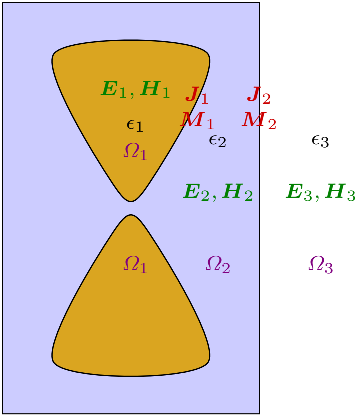
$$W = \frac{\mathrm{Re}}{4}\begin{pmatrix}\mathbf{I}\\ \mathbf{U}\end{pmatrix}^{\mathsf{H}}\begin{pmatrix}\mu_0(\mathbf{L} + \mathrm{j}\omega\mathbf{L}') & \mathbf{0}\\ \mathbf{0} & \epsilon_0(\mathbf{C} + \mathrm{j}\omega\mathbf{C}')\end{pmatrix}\begin{pmatrix}\mathbf{I}\\ \mathbf{U}\end{pmatrix}$$

$$\simeq \frac{\mathrm{Re}}{4}\mathbf{I}^{\mathsf{H}}\big(\mu_0(\mathbf{L} + \mathrm{j}\omega\mathbf{L}') + \frac{1}{\omega^2\epsilon_0}(\mathbf{C}_{\mathrm{i}} - \mathrm{j}\omega\mathbf{C}_{\mathrm{i}}')\big)\mathbf{I} = \frac{1}{4}\mathbf{I}^{\mathsf{H}}\frac{\partial\mathbf{X}}{\partial\omega}\mathbf{I}$$

Identical to (Harrington and Mautz 1972; Vandenbosch 2010).

Antennas in inhomogeneous dispersive media





- Antenna on/in the body. Debye/conductivity type dispersion.
- Optical antennas with Drude/Lorentz type dispersion.
- Stored energy. What is it?
- Note, the far field $\boldsymbol{F} = \boldsymbol{0}$ in a lossy background.

Stored energy for PEC antennas in Lorentz media I

The EFIE is valid in a homogeneous dispersive media

$$\mathbf{Z}\mathbf{I} = (s\mu\mathbf{L} + \frac{1}{s\epsilon}\mathbf{C}_{\mathrm{i}})\mathbf{I} = s\mu\mathbf{L}\mathbf{I} + \mathbf{U} = \mathbf{V}$$

Use the voltage state $\mathbf{U} = \frac{1}{s\epsilon}\mathbf{C}_{\mathrm{i}}\mathbf{I}$ as in the free-space case. The voltage is further rewritten to a first order system by introduction of the polarization state $\mathbf{P}$ and its temporal derivative $\dot{\mathbf{P}} = \beta^{-1}s\mathbf{P}$

$$\mathbf{I} = s\epsilon\mathbf{C}\mathbf{U} = (s\epsilon_\infty + \frac{s\alpha^2}{\beta^2 + \gamma s + \delta s^2})\mathbf{C}\mathbf{U} = s\epsilon_\infty\mathbf{C}\mathbf{U} + \alpha\dot{\mathbf{P}}$$

where $\mathbf{C} = \mathbf{C}_{\mathrm{i}}^{-1}$ is used for simplicity. The term $\frac{1}{\alpha\beta}(\beta^2 + \gamma s + \delta s^2)\mathbf{P} = \mathbf{C}\mathbf{U}$ is rewritten as

$$(\beta^2 + \gamma s + \delta s^2)\frac{1}{\beta}\mathbf{C}_{\mathrm{i}}\mathbf{P} = \beta\mathbf{C}_{\mathrm{i}}\mathbf{P} + (\gamma + \delta s)\mathbf{C}_{\mathrm{i}}\dot{\mathbf{P}} = \alpha\mathbf{U} = \frac{\alpha}{s\epsilon}\mathbf{C}_{\mathrm{i}}\mathbf{I}$$

Stored energy for PEC antennas in Lorentz media II

Rewrite as a linear system

$$\widetilde{\mathbf{Z}}\widetilde{\mathbf{I}} = \begin{pmatrix} s\mu\mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\mathbf{1} & s\epsilon_\infty\mathbf{C} & \mathbf{0} & \mathbf{1}\alpha \\ \mathbf{0} & \mathbf{0} & s\mathbf{C}_\mathrm{i} & -\beta\mathbf{C}_\mathrm{i} \\ \mathbf{0} & -\mathbf{1}\alpha & \beta\mathbf{C}_\mathrm{i} & (s\delta+\gamma)\mathbf{C}_\mathrm{i} \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{U} \\ \mathbf{P} \\ \dot{\mathbf{P}} \end{pmatrix} = \begin{pmatrix} \mathbf{V} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

This is a classical state-space representation (Willems 1972b) if the matrices $\mathbf{L}$ and $\mathbf{C}_\mathrm{i}$ are independent of s.
Approximate using frequency differentiation

$$\widetilde{\mathbf{Z}}' = \begin{pmatrix} \mu_\mathrm{r}\mathbf{L} + s\mu_\mathrm{r}\mathbf{L}' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \epsilon_\infty\mathbf{C} - s\epsilon_\infty\mathbf{C}\mathbf{C}_\mathrm{i}'\mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_\mathrm{i} + s\mathbf{C}_\mathrm{i}' & -\beta\mathbf{C}_\mathrm{i}' \\ \mathbf{0} & \mathbf{0} & \beta\mathbf{C}_\mathrm{i}' & \delta\mathbf{C}_\mathrm{i} + (s\delta+\gamma)\mathbf{C}_\mathrm{i}' \end{pmatrix}$$

Stored energy from the quadratic forms $\widetilde{\mathbf{I}}^\mathsf{H}\widetilde{\mathbf{X}}'\widetilde{\mathbf{I}}$.

Stored energy for PEC antennas in dispersive media III

The stored energy from the quadratic form $\widetilde{\mathbf{I}}^{\mathsf{H}}\widetilde{\mathbf{X}}'\widetilde{\mathbf{I}}$ can be simplified as

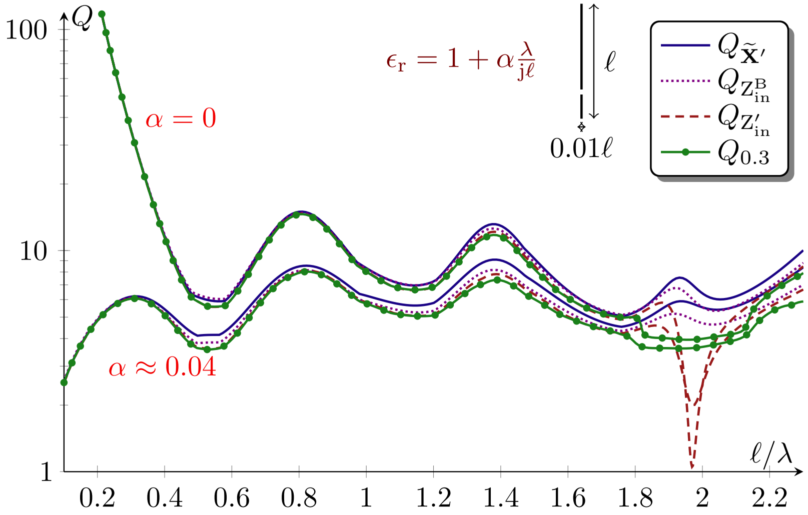
$$W = \frac{1}{4}\mathbf{I}^{\mathsf{H}}\bigg(\mu\mathbf{L} + \frac{\epsilon_\infty}{|\omega\epsilon|^2}\mathbf{C}_{\mathrm{i}} + \mathrm{j}\omega\mu\mathbf{L}' - \frac{\mathrm{j}\omega\epsilon_\infty}{|\omega\epsilon|^2}\mathbf{C}_{\mathrm{i}}' + \frac{\alpha^2}{|\omega\epsilon|^2|\chi|^2}\left((\beta^2+\omega^2\delta)\mathbf{C}_{\mathrm{i}} - \mathrm{j}\omega\chi\mathbf{C}_{\mathrm{i}}'\right)\bigg)\mathbf{I}$$

Note that the Lorentz model

$$\epsilon(s) = \epsilon_\infty + \frac{\alpha^2}{\beta^2+\gamma s+\delta s^2} = \epsilon_\infty + \frac{\alpha^2}{\chi(s)}$$

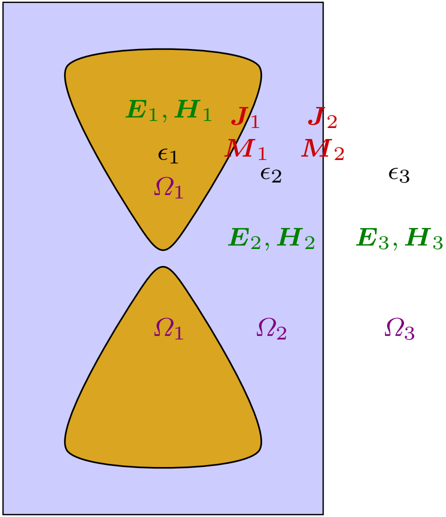
reduces to the conductivity ($\beta=\delta=0$), Debye ($\delta=0$), and Drude ($\beta=0$) models. Add additional states for multiple Lorentz terms and $\mu_{\mathrm{r}}(s)$.

Strip dipole in a conductive background media

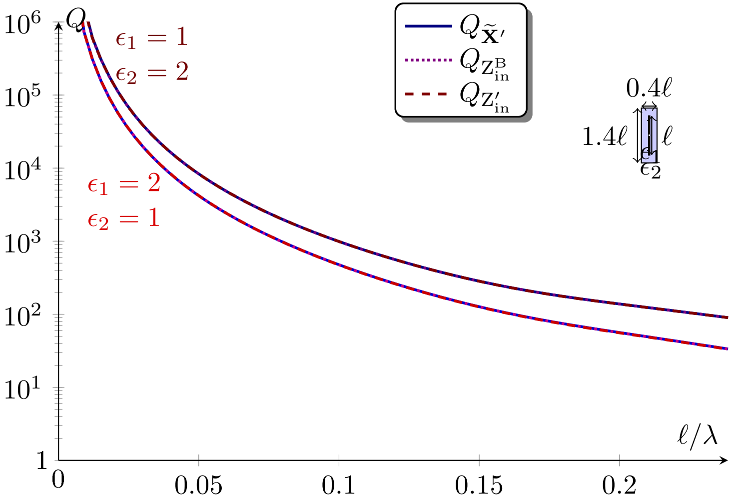


MoM for inhomogeneous media

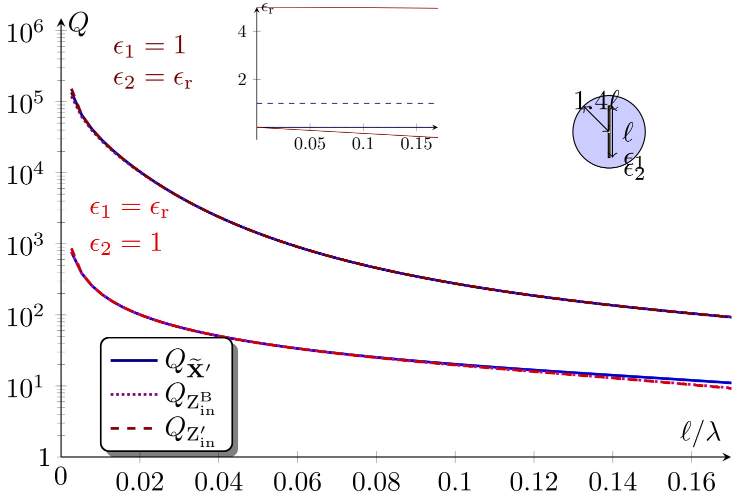
- Determine the equivalent surface currents $\boldsymbol{J}_n$ and $\boldsymbol{M}_n$ at the boundaries for Ω_m
- Müller or PMCHWT (Poggio, Miller, Chang, Harrington, Wu, Tsai) formulation.
- State-space approach for the stored energy.



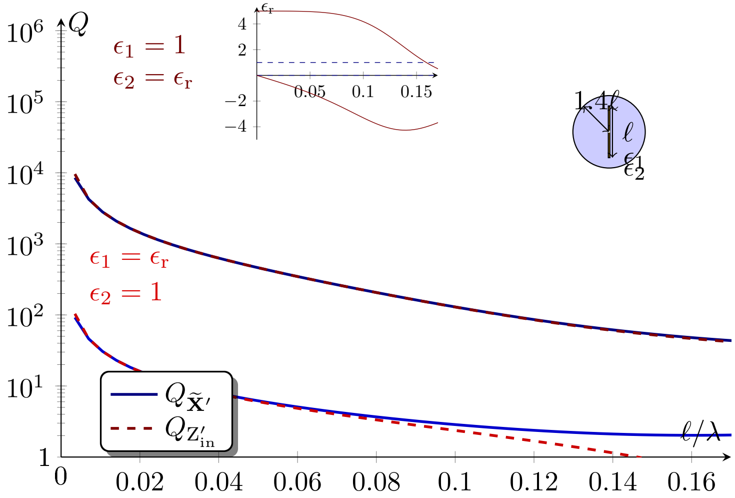
Cylindrical dipole in a cylindrical dielectric

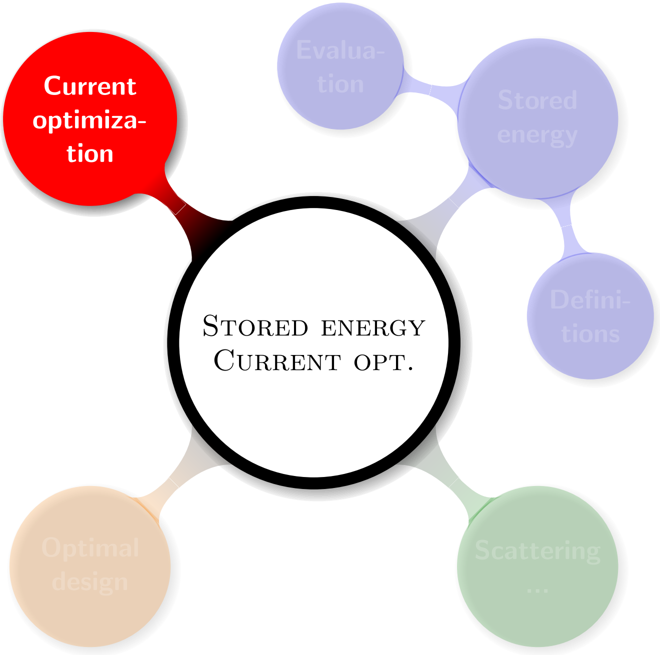


Cylindrical dipole in a spherical Debye region

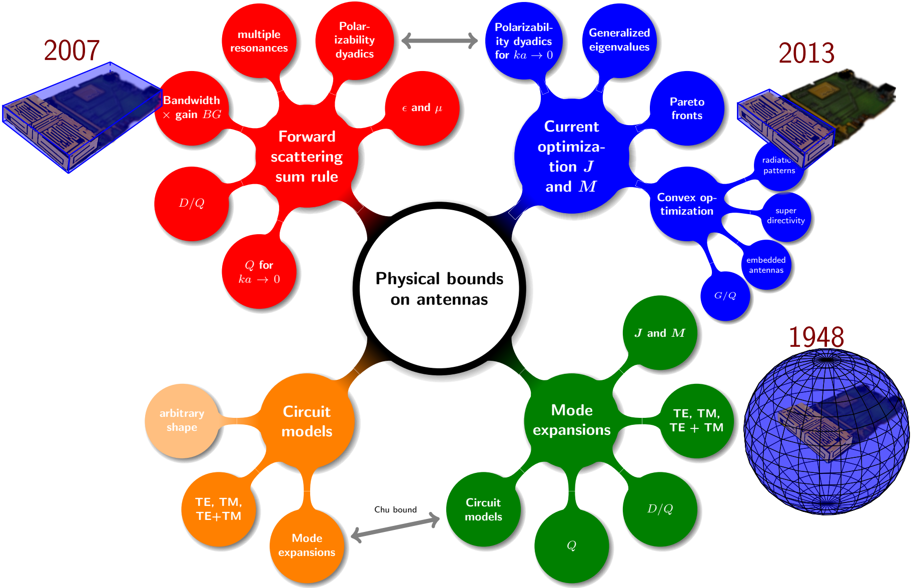


Cylindrical dipole in a spherical Lorentz region




Current optimiza-tion
Evalua-tion
Stored energy
Defini-tions
Stored energy
Current opt.
Optimal design
Scattering
...

Physical bounds on antennas: methods



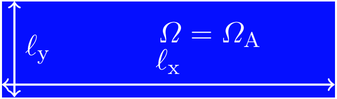
Gustafsson *etal*, Physical Bounds of Antennas, in Handbook of Antenna Technologies, Springer, 2015

Antenna and antenna current optimization

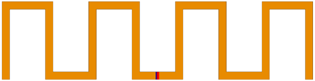
Device structure Ω with a maximal size for the antenna region Ω_{A}.

- **Antenna optimization:** determine the shape, material, and feed properties for optimal performance.
- **Antenna current optimization:** synthesize an optimal current distribution in the available geometry.

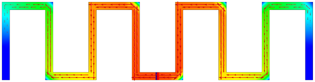
Maximal size of the antenna



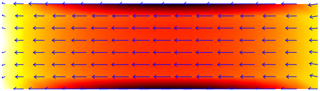
Antenna geometry with feed point



Current distribution on the antenna



Current distribution in the antenna antenna region



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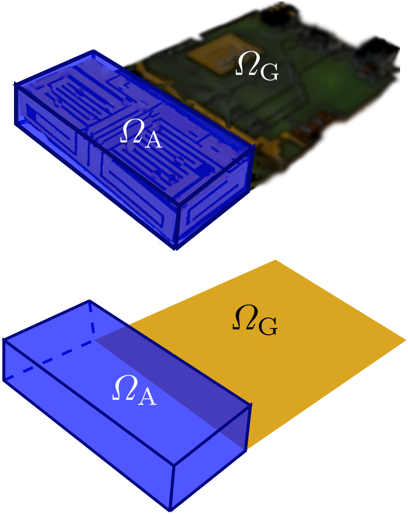


Optimization of antenna currents: examples

Gain over Q

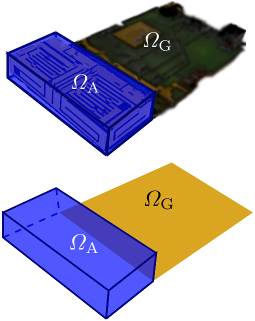
$$\begin{aligned} &\text{minimize} && \text{Stored energy} \\ &\text{subject to} && \text{Radiation intensity} = P_0 \end{aligned}$$

Q for superdirectivity $D \geq D_0$.

$$\begin{aligned} &\text{minimize} && \text{Stored energy} \\ &\text{subject to} && \text{Radiation intensity} = D_0 P_{\text{rad}}/(4\pi) \\ &&& \text{Radiated power} \leq P_{\text{rad}} \end{aligned}$$

Embedded structures

$$\begin{aligned} &\text{minimize} && \text{Stored energy} \\ &\text{subject to} && \text{Radiation intensity} = P_0 \\ &&& \text{Correct induced currents} \end{aligned}$$



Need to:

1. Express the *stored energy* in the current density $\boldsymbol{J}$.
2. Solve the optimization problems.

Matrix expressions for the stored EM energies

Method of Moments approximation (expand $\boldsymbol{J}$ in basis functions)

$$W_{\mathrm{e}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \quad \text{stored E-energy, } \mathbf{X}_{\mathrm{e}} \text{ electric reactance}$$

$$W_{\mathrm{m}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \quad \text{stored M-energy, } \mathbf{X}_{\mathrm{m}} \text{ magnetic reactance}$$

$$P_{\mathrm{rad}} \approx \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I} \quad \text{radiated power}$$

giving $\mathbf{Z} = \mathbf{R} + \mathrm{j}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})$. We also use

$$\hat{\boldsymbol{e}}^{*} \cdot \boldsymbol{F} \approx \mathbf{F}\mathbf{I} \quad \text{far field}$$

$$\boldsymbol{E} \approx \mathbf{N}\mathbf{I} \quad \text{near field}$$

$$\mathbf{I}_{\mathrm{G}} \approx \mathbf{C}\mathbf{I}_{\mathrm{A}} \quad \text{induced current on a PEC}$$

Matrix expressions for the stored EM energies

Method of Moments approximation (expand $\boldsymbol{J}$ in basis functions)

$$W_{\mathrm{e}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}$$ stored E-energy, $\mathbf{X}_{\mathrm{e}}$ electric reactance

$$W_{\mathrm{m}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}$$ stored M-energy, $\mathbf{X}_{\mathrm{m}}$ magnetic reactance

$$P_{\mathrm{rad}} \approx \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}$$ radiated power

giving $\mathbf{Z} = \mathbf{R} + \mathrm{j}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})$. We also use

$$\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F} \approx \mathbf{F}\mathbf{I}$$ far field

$$\boldsymbol{E} \approx \mathbf{N}\mathbf{I}$$ near field

$$\mathbf{I}_{\mathrm{G}} \approx \mathbf{C}\mathbf{I}_{\mathrm{A}}$$ induced current on a PEC

Pre-computed matrices used in the optimization.

Optimization of the current distribution

Characteristic modes

Modes with small Rayleigh quotients

$$\frac{\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}} = \frac{\mathbf{I}^{\mathsf{H}}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

Eigenvalue problem

$$(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})\mathbf{I} = \nu\mathbf{R}\mathbf{I}$$

- Modes with low reactive power.
- Resonances ($\nu = 0$)
- Does not enforce low stored energy.

Stored energy

Minimize the energy Rayleigh quotient

$$\frac{\mathbf{I}^{\mathsf{H}}(\mathbf{X}_{\mathrm{m}} + \mathbf{X}_{\mathrm{e}})\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

Eigenvalue problem

$$(\mathbf{X}_{\mathrm{m}} + \mathbf{X}_{\mathrm{e}})\mathbf{I} = \nu\mathbf{R}\mathbf{I}$$

- Modes with low stored energy.
- Does not enforce resonance.

Q-factor

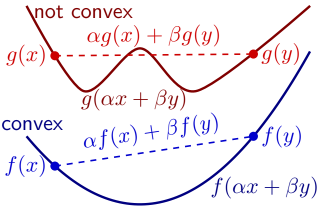
Minimize the Q-factor quotient

$$\frac{2\max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}\}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

- Currents with low Q-factors.
- Resonance by tuning.
- Need to solve these optimization problems $\Rightarrow$ convex optimization.

Chen and Wang 2015; Garbacz and Turpin 1971; Harrington and Mautz 1971

Convex optimization



$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0,\ i = 1, \ldots, N_1 \\ & \mathbf{Ax} = \mathbf{b} \end{aligned}$$

where $f_i(x)$ are convex, *i.e.*, $f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$ for $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

Solved with efficient standard algorithms. No risk of getting trapped in a local minimum. A problem is 'solved' if formulated as a convex optimization problem.

Antenna performance expressed in the current density $\boldsymbol{J}$, *e.g.*,

- Radiated field $\boldsymbol{F}(\hat{\boldsymbol{k}}) = -\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} \times \int_{\Omega} \boldsymbol{J}(\boldsymbol{r}) \mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{dV}$ is affine.
- Radiated power, stored electric and magnetic energies, and Ohmic losses are positive semi-definite quadratic forms in $\boldsymbol{J}$.

Currents for maximal G/Q

Determine a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume Ω that maximizes the partial-gain Q-factor quotient $G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})/Q$.

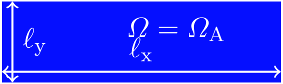
- Partial radiation intensity $P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})$

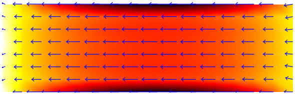
$$\frac{G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{\mathrm{c}_0 k \max\{W_{\mathrm{e}}, W_{\mathrm{m}}\}}.$$

- Scale $\boldsymbol{J}$ and reformulate $\max. P$ as $\max. \mathrm{Re}\{\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F}\}$.
- Convex optimization problem.

$$\begin{aligned} &\text{maximize} && \mathrm{Re}\{\mathbf{F}\mathbf{I}\} \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \le 1 \\ & && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \le 1 \end{aligned}$$

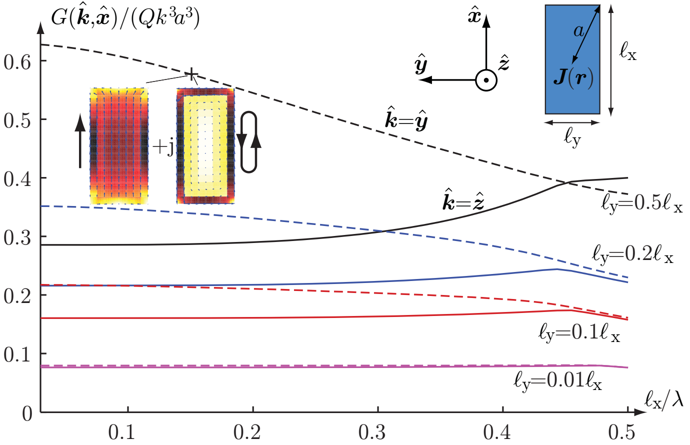
Maximal size of the antenna





Determines a current density $\boldsymbol{J}(\boldsymbol{r})$ in the region Ω with maximal partial radiation intensity and unit stored EM energy.

Maximal $G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{x}})/Q$ for planar rectangles



Solution for current densities confined to planar rectangles with side lengths ℓ_x and $\ell_y = \{0.01, 0.1, 0.2, 0.5\}\ell_x$.

Gustafsson and Nordebo 2013; Gustafsson et al. 2016

G/Q bounds

Typical (but not optimal) MATLAB code using CVX

```
cvx_begin
   variable I(n) complex;       % current density
   maximize(real(F*I))          % far-field
   subject to
      quad_form(I,Xe) <= 1;     % stored E energy
      quad_form(I,Xm) <= 1;     % stored M energy
cvx_end
```

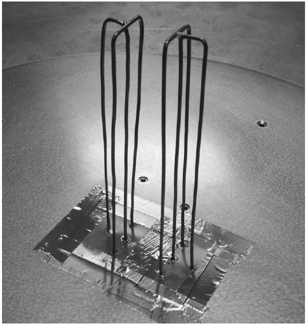
- Similar to the forward scattering bounds (2007) for TM.
- Can design 'optimal' electric dipole mode (TM) antennas.
- TE modes and TE+TM are not well understood.

We can reformulate the complex optimization problem to analyze superdirectivity, antennas with a prescribed radiation pattern, losses, and antennas embedded in a PEC structure.

M. Gustafsson, *etal*, Antenna current optimization using MATLAB and CVX, FERMAT, 2016

Superdirectivity

- A superdirective antenna has a directivity that is much higher than for a typical reference antenna.
- Often low efficiency (low gain) and narrow bandwidth.
- There is an interest in small superdirective antennas, *e.g.*, Best *etal.* 2008 and Arceo & Balanis 2011,



Best, *etal.*, An Impedance-Matched 2-Element Superdirective Array, IEEE-TAP, 2008

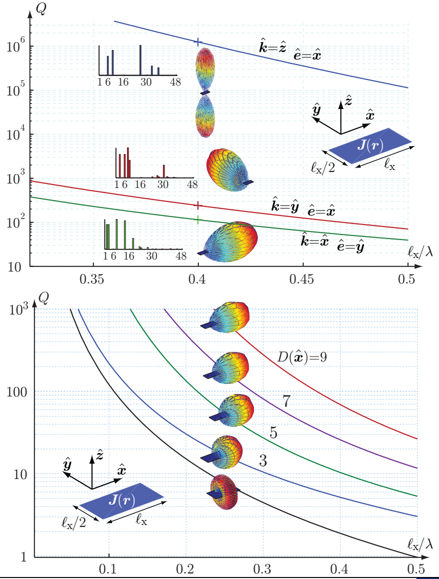
Here, we add the constraint $D \geq D_0$ to the convex optimization problem for G/Q to determine the minimum Q for superdirective lossless antennas. We can also add constraints on the losses.

Superdirectivity: min. G/Q s.t. $D \geq D_0$

Add the constraint $P_{\mathrm{rad}} \leq 4\pi D_0^{-1}$ the get the convex optimization problem

$$\begin{aligned} \text{min.} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{s.t.} \quad & \mathrm{Re}\{\mathbf{F}\mathbf{I}\} = 1 \\ & \mathbf{I}^{\mathsf{H}}\mathbf{P}\mathbf{I} \leq k^3 D_0^{-1} \end{aligned}$$

Example for current densities confined to planar rectangles with side lengths ℓ_{x} and $\ell_{\mathrm{y}} = 0.5\ell_{\mathrm{x}}$.

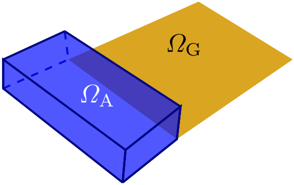


Currents for maximal G/Q for embedded antennas

Determine an optimal current density $\boldsymbol{J}_{\mathrm{A}}(\boldsymbol{r})$ in the region Ω_{A}. Assume that the ground plane $\Omega_{\mathrm{G}} = \Omega \setminus \Omega_{\mathrm{A}}$ is PEC.

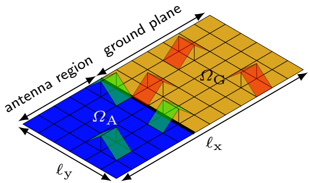
Can minimize the stored energy for given radiated field

$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \mathbf{F}\mathbf{I} = 1 \\ &&& \mathbf{I}_{\mathrm{G}} = \mathbf{C}\mathbf{I}_{\mathrm{A}} \end{aligned}$$

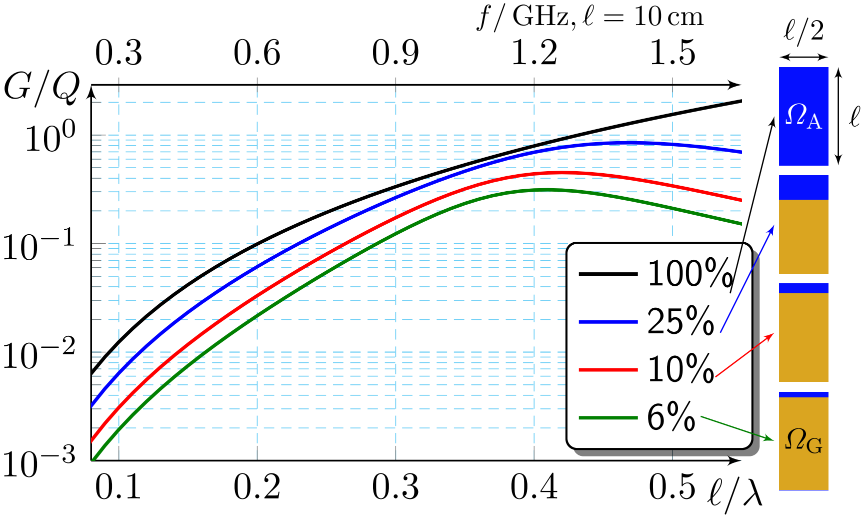


or maximize the radiated field for given stored energy

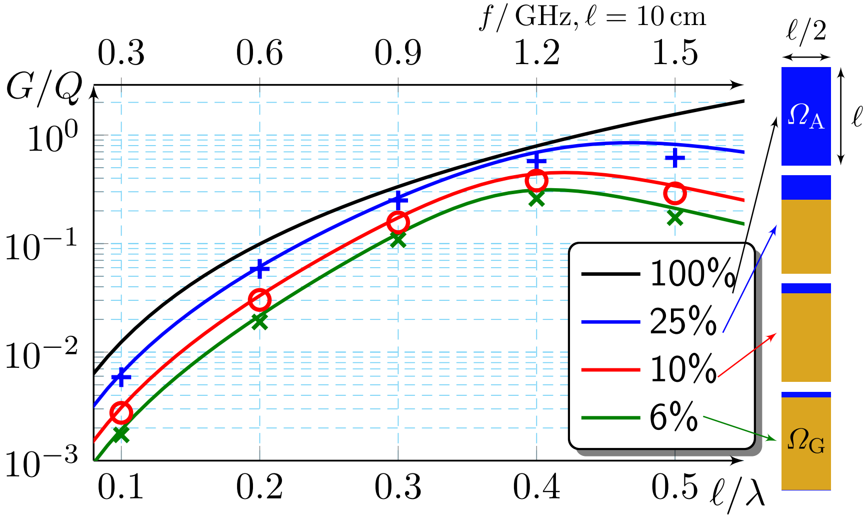
$$\begin{aligned} &\text{maximize} && \mathrm{Re}\{\mathbf{F}\mathbf{I}\} \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq 1 \\ &&& \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq 1 \\ &&& \mathbf{I}_{\mathrm{G}} = \mathbf{C}\mathbf{I}_{\mathrm{A}} \end{aligned}$$



Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region



Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region

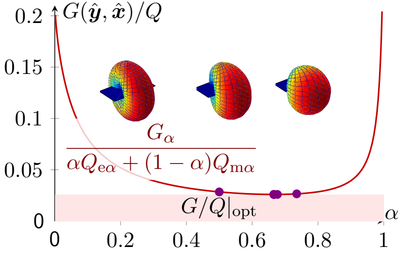


Why convex optimization: illustration

The upper bound on $G/Q|_{\text{opt}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (red) curve

$$\frac{G}{Q}\bigg|_{\text{opt}} \leq \frac{G_\alpha}{\alpha Q_{\text{e}\alpha} + (1-\alpha) Q_{\text{m}\alpha}}$$

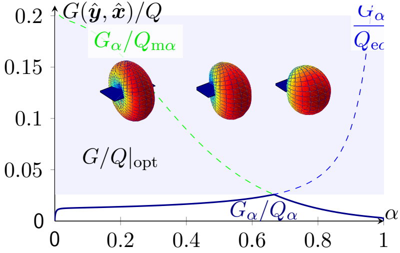
Efficiently solved with Newton iterations (cost $\mathbf{A}\mathbf{x} = \mathbf{b}$ per it).



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

The Newton iterations converge as $\alpha \approx 0.5$, 0.73536, 0.67677, 0.66629, 0.66602, 0.66602.

Why convex optimization: illustration



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

For free we also compute G/Q for the (dual) current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{\mathrm{e}\alpha}, Q_{\mathrm{m}\alpha}\}} \leq \left.\frac{G}{Q}\right|_{\mathrm{opt}}$$

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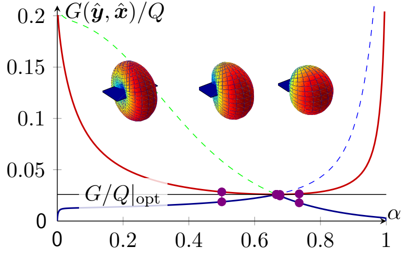

$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

The Newton iterations converge as $\alpha \approx 0.5$, 0.73536, 0.67677, 0.66629, 0.66602, 0.66602. Duality gap in G/Q approximately $10^{-\{2,2,3,4,8,16\}}$.

Why convex optimization: illustration

The upper bound on $G/Q|_{\mathrm{opt}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (red) curve

$$\left.\frac{G}{Q}\right|_{\mathrm{opt}} \leq \frac{G_\alpha}{\alpha Q_{\mathrm{e}\alpha} + (1-\alpha) Q_{\mathrm{m}\alpha}}$$

Efficiently solved with Newton iterations (cost $\mathbf{A}\mathbf{x} = \mathbf{b}$ per it).

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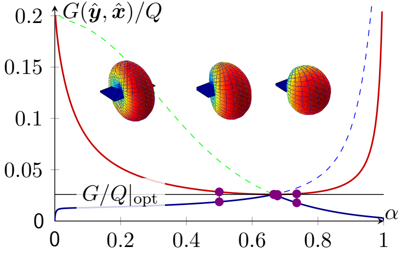
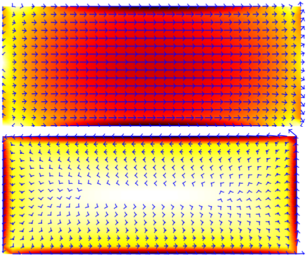


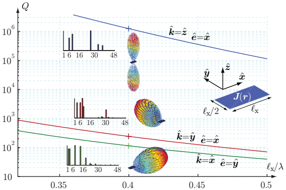
$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$



Simple optimization formulations

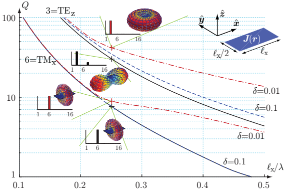
Superdirectivity:

$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \mathbf{F}\mathbf{I} = 1 \\ & && \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \leq 4\pi/(\eta_0 D_0) \end{aligned}$$



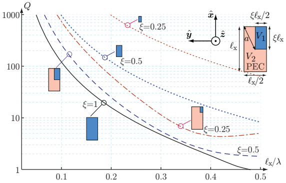
Prescribed far field:

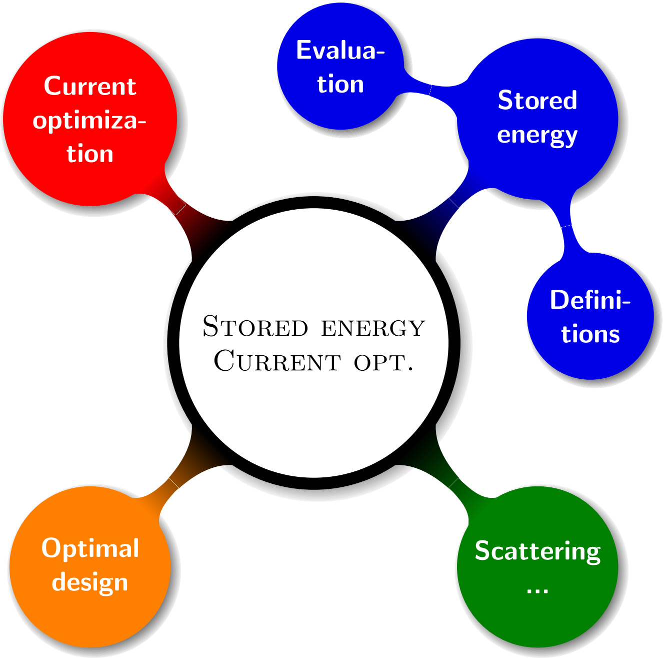
$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \int_{\Omega} |\boldsymbol{F}(\hat{\boldsymbol{k}}) - \boldsymbol{F}_0(\hat{\boldsymbol{k}})|^2 \, \mathrm{d}\Omega_{\hat{\boldsymbol{k}}} < \delta \end{aligned}$$



Embedded antennas:

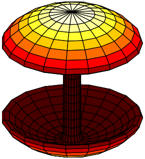
$$\begin{aligned} &\text{minimize} && \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} && \mathbf{F}\mathbf{I} = 1 \\ & && \mathbf{I}_{\mathrm{G}} = \mathbf{C}\mathbf{I}_{\mathrm{A}} \end{aligned}$$

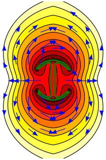



Stored energy
Current opt.
Current optimiza-tion
Evalua-tion
Stored energy
Defini-tions
Optimal design
Scattering
...

Summary

- Stored energy in the current density.
- State-space approach for temporal dispersion.
- Convex optimization for bounds and optimal currents: G/Q, superdirective, embedded, ...
- Physical bounds from spheres (Chu 1948) and arbitrary shapes (Gustafsson *etal* 2007) to embedded antennas...
- Non-Foster to overcome $B \sim 1/Q$ ...





M. Gustafsson *etal*, *Antenna current optimization using MATLAB and CVX*, FERMAT, 2016.

Slides: http://www.eit.lth.se/staff/mats.gustafsson

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