Physical bounds on the MIMO capacity for small antennas

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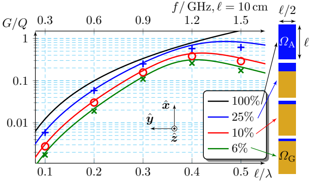
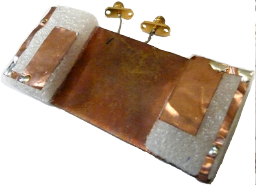
Physical bounds on antennas

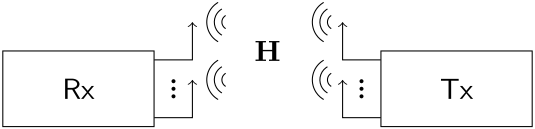





- very good understanding for small TM (electric dipole) antennas.
- good understanding for small TE, TE+TM antennas.
- partial understanding for radiation patterns, superdirectivity, efficiency, ...
- initial investigations of MIMO antennas and capacity [GGM11; GN06; Kun16; Mig08; TH12]. Mainly spherical structures.

Here, the approach in Ehrenborg and Gustafsson 2017 [EG17] based on antenna current optimization is presented.

MIMO system

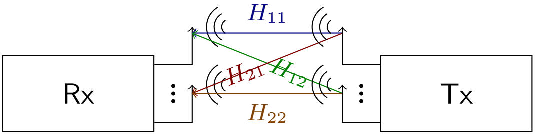


MIMO (multiple-input and multiple-output) systems have transmitting and receiving array antennas [PNG03]. Signal model

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & \cdots & H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{M1} & H_{M2} & \cdots & H_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_M \end{pmatrix}$$

- $\mathbf{y}$ is the $M \times 1$ receive matrix (Rx).
- $\mathbf{x}$ is the $N \times 1$ transmit matrix (Tx).
- $\mathbf{H}$ is the $M \times N$ channel matrix (H_{mn} connects x_n with y_m).
- $\mathbf{n}$ is the $M \times 1$ noise matrix (complex Gaussian).

MIMO system



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Capacity and spectral efficiency

The spectral efficiency for a channel $\mathbf{H}$ is

$$C = \max_{\mathrm{Tr}(\mathbf{RP})=P} \log_2 \det\left(\mathbf{1} + \frac{1}{N_0}\mathbf{HPH}^{\mathsf{H}}\right),$$

where $\mathbf{1}$ is the $M \times M$ identity matrix, N_0 is the noise spectral power density, and the covariance matrix of the transmitted signal [PNG03] is

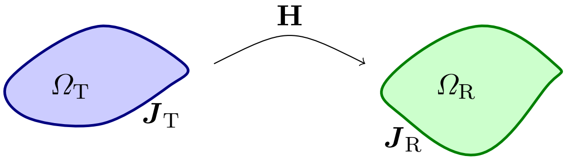
$$\mathbf{P} = \mathcal{E}\left\{\mathbf{x}\mathbf{x}^{\mathsf{H}}\right\} \quad \text{with the temporal average } \mathcal{E}\left\{\cdot\right\}$$

- Ergodic capacity for random channels.
- Multiply with bandwidth for capacity.

Information theoretical bound that expresses the maximum number of $\mathrm{bits}/(\mathrm{sHz})$ that can be transmitted over the channel.

How can we use the spectral efficiency (capacity) to quantify the performance of MIMO antennas?

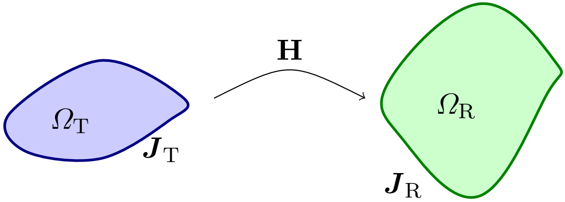
MIMO antennas and radio channel



Consider optimal antennas in regions Ω_T and Ω_R.

- Capacity depends on Ω_T, Ω_R, and properties of $\mathbf{H}$

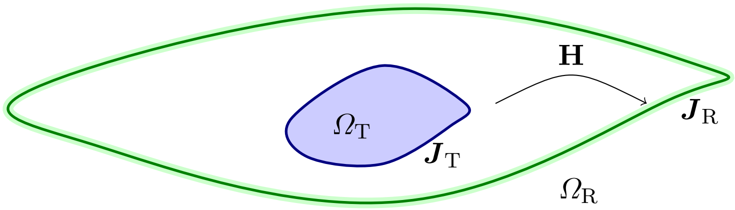
MIMO antennas and radio channel



Consider optimal antennas in regions Ω_{T} and Ω_{R}.

- Capacity depends on Ω_{T}, Ω_{R}, and properties of $\mathbf{H}$
- Idealize the receiver and channel to analyze the transmitter
  - Increase size and number of receiver elements
  - Channel

MIMO antennas and radio channel



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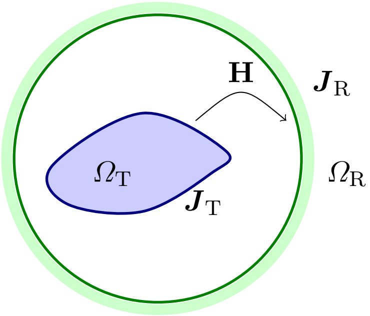
- Spherical (far field) receiver surrounding the transmitter

MIMO antennas and radio channel



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Spherical mode channel

Consider an idealized channel with receiver ports for each radiated spherical mode. Use current density $\boldsymbol{J}$ to express the radiated field and channel in outgoing spherical waves

$$\boldsymbol{E}(\boldsymbol{r}_1) = -\mathrm{j}\omega\mu \sum_{\tau\sigma ml} \mathbf{u}^{(4)}_{\tau\sigma ml}(k\boldsymbol{r}_1) \int_{\Omega_{\mathrm{T}}} \mathbf{u}^{(1)}_{\tau\sigma ml}(k\boldsymbol{r}_2) \cdot \boldsymbol{J}(\boldsymbol{r}_2)\,\mathrm{dV}_2.$$

Define a matrix $\mathbf{M}$ that maps the current to the spherical modes $\mathbf{y} = \mathbf{M}\mathbf{I}$, where the current $\mathbf{I}$ matrix contains the elements I_n from the expansion of the current density expanded in a basis $\boldsymbol{\psi}_n$

$$\boldsymbol{J}(\boldsymbol{r}) = \sum_n I_n \boldsymbol{\psi}_n(\boldsymbol{r}).$$

Channel (ports $\mathbf{x} \rightarrow$ current $\mathbf{I} \rightarrow$ modes=ports $\mathbf{y}$)

$$\mathbf{y} = \mathbf{M}\mathbf{I} = \mathbf{M}\mathbf{T}\mathbf{x} = \widehat{\mathbf{M}}\mathbf{x}$$

where $\mathbf{T}$ denotes the (linear) map from the excitation $\mathbf{x}$ to the current $\mathbf{I}$ matrix.

Radiated power and stored energy

The dissipated power and stored energy can be written in the current (density) $\mathbf{I}$ [Gus+16; Van10]

$$P = \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}, \quad W_{\mathrm{e}} = \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \quad \text{and } W_{\mathrm{m}} = \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}.$$

where $\mathbf{R}$ is the resistive part of the MoM impedance matrix, $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X} = \mathbf{R} + \mathrm{j}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})$.

The current depends linearly on the transmitted signal $\mathbf{I} = \mathbf{T}\mathbf{x}$.

Average transmitted power

$$P = \frac{1}{2}\mathcal{E}\left\{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}\right\} = \frac{1}{2}\mathcal{E}\left\{\mathbf{x}^{\mathsf{H}}\mathbf{T}^{\mathsf{H}}\mathbf{R}\mathbf{T}\mathbf{x}\right\}$$
$$= \frac{1}{2}\operatorname{Tr}\mathcal{E}\left\{\mathbf{T}^{\mathsf{H}}\mathbf{R}\mathbf{T}\mathbf{x}\mathbf{x}^{\mathsf{H}}\right\} = \frac{1}{2}\operatorname{Tr}\left(\mathbf{T}^{\mathsf{H}}\mathbf{R}\mathbf{T}\mathcal{E}\left\{\mathbf{x}\mathbf{x}^{\mathsf{H}}\right\}\right) = \frac{1}{2}\operatorname{Tr}(\widehat{\mathbf{R}}\mathbf{P}),$$

where $\widehat{\mathbf{R}} = \mathbf{T}^{\mathsf{H}}\mathbf{R}\mathbf{T}$, $\mathbf{P} = \mathcal{E}\left\{\mathbf{x}\mathbf{x}^{\mathsf{H}}\right\}$ is the covariance matrix of the transmitted signal [PNG03], $\mathcal{E}\{\cdot\}$ the temporal average, and Tr the trace $\operatorname{Tr}\mathbf{A} = \sum_n A_{nn}$.

Capacity

The capacity, expressed as spectral efficiency ($\mathrm{b/(s\,Hz)}$), of this channel is given by [PNG03]

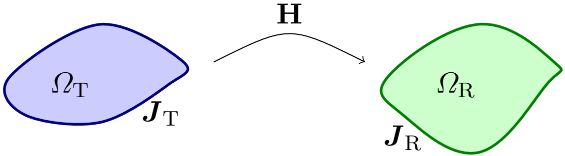
$$C = \max_{\mathrm{Tr}(\widehat{\mathbf{R}}\mathbf{P})=P} \log_2 \det\left(\mathbf{1} + \frac{1}{N_0}\widehat{\mathbf{M}}\mathbf{P}\widehat{\mathbf{M}}^{\mathsf{H}}\right),$$

where $\mathbf{1}$ is the $M \times M$ identity matrix, and N_0 is the noise spectral power density.
Semidefinite optimization program (convex optimization)

$$\begin{aligned} \text{maximize} \quad & \log_2 \det(\mathbf{1} + \gamma\widehat{\mathbf{M}}\mathbf{P}\widehat{\mathbf{M}}^{\mathsf{H}}) \\ \text{subject to} \quad & \mathrm{Tr}(\widehat{\mathbf{R}}\mathbf{P}) = 1 \\ & \mathbf{P} \succeq 0, \end{aligned}$$

where the unit transmitted power is considered, and γ is the total SNR. Unbounded as the dimension ($\min\{M, N\}$) increases.

Maximum capacity for MIMO antennas



Add constrains on the stored energy to get [EG17]

$$
\begin{aligned}
\text{maximize} \quad & \log_2 \det(\mathbf{1} + \gamma \widehat{\mathbf{M}} \mathbf{P} \widehat{\mathbf{M}}^{\mathsf{H}}) \\
\text{subject to} \quad & \mathrm{Tr}(\widehat{\mathbf{X}}_{\mathrm{e}} \mathbf{P}) \leq Q \\
& \mathrm{Tr}(\widehat{\mathbf{X}}_{\mathrm{m}} \mathbf{P}) \leq Q \\
& \mathrm{Tr}(\widehat{\mathbf{R}} \mathbf{P}) = 1 \\
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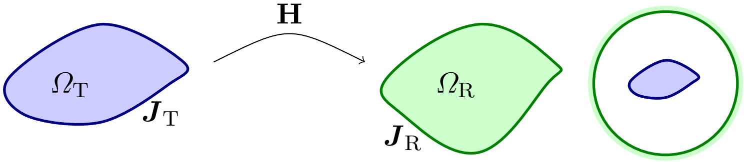
\end{aligned}
$$

Total stored energy. Note, there is no (known) simple relation between bandwidth and Q-factors for multiport antennas.

Maximum capacity for MIMO antennas



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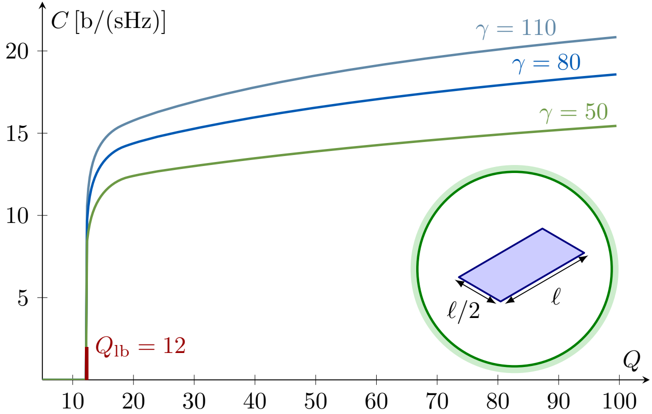
Maximum capacity for a planar rectangle



Capacity in $\mathrm{bits}/(\mathrm{s\,Hz})$ for fixed noise level [EG17].

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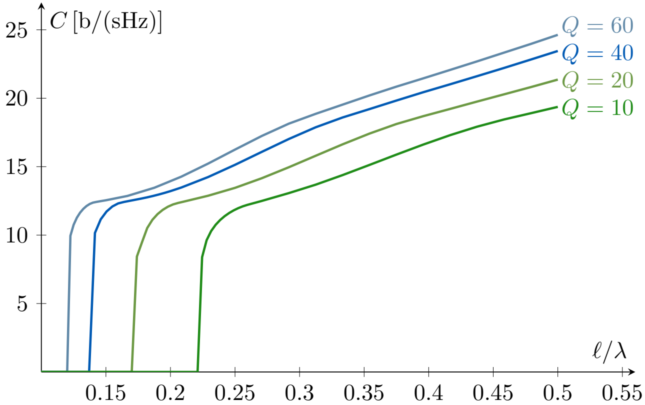

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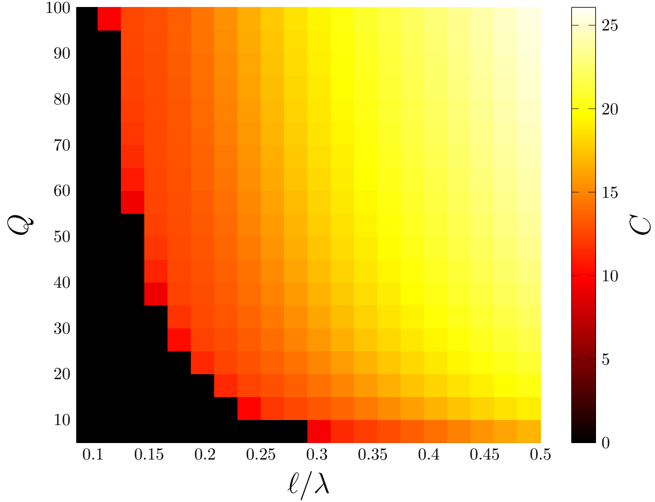

Capacity in $\mathrm{bits/(s\,Hz)}$ for fixed noise level [EG17].

Maximum capacity for MIMO antennas: efficiency and Q

Add constraints on the dissipated power for efficiency [EG17].

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& \mathrm{Tr}(\widehat{\mathbf{X}}_{\mathrm{m}} \mathbf{P}) \leq Q \\
& \mathrm{Tr}(\widehat{\mathbf{R}}_{\Omega} \mathbf{P}) = 1 - \eta \\
& \mathrm{Tr}(\widehat{\mathbf{R}} \mathbf{P}) = 1 \\
& \mathbf{P} \succeq \mathbf{0}
\end{aligned}
$$

Total stored energy and total dissipated power.

Maximum capacity for MIMO antennas: efficiency and Q

Add constraints on the dissipated power for efficiency [EG17].
Equality for resonance.

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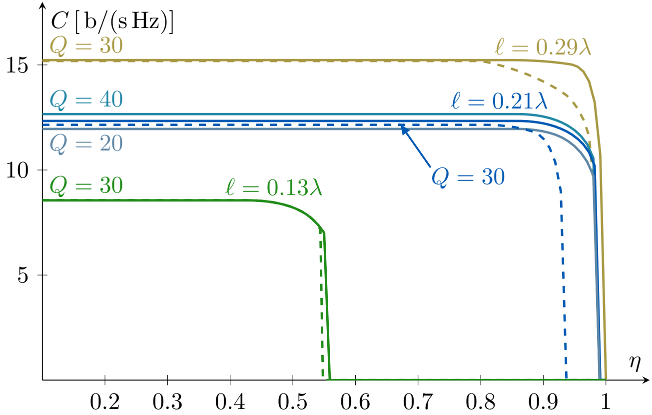
Total stored energy and total dissipated power.

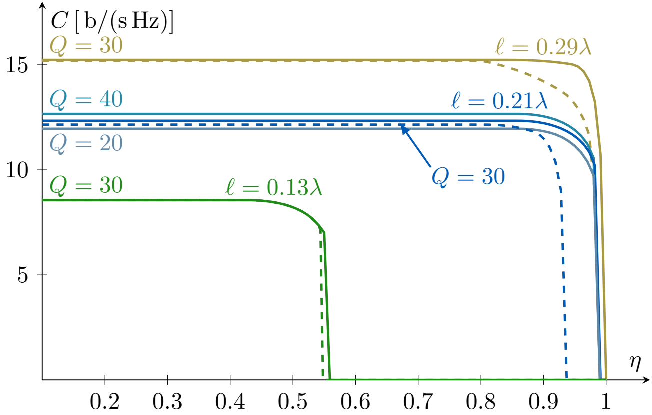
The SDP problem has a unique solution and there are many standard algorithms (convex optimization). However, computationally challenging (memory and time) for large problems.

Maximum capacity for a planar rectangle vs efficiency



Dashed (solid) curves with (without) resonance $\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I} = 0$ constraint enforced. Losses modeled as a resistive sheet with $R = 0.2\,\Omega/\square$.

Maximum capacity for a planar rectangle vs efficiency



Dashed (solid) curves with (without) resonance $\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I} = 0$ constraint enforced. Losses modeled as a resistive sheet with $R = 0.2\,\Omega/\square$.

The cut-off levels are also determined by solving a convex optimization problem for maximal efficiency [EG17].

Some computational challenges

Compare the quadratic form with its semi-definite programming (SDP) formulation

$$\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I} = \mathrm{Tr}\{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}\} = \mathrm{Tr}\{\mathbf{I}\mathbf{I}^{\mathsf{H}}\mathbf{R}\} = \mathrm{Tr}\{\mathbf{P}\mathbf{R}\}$$

where

- $\mathbf{R}$ is of size $N \times N$ (size of MoM impedance matrix)
- $\mathbf{I}$ is of size $N \times 1$ (number of unknowns (current))
- $\mathbf{P}$ is of size $N \times N$ (dropped the rank one constraint)

Solving an optimization problem using SDP increases the number of unknowns from N to $\approx N^2/2$ (symmetric).

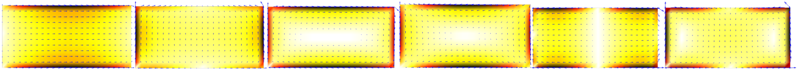
A planar rectangle with $64 \times 32 \approx 2000$ elements and $N \approx 4000$ unknowns compared to $8\ 10^6$ unknowns in SDP.

Desired (necessary) to reduce the number of unknowns using some model order reduction [Gus+16].

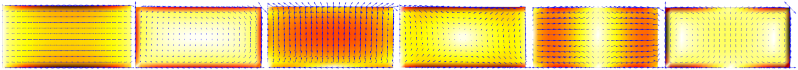
Model order reduction

Can expand in higher order (global) basis functions. Many possibilities. Can *e.g.*, use modes (combination of) from eigenvalue problems based on the MoM impedance matrix
$\mathbf{Z} = \mathbf{R}_{\mathrm{r}} + \mathbf{R}_{\Omega} + \mathrm{j}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})$

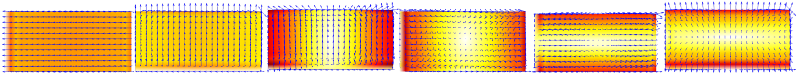
- Characteristic modes $\mathbf{X}\mathbf{I} = (\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})\mathbf{I} = \nu \mathbf{R}_{\mathrm{r}}\mathbf{I}$



- Energy modes $(\mathbf{X}_{\mathrm{m}} + \mathbf{X}_{\mathrm{e}})\mathbf{I} = \nu \mathbf{R}_{\mathrm{r}}\mathbf{I}$



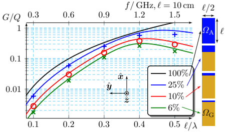
- Efficiency modes $\mathbf{R}_{\Omega}\mathbf{I} = \nu \mathbf{R}_{\mathrm{r}}\mathbf{I}$



6 first depicted. Also with $\mathbf{R}_{\mathrm{r}} + \mathbf{R}_{\Omega}$ and many other possibilities [Gus+16]. Here, we use combinations of them.

Summary

- Physical bounds on capacity.
- Convex optimization, Semi-definite programming (SDP).
- Model order reduction (characteristic modes, energy modes, efficiency modes, ...)



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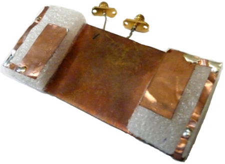
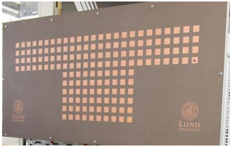


In progress

- Interpretation of the Q-factor for mutiport antennas.
- Comparison with MIMO antennas.
- Sub regions.
- Larger structures massive MIMO.



Slides: `http://www.eit.lth.se/staff/mats.gustafsson`

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