Physical bounds on small antennas as convex optimization problems

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IEEE-APS, Chicago, 2012-07-09

Design of small antennas

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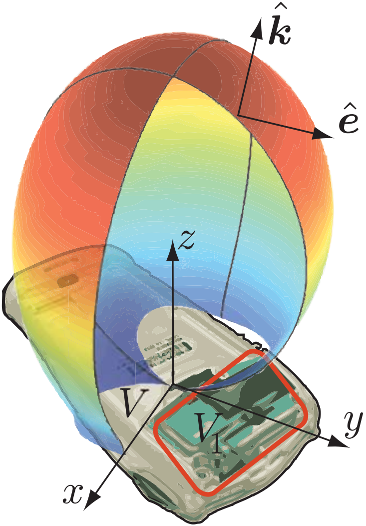
- There are many advanced methods to design small antennas.
- Performance often in bandwidth, matching, and efficiency.
- How can new designs, geometries, and materials improve performance?
- Here, what is the fundamental tradeoff between performance and size?

Tradeoff between performance and size

- Radiating (antenna) structure, V.
- Antenna volume, $V_1 \subset V$.
- Current density $\boldsymbol{J}_1$ in V_1.
- Radiated field, $\boldsymbol{F}(\hat{\boldsymbol{k}})$, in direction $\hat{\boldsymbol{k}}$ and polarization $\hat{\boldsymbol{e}}$.

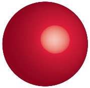


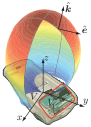
Questions analyzed here, $\boldsymbol{J}_1$ for:

- maximum $G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})/Q$.
- maximum $G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})/Q$ for $D(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}}) \geq D_0$ (superdirectivity).
- embedded antennas.
- also minimum Q for radiated field approximately $\boldsymbol{F}(\hat{\boldsymbol{k}})$ and effects of Ohmic losses.

Background

- 1947 Wheeler: *Bounds based on circuit models.*
- 1948 Chu: *Bounds on Q and D/Q for spheres.*
- 1964 Collin & Rothchild: *Closed form expressions of Q for arbitrary spherical modes, see also* Harrington, Collin, Fantes, Maclean, Gayi, Hansen, Hujanen, Sten, Thiele, Best, Yaghjian, Kildal... *(most are based on Chu's approach using spherical modes.)*
- 1999 Foltz & McLean, 2001 Sten, Koivisto, and Hujanen: *Attempts for bounds in spheroidal volumes.*
- 2006 Thal: *Bounds on Q for small hollow spherical antennas.*
- 2007 Gustafsson, Sohl & Kristensson: *Bounds on D/Q for arbitrary geometries (and Q for small antennas).*
- 2010 Yaghjian & Stuart: *Bounds on Q for dipole antennas in the limit* $ka \to 0$.
- 2011 Vandenbosch: *Bounds on Q for small (non-magnetic) antennas in the limit* $ka \to 0$.
- 2011 Chalas, Sertel & Volakis: *Bounds on Q using characteristic modes.*
- 2012 Gustafsson, Cismasu, & Jonsson: *Optimal charge and current distributions on antennas.*
- 2012 Bernland: *Physical Limitations on the Scattering of High Order Electromagnetic Vector Spherical Waves.*
- 2012 Gustafsson & Nordebo: *Optimal antenna Q, superdirectivity, and radiation patterns using convex optimization.*

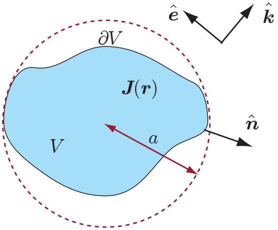




G/Q and D/Q

Partial gain expressed in the partial radiation intensity $P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})$ and total radiated P_{rad} and dissipated power P_{loss}

$$G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}}) = 4\pi\frac{P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{P_{\mathrm{rad}}+P_{\mathrm{loss}}}$$



Q-factor

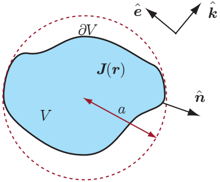
$$Q = \frac{2\omega W}{P_{\mathrm{rad}}+P_{\mathrm{loss}}} = \frac{2\mathrm{c}_0 kW}{P_{\mathrm{rad}}+P_{\mathrm{loss}}},$$

where $W=\max\{W_{\mathrm{e}},W_{\mathrm{m}}\}$ denotes the maximum of the stored electric and magnetic energies. The G/Q and (D/Q for lossless) quotient cancels $P_{\mathrm{rad}}+P_{\mathrm{loss}}$

$$\frac{G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q} = \frac{D(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{\mathrm{c}_0 kW}.$$

Stored EM energies from current densities $\boldsymbol{J}$ in V

Use the expressions by Vandenbosch (2010) (and Geyi (2003) for small antennas). Stored electric energy $\widetilde{W}^{(\mathrm{e})}_{\mathrm{vac}} = \frac{\mu_0}{16\pi k^2} w^{(\mathrm{e})}$



$$w^{(\mathrm{e})} = \int_V \int_V \nabla_1 \cdot \boldsymbol{J}_1 \nabla_2 \cdot \boldsymbol{J}_2^* \frac{\cos(kR_{12})}{R_{12}}$$
$$-\frac{k}{2}\left(k^2 \boldsymbol{J}_1 \cdot \boldsymbol{J}_2^* - \nabla_1 \cdot \boldsymbol{J}_1 \nabla_2 \cdot \boldsymbol{J}_2^*\right) \sin(kR_{12}) \, \mathrm{dV}_1 \, \mathrm{dV}_2,$$

where $\boldsymbol{J}_1 = \boldsymbol{J}(\boldsymbol{r}_1)$, $\boldsymbol{J}_2 = \boldsymbol{J}(\boldsymbol{r}_2)$, $R_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. Stored magnetic energy $\widetilde{W}^{(\mathrm{m})}_{\mathrm{vac}} = \frac{\mu_0}{16\pi k^2} w^{(\mathrm{m})}$, where

$$w^{(\mathrm{m})} = \int_V \int_V k^2 \boldsymbol{J}_1 \cdot \boldsymbol{J}_2^* \frac{\cos(kR_{12})}{R_{12}}$$
$$- \frac{k}{2}\left(k^2 \boldsymbol{J}_1 \cdot \boldsymbol{J}_2^* - \nabla_1 \cdot \boldsymbol{J}_1 \nabla_2 \cdot \boldsymbol{J}_2^*\right) \sin(kR_{12}) \, \mathrm{dV}_1 \, \mathrm{dV}_2.$$

Stored EM energies from current densities $\boldsymbol{J}$ in V II

Also the total radiated power $P_{\mathrm{rad}} = \frac{\eta_0}{8\pi k} p_{\mathrm{rad}}$ with

$$p_{\mathrm{rad}} = \int_V \int_V \left(k^2 \boldsymbol{J}_1 \cdot \boldsymbol{J}_2^* - \nabla_1 \cdot \boldsymbol{J}_1 \nabla_2 \cdot \boldsymbol{J}_2^*\right) \frac{\sin(kR_{12})}{R_{12}} \,\mathrm{dV}_1 \,\mathrm{dV}_2.$$

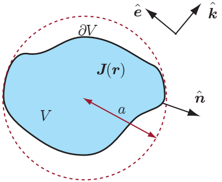
The normalized quantities $w^{(\mathrm{e})}, w^{(\mathrm{m})}$, and p_{rad} have dimensions given by volume, m^3, times the dimension of $|\boldsymbol{J}|^2$.

- Introduced by Vandenbosch in *Reactive energies, impedance, and Q factor of radiating structures*, IEEE-TAP 2010.
- In the limit $ka \to 0$ by Geyi in *Physical limitations of antenna*, IEEE-TAP 2003.
- Validation for wire antennas in Hazdra *etal*, *Radiation Q-factors of thin-wire dipole arrangements*, IEEE-AWPL 2011.
- Some issues with 'negative stored energy' for large structures in Gustafsson *etal*, IEEE-TAP 2012.

G/Q in the current density $\boldsymbol{J}$

The partial radiation intensity $P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})$ in direction $\hat{\boldsymbol{k}}$ and for the polarization $\hat{\boldsymbol{e}}$ is

$$P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}}) = \frac{\eta_0 k^2}{32\pi^2}\left|\int_V \hat{\boldsymbol{e}}^*\cdot\boldsymbol{J}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{dV}\right|^2$$



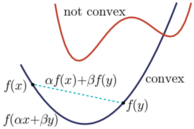
We have the G/Q quotient

$$\frac{G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q} = k^3\frac{\left|\int_V \hat{\boldsymbol{e}}^*\cdot\boldsymbol{J}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{dV}\right|^2}{\max\{w^{(\mathrm{e})}(\boldsymbol{J}),w^{(\mathrm{m})}(\boldsymbol{J})\}}$$
$$\leq \max_{\boldsymbol{J}} k^3\frac{\left|\int_V \hat{\boldsymbol{e}}^*\cdot\boldsymbol{J}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{dV}\right|^2}{\max\{w^{(\mathrm{e})}(\boldsymbol{J}),w^{(\mathrm{m})}(\boldsymbol{J})\}}$$

Solve the optimization problem. Closed form solutions in the limit $ka\to 0$ and convex optimization for larger (but small) structures.

Convex optimization

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, ..., N_1 \\ & \mathbf{Ax} = \mathbf{b} \end{aligned}$$



where $f_i(x)$ are convex, *i.e.*, $f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$ for $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

Solved with efficient standard algorithms. No risk of getting trapped in a local minimum. A problem is 'solved' if formulated as a convex optimization problem.

Can be used in many convex formulations for antenna performance expressed in the current density $\boldsymbol{J}$, *e.g.*,

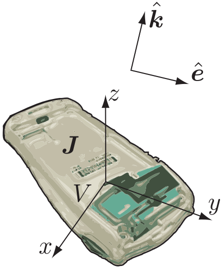
- Radiated field $\boldsymbol{F}(\hat{\boldsymbol{k}}) = -\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} \times \int_V \boldsymbol{J}(\boldsymbol{r}) \mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}} \, \mathrm{dV}$ is affine.
- Radiated power, stored electric and magnetic energies, and Ohmic losses are positive semi-definite quadratic forms in $\boldsymbol{J}$.

Currents for maximal G/Q

Determine a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V that maximizes the partial-gain Q-factor quotient $G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})/Q$.

- Partial radiation intensity $P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})$

$$\frac{G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{\mathrm{c}_0 k \max\{W_{\mathrm{e}},W_{\mathrm{m}}\}}.$$



- Scale $\boldsymbol{J}$ and reformulate $P=1$ as $\mathrm{Re}\{\hat{\boldsymbol{e}}^{*}\cdot\boldsymbol{F}\}=1$.
- Convex optimization problem.

$$\begin{aligned} &\text{minimize} && \max\{\mathbf{J}^{\mathsf{H}}\mathbf{W}_{\mathrm{e}}\mathbf{J},\mathbf{J}^{\mathsf{H}}\mathbf{W}_{\mathrm{m}}\mathbf{J}\} \\ &\text{subject to} && \mathrm{Re}\{\mathbf{F}^{\mathsf{H}}\mathbf{J}\}=1 \end{aligned}$$

Determines a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume V with minimal stored EM energy and unit partial radiation intensity in $\{\hat{\boldsymbol{k}},\hat{\boldsymbol{e}}\}$.

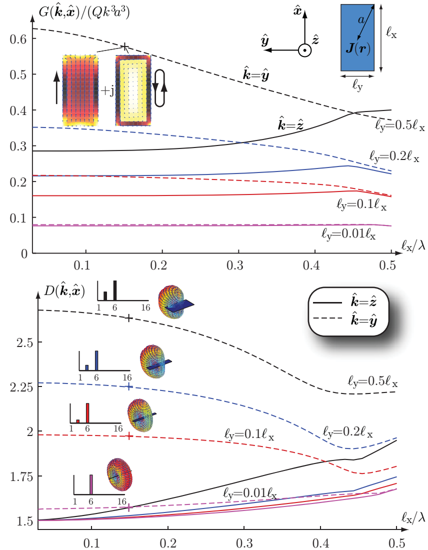
Maximal $G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{x}})/Q$ for planar rectangles

Solution of the convex optimization problem

$$\begin{aligned} \text{min.} \quad & \max\{\mathbf{J}^{\mathsf{H}}\mathbf{W}_{\mathrm{e}}\mathbf{J}, \mathbf{J}^{\mathsf{H}}\mathbf{W}_{\mathrm{m}}\mathbf{J}\} \\ \text{s.t.} \quad & \mathrm{Re}\{\mathbf{F}^{\mathsf{H}}\mathbf{J}\} = 1 \end{aligned}$$

for current densities confined to planar rectangles with side lengths ℓ_{x} and $\ell_{\mathrm{y}} = \{0.01, 0.1, 0.2, 0.5\}\ell_{\mathrm{x}}$.

Note $\ell_{\mathrm{x}}/\lambda = k\ell_{\mathrm{x}}/(2\pi)$, giving $\ell_{\mathrm{x}} = \lambda/2 \to k\ell_{\mathrm{x}} = \pi \to ka \geq \pi/2$.



D/Q (or G/Q) bounds

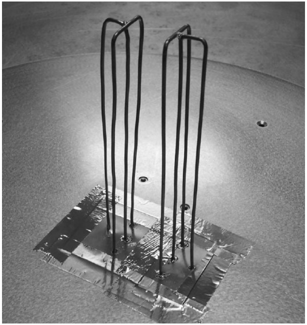
- Similar to the forward scattering bounds for TM.
- Can design 'optimal' electric dipole mode (TM) antennas.
- TE modes and TE+TM are not well understood.
- Typical matlab code using CVX

```
cvx_begin
    variable J(n) complex;
    dual variables We Wm
    maximize(real(F'*J))
    We: quad_form(J,Ze) <= 1;
    Wm: quad_form(J,Zm) <= 1;
cvx_end
```

We now reformulate the complex optimization problem to analyze superdirectivity, antennas with a prescribed radiation pattern, losses, and antennas embedded in a PEC structure.

Superdirectivity

- A superdirective antenna has a directivity that is much larger than for a typical reference antenna.
- Often low efficiency (low gain) and narrow bandwidth.
- There is an interest in small superdirective antennas, *e.g.*, Best *etal.* 2008 and Arceo & Balanis 2011,



Best, *etal.*, An Impedance-Matched 2-Element Superdirective Array, IEEE-TAP, 2008

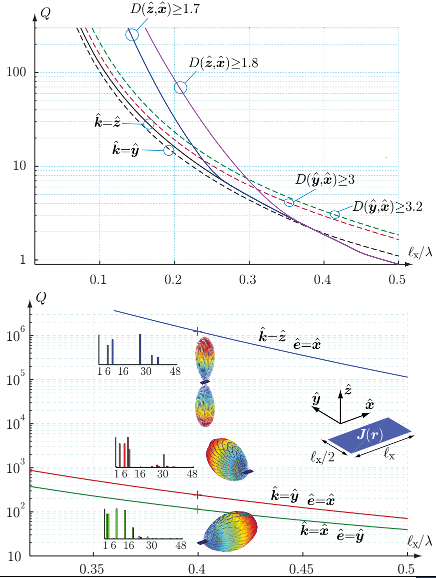
Here, we add the constraint $D \geq D_0$ to the convex optimization problem for G/Q to determine the minimum Q for superdirective lossless antennas. We can also add constraints on the losses.

Superdirectivity

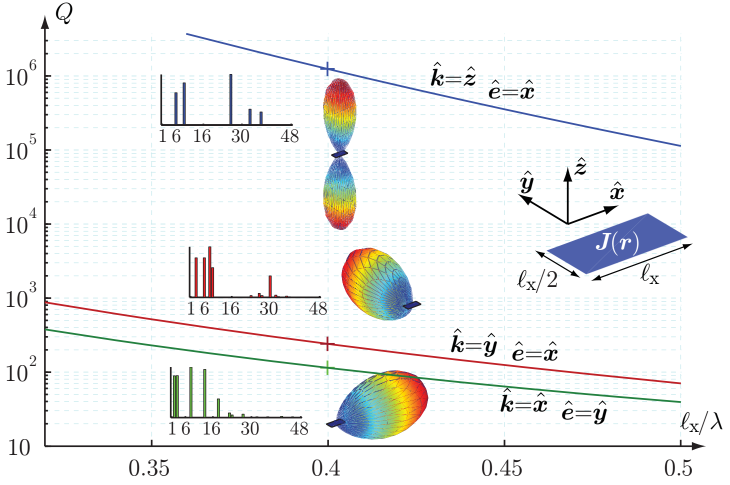
Add the constraint $P_{\text{rad}} \leq 4\pi D_0^{-1}$ the get the convex optimization problem

$$\begin{aligned} \text{min.} \quad & \max\{\mathbf{J}^{\mathsf{H}}\mathbf{W}_{\text{e}}\mathbf{J}, \mathbf{J}^{\mathsf{H}}\mathbf{W}_{\text{m}}\mathbf{J}\} \\ \text{s.t.} \quad & \operatorname{Re}\{\mathbf{F}^{\mathsf{H}}\mathbf{J}\} = 1 \\ & \mathbf{J}^{\mathsf{H}}\mathbf{P}\mathbf{J} \leq k^3 D_0^{-1} \end{aligned}$$

Example for current densities confined to planar rectangles with side lengths ℓ_{x} and $\ell_{\text{y}} = 0.5\ell_{\text{x}}$.



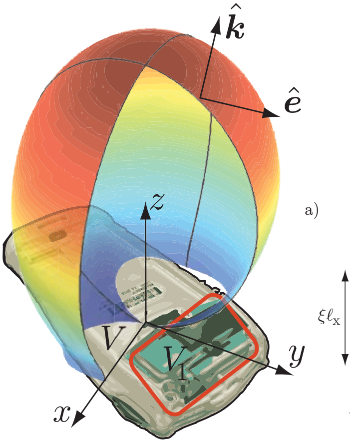
Superdirectivity with $D \geq D_0 = 10$



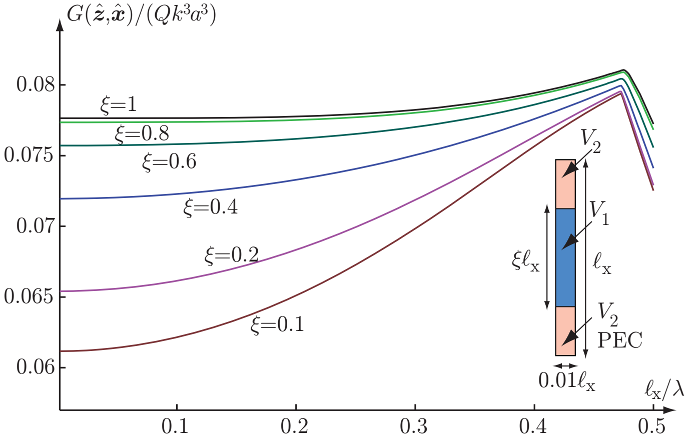
Note, it gives a bound on Q as D is known.

Optimal performance for embedded antennas

- It is common with antennas embedded in metallic structures.
- The induced currents radiate but they are not arbitrary.
- Linear map from the antenna region adds a (convex) constraint.
- Here, we assume that the surrounding structure is PEC and add a constraint to account for the induced currents on the surrounding structure in the G/Q formulation.

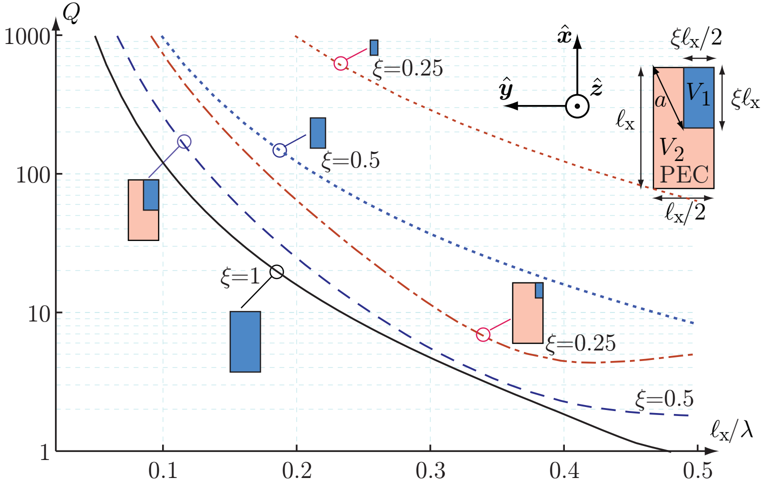


Center fed strip dipole



Almost independent of the feed width at the resonance just below $\ell_x = 0.5\lambda$.

Embedded antennas in a planar rectangle



Conclusions

- Closed form solution for small antennas.
  - Optimal current distributions. Spherical dipole, capped dipole, and folded spherical helix. More in IF46, Small Antennas: Designs and Applications on Thursday.
- Convex optimization to determine bounds and optimal currents for larger structures:
  - D/Q and G/Q.
  - Q for superdirective antennas.
  - Embedded antennas in PEC structures.
  - Q for antennas with prescribed far fields.

See also *Physical bounds and optimal currents on antennas* IEEE TAP, 60, 6, pp. 2672-2681, 2012 and *Antenna currents for optimal Q, superdirectivity, and radiation patterns using convex optimization* (www.eit.lth.se/staff/mats.gustafsson)