Physical bounds on small antennas as convex optimization problems

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There are many advanced methods to design small antennas.
Performance often in bandwidth, matching, and efficiency.
How can new designs, geometries, and materials improve performance?
Here, what is the fundamental tradeoff between performance and size?
Tradeoff between performance and size

- Radiating (antenna) structure, $V$.
- Antenna volume, $V_1 \subset V$.
- Current density $J_1$ in $V_1$.
- Radiated field, $F(\hat{k})$, in direction $\hat{k}$ and polarization $\hat{e}$.

Questions analyzed here, $J_1$ for:
- maximum $G(\hat{k}, \hat{e})/Q$.
- maximum $G(\hat{k}, \hat{e})/Q$ for $D(\hat{k}, \hat{e}) \geq D_0$ (superdirectivity).
- embedded antennas.
- also minimum $Q$ for radiated field approximately $F(\hat{k})$ and effects of Ohmic losses.

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Background

- 1947 Wheeler: *Bounds based on circuit models.*
- 1948 Chu: *Bounds on Q and D/Q for spheres.*
- 1964 Collin & Rothchild: *Closed form expressions of Q for arbitrary spherical modes, see also Harrington, Collin, Fantes, Maclean, Gayi, Hansen, Hujanen, Sten, Thiele, Best, Yaghjian, Kildal... (most are based on Chu’s approach using spherical modes.)*
- 2006 Thal: *Bounds on Q for small hollow spherical antennas.*
- 2007 Gustafsson, Sohl & Kristensson: *Bounds on D/Q for arbitrary geometries (and Q for small antennas).*
- 2010 Yaghjian & Stuart: *Bounds on Q for dipole antennas in the limit ka → 0.*
- 2011 Vandenbosch: *Bounds on Q for small (non-magnetic) antennas in the limit ka → 0.*
- 2011 Chalas, Sertel & Volakis: *Bounds on Q using characteristic modes.*
- 2012 Gustafsson, Cismasu, & Jonsson: *Optimal charge and current distributions on antennas.*
$G/Q$ and $D/Q$

Partial gain expressed in the partial radiation intensity $P(\hat{k}, \hat{e})$ and total radiated $P_{\text{rad}}$ and dissipated power $P_{\text{loss}}$

$$G(\hat{k}, \hat{e}) = 4\pi \frac{P(\hat{k}, \hat{e})}{P_{\text{rad}} + P_{\text{loss}}}$$

Q-factor

$$Q = \frac{2\omega W}{P_{\text{rad}} + P_{\text{loss}}} = \frac{2c_0 kW}{P_{\text{rad}} + P_{\text{loss}}}$$

where $W = \max\{W_e, W_m\}$ denotes the maximum of the stored electric and magnetic energies. The $G/Q$ and $(D/Q$ for lossless) quotient cancels $P_{\text{rad}} + P_{\text{loss}}$

$$\frac{G(\hat{k}, \hat{e})}{Q} = \frac{D(\hat{k}, \hat{e})}{Q} = \frac{2\pi P(\hat{k}, \hat{e})}{c_0 kW}.$$

Stored electric energy

\[
\widetilde{W}_\text{vac}^{(e)} = \frac{\mu_0}{16\pi k^2} w^{(e)}
\]

\[
w^{(e)} = \int_V \int_V \nabla_1 \cdot J_1 \nabla_2 \cdot J_2^* \frac{\cos(kR_{12})}{R_{12}}
\]

\[
-\frac{k}{2} \left( k^2 J_1 \cdot J_2^* - \nabla_1 \cdot J_1 \nabla_2 \cdot J_2^* \right) \sin(kR_{12}) \, dV_1 \, dV_2,
\]

where \( J_1 = J(r_1) \), \( J_2 = J(r_2) \), \( R_{12} = |r_1 - r_2| \). Stored magnetic energy

\[
\widetilde{W}_\text{vac}^{(m)} = \frac{\mu_0}{16\pi k^2} w^{(m)}, \text{ where}
\]

\[
w^{(m)} = \int_V \int_V k^2 J_1 \cdot J_2^* \frac{\cos(kR_{12})}{R_{12}}
\]

\[
-\frac{k}{2} \left( k^2 J_1 \cdot J_2^* - \nabla_1 \cdot J_1 \nabla_2 \cdot J_2^* \right) \sin(kR_{12}) \, dV_1 \, dV_2.
\]
Stored EM energies from current densities $\mathbf{J}$ in $V$

Also the total radiated power $P_{\text{rad}} = \frac{\eta_0}{8\pi k} p_{\text{rad}}$ with

$$p_{\text{rad}} = \int_V \int_V \left( k^2 \mathbf{J}_1 \cdot \mathbf{J}_2^* - \nabla_1 \cdot \mathbf{J}_1 \nabla_2 \cdot \mathbf{J}_2^* \right) \frac{\sin(kR_{12})}{R_{12}} \, dV_1 \, dV_2.$$ 

The normalized quantities $w^{(e)}$, $w^{(m)}$, and $p_{\text{rad}}$ have dimensions given by volume, $m^3$, times the dimension of $|\mathbf{J}|^2$.

- In the limit $k\alpha \to 0$ by Geyi in *Physical limitations of antenna*, IEEE-TAP 2003.
- Some issues with 'negative stored energy' for large structures in Gustafsson et al, IEEE-TAP 2012.
The partial radiation intensity \( P(\hat{k}, \hat{e}) \) in direction \( \hat{k} \) and for the polarization \( \hat{e} \) is

\[
P(\hat{k}, \hat{e}) = \frac{\eta_0 k^2}{32\pi^2} \left| \int_V \hat{e}^* \cdot J(r) e^{jk\hat{k} \cdot r} \, dV \right|^2
\]

We have the \( G/Q \) quotient

\[
\frac{G(\hat{k}, \hat{e})}{Q} = k^3 \frac{\left| \int_V \hat{e}^* \cdot J(r) e^{jk\hat{k} \cdot r} \, dV \right|^2}{\max\{w^{(e)}(J), w^{(m)}(J)\}}
\]

\[
\leq \max_J k^3 \frac{\left| \int_V \hat{e}^* \cdot J(r) e^{jk\hat{k} \cdot r} \, dV \right|^2}{\max\{w^{(e)}(J), w^{(m)}(J)\}}
\]

Solve the optimization problem. Closed form solutions in the limit \( ka \to 0 \) and convex optimization for larger (but small) structures.
Convex optimization

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, N_1 \\
& \quad Ax = b
\end{align*}
\]

where \( f_i(x) \) are convex, i.e., \( f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y) \) for \( \alpha, \beta \in \mathbb{R}, \alpha + \beta = 1, \alpha, \beta \geq 0 \).

Solved with efficient standard algorithms. No risk of getting trapped in a local minimum. A problem is 'solved' if formulated as a convex optimization problem.

Can be used in many convex formulations for antenna performance expressed in the current density \( J \), e.g.,

- Radiated field \( F(\hat{k}) = -\hat{k} \times \hat{k} \times \int_V J(r) e^{jk \cdot r} \, dV \) is affine.
- Radiated power, stored electric and magnetic energies, and Ohmic losses are positive semi-definite quadratic forms in \( J \).
Determine a current density $J(r)$ in the volume $V$ that maximizes the partial-gain Q-factor quotient $G(\hat{k}, \hat{e})/Q$.

- Partial radiation intensity $P(\hat{k}, \hat{e})$

\[
\frac{G(\hat{k}, \hat{e})}{Q} = \frac{2\pi P(\hat{k}, \hat{e})}{c_0 k \max\{W_e, W_m\}}.
\]

- Scale $J$ and reformulate $P = 1$ as $\Re\{\hat{e}^* \cdot F\} = 1$.

- Convex optimization problem.

\[
\text{minimize} \quad \max\{J^H W_e J, J^H W_m J\}
\]
\[
\text{subject to} \quad \Re\{F^H J\} = 1
\]

Determines a current density $J(r)$ in the volume $V$ with minimal stored EM energy and unit partial radiation intensity in $\{\hat{k}, \hat{e}\}$.
Maximal $G(\hat{k}, \hat{x})/Q$ for planar rectangles

Solution of the convex optimization problem

$$
\min \quad \max \{ J^H W_e J, J^H W_m J \}
$$

s.t. \quad \text{Re}\{F^H J\} = 1

for current densities confined to planar rectangles with side lengths $\ell_x$ and $\ell_y = \{0.01, 0.1, 0.2, 0.5\} \ell_x$.

Note $\ell_x/\lambda = k \ell_x/(2\pi)$, giving

$\ell_x = \lambda/2 \rightarrow k \ell_x = \pi \rightarrow ka \geq \pi/2$. 

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$D/Q$ (or $G/Q$) bounds

- Similar to the forward scattering bounds for TM.
- Can design 'optimal' electric dipole mode (TM) antennas.
- TE modes and TE+TM are not well understood.
- Typical matlab code using CVX

```matlab
cvx_begin
    variable J(n) complex;
    dual variables We Wm
    maximize(real(F'*J))
    We: quad_form(J,Ze) <= 1;
    Wm: quad_form(J,Zm) <= 1;
cvx_end
```

We now reformulate the complex optimization problem to analyze superdirectivity, antennas with a prescribed radiation pattern, losses, and antennas embedded in a PEC structure.
Superdirectivity

- A superdirective antenna has a directivity that is much larger than for a typical reference antenna.
- Often low efficiency (low gain) and narrow bandwidth.
- There is an interest in small superdirective antennas, e.g., Best et al. 2008 and Arceo & Balanis 2011.

Here, we add the constraint $D \geq D_0$ to the convex optimization problem for $G/Q$ to determine the minimum $Q$ for superdirective lossless antennas. We can also add constraints on the losses.
Add the constraint $P_{\text{rad}} \leq 4\pi D_0^{-1}$ the get the convex optimization problem

$$\min \quad \max \{ J^H W_e J, J^H W_m J \}$$

s.t. \quad \text{Re}\{ F^H J \} = 1

$$J^H P J \leq k^3 D_0^{-1}$$

Example for current densities confined to planar rectangles with side lengths $\ell_x$ and $\ell_y = 0.5\ell_x$. 

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Superdirectivity with $D \geq D_0 = 10$

Note, it gives a bound on $Q$ as $D$ is known.
Optimal performance for embedded antennas

- It is common with antennas embedded in metallic structures.
- The induced currents radiate but they are not arbitrary.
- Linear map from the antenna region adds a (convex) constraint.
- Here, we assume that the surrounding structure is PEC and add a constraint to account for the induced currents on the surrounding structure in the $G/Q$ formulation.
Almost independent of the feed width at the resonance just below $l_x = 0.5\lambda$. 

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Embedded antennas in a planar rectangle

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Conclusions

- Closed form solution for small antennas.
- Convex optimization to determine bounds and optimal currents for larger structures:
  - $D/Q$ and $G/Q$.
  - $Q$ for superdirective antennas.
  - Embedded antennas in PEC structures.
  - $Q$ for antennas with prescribed far fields.

See also Physical bounds and optimal currents on antennas IEEE TAP, 60, 6, pp. 2672-2681, 2012 and Antenna currents for optimal $Q$, superdirectivity, and radiation patterns using convex optimization (www.eit.lth.se/staff/mats.gustafsson)