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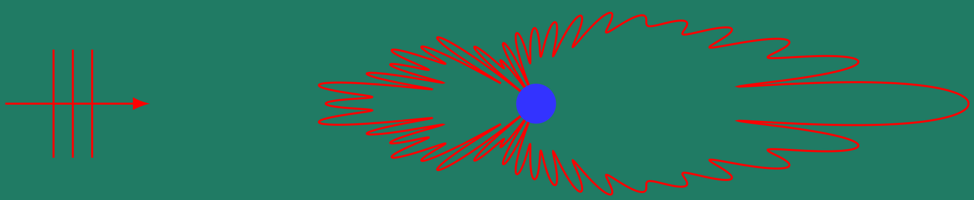
Supplement to SCATTERING OF ELECTROMAGNETIC WAVES BY OBSTACLES

I. ANTENNAS

January 24, 2024



Supplement to
Scattering of Electromagnetic Waves by Obstacles
I. Antennas

Gerhard Kristensson

This supplement is dedicated to the memory of my mother, Birgit

Contents

Preface

The original manuscript to the textbook: G. Kristensson, *Scattering by Electromagnetic Waves by Obstacles* [17] was at first much longer than the 750 pages the publisher allowed. I then decided to extract some of the chapters and add this material later as supplements, to be published on my home page. This is the first in that series. The notation follows the one in the main book, which is referred to as the textbook in this supplement, and references to the equations in the textbook are frequently made.

The author is indebted to Torleif Martin for many rewarding discussions on antennas. I am grateful if the reader reports misprints and give comments on this supplement to the email below. The supplement is updated regularly and no errata list is produced. Make sure you use the latest version of the supplement.

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Transmitting and receiving antennas

A receiving antenna is — from a scattering point of view — characterized as a passive scatterer, just like any other scatterer treated in Chapter 4 in the textbook [17]. It is therefore pertinent to deal with some general aspects of receiving antennas in this supplement. For completeness, some results for a transmitting antenna are also reviewed.

The procedure presented in this supplement bears many similarities with already existing treatments. In fact, the subject of scattering by loaded structures (*e.g.*, antennas) is old and dates back to the first half of the last century [5, 11, 14, 24]. Several articles, which adopt a S-parameter description of the exterior fields to the antenna, are found, see *e.g.*, [8–10, 12, 23]. The present treatment, however, emphasizes the far field amplitude and the exiting electric field as vectors and not as a series of spherical vector waves. The point here is that the decomposition into spherical vector waves (in the far field an expansion in vector spherical harmonics) can be done at a later stage, see Section 7.1 on page 28. We also prefer to work with the amplitudes of the travelling waves on the transmission line feed, rather than the equivalent voltage and currents. The equivalence between the to representations is presented in some detail in Section 9 on page 34.

This supplement is organized as follows: In Section 1, the concepts used in this supplement are defined. In particular, the reference plane is defined, and the fields on this plane are discussed. Moreover, a representation of the antenna in terms of the wave amplitudes on the feeding transmission line, the far field amplitude, and the exciting fields is introduced. A formal proof of this representation is found in the appendix on page 39. The particular conditions that hold for the transmitting and receiving antennas are presented in Sections 2 and 3, respectively. The two invariance principles in Section 4 — reciprocity and generalized power balance — are employed in Sections 5 and 6, respectively, to relate the generic quantities of the antenna to each other. A few elementary examples are analyzed in Section 7. As an application of the results, we derive Friis’ transmission formula in a general setting in Section 8. In a final section, Section 9, we present an alternative impedance representation of the antenna in terms of the voltage and the current in the transmission line feed.

1 Basic concepts and prerequisites

In general, two different sources excite the antenna. As a receiving antenna, one incident field as usual (often a plane wave excitation with polarization vector $\boldsymbol{E}_0$), but now also a source

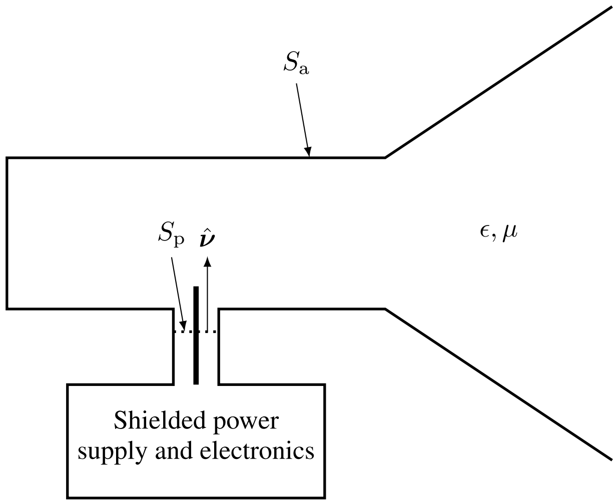

Figure 1: The generic geometry of an antenna and the reference plane $S_{\rm p}$. The material exterior to the antenna is parameterized by the relative permittivity ϵ and relative permeability μ.

inside the antenna. In the transmitting mode, there are only sources inside the antenna. To start, we first discuss the field distribution inside the antenna.

1.1 The field in the port

The basic difference between passive scatterers, which we deal with in the textbook [17], and a transmitting or a receiving antenna, is the presence of an antenna port or terminal. The antenna port is characterized by a reference plane $S_{\rm p}$ [6, 22, 27], see Figure 1, which illustrates a horn configuration with a coaxial feeding transmission line. Other antenna geometries are treated in a similar way.

In the reference plane $S_{\rm p}$, there exists a well-defined transverse electric field in terms of a TEM-mode $\boldsymbol{E}_m(\boldsymbol{r}_{\rm c})$.[1] The TEM-mode satisfies (we let the z axis be oriented along the direction of the transmission line) [4, 13, 20, 21]

$$\boldsymbol{E}_m(\boldsymbol{r}_{\rm c}) = -\nabla_{\rm t}\psi(\boldsymbol{r}_{\rm c})$$

where $\nabla_{\rm t} = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}$, $\boldsymbol{r}_{\rm c} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$, and where $\psi(\boldsymbol{r}_{\rm c})$ satisfies

$$\nabla_{\rm t}^2\psi(\boldsymbol{r}_{\rm c}) = 0$$

with boundary conditions $\psi =$ constant on every simply connected part of the boundary line of $S_{\rm p}$. The mode is $\boldsymbol{E}_m$ is a real-valued function and it is transverse, *i.e.*, $\hat{\boldsymbol{\nu}} \cdot \boldsymbol{E}_m = 0$, and,

[1]The analysis can be generalized to also include waveguide excitation, but to the expense of a more complex analysis.

moreover, we normalize the function ψ such that

$$\iint\limits_{S_{\mathrm{p}}} |\nabla_{\mathrm{t}}\psi(\boldsymbol{r}_{\mathrm{c}})|^2 \,\mathrm{d}S = 1$$

By definition, the components of the functions $\boldsymbol{E}_m$ have dimension m^{-1}.

In the case of a coaxial cable (inner radius a and outer radius b), the TEM-mode is

$$\nabla_{\mathrm{t}}\psi(\boldsymbol{r}_{\mathrm{c}}) = \frac{\boldsymbol{r}_{\mathrm{c}}}{2\pi r_{\mathrm{c}}^2 \ln(b/a)} = \frac{\hat{\boldsymbol{r}}_{\mathrm{c}}}{2\pi r_{\mathrm{c}} \ln(b/a)}$$

where and $\hat{\boldsymbol{r}}_{\mathrm{c}} = \boldsymbol{r}_{\mathrm{c}}/r_{\mathrm{c}}$ and $r_{\mathrm{c}} = \sqrt{x^2+y^2}$.

The general expressions of the electric and magnetic fields on S_{p}, then are [4, 13, 20, 21]

$$\begin{cases} \boldsymbol{E}(\boldsymbol{r}) = \left(A_+\mathrm{e}^{\mathrm{i}kz} + A_-\mathrm{e}^{-\mathrm{i}kz}\right)\boldsymbol{E}_m(\boldsymbol{r}_{\mathrm{c}}) \\ Z\boldsymbol{H}(\boldsymbol{r}) = \left(A_+\mathrm{e}^{\mathrm{i}kz} - A_-\mathrm{e}^{-\mathrm{i}kz}\right)\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_m(\boldsymbol{r}_{\mathrm{c}}) \end{cases} \quad \boldsymbol{r}\in S_{\mathrm{p}}$$

Here k and $Z = \sqrt{\mu\mu_0/\epsilon\epsilon_0}$ are the wave number and the wave impedance of the surrounding material, respectively.[2,3] The coefficients $A_\pm$ give the amplitude of the wave travelling in the positive and the negative direction, respectively. Since on S_{p} $z = z_0 =$ constant, the electric and magnetic fields at the reference plane then have the form

$$\begin{cases} \boldsymbol{E}(\boldsymbol{r}_{\mathrm{c}}) = (\alpha_+ + \alpha_-)\,\boldsymbol{E}_m(\boldsymbol{r}_{\mathrm{c}}) \\ Z\boldsymbol{H}(\boldsymbol{r}_{\mathrm{c}}) = (\alpha_+ - \alpha_-)\,\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_m(\boldsymbol{r}_{\mathrm{c}}) \end{cases} \quad \boldsymbol{r}_{\mathrm{c}}\in S_{\mathrm{p}} \tag{1.1}$$

where

$$\alpha_\pm = A_\pm \mathrm{e}^{\pm\mathrm{i}kz_0}$$

The coefficients $\alpha_\pm$ have the dimension Volt (V). Equation (1.1) gives the a general expression of the fields at the reference plane. The exact form of the waveguide mode, $\boldsymbol{E}_m$, is irrelevant for the analysis in this supplement, and it is enough to specify that the fields in the port have this particular form, and that $\boldsymbol{E}_m\cdot\hat{\boldsymbol{\nu}} = 0$, *i.e.*, the electric mode $\boldsymbol{E}_m$ is tangential to the reference plane S_{p}. Moreover, the mode is a real-valued function and the integral over the magnitude of the mode $\boldsymbol{E}_m$ is

$$\iint\limits_{S_{\mathrm{p}}} |\boldsymbol{E}_m|^2 \,\mathrm{d}S = \iint\limits_{S_{\mathrm{p}}} \left(\boldsymbol{E}_m\times(\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_m)\right)\cdot\hat{\boldsymbol{\nu}} \,\mathrm{d}S = \iint\limits_{S_{\mathrm{p}}} |\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_m|^2 \,\mathrm{d}S = 1 \tag{1.2}$$

The amplitudes of the waves, α_+ and α_-, act like input and output, respectively, to the antenna configuration, see Figure 2. In the port, we can also identify the voltage V and the current I defined by

$$\begin{cases} V^{\mathrm{r,t}} = \alpha_+^{\mathrm{r,t}} + \alpha_-^{\mathrm{r,t}} \\ ZI^{\mathrm{r,t}} = \alpha_+^{\mathrm{r,t}} - \alpha_-^{\mathrm{r,t}} \end{cases} \tag{1.3}$$

where the superscripts, r and t, are added to distinguish between the pure receiving mode, or the transmitting mode, respectively.

[2] Note the different notation for the wave impedance Z. In the textbook, we used η for the wave impedance. The reason for the change is that the symbol η is used for the absorption efficiency of the antenna below.

[3] We assume the material supporting the TEM-mode is the same as the material exterior of the antenna. This assumption can be generalized to two different media.

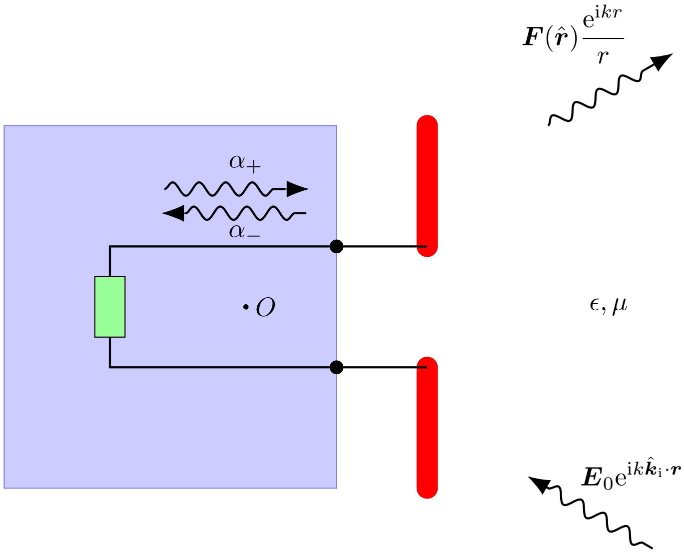

Figure 2: The generic geometry of an antenna (red), the supporting electronics (blue) with the detector, and its exciting fields α_+ and $\boldsymbol{E}_0$. The detector/generator is marked with a green box. The local origin is denoted O. The outputs are α_- and the far field amplitude $\boldsymbol{F}(\hat{\boldsymbol{r}})$.

The total power in the transmission line, as a receiving or as a transmitting antenna, respectively, is

$$\begin{aligned} P_{\mathrm{r,t}} &= \frac{1}{2}\,\mathrm{Re}\iint\limits_{S_{\mathrm{p}}} (\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r})) \cdot \hat{\boldsymbol{\nu}}\ \mathrm{d}S \\ &= \frac{1}{2Z}\,\mathrm{Re}\left(\alpha_+^{\mathrm{r,t}} + \alpha_-^{\mathrm{r,t}}\right)\left(\alpha_+^{\mathrm{r,t}} - \alpha_-^{\mathrm{r,t}}\right)^* \iint\limits_{S_{\mathrm{p}}} (\boldsymbol{E}_m(\boldsymbol{r}) \times (\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m(\boldsymbol{r}))) \cdot \hat{\boldsymbol{\nu}}\ \mathrm{d}S \\ &= \frac{1}{2Z}\left\{\left|\alpha_+^{\mathrm{r,t}}\right|^2 - \left|\alpha_-^{\mathrm{r,t}}\right|^2\right\} \end{aligned} \tag{1.4}$$

where the star * denotes the complex conjugate, and where we also used (1.1) and (1.2). The power flow shows that the amplitude $\alpha_-^{\mathrm{r,t}}$ quantifies the output power flow, as seen from the antenna, and $\alpha_+^{\mathrm{r,t}}$ quantifies the input power flow to the antenna.

1.2 The antenna as a two-port

Let O denote a local origin at the antenna, see Figure 2. This origin can be chosen at a point in the vicinity of the antenna or at the radiation center of the antenna [7]. For each fixed incident direction $\hat{\boldsymbol{k}}_{\mathrm{i}}$ and each fixed observation direction $\hat{\boldsymbol{r}}$ in the far field, the electromagnetic problem is linear and there exists a linear transformation between the input parameters, α_+ and $\boldsymbol{E}_0$, and the output wave in the antenna, α_-, and the far field amplitude $\boldsymbol{F}(\hat{\boldsymbol{r}})$. We prefer to express this relation as

$$\begin{cases} \alpha_- = S\alpha_+ + \dfrac{2\pi\mathrm{i}}{k}\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0 \\ \boldsymbol{F}(\hat{\boldsymbol{r}}) = \boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+ + \dfrac{2\pi\mathrm{i}}{k}\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0 \end{cases} \tag{1.5}$$

A formal derivation of this relation is provided in the appendix on page 39.[4] The antenna is characterized by the four dimension-less quantities S, $\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})$, $\boldsymbol{f}(\hat{\boldsymbol{r}})$, and $\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})$. These quantities are, in general, functions of frequency (the wave number k), but this dependence is suppressed, if it is not important for the analysis. The electric field $\boldsymbol{E}_0$ denotes the electric field at the origin of the antenna, which is assumed to have the form of an incident plane wave with propagation direction $\hat{\boldsymbol{k}}_\mathrm{i}$, *i.e.*, the electric field is

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_0\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}$$

If the incident field is a superposition of plane waves, the output becomes a superposition of the individual plane wave contributions. The reflection coefficient S in the first line is independent of the directions $\hat{\boldsymbol{k}}_\mathrm{i}$ and $\hat{\boldsymbol{r}}$, see the derivation in the appendix, and the second term in the first line quantifies the absorption of power in the antenna. In the second line, the first term is related to the radiation of electromagnetic power away from the structure, and the second term contains a the contribution to pure scattering. Note that we have introduced an extra factor $2\pi\mathrm{i}/k$ in each line, which makes $\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})$, $\boldsymbol{f}(\hat{\boldsymbol{r}})$, and $\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})$ dimension-less, and, as we discover below, leads to simple relations between these quantities.

The scalar S is the reflection coefficient of the incident amplitude, α_+, on the transmission line by the antenna, and $\boldsymbol{f}(\hat{\boldsymbol{r}})$ is the radiation pattern of the antenna.[5] If the antenna is perfectly matched to the transmission line $S = 0$. The reflection coefficient S lies inside the unit circle in the complex plane, *i.e.*, $|S| \leq 1$, see appendix for a proof. The dyadic $\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})$ quantifies the field scattered directly by the antenna without any influence of the shielded electronics.[6] The actual computation of these quantities for a general antenna can be a very difficult task indeed, see appendix on page 39.

Another, equivalent parametrization of the antenna is in terms of the current I and the voltage V at the antenna port. We adopt

$$\begin{cases} V = Z_\mathrm{in}I + \dfrac{2\pi\mathrm{i}}{k}\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0 \\ \boldsymbol{F}(\hat{\boldsymbol{r}}) = Z_\mathrm{in}\boldsymbol{f}_\mathrm{Z}(\hat{\boldsymbol{r}})I + \dfrac{2\pi\mathrm{i}}{k}\mathbf{t}_\mathrm{Z}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0 \end{cases} \tag{1.6}$$

This form of the relation between the input and output of the antenna is treated in more detail in Section 9 on page 34. With this alternative description, the characterization of the antenna is made with the parameters Z_in, $\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i})$, $\boldsymbol{f}_\mathrm{Z}(\hat{\boldsymbol{r}})$, and $\mathbf{t}_\mathrm{Z}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})$. These parameters can be expressed in terms of S, $\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})$, $\boldsymbol{f}(\hat{\boldsymbol{r}})$, and $\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})$. We postpone these relations to Section 9 on page 34.

[4]Several antennas, as in a MIMO configuration, can be accommodated in this relation as a matrix version of the relation in (1.5), but at the expense of clarity. We refrain from this extension here.

[5]This reflection coefficient is identical to the S-parameter S_{11} in microwave theory.

[6]In Section 3, we conclude that the dyadic $\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})$ quantifies the scattered far field with a matched load, see (3.8) on page 9.

2 Transmitting antenna

We start with the properties of the transmitting antenna. This case illustrates the most simple situation with only internal sources present. In a transmitting situation, the relation (1.5) becomes

$$\begin{cases} \alpha_-^{\mathrm{t}} = S\alpha_+^{\mathrm{t}} \\ \boldsymbol{F}^{\mathrm{t}}(\hat{\boldsymbol{r}}) = \boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+^{\mathrm{t}} \end{cases} \tag{2.1}$$

Note that, in the transmitting mode, the far field amplitude characterizes the total field and not the scattered field.

2.1 Power, directivity, and gain

The total power in the transmission line is, see (1.4)

$$P_{\mathrm{in}} = \frac{1}{2Z}\left(\left|\alpha_+^{\mathrm{t}}\right|^2 - \left|\alpha_-^{\mathrm{t}}\right|^2\right) = \frac{1}{2Z}\left(1 - |S|^2\right)\left|\alpha_+^{\mathrm{t}}\right|^2 \tag{2.2}$$

and it quantifies the power delivered to the antenna (accepted by the antenna port).

The power radiated or transmitted by the antenna in the direction $\hat{\boldsymbol{r}}$ per unit solid angle is, see [17, Sec. 4.2]

$$r^2 \left\langle \boldsymbol{S}_{\mathrm{t}}(t)\right\rangle(\hat{\boldsymbol{r}}) = \frac{\hat{\boldsymbol{r}}}{2Z}\left|\boldsymbol{F}^{\mathrm{t}}(\hat{\boldsymbol{r}})\right|^2 = \frac{\hat{\boldsymbol{r}}}{2Z}\left|\boldsymbol{f}(\hat{\boldsymbol{r}})\right|^2\left|\alpha_+^{\mathrm{t}}\right|^2$$

Similarly, the total radiated power by the antenna is, see [17, Eq. (4.23)]

$$P_{\mathrm{t}} = \frac{1}{2Z}\iint\limits_{\Omega}\left|\boldsymbol{F}^{\mathrm{t}}(\hat{\boldsymbol{r}})\right|^2\,\mathrm{d}\Omega = \frac{1}{2Z}\iint\limits_{\Omega}\left|\boldsymbol{f}(\hat{\boldsymbol{r}})\right|^2\,\mathrm{d}\Omega\left|\alpha_+^{\mathrm{t}}\right|^2 \tag{2.3}$$

The antenna directivity, $D(\hat{\boldsymbol{r}})$, and the antenna gain, $G(\hat{\boldsymbol{r}})$, can now be defined [2]. The (absolute) directivity $D(\hat{\boldsymbol{r}})$ is defined as the power radiated in the direction $\hat{\boldsymbol{r}}$ per unit solid angle divided by the average power radiated per unit solid angle, *i.e.*,

$$\begin{aligned} D(\hat{\boldsymbol{r}}) &= \frac{\text{power radiated/unit solid angle}}{\text{average power radiated/unit solid angle}} \\ &= \frac{\left|\boldsymbol{F}^{\mathrm{t}}(\hat{\boldsymbol{r}})\right|^2/(2Z)}{\iint\limits_{\Omega}\left|\boldsymbol{F}^{\mathrm{t}}(\hat{\boldsymbol{r}})\right|^2\,\mathrm{d}\Omega/(2Z)/(4\pi)} = \frac{4\pi\left|\boldsymbol{f}(\hat{\boldsymbol{r}})\right|^2}{\iint\limits_{\Omega}\left|\boldsymbol{f}(\hat{\boldsymbol{r}})\right|^2\,\mathrm{d}\Omega} \end{aligned} \tag{2.4}$$

The partial directivity $D(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}_{\mathrm{e}})$ is defined as the power radiated in the direction $\hat{\boldsymbol{r}}$ per unit solid angle corresponding to the polarization $\hat{\boldsymbol{p}}_{\mathrm{e}} = \boldsymbol{E}_0/|\boldsymbol{E}_0|$ divided by the average power radiated per unit solid angle, *i.e.*,

$$D(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}_{\mathrm{e}}) = \frac{4\pi\left|\boldsymbol{f}(\hat{\boldsymbol{r}})\cdot\hat{\boldsymbol{p}}_{\mathrm{e}}\right|^2}{\iint\limits_{\Omega}\left|\boldsymbol{f}(\hat{\boldsymbol{r}})\right|^2\,\mathrm{d}\Omega}$$

Notice that the unit vector $\hat{\boldsymbol{p}}_{\mathrm{e}}$ can be a complex vector, which is the case for a circular polarization.

The (absolute) gain $G(\hat{\boldsymbol{r}})$ is defined as the power radiated in the direction $\hat{\boldsymbol{r}}$ per unit solid angle divided by the power delivered to the antenna $P_{\rm in}$, *i.e.*,

$$G(\hat{\boldsymbol{r}}) = 4\pi\frac{\text{power radiated/unit solid angle}}{\text{power delivered to the antenna}} = \frac{\left|\boldsymbol{F}^{\rm t}(\hat{\boldsymbol{r}})\right|^2}{2Z}\frac{4\pi}{P_{\rm in}} = \frac{4\pi\left|\boldsymbol{f}(\hat{\boldsymbol{r}})\right|^2}{1-|S|^2} \tag{2.5}$$

We immediately observe that the relation between the directivity and the gain is

$$\left(1-|S|^2\right)G(\hat{\boldsymbol{r}}) = D(\hat{\boldsymbol{r}})\iint\limits_{\Omega}|\boldsymbol{f}(\hat{\boldsymbol{r}})|^2\,\mathrm{d}\Omega \tag{2.6}$$

Moreover, the realized gain $G(\hat{\boldsymbol{r}}, S)$ of the antenna in a direction $\hat{\boldsymbol{r}}$ is the gain reduced by the mismatch loss due to the reflection in the antenna port, *i.e.*,

$$G(\hat{\boldsymbol{r}}, S) = \left(1-|S|^2\right)G(\hat{\boldsymbol{r}}) = D(\hat{\boldsymbol{r}})\iint\limits_{\Omega}|\boldsymbol{f}(\hat{\boldsymbol{r}})|^2\,\mathrm{d}\Omega$$

The return loss RL is defined as fractional power (in dB) reflected by the antenna, *i.e.*,

$$RL = -20\log(|S|)\text{ dB}$$

A matched system, $S = 0$ (no power reflected) gives a return loss of $\text{RL} = \infty$, and total reflection, $|S| = 1$ (all power reflected), gives $\text{RL} = 0$.

The antenna parameters that we introduced here are frequently used in applications.

3 Receiving antenna

In the receiving mode, there are two sources — an amplitude $\alpha_+^{\rm r}$ and an incident field $\boldsymbol{E}_0$, and the relation (1.5) becomes

$$\begin{cases}\alpha_-^{\rm r} = S\alpha_+^{\rm r} + \dfrac{2\pi\mathrm{i}}{k}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\rm i})\cdot\boldsymbol{E}_0\\ \boldsymbol{F}^{\rm r}(\hat{\boldsymbol{r}}) = \boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+^{\rm r} + \dfrac{2\pi\mathrm{i}}{k}\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\rm i})\cdot\boldsymbol{E}_0\end{cases} \tag{3.1}$$

The load in the electronics provides a relation between $\alpha_\pm^{\rm r}$ on the transmission line, *i.e.*, $\alpha_+^{\rm r} = \Gamma_{\rm L}\alpha_-^{\rm r}$, where $\Gamma_{\rm L}$ is the load reflection coefficient as seen from the antenna port (reference plane).[7] We get

$$\alpha_-^{\rm r} = \frac{2\pi\mathrm{i}}{k}\frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\rm i})\cdot\boldsymbol{E}_0}{1-S\Gamma_{\rm L}} \tag{3.2}$$

and

$$\boldsymbol{F}^{\rm r}(\hat{\boldsymbol{r}}) = \frac{2\pi\mathrm{i}}{k}\left\{\frac{\Gamma_{\rm L}\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\rm i})}{1-S\Gamma_{\rm L}} + \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\rm i})\right\}\cdot\boldsymbol{E}_0 \tag{3.3}$$

[7] At another position on the transmission line, the reflection coefficient has to be transformed.

Load	**Reflection coefficient**
Open-circuit (oc)	$\Gamma_\mathrm{L} = 1$
Short-circuit (sc)	$\Gamma_\mathrm{L} = -1$
Matched (m)	$\Gamma_\mathrm{L} = 0$
Conjugate matched (cm)	$\Gamma_\mathrm{L} = S^*$

Table 1: The four canonical loads for the antenna.

This relation defines the total scattering dyadic $\mathbf{S}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i})$, see [17, Sec. 4.3]

$$\mathbf{S}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) = \frac{2\pi\mathrm{i}}{k}\left\{\frac{\Gamma_\mathrm{L}\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})}{1 - S\Gamma_\mathrm{L}} + \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i})\right\} \tag{3.4}$$

The total received (accepted) voltage, V^r, is, see (1.3)

$$V^\mathrm{r} = \alpha_+^\mathrm{r} + \alpha_-^\mathrm{r} = \frac{2\pi\mathrm{i}}{k}\frac{1 + \Gamma_\mathrm{L}}{1 - S\Gamma_\mathrm{L}}\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \tag{3.5}$$

The total power in the transmission line, which is the total power received in the antenna load, is, see (1.4)[8]

$$P_\mathrm{r} = -\frac{1}{2Z}\left(\left|\alpha_+^\mathrm{r}\right|^2 - \left|\alpha_-^\mathrm{r}\right|^2\right)$$

which we rewrite as

$$P_\mathrm{r} = \left(1 - |\Gamma_\mathrm{L}|^2\right)\frac{\left|\alpha_-^\mathrm{r}\right|^2}{2Z} = \frac{2\pi^2}{k^2 Z}\frac{1 - |\Gamma_\mathrm{L}|^2}{|1 - S\Gamma_\mathrm{L}|^2}\left|\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0\right|^2 \tag{3.6}$$

Expression (3.6) is also the power absorbed, and sometimes we use the notation $P_\mathrm{a} = P_\mathrm{r}$, which is related to the absorption cross section $\sigma_\mathrm{a}(\hat{\boldsymbol{k}}_\mathrm{i})$, see [17, Sec. 4.2]

$$\sigma_\mathrm{a}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{2ZP_\mathrm{a}}{|\boldsymbol{E}_0|^2} = \frac{4\pi^2}{k^2}\frac{1 - |\Gamma_\mathrm{L}|^2}{|1 - S\Gamma_\mathrm{L}|^2}\frac{\left|\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0\right|^2}{|\boldsymbol{E}_0|^2} \tag{3.7}$$

3.1 Different matching loads

Four special loads are of interest in practice, *viz.* open-circuited $\Gamma_\mathrm{L} = 1$, short-circuited $\Gamma_\mathrm{L} = -1$, matched $\Gamma_\mathrm{L} = 0$, and conjugate matched $\Gamma_\mathrm{L} = S^*$. We summarize the four different cases in Table 1. These four cases correspond to received amplitude, see (3.2)

$$\begin{cases} \alpha_-^{\mathrm{r,oc}} = \dfrac{2\pi\mathrm{i}}{k}\dfrac{\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0}{1 - S} \\ \alpha_-^{\mathrm{r,sc}} = \dfrac{2\pi\mathrm{i}}{k}\dfrac{\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0}{1 + S} \end{cases} \qquad \begin{cases} \alpha_-^{\mathrm{r,m}} = \dfrac{2\pi\mathrm{i}}{k}\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \\ \alpha_-^{\mathrm{r,cm}} = \dfrac{2\pi\mathrm{i}}{k}\dfrac{\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0}{1 - |S|^2} \end{cases}$$

[8]Note the change in sign, due to opposite reference direction.

and far field amplitude, see (3.3)

$$\begin{cases} \boldsymbol{F}^{\mathrm{r,oc}}(\hat{\boldsymbol{r}}) = \dfrac{2\pi\mathrm{i}}{k}\left\{\dfrac{\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})}{1-S} + \mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\right\}\cdot\boldsymbol{E}_0 \\ \boldsymbol{F}^{\mathrm{r,sc}}(\hat{\boldsymbol{r}}) = \dfrac{2\pi\mathrm{i}}{k}\left\{-\dfrac{\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})}{1+S} + \mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\right\}\cdot\boldsymbol{E}_0 \\ \boldsymbol{F}^{\mathrm{r,m}}(\hat{\boldsymbol{r}}) = \dfrac{2\pi\mathrm{i}}{k}\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0 \\ \boldsymbol{F}^{\mathrm{r,cm}}(\hat{\boldsymbol{r}}) = \dfrac{2\pi\mathrm{i}}{k}\left\{\dfrac{S^*\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})}{1-|S|^2} + \mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\right\}\cdot\boldsymbol{E}_0 \end{cases} \tag{3.8}$$

From (3.6), we conclude that the total power absorbed in the load in the different cases are

$$\begin{cases} P_{\mathrm{r,oc}} = P_{\mathrm{r,sc}} = 0 \\ P_{\mathrm{r,m}} = \dfrac{2\pi^2}{k^2 Z}\left|\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\right|^2 \\ P_{\mathrm{r,cm}} = \dfrac{2\pi^2}{k^2 Z}\dfrac{1-|S|^2}{(1-|S|^2)^2}\left|\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\right|^2 = \dfrac{P_{\mathrm{r,m}}}{1-|S|^2} \end{cases}$$

We also conclude that the total power absorbed in the load, (3.6), is maximized if the load is chosen as $\Gamma_\mathrm{L} = S^*$ (conjugate match)

$$P_{\mathrm{r,max}} = P_{\mathrm{r,cm}} = \frac{2\pi^2\left|\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\right|^2}{k^2 Z\left(1-|S|^2\right)}$$

In fact, this statement is seen be letting $\Gamma_\mathrm{L} = S^* + z$, where z is a complex number. Then

$$\frac{1}{1-|S|^2} - \frac{1-|\Gamma_\mathrm{L}|^2}{|1-S\Gamma_\mathrm{L}|^2} = \frac{\left|1-|S|^2-Sz\right|^2 - \left(1-|S|^2\right)\left(1-|S|^2-|z|^2-2\operatorname{Re}(Sz)\right)}{|1-|S|^2-Sz|^2\left(1-|S|^2\right)}$$

or, since $|S|<1$, for all $z\in\mathbb{C}$

$$\frac{1}{1-|S|^2} - \frac{1-|\Gamma_\mathrm{L}|^2}{|1-S\Gamma_\mathrm{L}|^2} = \frac{|z|^2}{|1-|S|^2-Sz|^2\left(1-|S|^2\right)} \geq 0 \tag{3.9}$$

The relations in (3.8) imply that

$$(1-S)\boldsymbol{F}^{\mathrm{r,oc}}(\hat{\boldsymbol{r}}) + (1+S)\boldsymbol{F}^{\mathrm{r,sc}}(\hat{\boldsymbol{r}}) = 2\boldsymbol{F}^{\mathrm{r,m}}(\hat{\boldsymbol{r}})$$

We also have for a general load Γ_L, see (3.3)

$$\boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}) = \boldsymbol{F}^{\mathrm{r,m}}(\hat{\boldsymbol{r}}) + \frac{2\pi\mathrm{i}}{k}\frac{\Gamma_\mathrm{L}\boldsymbol{f}(\hat{\boldsymbol{r}})\left(\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\right)}{1-S\Gamma_\mathrm{L}}$$

or

$$\boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}) = \boldsymbol{F}^{\mathrm{r,sc}}(\hat{\boldsymbol{r}}) + \frac{2\pi\mathrm{i}}{k}\boldsymbol{f}(\hat{\boldsymbol{r}})\left(\frac{\Gamma_{\mathrm{L}}}{1-S\Gamma_{\mathrm{L}}} + \frac{1}{1+S}\right)\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})\cdot\boldsymbol{E}_0$$

Moreover, the far field amplitude in the forward direction for a general load can be expressed in terms of the quantities $\boldsymbol{f}(\hat{\boldsymbol{k}}_{\mathrm{i}})$, $\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})$, and $\boldsymbol{F}^{\mathrm{r,sc}}(\hat{\boldsymbol{k}}_{\mathrm{i}})$.

$$\boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{k}}_{\mathrm{i}}) = \boldsymbol{F}^{\mathrm{r,m}}(\hat{\boldsymbol{k}}_{\mathrm{i}}) + \frac{2\pi\mathrm{i}}{k}\frac{\Gamma_{\mathrm{L}}\boldsymbol{f}(\hat{\boldsymbol{k}}_{\mathrm{i}})\left(\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})\cdot\boldsymbol{E}_0\right)}{1-S\Gamma_{\mathrm{L}}}$$

The optical theorem, see [17, Eq. (4.47) on page 230]

$$\sigma_{\mathrm{ext}} = \frac{4\pi}{k}\,\mathrm{Im}\left\{\frac{\boldsymbol{E}_0^*\cdot\boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{k}}_{\mathrm{i}})}{|\boldsymbol{E}_0|^2}\right\}$$

implies

$$\sigma_{\mathrm{ext}}(\hat{\boldsymbol{k}}_{\mathrm{i}}) = \sigma_{\mathrm{ext}}^{\mathrm{m}}(\hat{\boldsymbol{k}}_{\mathrm{i}}) + \frac{8\pi^2}{k^2}\,\mathrm{Re}\left\{\frac{\Gamma_{\mathrm{L}}}{1-S\Gamma_{\mathrm{L}}}\frac{\left(\boldsymbol{E}_0^*\cdot\boldsymbol{f}(\hat{\boldsymbol{k}}_{\mathrm{i}})\right)\left(\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})\cdot\boldsymbol{E}_0\right)}{|\boldsymbol{E}_0|^2}\right\}$$

Below, we have use of the current in the short-circuited case, see (1.3)

$$I^{\mathrm{r,sc}} = (\Gamma_{\mathrm{L}}-1)\,\frac{\alpha_-^{\mathrm{r,sc}}}{Z} = -2\frac{\alpha_-^{\mathrm{r,sc}}}{Z} = -\frac{4\pi\mathrm{i}}{kZ}\frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})\cdot\boldsymbol{E}_0}{1+S} \tag{3.10}$$

Example 3.1
The effective height of the antenna, $\boldsymbol{\ell}(\hat{\boldsymbol{k}}_{\mathrm{i}})$, is defined as $V^{\mathrm{r,oc}} = \boldsymbol{\ell}(\hat{\boldsymbol{k}}_{\mathrm{i}})\cdot\boldsymbol{E}_0$. We get from (3.5)

$$V^{\mathrm{r,oc}} = \frac{2\pi\mathrm{i}}{k}\frac{2}{1-S}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})\cdot\boldsymbol{E}_0$$

from which we find

$$\boldsymbol{\ell}(\hat{\boldsymbol{k}}_{\mathrm{i}}) = \frac{4\pi\mathrm{i}}{k}\frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})}{1-S}$$

The effective height of the antenna is a measure of the receiving capacity of the antenna. ■

3.2 Reradiated power

The power that is reradiated by the antenna consists of two parts — a direct scattered part and one part due to reflection of the received amplitude by the load.

The total scattered power is, see [17, Eq. (4.23)] and (3.3)

$$P_{\mathrm{s}} = \frac{1}{2Z}\iint\limits_{\Omega}\left|\boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}})\right|^2\,\mathrm{d}\Omega = \frac{2\pi^2}{k^2Z}\iint\limits_{\Omega}\left|\frac{\Gamma_{\mathrm{L}}\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})\cdot\boldsymbol{E}_0}{1-S\Gamma_{\mathrm{L}}} + \mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\mathrm{i}})\cdot\boldsymbol{E}_0\right|^2\,\mathrm{d}\Omega \tag{3.11}$$

or

$$P_{\mathrm{s}} = \frac{2\pi^2}{k^2 Z} \left\{ \left| \frac{\Gamma_{\mathrm{L}} \boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0}{1 - S\Gamma_{\mathrm{L}}} \right|^2 \iint\limits_{\Omega} |\boldsymbol{f}(\hat{\boldsymbol{r}})|^2 \,\mathrm{d}\Omega + \iint\limits_{\Omega} \left| \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \right|^2 \,\mathrm{d}\Omega \right.$$
$$\left. + 2\,\mathrm{Re} \left\{ \frac{\Gamma_{\mathrm{L}} \boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0}{1 - S\Gamma_{\mathrm{L}}} \iint\limits_{\Omega} \boldsymbol{E}_0^* \cdot \mathbf{t}^\dagger(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{f}(\hat{\boldsymbol{r}}) \,\mathrm{d}\Omega \right\} \right\}$$

For a conjugate matched load, $\Gamma_{\mathrm{L}} = S^*$, the absorbed and reradiated powers are, see (3.6)

$$P_{\mathrm{r}}^{\mathrm{cm}} = P_{\mathrm{a}}^{\mathrm{cm}} = \frac{2\pi^2}{k^2 Z} \frac{\left| \boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \right|^2}{1 - |S|^2}$$

and

$$P_{\mathrm{s}}^{\mathrm{cm}} = \frac{2\pi^2}{k^2 Z} \left\{ \frac{|S|^2 \left| \boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \right|^2}{\left(1 - |S|^2\right)^2} \iint\limits_{\Omega} |\boldsymbol{f}(\hat{\boldsymbol{r}})|^2 \,\mathrm{d}\Omega + \iint\limits_{\Omega} \left| \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \right|^2 \,\mathrm{d}\Omega \right.$$
$$\left. + 2\,\mathrm{Re} \left\{ \frac{S^* \boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0}{1 - |S|^2} \iint\limits_{\Omega} \boldsymbol{E}_0^* \cdot \mathbf{t}^\dagger(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{f}(\hat{\boldsymbol{r}}) \,\mathrm{d}\Omega \right\} \right\}$$

Similarly, for a matched load, $\Gamma_{\mathrm{L}} = 0$, the absorbed and reradiated powers are

$$P_{\mathrm{r}}^{\mathrm{m}} = P_{\mathrm{a}}^{\mathrm{m}} = \frac{2\pi^2}{k^2 Z} \left| \boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \right|^2$$

and

$$P_{\mathrm{s}}^{\mathrm{m}} = \frac{2\pi^2}{k^2 Z} \iint\limits_{\Omega} \left| \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \right|^2 \,\mathrm{d}\Omega$$

4 Invariance principles

Assume the Maxwell equations hold in a source-free region V in space, see Figure 3, and that the material parameters ϵ and μ are real-valued (lossless) in this region. Then, two different solutions of the Maxwell equations, $\boldsymbol{E}^{\mathrm{a}}$, $\boldsymbol{H}^{\mathrm{a}}$,and $\boldsymbol{E}^{\mathrm{b}}$, $\boldsymbol{H}^{\mathrm{b}}$, satisfy

$$\begin{cases} \nabla \cdot \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \times \boldsymbol{H}^{\mathrm{a}} \right\} = 0 \\ \nabla \cdot \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}^*} + \boldsymbol{E}^{\mathrm{b}^*} \times \boldsymbol{H}^{\mathrm{a}} \right\} = 0 \end{cases} \quad \boldsymbol{r} \in V \tag{4.1}$$

where * denotes the complex conjugate. These relations are easily derived from the Maxwell equations in a homogeneous, isotropic material

$$\begin{cases} \nabla \times \boldsymbol{E}^{\mathrm{a,b}}(\boldsymbol{r}) = \mathrm{i}kZ\boldsymbol{H}^{\mathrm{a,b}}(\boldsymbol{r}) \\ \nabla \times \boldsymbol{H}^{\mathrm{a,b}}(\boldsymbol{r}) = -\dfrac{\mathrm{i}k}{Z}\boldsymbol{E}^{\mathrm{a,b}}(\boldsymbol{r}) \end{cases} \quad \boldsymbol{r} \in V$$

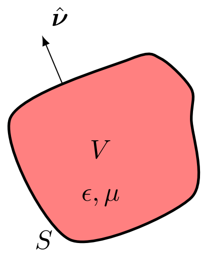

Figure 3: Geometry of the invariance principles.

The first invariance principle is easily expanded

$$\begin{aligned}
&\nabla\cdot\left\{\boldsymbol{E}^{\mathrm{a}}\times\boldsymbol{H}^{\mathrm{b}}-\boldsymbol{E}^{\mathrm{b}}\times\boldsymbol{H}^{\mathrm{a}}\right\}\\
&\quad=(\nabla\cdot\boldsymbol{E}^{\mathrm{a}})\cdot\boldsymbol{H}^{\mathrm{b}}-\boldsymbol{E}^{\mathrm{a}}\cdot\left(\nabla\times\boldsymbol{H}^{\mathrm{b}}\right)-\left(\nabla\cdot\boldsymbol{E}^{\mathrm{b}}\right)\cdot\boldsymbol{H}^{\mathrm{a}}+\boldsymbol{E}^{\mathrm{b}}\cdot(\nabla\times\boldsymbol{H}^{\mathrm{a}})\\
&\qquad=\mathrm{i}kZ\boldsymbol{H}^{\mathrm{a}}\cdot\boldsymbol{H}^{\mathrm{b}}+\frac{\mathrm{i}k}{Z}\boldsymbol{E}^{\mathrm{a}}\cdot\boldsymbol{E}^{\mathrm{b}}-\mathrm{i}kZ\boldsymbol{H}^{\mathrm{b}}\cdot\boldsymbol{H}^{\mathrm{a}}-\frac{\mathrm{i}k}{Z}\boldsymbol{E}^{\mathrm{b}}\cdot\boldsymbol{E}^{\mathrm{a}}=0
\end{aligned}$$

and similarly, the second one is expanded under the assumption that k and Z are real-valued

$$\begin{aligned}
&\nabla\cdot\left\{\boldsymbol{E}^{\mathrm{a}}\times\boldsymbol{H}^{\mathrm{b}^*}+\boldsymbol{E}^{\mathrm{b}^*}\times\boldsymbol{H}^{\mathrm{a}}\right\}\\
&\quad=(\nabla\cdot\boldsymbol{E}^{\mathrm{a}})\cdot\boldsymbol{H}^{\mathrm{b}^*}-\boldsymbol{E}^{\mathrm{a}}\cdot\left(\nabla\times\boldsymbol{H}^{\mathrm{b}^*}\right)+\left(\nabla\cdot\boldsymbol{E}^{\mathrm{b}^*}\right)\cdot\boldsymbol{H}^{\mathrm{a}}-\boldsymbol{E}^{\mathrm{b}^*}\cdot(\nabla\times\boldsymbol{H}^{\mathrm{a}})\\
&\qquad=\mathrm{i}kZ\boldsymbol{H}^{\mathrm{a}}\cdot\boldsymbol{H}^{\mathrm{b}^*}-\frac{\mathrm{i}k}{Z}\boldsymbol{E}^{\mathrm{a}}\cdot\boldsymbol{E}^{\mathrm{b}^*}-\mathrm{i}kZ\boldsymbol{H}^{\mathrm{b}^*}\cdot\boldsymbol{H}^{\mathrm{a}}+\frac{\mathrm{i}k}{Z}\boldsymbol{E}^{\mathrm{b}^*}\cdot\boldsymbol{E}^{\mathrm{a}}=0
\end{aligned}$$

The first one is well known and leads to the Lorentz reciprocity principle. The second has a special case when case a and case b are identical. The relation then expresses the conservation of power in V. We are going to use both these principles in the two following sections.

5 Reciprocity

The Lorentz reciprocity theorem, see also Section 4.3.6 in the textbook, provides us with a fundamental relation between two different solutions to the Maxwell equations in a source-free region bounded by a closed surface S. We label the solutions a and b, respectively, and we get by (4.1) and the divergence (Gauss) theorem

$$\iint_S\left\{\boldsymbol{E}^{\mathrm{a}}\times\boldsymbol{H}^{\mathrm{b}}-\boldsymbol{E}^{\mathrm{b}}\times\boldsymbol{H}^{\mathrm{a}}\right\}\cdot\hat{\boldsymbol{\nu}}\,\mathrm{d}S=0$$

where S denotes the bounding surface of the source-free, isotropic region, and $\hat{\boldsymbol{\nu}}$ the outward pointing unit normal vector to S.

We apply this identity to different transmitting (no sources outside the antenna) and receiving configurations, see Figure 4. The surface S_{p} is the reference plane in the antenna, see

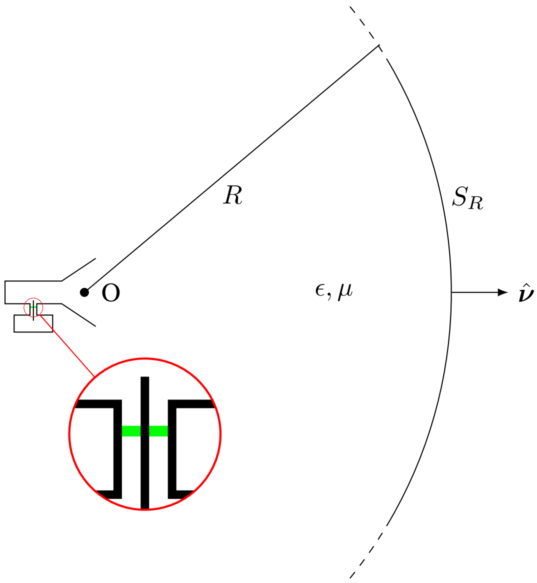

Figure 4: The geometry of the configuration for the invariance principles. The reference plane is depicted by a green line.

Figure 1, and S_R is a large sphere centered at the origin with radius R. The contributions of the integrals on the antennas vanish due to the assumed boundary conditions on the antennas, *i.e.*, $\hat{\boldsymbol{\nu}} \times \boldsymbol{E}^{\mathrm{a,b}} = \mathbf{0}$, and we get

$$\iint_{S_{\mathrm{p}}} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S = \iint_{S_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{r}} \, \mathrm{d}S$$

The surface integral on the reference plane in this expressions becomes

$$\begin{aligned}
&\iint_{S_{\mathrm{p}}} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S \\
&\qquad = \frac{1}{Z} \left\{ \left(\alpha_+^{\mathrm{a}} + \alpha_-^{\mathrm{a}}\right) \left(\alpha_+^{\mathrm{b}} - \alpha_-^{\mathrm{b}}\right) - \left(\alpha_+^{\mathrm{b}} + \alpha_-^{\mathrm{b}}\right) \left(\alpha_+^{\mathrm{a}} - \alpha_-^{\mathrm{a}}\right) \right\} \\
&\qquad\qquad \cdot \iint_{S_{\mathrm{p}}} \left(\boldsymbol{E}_m \times (\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)\right) \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S = \frac{2}{Z} \left(\alpha_+^{\mathrm{b}} \alpha_-^{\mathrm{a}} - \alpha_-^{\mathrm{b}} \alpha_+^{\mathrm{a}}\right)
\end{aligned}$$

where we used (1.2) and the fields at the reference plane, see (1.1)

$$\begin{cases} \boldsymbol{E}^{\mathrm{a,b}}(\boldsymbol{r}_{\mathrm{c}}) = \left(\alpha_+^{\mathrm{a,b}} + \alpha_-^{\mathrm{a,b}}\right) \boldsymbol{E}_m(\boldsymbol{r}_{\mathrm{c}}) \\ Z\boldsymbol{H}^{\mathrm{a,b}}(\boldsymbol{r}_{\mathrm{c}}) = \left(\alpha_+^{\mathrm{a,b}} - \alpha_-^{\mathrm{a,b}}\right) \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m(\boldsymbol{r}_{\mathrm{c}}) \end{cases} \quad \boldsymbol{r}_{\mathrm{c}} \in S_{\mathrm{p}}$$

We are left with

$$\iint_{S_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{r}} \, \mathrm{d}S = \frac{2}{Z} \left(\alpha_+^{\mathrm{b}} \alpha_-^{\mathrm{a}} - \alpha_-^{\mathrm{b}} \alpha_+^{\mathrm{a}} \right) \tag{5.1}$$

We now apply this invariance principle to three different antenna configurations.

5.1 Comparing two transmitting configurations

The configuration of two different transmitting configurations is the simplest case of application of (5.1). The two transmitting situations with the same antenna have far fields of the form

$$\begin{cases} \boldsymbol{E}^{\mathrm{a,b}}(\boldsymbol{r}) \sim \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r} \boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}}) \\ Z\boldsymbol{H}^{\mathrm{a,b}}(\boldsymbol{r}) \sim \hat{\boldsymbol{r}} \times \boldsymbol{E}^{\mathrm{a,b}}(\boldsymbol{r}) \end{cases} \qquad r \to \infty$$

where the far field amplitudes $\boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}})$ and the amplitude of the reflected wave in the transmission line $\alpha_-^{\mathrm{a,b}}$ are related to the excitation $\alpha_+^{\mathrm{a,b}}$, see (2.1)

$$\begin{cases} \alpha_-^{\mathrm{a,b}} = S\alpha_+^{\mathrm{a,b}} \\ \boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}}) = \boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+^{\mathrm{a,b}} \end{cases}$$

Both the left- and the right-hand side in equation (5.1) evaluates to zero, and the identity is trivially satisfied. In the next section, we analyze the first non-trivial situation of one receiving and one transmitting antenna.

5.2 Comparing one receiving and one transmitting configuration

We now prove that the vector-valued fields $\boldsymbol{f}$ and $\boldsymbol{s}$ are basically the same fields. To accomplish this, we apply the reciprocity theorem by Lorentz, (5.1), to a transmitting configuration (no sources outside the antenna) and a receiving configuration. The antenna in both configurations is the same, *i.e.*, we compare the receiving and transmitting mode with same antenna.We have

$$\iint_{S_R} \left\{ \boldsymbol{E}^{\mathrm{t}} \times \boldsymbol{H}^{\mathrm{r}} - \boldsymbol{E}^{\mathrm{r}} \times \boldsymbol{H}^{\mathrm{t}} \right\} \cdot \hat{\boldsymbol{r}} \, \mathrm{d}S = \frac{2}{Z} \left(\alpha_+^{\mathrm{r}} \alpha_-^{\mathrm{t}} - \alpha_-^{\mathrm{r}} \alpha_+^{\mathrm{t}} \right) \tag{5.2}$$

The source in the receiving configuration is an incident plane wave $\boldsymbol{E}_{\mathrm{i}}(\boldsymbol{r})$. Far away from the antenna, the total field of the receiving field has the form

$$\begin{cases} \boldsymbol{E}^{\mathrm{r}}(\boldsymbol{r}) \sim \boldsymbol{E}_{\mathrm{i}}(\boldsymbol{r}) + \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r} \boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}) \\ Z\boldsymbol{H}^{\mathrm{r}}(\boldsymbol{r}) \sim Z\boldsymbol{H}_{\mathrm{i}}(\boldsymbol{r}) + \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r} \hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}) \end{cases} \qquad r \to \infty$$

and the transmitting field has the form

$$\begin{cases} \boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r}) \sim \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r} \boldsymbol{F}^{\mathrm{t}}(\hat{\boldsymbol{r}}) \\ Z\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r}) \sim \hat{\boldsymbol{r}} \times \boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r}) \end{cases} \qquad r \to \infty$$

where the incident field is

$$\begin{cases} \boldsymbol{E}_\mathrm{i}(\boldsymbol{r}) = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}} \\ Z\boldsymbol{H}_\mathrm{i}(\boldsymbol{r}) = \dfrac{1}{\mathrm{i}k}\nabla\times\boldsymbol{E}_\mathrm{i} = \hat{\boldsymbol{k}}_\mathrm{i}\times\boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}} \end{cases}$$

The left-hand side of the reciprocity theorem is

$$\begin{aligned} &Z\iint\limits_{S_R} \left\{\boldsymbol{E}^\mathrm{t}\times\boldsymbol{H}^\mathrm{r} - \boldsymbol{E}^\mathrm{r}\times\boldsymbol{H}^\mathrm{t}\right\}\cdot\hat{\boldsymbol{r}}\ \mathrm{d}S \\ &\qquad = R\iint\limits_{\Omega} \left\{\boldsymbol{F}^\mathrm{t}(\hat{\boldsymbol{r}})\times\left(\hat{\boldsymbol{k}}_\mathrm{i}\times\boldsymbol{E}_0\right) - \boldsymbol{E}_0\times\left(\hat{\boldsymbol{r}}\times\boldsymbol{F}^\mathrm{t}(\hat{\boldsymbol{r}})\right)\right\}\cdot\hat{\boldsymbol{r}}\mathrm{e}^{\mathrm{i}kR(1+\hat{\boldsymbol{k}}_\mathrm{i}\cdot\hat{\boldsymbol{r}})}\ \mathrm{d}\Omega \end{aligned}$$

since $\boldsymbol{F}^{\mathrm{r,t}}(\hat{\boldsymbol{r}})\cdot\hat{\boldsymbol{r}} = 0$, and consequently the expression $\boldsymbol{F}^\mathrm{t}(\hat{\boldsymbol{r}})\times(\hat{\boldsymbol{r}}\times\boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{r}})) - \boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{r}})\times\left(\hat{\boldsymbol{r}}\times\boldsymbol{F}^\mathrm{t}(\hat{\boldsymbol{r}})\right) = \boldsymbol{0}$, and where we used $\mathrm{d}S = R^2\ \mathrm{d}\Omega$ (Ω denotes the unit sphere). We simplify further

$$\begin{aligned} Z\iint\limits_{S_R} \left\{\boldsymbol{E}^\mathrm{t}\times\boldsymbol{H}^\mathrm{r} - \boldsymbol{E}^\mathrm{r}\times\boldsymbol{H}^\mathrm{t}\right\}\cdot\hat{\boldsymbol{r}}\ \mathrm{d}S = R\mathrm{e}^{\mathrm{i}kR}\iint\limits_{\Omega} &\left\{\left(\hat{\boldsymbol{k}}_\mathrm{i}\cdot\hat{\boldsymbol{r}} - 1\right)\boldsymbol{F}^\mathrm{t}(\hat{\boldsymbol{r}})\cdot\boldsymbol{E}_0\right. \\ &\left. - \boldsymbol{E}_0\cdot\hat{\boldsymbol{r}}\left(\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{F}^\mathrm{t}(\hat{\boldsymbol{r}}) - \hat{\boldsymbol{r}}\cdot\boldsymbol{F}^\mathrm{t}(\hat{\boldsymbol{r}})\right)\right\}\mathrm{e}^{\mathrm{i}kR\hat{\boldsymbol{k}}_\mathrm{i}\cdot\hat{\boldsymbol{r}}}\ \mathrm{d}\Omega \end{aligned}$$

From this expression, it is straightforward to show[9]

$$\iint\limits_{S_R} \left\{\boldsymbol{E}^\mathrm{t}\times\boldsymbol{H}^\mathrm{r} - \boldsymbol{E}^\mathrm{r}\times\boldsymbol{H}^\mathrm{t}\right\}\cdot\hat{\boldsymbol{r}}\ \mathrm{d}S \to -\frac{4\pi\mathrm{i}}{kZ}\boldsymbol{F}^\mathrm{t}(-\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0, \qquad \text{as } R\to\infty$$

since $\boldsymbol{E}_0\cdot\hat{\boldsymbol{k}}_\mathrm{i} = 0$. The reciprocity equation (5.2) then becomes

$$-\frac{4\pi\mathrm{i}}{kZ}\boldsymbol{F}^\mathrm{t}(-\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0 = \frac{2}{Z}\left(\alpha_+^\mathrm{r}\alpha_-^\mathrm{t} - \alpha_-^\mathrm{r}\alpha_+^\mathrm{t}\right)$$

We get by the use of (2.1) and (3.1) on page 6 and 7, respectively.

$$\begin{aligned} -\frac{4\pi\mathrm{i}}{k}\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\alpha_+^\mathrm{t} &= 2\left(\alpha_+^\mathrm{r}\alpha_-^\mathrm{t} - \alpha_-^\mathrm{r}\alpha_+^\mathrm{t}\right) \\ &= 2\left(S\alpha_+^\mathrm{r}\alpha_+^\mathrm{t} - \left(S\alpha_+^\mathrm{r} + \frac{2\pi\mathrm{i}}{k}\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\right)\alpha_+^\mathrm{t}\right) = -\frac{4\pi\mathrm{i}}{k}\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\alpha_+^\mathrm{t} \end{aligned}$$

This gives, since this relation must hold for all α_+^t and all $\boldsymbol{E}_0$

$$\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) = \boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i}) \tag{5.3}$$

[9] The asymptotic evaluation of the following integral is useful (obtained by integration by parts in the polar angle $\arccos(\hat{\boldsymbol{k}}_\mathrm{i}\cdot\hat{\boldsymbol{r}})$, see Appendix G.1 in [17])

$$\iint\limits_{\Omega} G(\hat{\boldsymbol{r}})\mathrm{e}^{\mathrm{i}\xi\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}}}\ \mathrm{d}\Omega = \frac{2\pi}{\mathrm{i}\xi}G(\hat{\boldsymbol{k}})\mathrm{e}^{\mathrm{i}\xi} - \frac{2\pi}{\mathrm{i}\xi}G(-\hat{\boldsymbol{k}})\mathrm{e}^{-\mathrm{i}\xi} + O(1/\xi^2), \quad \text{as } \xi\to\infty$$

which proves that for a reciprocal system, the vectors $\boldsymbol{f}$ and $\boldsymbol{s}$ are the same, only evaluated at different directions. The scattering dyadic for the receiving antenna in a reciprocal system becomes, see (3.4) on page 8

$$\mathbf{S}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{2\pi\mathrm{i}}{k}\left\{\frac{\Gamma_\mathrm{L}\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})}{1-S\Gamma_\mathrm{L}} + \mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\right\}$$

Example 5.1

For a reciprocal antenna system, there is simple connection between the effective receiving area or aperture $A_\mathrm{e}(\hat{\boldsymbol{k}}_\mathrm{i})$ defined as

$$P_\mathrm{r} = A_\mathrm{e}(\hat{\boldsymbol{k}}_\mathrm{i})P_\mathrm{i}(\hat{\boldsymbol{k}}_\mathrm{i})$$

where P_r is the received power, see (3.6)

$$P_\mathrm{r} = \frac{2\pi^2}{k^2 Z}\frac{1-|\Gamma_\mathrm{L}|^2}{|1-S\Gamma_\mathrm{L}|^2}\left|\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\right|^2$$

and $P_\mathrm{i}(\hat{\boldsymbol{k}}_\mathrm{i})$ is the incident power flux

$$P_\mathrm{i}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{1}{2Z}|\boldsymbol{E}_0|^2$$

and the gain $G(-\hat{\boldsymbol{k}}_\mathrm{i})$ of the transmitting antenna in the opposite direction given by (2.5)

$$G(-\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{4\pi\left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\right|^2}{1-|S|^2}$$

Combine these expressions, and we get using the reciprocity result (5.3)

$$A_\mathrm{e}(\hat{\boldsymbol{k}}_\mathrm{i})|\boldsymbol{E}_0|^2 = \frac{4\pi^2}{k^2}\frac{1-|\Gamma_\mathrm{L}|^2}{|1-S\Gamma_\mathrm{L}|^2}\left|\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\right|^2 = \frac{4\pi^2}{k^2}\frac{1-|\Gamma_\mathrm{L}|^2}{|1-S\Gamma_\mathrm{L}|^2}\left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\right|^2$$

and the general expression of the effective area becomes

$$A_\mathrm{e}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{4\pi^2}{k^2|\boldsymbol{E}_0|^2}\frac{1-|\Gamma_\mathrm{L}|^2}{|1-S\Gamma_\mathrm{L}|^2}\left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\right|^2$$

For matched load conditions, $\Gamma_\mathrm{L} = 0$, and for optimal polarization match on the antenna, *i.e.*, $\left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\right| = \left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\right||\boldsymbol{E}_0|$, we obtain

$$A_\mathrm{e}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{4\pi^2}{k^2}\left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\right|^2 = (1-|S|^2)\frac{\lambda^2}{4\pi}G(-\hat{\boldsymbol{k}}_\mathrm{i})$$

which for a matched antenna, $S = 0$, reads

$$A_\mathrm{e}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{\lambda^2}{4\pi}G(-\hat{\boldsymbol{k}}_\mathrm{i})$$

Similarly, for a conjugate match load, $\Gamma_\mathrm{L} = S^*$, and optimal polarization match, $\left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\right| = \left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\right||\boldsymbol{E}_0|$, we obtain

$$A_\mathrm{e}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{4\pi^2}{k^2}\frac{\left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\right|^2}{1-|S|^2} = \frac{\lambda^2}{4\pi}G(-\hat{\boldsymbol{k}}_\mathrm{i})$$

which is result found in many textbooks on antennas [6, 16, 25]. ■

5.3 Comparing two receiving configurations

We can also apply the Lorentz reciprocity theorem to two different receiving modes. It is the same antenna, but the two receiving situations differ by the way they are loaded and excited. Again, (5.1) is applied

$$\iint_{S_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{r}} \, \mathrm{d}S = \frac{2}{Z} \left(\alpha_+^{\mathrm{b}} \alpha_-^{\mathrm{a}} - \alpha_-^{\mathrm{b}} \alpha_+^{\mathrm{a}} \right) \tag{5.4}$$

The total fields are then

$$\begin{cases} \boldsymbol{E}^{\mathrm{a,b}}(\boldsymbol{r}) \sim \boldsymbol{E}_{\mathrm{i}}^{\mathrm{a,b}}(\boldsymbol{r}) + \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r} \boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}}) \\ Z\boldsymbol{H}^{\mathrm{a,b}}(\boldsymbol{r}) \sim Z\boldsymbol{H}_{\mathrm{i}}^{\mathrm{a,b}}(\boldsymbol{r}) + \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r} \hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}}) \end{cases} \qquad r \to \infty$$

where the incident fields are

$$\begin{cases} \boldsymbol{E}_{\mathrm{i}}^{\mathrm{a,b}}(\boldsymbol{r}) = \boldsymbol{E}_0^{\mathrm{a,b}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}} \cdot \boldsymbol{r}} \\ Z\boldsymbol{H}_{\mathrm{i}}^{\mathrm{a,b}}(\boldsymbol{r}) = \dfrac{1}{\mathrm{i}k} \nabla \times \boldsymbol{E}_{\mathrm{i}}^{\mathrm{a,b}} = \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}} \times \boldsymbol{E}_0^{\mathrm{a,b}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}} \cdot \boldsymbol{r}} \end{cases}$$

and the far field amplitudes are, see (3.2) and (3.3) on page 7

$$\begin{cases} \alpha_-^{\mathrm{a,b}} = \dfrac{2\pi \mathrm{i}}{k} \dfrac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}})}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a,b}}} \cdot \boldsymbol{E}_0^{\mathrm{a,b}} \\ \boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}}) = \dfrac{2\pi \mathrm{i}}{k} \left\{ \dfrac{\Gamma_{\mathrm{L}}^{\mathrm{a,b}} \boldsymbol{f}(\hat{\boldsymbol{r}}) \boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}})}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a,b}}} + \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}}) \right\} \cdot \boldsymbol{E}_0^{\mathrm{a,b}} \end{cases} \tag{5.5}$$

The asymptotic evaluation of the integral on the left-hand side in (5.4) shows many similarities to the one above for one receiving and one transmitting configuration. We have

$$\begin{aligned} &\iint_{S_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{r}} \, \mathrm{d}S \\ &= \frac{1}{Z} \iint_{S_R} \left\{ \left(\boldsymbol{E}_0^{\mathrm{a}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}} \cdot \boldsymbol{r}} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \right) \times \left(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}} \times \boldsymbol{E}_0^{\mathrm{b}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}} \cdot \boldsymbol{r}} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}}) \right) \right. \\ &\qquad \left. - \left(\boldsymbol{E}_0^{\mathrm{b}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}} \cdot \boldsymbol{r}} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}}) \right) \times \left(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}} \times \boldsymbol{E}_0^{\mathrm{a}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}} \cdot \boldsymbol{r}} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \right) \right\} \cdot \hat{\boldsymbol{r}} \, \mathrm{d}S \end{aligned}$$

There are four terms in the integral on the right-hand side, out of which two give zero contributions. In fact

$$\iint_{S_R} \left\{ \boldsymbol{E}_{\mathrm{i}}^{\mathrm{a}} \times \boldsymbol{H}_{\mathrm{i}}^{\mathrm{b}} - \boldsymbol{E}_{\mathrm{i}}^{\mathrm{b}} \times \boldsymbol{H}_{\mathrm{i}}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{r}} \, \mathrm{d}S = 0$$

due to reciprocity, since $\boldsymbol{E}_{\mathrm{i}}^{\mathrm{a}}$ and $\boldsymbol{E}_{\mathrm{i}}^{\mathrm{b}}$ satisfy the Maxwell equations inside S_R. Moreover, $\boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \times \left(\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}})\right) - \boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}}) \times (\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}})) = \boldsymbol{0}$, since $\boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}}) \cdot \hat{\boldsymbol{r}} = 0$. The terms that remain are ($\mathrm{d}S = R^2\ \mathrm{d}\Omega$, Ω denotes the unit sphere)

$$
\begin{aligned}
&\iint\limits_{S_R} \left\{\boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \times \boldsymbol{H}^{\mathrm{a}}\right\} \cdot \hat{\boldsymbol{r}}\ \mathrm{d}S \\
&\quad = \frac{R\mathrm{e}^{\mathrm{i}kR}}{Z} \iint\limits_{\Omega} \Big\{\boldsymbol{E}_0^{\mathrm{a}} \times \left(\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}})\right) \mathrm{e}^{\mathrm{i}kR\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}\cdot\hat{\boldsymbol{r}}} + \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \times \left(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}} \times \boldsymbol{E}_0^{\mathrm{b}}\right) \mathrm{e}^{\mathrm{i}kR\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}\cdot\hat{\boldsymbol{r}}} \\
&\qquad - \boldsymbol{E}_0^{\mathrm{b}} \times (\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}))\, \mathrm{e}^{\mathrm{i}kR\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}\cdot\hat{\boldsymbol{r}}} - \boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}}) \times \left(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}} \times \boldsymbol{E}_0^{\mathrm{a}}\right) \mathrm{e}^{\mathrm{i}kR\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}\cdot\hat{\boldsymbol{r}}}\Big\} \cdot \hat{\boldsymbol{r}}\ \mathrm{d}\Omega \\
&\quad = \frac{R}{Z}\mathrm{e}^{\mathrm{i}kR} \iint\limits_{\Omega} \Big\{\boldsymbol{E}_0^{\mathrm{a}} \cdot \boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}}) \left(1 - \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}} \cdot \hat{\boldsymbol{r}}\right) \mathrm{e}^{\mathrm{i}kR\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}\cdot\hat{\boldsymbol{r}}} \\
&\qquad - \left(\boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \cdot \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}\right) \boldsymbol{E}_0^{\mathrm{b}} \cdot \hat{\boldsymbol{r}}\mathrm{e}^{\mathrm{i}kR\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}\cdot\hat{\boldsymbol{r}}} + \left(\boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}}) \cdot \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}\right) \boldsymbol{E}_0^{\mathrm{a}} \cdot \hat{\boldsymbol{r}}\mathrm{e}^{\mathrm{i}kR\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}\cdot\hat{\boldsymbol{r}}} \\
&\qquad - \boldsymbol{E}_0^{\mathrm{b}} \cdot \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \left(1 - \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}} \cdot \hat{\boldsymbol{r}}\right) \mathrm{e}^{\mathrm{i}kR\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}\cdot\hat{\boldsymbol{r}}}\Big\}\ \mathrm{d}\Omega
\end{aligned}
$$

In the limit $R \to \infty$, we get, see Footnote 9 on page 15

$$
\begin{aligned}
&\iint\limits_{S_R} \left\{\boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \times \boldsymbol{H}^{\mathrm{a}}\right\} \cdot \hat{\boldsymbol{r}}\ \mathrm{d}S \\
&\qquad \to \frac{2\pi\mathrm{i}}{kZ} \left\{2\boldsymbol{E}_0^{\mathrm{a}} \cdot \boldsymbol{F}^{\mathrm{b}}(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) - 2\boldsymbol{E}_0^{\mathrm{b}} \cdot \boldsymbol{F}^{\mathrm{a}}(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})\right\}, \quad R \to \infty
\end{aligned}
$$

The right-hand side of reciprocity identity is (use (5.5) and $\alpha_+^{\mathrm{a,b}} - = \Gamma_{\mathrm{L}}^{\mathrm{a,b}}\alpha_-^{\mathrm{a,b}}$)

$$
\frac{2}{Z}\left(\alpha_+^{\mathrm{b}}\alpha_-^{\mathrm{a}} - \alpha_-^{\mathrm{b}}\alpha_+^{\mathrm{a}}\right) = -\frac{8\pi^2}{k^2 Z} \frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) \cdot \boldsymbol{E}_0^{\mathrm{a}}}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a}}} \frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}) \cdot \boldsymbol{E}_0^{\mathrm{b}}}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{b}}} \left(\Gamma_{\mathrm{L}}^{\mathrm{b}} - \Gamma_{\mathrm{L}}^{\mathrm{a}}\right)
$$

In the limit $R \to \infty$, the identity (5.4) then becomes

$$
\boldsymbol{E}_0^{\mathrm{a}} \cdot \boldsymbol{F}^{\mathrm{b}}(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) - \boldsymbol{E}_0^{\mathrm{b}} \cdot \boldsymbol{F}^{\mathrm{a}}(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}) = \frac{2\pi\mathrm{i}}{k} \frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) \cdot \boldsymbol{E}_0^{\mathrm{a}}}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a}}} \frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}) \cdot \boldsymbol{E}_0^{\mathrm{b}}}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{b}}} \left(\Gamma_{\mathrm{L}}^{\mathrm{b}} - \Gamma_{\mathrm{L}}^{\mathrm{a}}\right)
$$

or (use (5.5))

$$
\begin{aligned}
&\boldsymbol{E}_0^{\mathrm{b}} \cdot \left\{\frac{\Gamma_{\mathrm{L}}^{\mathrm{a}}\boldsymbol{f}(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a}}} + \mathbf{t}(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\right\} \cdot \boldsymbol{E}_0^{\mathrm{a}} \\
&\qquad - \boldsymbol{E}_0^{\mathrm{a}} \cdot \left\{\frac{\Gamma_{\mathrm{L}}^{\mathrm{b}}\boldsymbol{f}(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{b}}} + \mathbf{t}(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})\right\} \cdot \boldsymbol{E}_0^{\mathrm{b}} \\
&\qquad\qquad = \frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) \cdot \boldsymbol{E}_0^{\mathrm{a}}}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a}}} \frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}) \cdot \boldsymbol{E}_0^{\mathrm{b}}}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{b}}} \left(\Gamma_{\mathrm{L}}^{\mathrm{a}}\left(1 - S\Gamma_{\mathrm{L}}^{\mathrm{b}}\right) - \Gamma_{\mathrm{L}}^{\mathrm{b}}\left(1 - S\Gamma_{\mathrm{L}}^{\mathrm{a}}\right)\right)
\end{aligned}
$$

Using (5.3), we conclude

$$\boldsymbol{E}_0^{\mathrm{b}} \cdot \mathbf{t}(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) \cdot \boldsymbol{E}_0^{\mathrm{a}} = \boldsymbol{E}_0^{\mathrm{a}} \cdot \mathbf{t}(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}) \cdot \boldsymbol{E}_0^{\mathrm{b}}$$

or sin ce this relation holds for all $\boldsymbol{E}_0^{\mathrm{a,b}}$

$$\mathbf{t}(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) = \mathbf{t}^t(-\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})$$

where the superscript t denotes the transpose (distinguish between the transpose t and the transmitting index t). We rewrite this expression as

$$\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) = \mathbf{t}^t(-\hat{\boldsymbol{k}}_{\mathrm{i}}, -\hat{\boldsymbol{r}})$$

In summary, in a reciprocal system the antenna parameters satisfy

$$\begin{cases} \boldsymbol{f}(\hat{\boldsymbol{r}}) = \boldsymbol{s}(-\hat{\boldsymbol{r}}) \\ \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) = \mathbf{t}^t(-\hat{\boldsymbol{k}}_{\mathrm{i}}, -\hat{\boldsymbol{r}}) \end{cases} \tag{5.6}$$

6 Generalized power balance

In the previous section, we utilized the reciprocity properties of the exterior material to obtain the connection between the receiving and transmitting properties of an antenna. The results are summarized in (5.6).

In this section, we exploit a second principle related to the power conservation of the system. The principle is somewhat more general than the usual power conservation in that it allows us to compare two receiving or transmitting configurations.

We apply the generalized power identity (use (4.1) and the divergence (Gauss) theorem)

$$\iint_S \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}^*} + \boldsymbol{E}^{\mathrm{b}^*} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S = 0$$

to the volume between the antenna and a large sphere centered at the origin of radius R, see Figure 4. We get

$$\iint_{S_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}^*} + \boldsymbol{E}^{\mathrm{b}^*} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{r}} \, \mathrm{d}S = \iint_{S_{\mathrm{p}}} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}^*} + \boldsymbol{E}^{\mathrm{b}^*} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S$$

The surface S_{p} is the reference plane in the antenna, see Figure 1, and S_R is a large circumscribing sphere of radius R (the contributions of the integrals on the antennas vanish due to the assumed boundary conditions on the antennas). Evaluation of the surface integral over the reference plane S_{p} gives

$$\begin{aligned} &\iint_{S_{\mathrm{p}}} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}^*} + \boldsymbol{E}^{\mathrm{b}^*} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S \\ &\qquad = \frac{1}{Z} \left\{ \left(\alpha_+^{\mathrm{a}} + \alpha_-^{\mathrm{a}}\right) \left(\alpha_+^{\mathrm{b}} - \alpha_-^{\mathrm{b}}\right)^* + \left(\alpha_+^{\mathrm{b}} + \alpha_-^{\mathrm{b}}\right)^* \left(\alpha_+^{\mathrm{a}} - \alpha_-^{\mathrm{a}}\right) \right\} \\ &\qquad\qquad \cdot \iint_{S_{\mathrm{p}}} \left(\boldsymbol{E}_m \times (\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)\right) \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S = \frac{2}{Z} \left(\alpha_+^{\mathrm{a}} {\alpha_+^{\mathrm{b}}}^* - \alpha_-^{\mathrm{a}} {\alpha_-^{\mathrm{b}}}^* \right) \end{aligned}$$

We get

$$\iint_{S_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}^*} + \boldsymbol{E}^{\mathrm{b}^*} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{r}} \, \mathrm{d}S = \frac{2}{Z} \left(\alpha_+^{\mathrm{a}} \alpha_+^{\mathrm{b}\,*} - \alpha_-^{\mathrm{a}} \alpha_-^{\mathrm{b}\,*} \right) \tag{6.1}$$

6.1 Comparing two transmitting configurations

The configuration of two different transmitting configurations is the simplest case that we now apply to (6.1). The two transmitting fields have the form

$$\begin{cases} \boldsymbol{E}^{\mathrm{a,b}}(\boldsymbol{r}) \sim \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r} \boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}}) \\ Z\boldsymbol{H}^{\mathrm{a,b}}(\boldsymbol{r}) \sim \hat{\boldsymbol{r}} \times \boldsymbol{E}^{\mathrm{a,b}}(\boldsymbol{r}) \end{cases} \qquad r \to \infty$$

where the far field amplitudes $\boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}})$ and the amplitude of the reflected wave in the transmission line $\alpha_-^{\mathrm{a,b}}$ are, see (2.1) on page 6

$$\begin{cases} \alpha_-^{\mathrm{a,b}} = S\alpha_+^{\mathrm{a,b}} \\ \boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}}) = \boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+^{\mathrm{a,b}} \end{cases}$$

The left- and right-hand sides of equation (6.1) are easily evaluated. We get

$$\iint_{S_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}^*} + \boldsymbol{E}^{\mathrm{b}^*} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{r}} \, \mathrm{d}S = \frac{2}{Z} \iint_{\Omega} |\boldsymbol{f}(\hat{\boldsymbol{r}})|^2 \, \mathrm{d}\Omega \alpha_+^{\mathrm{a}} \alpha_+^{\mathrm{b}\,*}$$

and

$$\frac{2}{Z} \left(\alpha_+^{\mathrm{a}} \alpha_+^{\mathrm{b}\,*} - \alpha_-^{\mathrm{a}} \alpha_-^{\mathrm{b}\,*} \right) = \frac{2}{Z} \left(1 - |S|^2 \right) \alpha_+^{\mathrm{a}} \alpha_+^{\mathrm{b}\,*}$$

and we get

$$\iint_{\Omega} |\boldsymbol{f}(\hat{\boldsymbol{r}})|^2 \, \mathrm{d}\Omega = 1 - |S|^2 \tag{6.2}$$

and, see (2.6) on page 7, the directivity and the gain are identical, *i.e.*, $D(\hat{\boldsymbol{r}}) = G(\hat{\boldsymbol{r}})$.

This result is identical to the power conservation result. The total power delivered to the antenna is given by (2.2) on page 6, *i.e.*,

$$P_{\mathrm{in}} = \frac{1}{2Z} \left(1 - |S|^2 \right) \left| \alpha_+^{\mathrm{t}} \right|^2$$

Similarly, the total power transmitted is, see (2.3) on page 6

$$P_{\mathrm{t}} = \frac{1}{2Z} \iint_{\Omega} |\boldsymbol{f}(\hat{\boldsymbol{r}})|^2 \, \mathrm{d}\Omega \left| \alpha_+^{\mathrm{t}} \right|^2$$

If no power is dissipated (lossless antenna) the input power and the radiated power balance, *i.e.*, $P_{\mathrm{t}} = P_{\mathrm{in}}$, which implies (6.2).

6.2 Comparing one receiving and one transmitting configuration

The analysis of comparing one receiving and one transmitting configuration follows the corresponding reciprocity analysis closely.

The source in the receiving configuration is an incident plane wave. Far away from the antennas the total field of the receiving field has the form

$$\begin{cases} \boldsymbol{E}^{\mathrm{r}}(\boldsymbol{r}) \sim \boldsymbol{E}_{\mathrm{i}}(\boldsymbol{r}) + \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r}\boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}) \\ Z\boldsymbol{H}^{\mathrm{r}}(\boldsymbol{r}) \sim Z\boldsymbol{H}_{\mathrm{i}}(\boldsymbol{r}) + \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r}\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}) \end{cases} \qquad r \to \infty$$

and the transmitting field has the form

$$\begin{cases} \boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r}) \sim \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r}\boldsymbol{F}^{\mathrm{t}}(\hat{\boldsymbol{r}}) \\ Z\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r}) \sim \hat{\boldsymbol{r}} \times \boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r}) \end{cases} \qquad r \to \infty$$

where the incident field is

$$\begin{cases} \boldsymbol{E}_{\mathrm{i}}(\boldsymbol{r}) = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}\cdot\boldsymbol{r}} \\ Z\boldsymbol{H}_{\mathrm{i}}(\boldsymbol{r}) = \dfrac{1}{\mathrm{i}k}\nabla \times \boldsymbol{E}_{\mathrm{i}} = \hat{\boldsymbol{k}}_{\mathrm{i}} \times \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}\cdot\boldsymbol{r}} \end{cases}$$

The left-hand side of the identity (6.1) then becomes

$$\begin{aligned} Z\iint_{S_R} &\left\{\boldsymbol{E}^{\mathrm{r}} \times \boldsymbol{H}^{\mathrm{t}*} + \boldsymbol{E}^{\mathrm{t}*} \times \boldsymbol{H}^{\mathrm{r}}\right\} \cdot \hat{\boldsymbol{r}} \,\mathrm{d}S \\ &= R\iint_{\Omega}\left\{\boldsymbol{E}_0 \times (\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{t}*}(\hat{\boldsymbol{r}})) + \boldsymbol{F}^{\mathrm{t}*}(\hat{\boldsymbol{r}}) \times \left(\hat{\boldsymbol{k}}_{\mathrm{i}} \times \boldsymbol{E}_0\right)\right\} \cdot \hat{\boldsymbol{r}}\mathrm{e}^{\mathrm{i}kR(-1+\hat{\boldsymbol{k}}_{\mathrm{i}}\cdot\hat{\boldsymbol{r}})} \,\mathrm{d}\Omega \\ &\qquad + \iint_{\Omega}\left\{\boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}) \times \left(\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{t}*}(\hat{\boldsymbol{r}})\right) + \boldsymbol{F}^{\mathrm{t}*}(\hat{\boldsymbol{r}}) \times (\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}))\right\} \cdot \hat{\boldsymbol{r}} \,\mathrm{d}\Omega \end{aligned}$$

where we used $\mathrm{d}S = R^2 \,\mathrm{d}\Omega$ (Ω denotes the unit sphere). We simplify with use of the BAC-CAB rule and $\hat{\boldsymbol{r}} \cdot \boldsymbol{F}^{\mathrm{t}}(\hat{\boldsymbol{r}}) = 0$.

$$\begin{aligned} Z\iint_{S_R} \left\{\boldsymbol{E}^{\mathrm{r}} \times \boldsymbol{H}^{\mathrm{t}*} + \boldsymbol{E}^{\mathrm{t}*} \times \boldsymbol{H}^{\mathrm{r}}\right\} \cdot \hat{\boldsymbol{r}} \,\mathrm{d}S &= 2\iint_{\Omega} \boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{F}^{\mathrm{t}*}(\hat{\boldsymbol{r}}) \,\mathrm{d}\Omega \\ + R\iint_{\Omega}\Big\{\boldsymbol{E}_0 \cdot \boldsymbol{F}^{\mathrm{t}*}(\hat{\boldsymbol{r}})\left(1 + \hat{\boldsymbol{k}}_{\mathrm{i}} \cdot \hat{\boldsymbol{r}}\right) &- \boldsymbol{F}^{\mathrm{t}*}(\hat{\boldsymbol{r}}) \cdot \hat{\boldsymbol{k}}_{\mathrm{i}}\,(\hat{\boldsymbol{r}} \cdot \boldsymbol{E}_0)\Big\}\mathrm{e}^{\mathrm{i}kR(-1+\hat{\boldsymbol{k}}_{\mathrm{i}}\cdot\hat{\boldsymbol{r}})} \,\mathrm{d}\Omega \end{aligned}$$

The dominant contribution of the surface integral over S_R of this identity is determined by the limiting values as $R \to \infty$. We get, see Footnote 9 on page 15

$$Z\iint_{S_R} \left\{\boldsymbol{E}^{\mathrm{r}} \times \boldsymbol{H}^{\mathrm{t}*} + \boldsymbol{E}^{\mathrm{t}*} \times \boldsymbol{H}^{\mathrm{r}}\right\} \cdot \hat{\boldsymbol{r}} \,\mathrm{d}S = 2\iint_{\Omega} \boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{F}^{\mathrm{t}*}(\hat{\boldsymbol{r}}) \,\mathrm{d}\Omega - \frac{4\pi\mathrm{i}}{k}\boldsymbol{F}^{\mathrm{t}*}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0$$

Equation (6.1) then becomes

$$\alpha^{\mathrm{r}}_{+}{\alpha^{\mathrm{t}}_{+}}^{*} - {\alpha^{\mathrm{t}}_{-}}^{*}\alpha^{\mathrm{r}}_{-} = \iint_{\Omega} \boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{F}^{\mathrm{t}*}(\hat{\boldsymbol{r}}) \,\mathrm{d}\Omega - \frac{2\pi\mathrm{i}}{k}\boldsymbol{F}^{\mathrm{t}*}(\hat{\boldsymbol{k}}) \cdot \boldsymbol{E}_0$$

We insert, see (3.2) and (3.3) on page 7

$$\begin{cases} \alpha^{\mathrm{r}}_{-} = \dfrac{2\pi\mathrm{i}}{k}\dfrac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0}{1 - S\Gamma_{\mathrm{L}}} \\ \alpha^{\mathrm{r}}_{+} = \Gamma_{\mathrm{L}}\alpha^{\mathrm{r}}_{-} \\ \boldsymbol{F}^{\mathrm{r}}(\hat{\boldsymbol{r}}) = \dfrac{2\pi\mathrm{i}}{k}\left\{\dfrac{\Gamma_{\mathrm{L}}\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})}{1 - S\Gamma_{\mathrm{L}}} + \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}})\right\} \cdot \boldsymbol{E}_0 \end{cases}$$

and, see (2.1) on page 6

$$\begin{cases} \alpha^{\mathrm{t}}_{-} = S\alpha^{\mathrm{t}}_{+} \\ \boldsymbol{F}^{\mathrm{t}}(\hat{\boldsymbol{r}}) = \boldsymbol{f}(\hat{\boldsymbol{r}})\alpha^{\mathrm{t}}_{+} \end{cases}$$

We get expressed in the input quantities α^{t}_{+} and $\boldsymbol{E}_0$

$$\begin{aligned} &\frac{\Gamma_{\mathrm{L}} - S^*}{1 - S\Gamma_{\mathrm{L}}}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0{\alpha^{\mathrm{t}}_{+}}^{*} + \boldsymbol{f}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0{\alpha^{\mathrm{t}}_{+}}^{*} \\ &\qquad = \iint_{\Omega} \boldsymbol{f}^*(\hat{\boldsymbol{r}}) \cdot \left(\frac{\Gamma_{\mathrm{L}}\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})}{1 - S\Gamma_{\mathrm{L}}} + \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}})\right) \mathrm{d}\Omega \cdot \boldsymbol{E}_0{\alpha^{\mathrm{t}}_{+}}^{*} \end{aligned}$$

and consequently ($\boldsymbol{E}_0$ and α^{t}_{+} are arbitrary)

$$\frac{\Gamma_{\mathrm{L}} - S^*}{1 - S\Gamma_{\mathrm{L}}}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) + \boldsymbol{f}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}) = \iint_{\Omega} \boldsymbol{f}^*(\hat{\boldsymbol{r}}) \cdot \left(\frac{\Gamma_{\mathrm{L}}\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})}{1 - S\Gamma_{\mathrm{L}}} + \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}})\right) \mathrm{d}\Omega$$

This relation can be rephrased as

$$\begin{aligned} &(\Gamma_{\mathrm{L}} - S^*)\,\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) + (1 - S\Gamma_{\mathrm{L}})\,\boldsymbol{f}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}) \\ &\qquad = \Gamma_{\mathrm{L}}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \iint_{\Omega} |\boldsymbol{f}(\hat{\boldsymbol{r}})|^2 \,\mathrm{d}\Omega + (1 - S\Gamma_{\mathrm{L}}) \iint_{\Omega} \boldsymbol{f}^*(\hat{\boldsymbol{r}}) \cdot \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \,\mathrm{d}\Omega \end{aligned}$$

from which we infer (by the use of (6.2))

$$-S^*\,(1 - S\Gamma_{\mathrm{L}})\,\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) + (1 - S\Gamma_{\mathrm{L}})\,\boldsymbol{f}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}) = (1 - S\Gamma_{\mathrm{L}}) \iint_{\Omega} \boldsymbol{f}^*(\hat{\boldsymbol{r}}) \cdot \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \,\mathrm{d}\Omega$$

or

$$\boldsymbol{f}(\hat{\boldsymbol{k}}_{\mathrm{i}}) - S\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}) = \iint_{\Omega} \mathbf{t}^{\dagger}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{f}(\hat{\boldsymbol{r}}) \,\mathrm{d}\Omega \tag{6.3}$$

Further simplifications are made if the system is reciprocal, see (5.3) on page 15

$$\boldsymbol{f}(\hat{\boldsymbol{k}}_{\mathrm{i}}) - S\boldsymbol{f}^*(-\hat{\boldsymbol{k}}_{\mathrm{i}}) = \iint_{\Omega} \mathbf{t}^{\dagger}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{f}(\hat{\boldsymbol{r}}) \,\mathrm{d}\Omega$$

which, for given $\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}})$, is an equation in the unknown $\boldsymbol{f}(\hat{\boldsymbol{r}})$.

6.3 Comparing two receiving configurations

We now examine two different receiving situations, excited with two different plane waves and loads. The total fields are then

$$\begin{cases} \boldsymbol{E}^{\mathrm{a,b}}(\boldsymbol{r}) \sim \boldsymbol{E}_{\mathrm{i}}^{\mathrm{a,b}}(\boldsymbol{r}) + \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r}\boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}}) \\ Z\boldsymbol{H}^{\mathrm{a,b}}(\boldsymbol{r}) \sim Z\boldsymbol{H}_{\mathrm{i}}^{\mathrm{a,b}}(\boldsymbol{r}) + \dfrac{\mathrm{e}^{\mathrm{i}kr}}{r}\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}}) \end{cases} \qquad r \to \infty$$

where the incident fields are

$$\begin{cases} \boldsymbol{E}_{\mathrm{i}}^{\mathrm{a,b}}(\boldsymbol{r}) = \boldsymbol{E}_0^{\mathrm{a,b}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}}\cdot\boldsymbol{r}} \\ Z\boldsymbol{H}_{\mathrm{i}}^{\mathrm{a,b}}(\boldsymbol{r}) = \dfrac{1}{\mathrm{i}k}\nabla \times \boldsymbol{E}_{\mathrm{i}}^{\mathrm{a,b}} = \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}} \times \boldsymbol{E}_0^{\mathrm{a,b}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}}\cdot\boldsymbol{r}} \end{cases}$$

and the far field amplitudes are, see (3.2) and (3.3) on page 7

$$\begin{cases} \alpha_-^{\mathrm{a,b}} = \dfrac{2\pi\mathrm{i}}{k}\dfrac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}}) \cdot \boldsymbol{E}_0^{\mathrm{a,b}}}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a,b}}} \\ \boldsymbol{F}^{\mathrm{a,b}}(\hat{\boldsymbol{r}}) = \dfrac{2\pi\mathrm{i}}{k}\left\{\dfrac{\Gamma_{\mathrm{L}}^{\mathrm{a,b}}\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}})}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a,b}}} + \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a,b}})\right\} \cdot \boldsymbol{E}_0^{\mathrm{a,b}} \end{cases}$$

The evaluation of the integral in equation (6.1) follows the same derivation as above. The integral over the large sphere S_R has a dominant contribution, which is

$$\begin{aligned} &Z\iint_{S_R} \left\{\boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}^*} + \boldsymbol{E}^{\mathrm{b}^*} \times \boldsymbol{H}^{\mathrm{a}}\right\} \cdot \hat{\boldsymbol{r}}\ \mathrm{d}S \\ &\quad = \iint_{S_R} \Bigg\{ \left(\boldsymbol{E}_0^{\mathrm{a}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}\cdot\boldsymbol{r}} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}})\right) \times \left(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}} \times \boldsymbol{E}_0^{\mathrm{b}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}\cdot\boldsymbol{r}} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}})\right)^* \\ &\qquad + \left(\boldsymbol{E}_0^{\mathrm{b}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}\cdot\boldsymbol{r}} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}})\right)^* \times \left(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}} \times \boldsymbol{E}_0^{\mathrm{a}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}\cdot\boldsymbol{r}} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}})\right) \Bigg\} \cdot \hat{\boldsymbol{r}}\ \mathrm{d}S \end{aligned}$$

There are four terms in the integral on the right-hand side, out of which one gives zero contribution. In fact

$$\iint_{S_R} \left\{\boldsymbol{E}_{\mathrm{i}}^{\mathrm{a}} \times \boldsymbol{H}_{\mathrm{i}}^{\mathrm{b}^*} + \boldsymbol{E}_{\mathrm{i}}^{\mathrm{b}^*} \times \boldsymbol{H}_{\mathrm{i}}^{\mathrm{a}}\right\} \cdot \hat{\boldsymbol{r}}\ \mathrm{d}S = 0$$

due to generalized power balance, since $\boldsymbol{E}_{\mathrm{i}}^{\mathrm{a}}$ and $\boldsymbol{E}_{\mathrm{i}}^{\mathrm{b}}$ satisfy the Maxwell equations inside S_R. Moreover, $\left\{\boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \times \left(\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}})\right)^* + \boldsymbol{F}^{\mathrm{b}^*}(\hat{\boldsymbol{r}}) \times (\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}))\right\} \cdot \hat{\boldsymbol{r}} = 2\boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{F}^{\mathrm{b}^*}(\hat{\boldsymbol{r}})$. The remaining three terms are ($\mathrm{d}S = R^2\,\mathrm{d}\Omega$, where Ω is the unit sphere)

$$
\begin{aligned}
Z \iint\limits_{S_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}*} + \boldsymbol{E}^{\mathrm{b}*} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{r}}\,\mathrm{d}S &= 2 \iint\limits_{\Omega} \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{F}^{\mathrm{b}*}(\hat{\boldsymbol{r}})\,\mathrm{d}\Omega \\
+ R \iint\limits_{\Omega} \Big\{ \boldsymbol{E}_0^{\mathrm{a}} \times \left(\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{b}}(\hat{\boldsymbol{r}})\right)^{*} \mathrm{e}^{\mathrm{i}kR(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}} \cdot \hat{\boldsymbol{r}} - 1)} &+ \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \times \left(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}} \times \boldsymbol{E}_0^{\mathrm{b}}\right)^{*} \mathrm{e}^{\mathrm{i}kR(1 - \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}} \cdot \hat{\boldsymbol{r}})} \\
+ \boldsymbol{E}_0^{\mathrm{b}*} \times (\hat{\boldsymbol{r}} \times \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}))\,\mathrm{e}^{\mathrm{i}kR(1 - \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}} \cdot \hat{\boldsymbol{r}})} &+ \boldsymbol{F}^{\mathrm{b}*}(\hat{\boldsymbol{r}}) \times \left(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}} \times \boldsymbol{E}_0^{\mathrm{a}}\right) \mathrm{e}^{\mathrm{i}kR(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}} \cdot \hat{\boldsymbol{r}} - 1)} \Big\} \cdot \hat{\boldsymbol{r}}\,\mathrm{d}\Omega
\end{aligned}
$$

The BAC-CAB rule simplifies the integrand. We get

$$
\begin{aligned}
Z \iint\limits_{S_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}*} + \boldsymbol{E}^{\mathrm{b}*} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{r}}\,\mathrm{d}S &= 2 \iint\limits_{\Omega} \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{F}^{\mathrm{b}*}(\hat{\boldsymbol{r}})\,\mathrm{d}\Omega \\
+ R \iint\limits_{\Omega} \Big\{ \Big(\boldsymbol{E}_0^{\mathrm{a}} \cdot \boldsymbol{F}^{\mathrm{b}*}(\hat{\boldsymbol{r}}) \left(1 + \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}} \cdot \hat{\boldsymbol{r}}\right) &- \left(\boldsymbol{F}^{\mathrm{b}*}(\hat{\boldsymbol{r}}) \cdot \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}\right) (\boldsymbol{E}_0^{\mathrm{a}} \cdot \hat{\boldsymbol{r}}) \Big) \mathrm{e}^{\mathrm{i}kR(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}} \cdot \hat{\boldsymbol{r}} - 1)} \\
+ \Big(\boldsymbol{E}_0^{\mathrm{b}*} \cdot \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \left(1 + \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}} \cdot \hat{\boldsymbol{r}}\right) &- \left(\boldsymbol{E}_0^{\mathrm{b}*} \cdot \hat{\boldsymbol{r}}\right) \left(\boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \cdot \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}\right) \Big) \mathrm{e}^{\mathrm{i}kR(1 - \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}} \cdot \hat{\boldsymbol{r}})} \Big\}\,\mathrm{d}\Omega
\end{aligned}
$$

The dominant contribution of the surface integral over S_R of this identity is determined by the limiting values as $R \to \infty$. We get, see Footnote 9 on page 15

$$
\begin{aligned}
Z \iint\limits_{S_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}*} + \boldsymbol{E}^{\mathrm{b}*} \times \boldsymbol{H}^{\mathrm{a}} \right\} \cdot \hat{\boldsymbol{r}}\,\mathrm{d}S &\to 2 \iint\limits_{\Omega} \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{F}^{\mathrm{b}*}(\hat{\boldsymbol{r}})\,\mathrm{d}\Omega \\
&+ \frac{4\pi}{\mathrm{i}k} \left(\boldsymbol{E}_0^{\mathrm{a}} \cdot \boldsymbol{F}^{\mathrm{b}*}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) - \boldsymbol{E}_0^{\mathrm{b}*} \cdot \boldsymbol{F}^{\mathrm{a}}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}) \right), \quad R \to \infty
\end{aligned}
$$

The right-hand side of the integral in equation (6.1) is

$$
2\left(a_+^{\mathrm{a}} a_+^{\mathrm{b}*} - a_-^{\mathrm{a}} a_-^{\mathrm{b}*}\right) = \frac{8\pi^2}{k^2} \left(\Gamma_{\mathrm{L}}^{\mathrm{a}} \Gamma_{\mathrm{L}}^{\mathrm{b}*} - 1\right) \frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) \cdot \boldsymbol{E}_0^{\mathrm{a}}}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a}}} \frac{\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}) \cdot \boldsymbol{E}_0^{\mathrm{b}*}}{1 - S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}}
$$

We can now put the right- and left-hand together. The generalized power balance for two receiving antenna then becomes

$$
\begin{aligned}
&\frac{8\pi^2}{k^2} \left(\Gamma_{\mathrm{L}}^{\mathrm{a}} \Gamma_{\mathrm{L}}^{\mathrm{b}*} - 1\right) \frac{\boldsymbol{E}_0^{\mathrm{a}} \cdot \boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a}}} \frac{\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}) \cdot \boldsymbol{E}_0^{\mathrm{b}*}}{1 - S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}} \\
&\quad = -\frac{8\pi^2}{k^2} \boldsymbol{E}_0^{\mathrm{a}} \cdot \Bigg\{ \frac{\Gamma_{\mathrm{L}}^{\mathrm{b}*} \boldsymbol{f}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) \boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1 - S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}} + \frac{\Gamma_{\mathrm{L}}^{\mathrm{a}} \boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) \boldsymbol{f}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a}}} + \mathbf{t}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}) + \mathbf{t}^t(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) \\
&\qquad - \iint\limits_{\Omega} \left(\frac{\Gamma_{\mathrm{L}}^{\mathrm{a}} \boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) \boldsymbol{f}(\hat{\boldsymbol{r}})}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{a}}} + \mathbf{t}^t(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) \right) \cdot \left(\frac{\Gamma_{\mathrm{L}}^{\mathrm{b}} \boldsymbol{f}(\hat{\boldsymbol{r}}) \boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1 - S\Gamma_{\mathrm{L}}^{\mathrm{b}}} + \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}}) \right)^{*} \mathrm{d}\Omega \Bigg\} \cdot \boldsymbol{E}_0^{\mathrm{b}*}
\end{aligned}
$$

The vector fields, $\boldsymbol{E}_0^{\mathrm{a,b}}$, are arbitrary. This implies

$$\begin{aligned}\left(1-\Gamma_{\mathrm{L}}^{\mathrm{a}}\Gamma_{\mathrm{L}}^{\mathrm{b}*}\right)\frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})}{1-S\Gamma_{\mathrm{L}}^{\mathrm{a}}}\frac{\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1-S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}} &= \frac{\Gamma_{\mathrm{L}}^{\mathrm{b}*}\boldsymbol{f}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1-S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}}+\frac{\Gamma_{\mathrm{L}}^{\mathrm{a}}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\boldsymbol{f}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1-S\Gamma_{\mathrm{L}}^{\mathrm{a}}} \\ &\quad+\mathbf{t}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})+\mathbf{t}^t(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}}) \\ &\quad-\iint\limits_{\Omega}\left(\frac{\Gamma_{\mathrm{L}}^{\mathrm{a}}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\boldsymbol{f}(\hat{\boldsymbol{r}})}{1-S\Gamma_{\mathrm{L}}^{\mathrm{a}}}+\mathbf{t}^t(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\right)\cdot\left(\frac{\Gamma_{\mathrm{L}}^{\mathrm{b}}\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1-S\Gamma_{\mathrm{L}}^{\mathrm{b}}}+\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})\right)^*\mathrm{d}\Omega\end{aligned}$$

This relation is now simplified with the use of the two already existing relations for a lossless antenna configuration given in (6.2) and (6.3). To fit our expression, we first rewrite the relation in (6.3) as

$$\iint\limits_{\Omega}\boldsymbol{f}(\hat{\boldsymbol{r}})\cdot\mathbf{t}^*(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\mathrm{i}})\,\mathrm{d}\Omega=\boldsymbol{f}(\hat{\boldsymbol{k}}_{\mathrm{i}})-S\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}})$$

and

$$\iint\limits_{\Omega}\mathbf{t}^t(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\mathrm{i}})\cdot\boldsymbol{f}^*(\hat{\boldsymbol{r}})\,\mathrm{d}\Omega=\boldsymbol{f}^*(\hat{\boldsymbol{k}}_{\mathrm{i}})-S^*\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}})$$

These relations and (6.2) are used to simplify three of the integrals on the right-hand side in the expression above. We get

$$\begin{aligned}\left(1-\Gamma_{\mathrm{L}}^{\mathrm{a}}\Gamma_{\mathrm{L}}^{\mathrm{b}*}\right)\frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})}{1-S\Gamma_{\mathrm{L}}^{\mathrm{a}}}\frac{\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1-S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}} &= \frac{\Gamma_{\mathrm{L}}^{\mathrm{b}*}\boldsymbol{f}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1-S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}}+\frac{\Gamma_{\mathrm{L}}^{\mathrm{a}}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\boldsymbol{f}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1-S\Gamma_{\mathrm{L}}^{\mathrm{a}}} \\ &\quad+\mathbf{t}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})+\mathbf{t}^t(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})-\frac{\Gamma_{\mathrm{L}}^{\mathrm{a}}\Gamma_{\mathrm{L}}^{\mathrm{b}*}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{(1-S\Gamma_{\mathrm{L}}^{\mathrm{a}})\left(1-S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}\right)}\left(1-|S|^2\right) \\ &\quad-\frac{\Gamma_{\mathrm{L}}^{\mathrm{a}}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\left(\boldsymbol{f}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})-S\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})\right)}{1-S\Gamma_{\mathrm{L}}^{\mathrm{a}}}-\frac{\Gamma_{\mathrm{L}}^{\mathrm{b}*}\left(\boldsymbol{f}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})-S^*\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\right)\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{1-S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}} \\ &\quad-\iint\limits_{\Omega}\mathbf{t}^t(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\cdot\mathbf{t}^*(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})\,\mathrm{d}\Omega\end{aligned}$$

Several factors cancel. We get

$$\begin{aligned}&\frac{\left(1-|S|^2\Gamma_{\mathrm{L}}^{\mathrm{a}}\Gamma_{\mathrm{L}}^{\mathrm{b}*}-S\Gamma_{\mathrm{L}}^{\mathrm{a}}\left(1-S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}\right)-S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}\left(1-S\Gamma_{\mathrm{L}}^{\mathrm{a}}\right)\right)\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})}{(1-S\Gamma_{\mathrm{L}}^{\mathrm{a}})\left(1-S^*\Gamma_{\mathrm{L}}^{\mathrm{b}*}\right)} \\ &\qquad=\mathbf{t}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})+\mathbf{t}^t(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})-\iint\limits_{\Omega}\mathbf{t}^t(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\cdot\mathbf{t}^*(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})\,\mathrm{d}\Omega\end{aligned}$$

or simplified (and take the transpose)

$$\boldsymbol{s}^*(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})=\mathbf{t}^\dagger(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})+\mathbf{t}(\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})-\iint\limits_{\Omega}\mathbf{t}^\dagger(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{b}})\cdot\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\mathrm{i}}^{\mathrm{a}})\,\mathrm{d}\Omega$$

This relation is equivalent to the generalized optical theorem. For more details on this relations, see [17, Sec. 7.8.3] and [26, p. 301].

In summary, a lossless system the antenna parameters satisfy

$$\begin{cases} \displaystyle\iint_\Omega |\boldsymbol{f}(\hat{\boldsymbol{r}})|^2 \, \mathrm{d}\Omega = 1 - |S|^2 \\ \displaystyle\boldsymbol{f}(\hat{\boldsymbol{k}}_\mathrm{i}) - S\boldsymbol{s}^*(\hat{\boldsymbol{k}}_\mathrm{i}) = \iint_\Omega \mathbf{t}^\dagger(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{f}(\hat{\boldsymbol{r}}) \, \mathrm{d}\Omega \\ \displaystyle\boldsymbol{s}^*(\hat{\boldsymbol{k}}_\mathrm{i}^\mathrm{b})\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}^\mathrm{a}) = \mathbf{t}^\dagger(\hat{\boldsymbol{k}}_\mathrm{i}^\mathrm{a}, \hat{\boldsymbol{k}}_\mathrm{i}^\mathrm{b}) + \mathbf{t}(\hat{\boldsymbol{k}}_\mathrm{i}^\mathrm{b}, \hat{\boldsymbol{k}}_\mathrm{i}^\mathrm{a}) - \iint_\Omega \mathbf{t}^\dagger(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}^\mathrm{b}) \cdot \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}^\mathrm{a}) \, \mathrm{d}\Omega \end{cases} \tag{6.4}$$

In particular, if the two cases a and b are the same, *i.e.*, $\hat{\boldsymbol{k}}_\mathrm{i} = \hat{\boldsymbol{k}}_\mathrm{i}^\mathrm{a} = \hat{\boldsymbol{k}}_\mathrm{i}^\mathrm{b}$, the last identity reads

$$\boldsymbol{s}^*(\hat{\boldsymbol{k}}_\mathrm{i})\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) = \mathbf{t}^\dagger(\hat{\boldsymbol{k}}_\mathrm{i}, \hat{\boldsymbol{k}}_\mathrm{i}) + \mathbf{t}(\hat{\boldsymbol{k}}_\mathrm{i}, \hat{\boldsymbol{k}}_\mathrm{i}) - \iint_\Omega \mathbf{t}^\dagger(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) \cdot \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) \, \mathrm{d}\Omega \tag{6.5}$$

Example 6.1

The results in this section can be used to verify the optical theorem [17, Sec. 4.4].

$$\sigma_\mathrm{a}(\hat{\boldsymbol{k}}_\mathrm{i}) + \sigma_\mathrm{s}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{4\pi}{k|\boldsymbol{E}_0|^2} \operatorname{Im}\left\{\boldsymbol{E}_0^* \cdot \boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{k}}_\mathrm{i})\right\}$$

which is an energy conservation statement. We verify this identity by evaluating the left-hand side. The absorption cross section $\sigma_\mathrm{a}(\hat{\boldsymbol{k}}_\mathrm{i})$ is in (3.7) on page 8, *i.e.*,

$$\sigma_\mathrm{a}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{4\pi^2}{k^2} \frac{1 - |\Gamma_\mathrm{L}|^2}{|1 - S\Gamma_\mathrm{L}|^2} \frac{\left|\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0\right|^2}{|\boldsymbol{E}_0|^2}$$

and the scattering cross section $\sigma_\mathrm{s}(\hat{\boldsymbol{k}}_\mathrm{i})$ in terms of the scattered or reradiated power P_s is, see (3.11) on page 10

$$\sigma_\mathrm{s}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{2ZP_\mathrm{s}}{|\boldsymbol{E}_0|^2} = \frac{4\pi^2}{k^2\,|\boldsymbol{E}_0|^2} \iint_\Omega \left| \frac{\Gamma_\mathrm{L}\boldsymbol{f}(\hat{\boldsymbol{r}})\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0}{1 - S\Gamma_\mathrm{L}} + \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \right|^2 \mathrm{d}\Omega$$

We expand the integral in the scattering cross section and use (6.4) and (6.5). We obtain

$$\begin{aligned} \sigma_\mathrm{s}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{4\pi^2}{k^2\,|\boldsymbol{E}_0|^2} \Bigg\{ & \left(1 - |S|^2\right) \left| \frac{\Gamma_\mathrm{L}\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0}{1 - S\Gamma_\mathrm{L}} \right|^2 \\ & - \boldsymbol{E}_0^* \cdot \left(\boldsymbol{s}^*(\hat{\boldsymbol{k}}_\mathrm{i})\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) - \mathbf{t}^\dagger(\hat{\boldsymbol{k}}_\mathrm{i}, \hat{\boldsymbol{k}}_\mathrm{i}) - \mathbf{t}(\hat{\boldsymbol{k}}_\mathrm{i}, \hat{\boldsymbol{k}}_\mathrm{i})\right) \cdot \boldsymbol{E}_0 \\ & + 2\operatorname{Re} \frac{\Gamma_\mathrm{L}\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0}{1 - S\Gamma_\mathrm{L}} \boldsymbol{E}_0^* \cdot \left(\boldsymbol{f}(\hat{\boldsymbol{k}}_\mathrm{i}) - S\boldsymbol{s}^*(\hat{\boldsymbol{k}}_\mathrm{i})\right) \Bigg\} \end{aligned}$$

or

$$\sigma_{\rm s}(\hat{\boldsymbol{k}}_{\rm i}) = \frac{4\pi^2}{k^2\left|\boldsymbol{E}_0\right|^2}\left\{\left(\left(1-|S|^2\right)\left|\frac{\Gamma_{\rm L}}{1-S\Gamma_{\rm L}}\right|^2 - 1 - 2\,\mathrm{Re}\,\frac{S\Gamma_{\rm L}}{1-S\Gamma_{\rm L}}\right)\left|\boldsymbol{s}(\hat{\boldsymbol{k}}_{\rm i})\cdot\boldsymbol{E}_0\right|^2 + 2\,\mathrm{Re}\,\boldsymbol{E}_0^*\cdot\left(\mathbf{t}(\hat{\boldsymbol{k}}_{\rm i},\hat{\boldsymbol{k}}_{\rm i}) + \frac{\Gamma_{\rm L}\boldsymbol{f}(\hat{\boldsymbol{k}}_{\rm i})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\rm i})}{1-S\Gamma_{\rm L}}\right)\cdot\boldsymbol{E}_0\right\}$$

The parenthesis in the first term on the right-hand side simplifies, and we left with

$$\sigma_{\rm s}(\hat{\boldsymbol{k}}_{\rm i}) = \frac{4\pi^2}{k^2\left|\boldsymbol{E}_0\right|^2}\left\{\frac{|\Gamma_{\rm L}|^2-1}{|1-S\Gamma_{\rm L}|^2}\left|\boldsymbol{s}(\hat{\boldsymbol{k}}_{\rm i})\cdot\boldsymbol{E}_0\right|^2 + 2\,\mathrm{Re}\,\boldsymbol{E}_0^*\cdot\left(\mathbf{t}(\hat{\boldsymbol{k}}_{\rm i},\hat{\boldsymbol{k}}_{\rm i}) + \frac{\Gamma_{\rm L}\boldsymbol{f}(\hat{\boldsymbol{k}}_{\rm i})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\rm i})}{1-S\Gamma_{\rm L}}\right)\cdot\boldsymbol{E}_0\right\}$$

which implies

$$\sigma_{\rm a}(\hat{\boldsymbol{k}}_{\rm i}) + \sigma_{\rm s}(\hat{\boldsymbol{k}}_{\rm i}) = \frac{8\pi^2}{k^2\left|\boldsymbol{E}_0\right|^2}\,\mathrm{Re}\,\boldsymbol{E}_0^*\cdot\left(\mathbf{t}(\hat{\boldsymbol{k}}_{\rm i},\hat{\boldsymbol{k}}_{\rm i}) + \frac{\Gamma_{\rm L}\boldsymbol{f}(\hat{\boldsymbol{k}}_{\rm i})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\rm i})}{1-S\Gamma_{\rm L}}\right)\cdot\boldsymbol{E}_0 = \frac{4\pi}{k|\boldsymbol{E}_0|^2}\,\mathrm{Im}\left\{\boldsymbol{E}_0^*\cdot\boldsymbol{F}^{\rm r}(\hat{\boldsymbol{k}}_{\rm i})\right\}$$

where, see (3.3) on page 7

$$\boldsymbol{E}_0^*\cdot\boldsymbol{F}^{\rm r}(\hat{\boldsymbol{k}}_{\rm i}) = \frac{2\pi\mathrm{i}}{k}\boldsymbol{E}_0^*\cdot\left\{\frac{\Gamma_{\rm L}\boldsymbol{f}(\hat{\boldsymbol{k}}_{\rm i})\boldsymbol{s}(\hat{\boldsymbol{k}}_{\rm i})}{1-S\Gamma_{\rm L}} + \mathbf{t}(\hat{\boldsymbol{k}}_{\rm i},\hat{\boldsymbol{k}}_{\rm i})\right\}\cdot\boldsymbol{E}_0$$

which completes the verification of the optical theorem. Notice that we have used all three relations in (6.4) to verify the optical theorem, and that the result holds irrespectively of the load $\Gamma_{\rm L}$. ∎

7 Examples

The conditions for a reciprocal antenna system are summarized in (5.6) on page 19, and the conditions for a lossless system are summarized in (6.4). If both these conditions are satisfied we can eliminate the vector $\boldsymbol{s}$ and replace it with the vector $\boldsymbol{f}$. We have

$$\begin{cases}\boldsymbol{f}(\hat{\boldsymbol{r}}) = \boldsymbol{s}(-\hat{\boldsymbol{r}})\\ \mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\rm i}) = \mathbf{t}^t(-\hat{\boldsymbol{k}}_{\rm i},-\hat{\boldsymbol{r}})\\ \displaystyle\iint_\Omega \left|\boldsymbol{f}(\hat{\boldsymbol{r}})\right|^2\,\mathrm{d}\Omega = 1-|S|^2\\ \boldsymbol{f}(\hat{\boldsymbol{k}}_{\rm i}) - S\boldsymbol{f}^*(-\hat{\boldsymbol{k}}_{\rm i}) = \displaystyle\iint_\Omega \mathbf{t}^\dagger(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\rm i})\cdot\boldsymbol{f}(\hat{\boldsymbol{r}})\,\mathrm{d}\Omega\\ \boldsymbol{f}(-\hat{\boldsymbol{k}}_{\rm i}^{\rm b})\boldsymbol{f}^*(-\hat{\boldsymbol{k}}_{\rm i}^{\rm a}) = \mathbf{t}^\dagger(\hat{\boldsymbol{k}}_{\rm i}^{\rm a},\hat{\boldsymbol{k}}_{\rm i}^{\rm b}) + \mathbf{t}(\hat{\boldsymbol{k}}_{\rm i}^{\rm b},\hat{\boldsymbol{k}}_{\rm i}^{\rm a}) - \displaystyle\iint_\Omega \mathbf{t}^\dagger(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\rm i}^{\rm b})\cdot\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\rm i}^{\rm a})\,\mathrm{d}\Omega\end{cases}\tag{7.1}$$

In this section, we apply these results and the analysis above to a specific representation of the antenna parameters $\boldsymbol{f}(\hat{\boldsymbol{r}})$ and $\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_{\rm i})$ by a series expansion in vector spherical harmonics. We also analyze bounds on the absorption efficiency of the antenna.

7.1 Vector spherical harmonic representation

The radiation pattern $\boldsymbol{f}(\hat{\boldsymbol{r}})$ and the scattering contribution $\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i})$ are most conveniently expanded in vector spherical harmonics $\boldsymbol{A}_n(\hat{\boldsymbol{r}})$, which is a complete set of orthonormal vector-valued functions on the unit sphere. The definition of these functions are given in [17]. The index n is a multi-index $n = \{\tau, \sigma, m, l\}$, where $\tau = 1, 2$, $\sigma = \mathrm{e}, \mathrm{o}$ (even or odd functions in the azimuthal angle), $m = 0, 1, 2, \ldots, l$, $l = 1, 2, \ldots$. We adopt

$$\begin{cases} \boldsymbol{f}(\hat{\boldsymbol{r}}) = \sum\limits_n f_n \boldsymbol{A}_n(\hat{\boldsymbol{r}}) \\ \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) = \sum\limits_{nn'} t_{nn'} \boldsymbol{A}_n(\hat{\boldsymbol{r}}) \boldsymbol{A}_{n'}(\hat{\boldsymbol{k}}_\mathrm{i}) \end{cases}$$

The coefficients f_n and $t_{nn'}$ are obtained by the use of orthonormality of the vector spherical harmonics over the unit sphere Ω, *i.e.*,

$$\begin{cases} f_n = \iint\limits_{\Omega} \boldsymbol{f}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{A}_n(\hat{\boldsymbol{r}}) \, \mathrm{d}\Omega \\ t_{nn'} = \iint\limits_{\Omega} \iint\limits_{\Omega} \boldsymbol{A}_n(\hat{\boldsymbol{r}}) \cdot \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{r}}') \cdot \boldsymbol{A}_{n'}(\hat{\boldsymbol{r}}') \, \mathrm{d}\Omega \, \mathrm{d}\Omega' \end{cases}$$

In fact, the expansion of the dyadic $\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i})$ is non-standard. We derive this expansion by expanding $\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{r}}') \cdot \boldsymbol{a}$ first as a function in $\hat{\boldsymbol{r}}$ for fixed direction $\hat{\boldsymbol{r}}'$ and arbitrary fixed vector $\boldsymbol{a}$. We have

$$\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{r}}') \cdot \boldsymbol{a} = \sum_n t_n(\hat{\boldsymbol{r}}') \boldsymbol{A}_n(\hat{\boldsymbol{r}}), \quad t_n(\hat{\boldsymbol{r}}') = \iint\limits_{\Omega} \boldsymbol{A}_n(\hat{\boldsymbol{r}}) \cdot \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{r}}') \, \mathrm{d}\Omega \cdot \boldsymbol{a}$$

We also expand $\boldsymbol{A}_n(\hat{\boldsymbol{r}}) \cdot \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{r}}')$ as a function in $\hat{\boldsymbol{r}}'$, *i.e.*,

$$\begin{cases} \boldsymbol{A}_n(\hat{\boldsymbol{r}}) \cdot \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{r}}') = \sum\limits_{n'} t_{nn'}(\hat{\boldsymbol{r}}) \boldsymbol{A}_{n'}(\hat{\boldsymbol{r}}') \\ t_{nn'}(\hat{\boldsymbol{r}}) = \iint\limits_{\Omega} \boldsymbol{A}_n(\hat{\boldsymbol{r}}) \cdot \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{r}}') \cdot \boldsymbol{A}_{n'}(\hat{\boldsymbol{r}}') \, \mathrm{d}\Omega' \end{cases}$$

Now combine these expansions

$$t_n(\hat{\boldsymbol{r}}') = \sum_{n'} \iint\limits_{\Omega} t_{nn'}(\hat{\boldsymbol{r}}) \, \mathrm{d}\Omega \boldsymbol{A}_{n'}(\hat{\boldsymbol{r}}') \cdot \boldsymbol{a} = \sum_{n'} t_{nn'} \boldsymbol{A}_{n'}(\hat{\boldsymbol{r}}') \cdot \boldsymbol{a}$$

where

$$t_{nn'} = \iint\limits_{\Omega} t_{nn'}(\hat{\boldsymbol{r}}) \, \mathrm{d}\Omega = \iint\limits_{\Omega} \iint\limits_{\Omega} \boldsymbol{A}_n(\hat{\boldsymbol{r}}) \cdot \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{r}}') \cdot \boldsymbol{A}_{n'}(\hat{\boldsymbol{r}}') \, \mathrm{d}\Omega \, \mathrm{d}\Omega'$$

Finally, we get

$$\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{r}}') \cdot \boldsymbol{a} = \sum_{nn'} \boldsymbol{A}_n(\hat{\boldsymbol{r}}) t_{nn'} \boldsymbol{A}_{n'}(\hat{\boldsymbol{r}}') \cdot \boldsymbol{a}$$

and the result follows, since $\boldsymbol{a}$ is arbitrary.

Equations (7.1) then imply

$$\begin{cases} t_{n'n} = (-1)^{l+l'+\tau+\tau'} t_{nn'} \\ \sum_n f_n f_n^* = 1 - |S|^2 \\ f_n - (-1)^{l+\tau-1} S f_n^* = \sum_{n'} t_{n'n}^* f_{n'} \\ (-1)^{l+l'+\tau+\tau'} f_n f_{n'}^* = t_{n'n}^* + t_{nn'} - \sum_{n''} t_{n''n}^* t_{n''n'} \end{cases} \tag{7.2}$$

since $\boldsymbol{A}_n(-\hat{\boldsymbol{r}}) = (-1)^{l+\tau-1} \boldsymbol{A}_n(\hat{\boldsymbol{r}})$.

Example 7.1

The simplest application of these expansions is an elementary dipole. A simple elementary dipole radiates as

$$\boldsymbol{f}(\boldsymbol{r}) = f \boldsymbol{A}_n(\hat{\boldsymbol{r}})$$

where f is an arbitrary complex number. The index n is here the combination $n = 2\mathrm{e}01$, which gives the radiation pattern of the elementary vertical electric dipole (oriented along the z axis). Similarly, $n = 1\mathrm{e}01$ gives the radiation pattern of the elementary vertical magnetic dipole. The natural Anzats for the scattering contribution $\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i})$ is

$$\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) = t \boldsymbol{A}_n(\hat{\boldsymbol{r}}) \boldsymbol{A}_n(\hat{\boldsymbol{k}}_\mathrm{i})$$

The a given value of f, the two constants S and t for a reciprocal system satisfy, see (7.2)

$$\begin{cases} |f|^2 = 1 - |S|^2 \\ f - (-1)^\tau S f^* = t^* f \\ |f|^2 = t + t^* - |t|^2 \end{cases}$$

Note that the second relation depends on whether $\tau = 1, 2$ (electric or magnetic elementary dipole). The solution is

$$\begin{cases} |S|^2 = 1 - |f|^2 \\ t = 1 - (-1)^\tau \dfrac{S^* f}{f^*} \end{cases}$$

and the phase of the reflection coefficient S is undetermined. Note that if $f = 1$, then $S = 0$ and $t = 1$.

The absorbed and the reradiated scattered power for this elementary dipole are, see (3.6) and (3.11) on page 8 and 10, respectively

$$P_\mathrm{r} = \frac{2\pi^2}{k^2 Z} \frac{1 - |\Gamma_\mathrm{L}|^2}{|1 - S\Gamma_\mathrm{L}|^2} \left| f \boldsymbol{A}_{\tau\mathrm{e}01}(-\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \right|^2 = \frac{2\pi^2 |f|^2}{k^2 Z} \frac{1 - |\Gamma_\mathrm{L}|^2}{|1 - S\Gamma_\mathrm{L}|^2} \left| \boldsymbol{A}_{\tau\mathrm{e}01}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \right|^2$$

and

$$\begin{aligned} P_\mathrm{s} = \frac{2\pi^2}{k^2 Z} \iint_\Omega & \left| \frac{\Gamma_\mathrm{L} f^2 \boldsymbol{A}_{\tau\mathrm{e}01}(\hat{\boldsymbol{r}}) \boldsymbol{A}_{\tau\mathrm{e}01}(-\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0}{1 - S\Gamma_\mathrm{L}} + t \boldsymbol{A}_{\tau\mathrm{e}01}(\hat{\boldsymbol{r}}) \boldsymbol{A}_{\tau\mathrm{e}01}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \right|^2 \mathrm{d}\Omega \\ &= \frac{2\pi^2}{k^2 Z} \left| \frac{(-1)^\tau \Gamma_\mathrm{L} f^2}{1 - S\Gamma_\mathrm{L}} + 1 - (-1)^\tau \frac{S^* f}{f^*} \right|^2 \left| \boldsymbol{A}_{\tau\mathrm{e}01}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \right|^2 \end{aligned}$$

where

$$\left|\boldsymbol{A}_{\tau e01}(\hat{\boldsymbol{k}}_{\rm i})\cdot\boldsymbol{E}_0\right|^2=\frac{3}{8\pi}\left|\left(\hat{\boldsymbol{k}}_{\rm i}\times\hat{\boldsymbol{z}}\right)\cdot\boldsymbol{E}_0\right|^2,\quad \tau=1$$

and

$$\left|\boldsymbol{A}_{\tau e01}(\hat{\boldsymbol{k}}_{\rm i})\cdot\boldsymbol{E}_0\right|^2=\frac{3}{8\pi}\left|\left(\hat{\boldsymbol{k}}_{\rm i}\times\left(\hat{\boldsymbol{k}}_{\rm i}\times\hat{\boldsymbol{z}}\right)\right)\cdot\boldsymbol{E}_0\right|^2=\frac{3}{8\pi}\left|\hat{\boldsymbol{z}}\cdot\boldsymbol{E}_0\right|^2,\quad \tau=2$$

For a conjugate matched load $\Gamma_{\rm L}=S^*$ these expressions become

$$P_{\rm r}=\frac{3\pi}{4k^2Z}\left|\left(\hat{\boldsymbol{k}}_{\rm i}\times\hat{\boldsymbol{z}}\right)\cdot\boldsymbol{E}_0\right|^2,\quad \tau=1,\qquad P_{\rm r}=\frac{3\pi}{4k^2Z}\left|\hat{\boldsymbol{z}}\cdot\boldsymbol{E}_0\right|^2,\quad \tau=2$$

and

$$P_{\rm s}=\frac{3\pi}{4k^2Z}\left|-\frac{S^*f^2}{1-|S|^2}+1+\frac{S^*f^2}{|f|^2}\right|^2\left|\left(\hat{\boldsymbol{k}}_{\rm i}\times\hat{\boldsymbol{z}}\right)\cdot\boldsymbol{E}_0\right|^2=\frac{3\pi}{4k^2Z}\left|\left(\hat{\boldsymbol{k}}_{\rm i}\times\hat{\boldsymbol{z}}\right)\cdot\boldsymbol{E}_0\right|^2,\quad \tau=1$$

and

$$P_{\rm s}=\frac{3\pi}{4k^2Z}\left|\frac{S^*f^2}{1-|S|^2}+1-\frac{S^*f^2}{|f|^2}\right|^2\left|\hat{\boldsymbol{z}}\cdot\boldsymbol{E}_0\right|^2=\frac{3\pi}{4k^2Z}\left|\hat{\boldsymbol{z}}\cdot\boldsymbol{E}_0\right|^2,\quad \tau=2$$

and the elementary dipole absorbs and reradiates the same amount for a given incident direction $\hat{\boldsymbol{k}}_{\rm i}$, if the load is conjugate matched. ■

7.2 Absorption efficiency

The absorption efficiency η is a measure of how much the receiving antenna absorbs compared to how much it scatters or reradiates. The definition of absorption efficiency is [1][10]

$$\eta=\frac{\sigma_{\rm a}(\hat{\boldsymbol{k}}_{\rm i})}{\sigma_{\rm a}(\hat{\boldsymbol{k}}_{\rm i})+\sigma_{\rm s}(\hat{\boldsymbol{k}}_{\rm i})}$$

where the absorption cross section $\sigma_{\rm a}(\hat{\boldsymbol{k}}_{\rm i})$ in terms of the absorbed power $P_{\rm a}$ is, see (3.7) on page 8 and (7.1)

$$\sigma_{\rm a}(\hat{\boldsymbol{k}}_{\rm i})=\frac{2ZP_{\rm a}}{|\boldsymbol{E}_0|^2}=\frac{4\pi^2}{k^2}\frac{1-|\Gamma_{\rm L}|^2}{|1-S\Gamma_{\rm L}|^2}\frac{\left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_{\rm i})\cdot\boldsymbol{E}_0\right|^2}{|\boldsymbol{E}_0|^2}=\frac{4\pi^2p_{\rm L}p_{\rm a}}{k^2}\frac{\left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_{\rm i})\right|^2}{1-|S|^2}$$

where we have introduced two mismatch factors, the absorption loss factor $p_{\rm a}$ and the load mismatch factor $p_{\rm L}$, defined as

$$p_{\rm a}=\frac{\left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_{\rm i})\cdot\boldsymbol{E}_0\right|^2}{\left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_{\rm i})\right|^2|\boldsymbol{E}_0|^2},\quad p_{\rm L}=\left(1-|S|^2\right)\frac{1-|\Gamma_{\rm L}|^2}{|1-S\Gamma_{\rm L}|^2}\tag{7.3}$$

Both these factors is number in the interval $[0,1]$ — perfect match is given by 1, *cf.* (3.9) on page 9.

The scattering cross section $\sigma_{\rm s}(\hat{\boldsymbol{k}}_{\rm i})$ in terms of the scattered or reradiated power $P_{\rm s}$ is, see (3.11) on page 10

$$\sigma_{\rm s}(\hat{\boldsymbol{k}}_{\rm i})=\frac{2ZP_{\rm s}}{|\boldsymbol{E}_0|^2}=\frac{\iint_\Omega|\boldsymbol{F}^{\rm r}(\hat{\boldsymbol{r}})|^2\,{\rm d}\Omega}{|\boldsymbol{E}_0|^2}$$

[10]Confer the similar definition of albedo α in [17]. We have $\alpha+\eta=1$.

Two other quantities are also relevant in this section. The directivity of the antenna towards the source, $-\hat{\boldsymbol{k}}_\mathrm{i}$, is, see (2.4) on page 6

$$D_\mathrm{a} = D(-\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{4\pi \left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\right|^2}{\iint\limits_{\Omega} |\boldsymbol{f}(\hat{\boldsymbol{r}})|^2 \,\mathrm{d}\Omega} = \frac{4\pi \left|\boldsymbol{f}(-\hat{\boldsymbol{k}}_\mathrm{i})\right|^2}{1-|S|^2}$$

where we also used (7.1). The absorption cross section then becomes

$$\sigma_\mathrm{a}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{\pi p_\mathrm{L} p_\mathrm{a}}{k^2} D_\mathrm{a}$$

The directivity of the scattered pattern in the forward direction, $\hat{\boldsymbol{k}}_\mathrm{i}$, is denoted D_s, and defined as [1]

$$D_\mathrm{s} = \frac{4\pi |\boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{k}}_\mathrm{i})|^2}{\iint\limits_{\Omega} |\boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{r}})|^2 \,\mathrm{d}\Omega} = \frac{4\pi |\boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{k}}_\mathrm{i})|^2}{|\boldsymbol{E}_0|^2 \sigma_\mathrm{s}(\hat{\boldsymbol{k}}_\mathrm{i})}$$

Note that the directivity of the scattered field in general is different from the directivity in (2.4) on page 6, since it also contains the scattered contribution and not just the reradiated field.

As a receiver, the antenna acts like a passive scatterer and consequently the optical theorem holds.

$$\sigma_\mathrm{a}(\hat{\boldsymbol{k}}_\mathrm{i}) + \sigma_\mathrm{s}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{4\pi}{k|\boldsymbol{E}_0|^2} \operatorname{Im}\left\{\boldsymbol{E}_0^* \cdot \boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{k}}_\mathrm{i})\right\}$$

Since the imaginary part always is smaller than its absolute value, we get

$$\sigma_\mathrm{a}(\hat{\boldsymbol{k}}_\mathrm{i}) + \sigma_\mathrm{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \le \frac{4\pi}{k|\boldsymbol{E}_0|^2} \left|\boldsymbol{E}_0^* \cdot \boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{k}}_\mathrm{i})\right| \le \frac{4\pi|\boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{k}}_\mathrm{i})|}{k|\boldsymbol{E}_0|} = 2\sqrt{\alpha \sigma_\mathrm{s}(\hat{\boldsymbol{k}}_\mathrm{i})}$$

where $\alpha = \pi D_\mathrm{s}/k^2$. Square the relation and rewrite as

$$(\sigma_\mathrm{s} + \sigma_\mathrm{a} - 2\alpha)^2 - 4(\alpha - \sigma_\mathrm{a})\alpha = (\sigma_\mathrm{s} + \sigma_\mathrm{a})^2 - 4\alpha\sigma_\mathrm{s} \le 0$$

Note that the inequality also implies that $\alpha \ge \sigma_\mathrm{a}$ since $0 \le (\sigma_\mathrm{s} + \sigma_\mathrm{a} - 2\alpha)^2 \le 4(\alpha - \sigma_\mathrm{a})\alpha$. The above inequality implies

$$2\alpha - 2\sqrt{(\alpha - \sigma_\mathrm{a})\alpha} \le \sigma_\mathrm{a} + \sigma_\mathrm{s} \le 2\alpha + 2\sqrt{(\alpha - \sigma_\mathrm{a})\alpha}$$

or in terms of the absorption efficiency η

$$\frac{1}{2}\frac{\sigma_\mathrm{a}/\alpha}{1+\sqrt{1-\sigma_\mathrm{a}/\alpha}} \le \eta \le \frac{1}{2}\frac{\sigma_\mathrm{a}/\alpha}{1-\sqrt{1-\sigma_\mathrm{a}/\alpha}}$$

which we simplify to

$$\frac{1}{2}\left(1-\sqrt{1-\sigma_\mathrm{a}/\alpha}\right) \le \eta \le \frac{1}{2}\left(1+\sqrt{1-\sigma_\mathrm{a}/\alpha}\right)$$

Denoting $\beta = \sigma_\mathrm{a}/\alpha = p_\mathrm{L} p_\mathrm{a} D_\mathrm{a}/D_\mathrm{s}$, which is restricted to the interval $[0,1]$ ($\alpha \ge \sigma_\mathrm{a}$), and the following inequalities hold

$$\frac{1}{2}\left(1-\sqrt{1-\beta}\right) \le \eta \le \frac{1}{2}\left(1+\sqrt{1-\beta}\right)$$

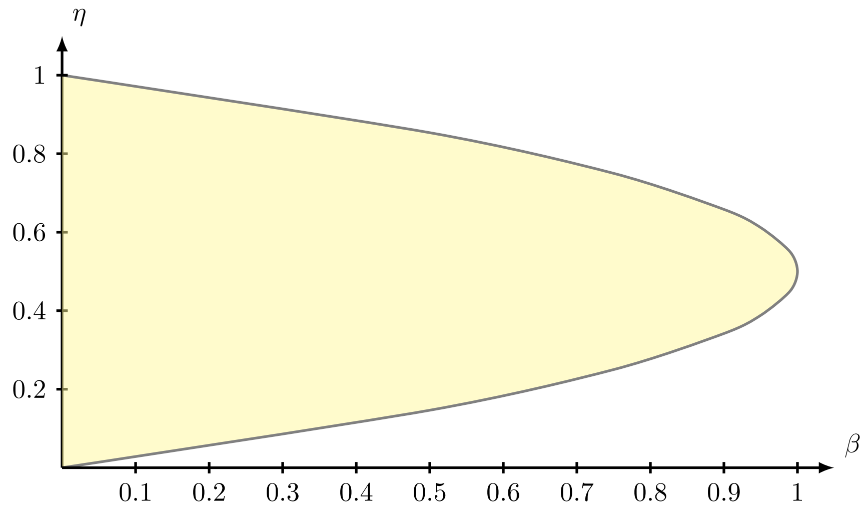

Figure 5: Upper and lower bounds of the absorption efficiency η as a function of β. Every antenna falls in the yellow domain.

and we conclude that the absorption efficiency η is bounded by the curves $\left(1 \pm \sqrt{1-\beta}\right)/2$ as a function of β, see Figure 5.

Example 7.2
In Example 7.1, we concluded that the absorbed power and the scattered power of an elementary dipole with a conjugate matched load were equal, *i.e.*, the absorption efficiency $\eta = 1/2$. The vertical dipole is located along the horizontal line between (0,0.5) and (1,0.5), depending on the absorption loss factor $p_{\mathrm{a}} \in [0, 1]$ ($p_{\mathrm{L}} = 1$ since the load is conjugate matched). ■

8 Friis’ transmission formula

We are now ready to apply the results in the previous sections to a transmission problem. One transmitting antenna (Tx) and one receiving antenna (Rx) are separated by a distance R in free space, see Figure 6. The distance R is large, so the receiving antenna is in the far zone of the transmitting antenna, and no other obstacles than the receiving antenna interacts with the wave propagation generated by the transmitting antenna.[11]

The transmitting antenna delivers the power P_{in} to the antenna, see (2.2) on page 6

$$P_{\mathrm{in}} = \frac{1}{2Z}\left(1 - |S_{\mathrm{t}}|^2\right)\left|\alpha_+^{\mathrm{t}}\right|^2$$

At the receiving antenna the electric field, to leading order, is a spherical wave

$$\boldsymbol{E}(\boldsymbol{r}_{\mathrm{r}}) = \boldsymbol{F}^{\mathrm{t}}(\hat{\boldsymbol{R}})\frac{\mathrm{e}^{\mathrm{i}kR}}{R} = \boldsymbol{f}_{\mathrm{t}}(\hat{\boldsymbol{R}})\alpha_+^{\mathrm{t}}\frac{\mathrm{e}^{\mathrm{i}kR}}{R}$$

[11] We are talking about line of sight (LOS) connection between transmitting and receiving antennas.

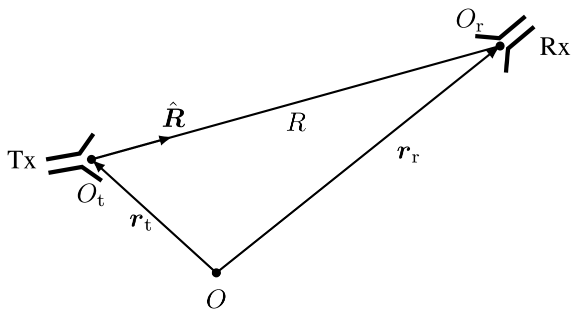

Figure 6: The geometry of the transmission problem in free space with local origins O_t and O_r.

and we can identify an incident direction $\hat{\boldsymbol{k}}_\mathrm{i} = \hat{\boldsymbol{R}}$ and an amplitude of the wave. The amplitude is

$$\boldsymbol{E}_0 = \boldsymbol{f}_\mathrm{t}(\hat{\boldsymbol{R}})\frac{\alpha_+^\mathrm{t}}{R}$$

The power received by the receiving antenna then is (load Γ_L), see (3.6) on page 8

$$P_\mathrm{r} = \frac{2\pi^2}{k^2 Z}\frac{1-|\Gamma_\mathrm{L}|^2}{|1-S_\mathrm{r}\Gamma_\mathrm{L}|^2}\left|\boldsymbol{s}_\mathrm{r}(\hat{\boldsymbol{R}})\cdot\boldsymbol{E}_0\right|^2 = \frac{2\pi^2}{k^2 Z R^2}\frac{1-|\Gamma_\mathrm{L}|^2}{|1-S_\mathrm{r}\Gamma_\mathrm{L}|^2}\left|\boldsymbol{s}_\mathrm{r}(\hat{\boldsymbol{R}})\cdot\boldsymbol{f}_\mathrm{t}(\hat{\boldsymbol{R}})\right|^2\left|\alpha_+^\mathrm{t}\right|^2$$

which for a reciprocal system is

$$P_\mathrm{r} = \frac{2\pi^2}{k^2 Z R^2}\frac{1-|\Gamma_\mathrm{L}|^2}{|1-S_\mathrm{r}\Gamma_\mathrm{L}|^2}\left|\boldsymbol{f}_\mathrm{r}(-\hat{\boldsymbol{R}})\cdot\boldsymbol{f}_\mathrm{t}(\hat{\boldsymbol{R}})\right|^2\left|\alpha_+^\mathrm{t}\right|^2$$

The gains of the transmitting and receiving antennas are, see (2.5) on page 7

$$G_\mathrm{t}(\hat{\boldsymbol{r}}) = \frac{4\pi\left|\boldsymbol{f}_\mathrm{t}(\hat{\boldsymbol{r}})\right|^2}{1-|S_\mathrm{t}|^2}, \qquad G_\mathrm{r}(\hat{\boldsymbol{r}}) = \frac{4\pi\left|\boldsymbol{f}_\mathrm{r}(\hat{\boldsymbol{r}})\right|^2}{1-|S_\mathrm{r}|^2}$$

The received power is now in terms of the gains and P_in

$$P_\mathrm{r} = \frac{pP_\mathrm{in}G_\mathrm{t}(\hat{\boldsymbol{R}})G_\mathrm{r}(-\hat{\boldsymbol{R}})}{4k^2R^2}\frac{\left(1-|\Gamma_\mathrm{L}|^2\right)\left(1-|S_\mathrm{r}|^2\right)}{|1-S_\mathrm{r}\Gamma_\mathrm{L}|^2} = \frac{pp_\mathrm{L}P_\mathrm{in}G_\mathrm{t}(\hat{\boldsymbol{R}})G_\mathrm{r}(-\hat{\boldsymbol{R}})}{4k^2R^2}$$

where we used (7.3) on page 30, the load mismatch factor of the receiving antenna, and we introduced the polarization loss factor (PLF) p or polarization-mismatch factor, which is a real number between 0 and 1, defined as

$$p = \frac{\left|\boldsymbol{f}_\mathrm{r}(-\hat{\boldsymbol{R}})\cdot\boldsymbol{f}_\mathrm{t}(\hat{\boldsymbol{R}})\right|^2}{\left|\boldsymbol{f}_\mathrm{r}(-\hat{\boldsymbol{R}})\right|^2\left|\boldsymbol{f}_\mathrm{t}(\hat{\boldsymbol{R}})\right|^2}$$

For a perfect polarization match this factor is 1, and for a total mismatch of the polarization, the factor is 0.

If the load at the receiving antenna is conjugate matched $\Gamma_{\mathrm{L}} = S_{\mathrm{r}}^*$ ($p_{\mathrm{L}} = 1$), we get

$$P_{\mathrm{r}} = \frac{pP_{\mathrm{in}}G_{\mathrm{t}}(\hat{\boldsymbol{R}})G_{\mathrm{r}}(-\hat{\boldsymbol{R}})}{4k^2R^2}$$

If polarization loss factor $p = 1$ (polarization match), then the result is the traditional Friis' formula [6, 25].

$$P_{\mathrm{r}} = \frac{P_{\mathrm{in}}G_{\mathrm{t}}(\hat{\boldsymbol{R}})G_{\mathrm{r}}(-\hat{\boldsymbol{R}})\lambda^2}{(4\pi R)^2}$$

where λ is the wavelength.

9 Power terms in the receive case — impedance representation

In the previous sections, we used the two-port description of the antenna of (1.5) on page 5, *i.e.*,

$$\begin{cases} \alpha_- = S\alpha_+ + \dfrac{2\pi\mathrm{i}}{k}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \\ \boldsymbol{F}(\hat{\boldsymbol{r}}) = \boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+ + \dfrac{2\pi\mathrm{i}}{k}\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \end{cases}$$

Another, equivalent description employs the current I and the voltage V as input parameters on the transmission line instead of the amplitude α_+ and α_-. We then have, see (1.6) on page 5

$$\begin{cases} V = Z_{\mathrm{in}}I + \dfrac{2\pi\mathrm{i}}{k}\boldsymbol{s}_{\mathrm{Z}}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \\ \boldsymbol{F}(\hat{\boldsymbol{r}}) = Z_{\mathrm{in}}\boldsymbol{f}_{\mathrm{Z}}(\hat{\boldsymbol{r}})I + \dfrac{2\pi\mathrm{i}}{k}\mathbf{t}_{\mathrm{Z}}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \end{cases} \tag{9.1}$$

where the current I and voltage V on the transmission line are defined as functions of the amplitudes $\alpha_\pm$ as, see (1.3) on page 3

$$\begin{cases} V = \alpha_+ + \alpha_- \\ I = (\alpha_+ - \alpha_-)/Z \end{cases}$$

with inverse

$$\begin{cases} \alpha_+ = (V + ZI)/2 \\ \alpha_- = (V - ZI)/2 \end{cases}$$

We now derive the connection between the two descriptions. In the first description, replace $\alpha_\pm$ with the voltage V and the current I. We get

$$\begin{cases} (V - ZI)/2 = S(V + ZI)/2 + \dfrac{2\pi\mathrm{i}}{k}\boldsymbol{s}(\hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \\ \boldsymbol{F}(\hat{\boldsymbol{r}}) = \boldsymbol{f}(\hat{\boldsymbol{r}})(V + ZI)/2 + \dfrac{2\pi\mathrm{i}}{k}\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_{\mathrm{i}}) \cdot \boldsymbol{E}_0 \end{cases}$$

We solve for V and rewrite these relations.

$$\begin{cases} V = \dfrac{1+S}{1-S} ZI + \dfrac{4\pi\mathrm{i}}{k(1-S)} \boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \\ \boldsymbol{F}(\hat{\boldsymbol{r}}) = \dfrac{\boldsymbol{f}(\hat{\boldsymbol{r}})}{1-S} ZI + \dfrac{2\pi\mathrm{i}}{k} \left(\dfrac{\boldsymbol{f}(\hat{\boldsymbol{r}}) \boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})}{1-S} + \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) \right) \cdot \boldsymbol{E}_0 \end{cases}$$

Comparison with the alternative description implies

$$\begin{cases} Z_\mathrm{in} = Z \dfrac{1+S}{1-S} \\ \boldsymbol{f}_Z(\hat{\boldsymbol{r}}) = \dfrac{\boldsymbol{f}(\hat{\boldsymbol{r}})}{1-S} \end{cases} \qquad \begin{cases} \boldsymbol{s}_Z(\hat{\boldsymbol{k}}_\mathrm{i}) = \dfrac{2\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})}{1-S} \\ \mathbf{t}_Z(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) = \dfrac{\boldsymbol{f}(\hat{\boldsymbol{r}}) \boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})}{1-S} + \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) \end{cases}$$

which gives the input impedance of the antenna, Z_in, the vectors $\boldsymbol{f}_Z(\hat{\boldsymbol{r}})$, $\boldsymbol{s}_Z(\hat{\boldsymbol{k}}_\mathrm{i})$, and the dyadic $\mathbf{t}_Z(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i})$ in terms of reflection coefficient, S, and the vectors $\boldsymbol{f}(\hat{\boldsymbol{r}})$, $\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})$, and the dyadic $\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i})$. The inverse of these connections is also useful. The inverse is[12]

$$\begin{cases} S = \dfrac{Z_\mathrm{in} - Z}{Z_\mathrm{in} + Z} \\ \boldsymbol{f}(\hat{\boldsymbol{r}}) = \dfrac{2Z \boldsymbol{f}_Z(\hat{\boldsymbol{r}})}{Z_\mathrm{in} + Z} \end{cases} \qquad \begin{cases} \boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) = \dfrac{Z \boldsymbol{s}_Z(\hat{\boldsymbol{k}}_\mathrm{i})}{Z_\mathrm{in} + Z} \\ \mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) = \mathbf{t}_Z(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) - \dfrac{Z \boldsymbol{f}_Z(\hat{\boldsymbol{r}}) \boldsymbol{s}_Z(\hat{\boldsymbol{k}}_\mathrm{i})}{Z_\mathrm{in} + Z} \end{cases}$$

With these transformation formulas, it is easy to transform the various expressions, we have derived above, between the old parameters S, $\boldsymbol{f}(\hat{\boldsymbol{r}})$, $\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})$, and $\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i})$ and the new parameters Z_in, $\boldsymbol{f}_Z(\hat{\boldsymbol{r}})$, $\boldsymbol{s}_Z(\hat{\boldsymbol{k}}_\mathrm{i})$, and $\mathbf{t}_Z(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i})$. Specifically, the absorbed power in the transmission line in (3.6) on page 8

$$P_\mathrm{a} = \frac{2\pi^2}{k^2 Z} \frac{1 - |\Gamma_\mathrm{L}|^2}{|1 - S\Gamma_\mathrm{L}|^2} \left| \boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \right|^2 = \frac{2\pi^2}{k^2 Z} \frac{|Z_\mathrm{L} + Z|^2 - |Z_\mathrm{L} - Z|^2}{|Z_\mathrm{L} + Z - S(Z_\mathrm{L} - Z)|^2} \left| \boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \right|^2$$

where we introduced the load impedance, Z_L, which is related to Γ_L by

$$\Gamma_\mathrm{L} = \frac{Z_\mathrm{L} - Z}{Z_\mathrm{L} + Z}, \qquad Z_\mathrm{L} = Z \frac{1 + \Gamma_\mathrm{L}}{1 - \Gamma_\mathrm{L}}$$

Simplify the expression. We get (Z is assumed real)

$$P_\mathrm{a} = \frac{2\pi^2}{k^2} \frac{4 \operatorname{Re} Z_\mathrm{L} \left| \boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \right|^2}{|Z_\mathrm{L} + Z - S(Z_\mathrm{L} - Z)|^2} = \frac{1}{2} \frac{Z^2 |1 + S|^2 \operatorname{Re} Z_\mathrm{L} \left| \frac{4\pi\mathrm{i}}{kZ} \frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0}{1+S} \right|^2}{|Z_\mathrm{L} + Z - S(Z_\mathrm{L} - Z)|^2}$$

[12]The mismatch between the antenna and the transmission line is usually quantified by the voltage standing wave ratio (VSWR) defined as

$$\text{VSWR} = \frac{1 + |S|}{1 - |S|}$$

A voltage standing wave ratio < 1.5 ($|S| < 1/5$) is usually considered acceptable. This corresponds to a $< 4\%$ power reflection.

Introduce the short-circuited current, see (3.10) on page 10

$$I^{\mathrm{r,sc}} = -\frac{4\pi\mathrm{i}}{kZ}\frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0}{1+S} = -\frac{2\pi\mathrm{i}}{k}\frac{\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0}{Z_\mathrm{in}}$$

and we obtain

$$P_\mathrm{a} = \frac{1}{2}\frac{Z^2\left|1+S\right|^2\operatorname{Re}Z_\mathrm{L}\left|I^{\mathrm{r,sc}}\right|^2}{\left|Z_\mathrm{L}+Z-S(Z_\mathrm{L}-Z)\right|^2}$$

and since

$$\frac{Z^2\left|1+S\right|^2}{\left|Z_\mathrm{L}+Z-S(Z_\mathrm{L}-Z)\right|^2} = \frac{Z^2\left|Z_\mathrm{in}+Z\right|^2\left|\frac{2Z_\mathrm{in}}{Z_\mathrm{in}+Z}\right|^2}{\left|(Z_\mathrm{in}+Z)(Z_\mathrm{L}+Z)-(Z_\mathrm{in}-Z)(Z_\mathrm{L}-Z)\right|^2} = \frac{\left|Z_\mathrm{in}\right|^2}{\left|Z_\mathrm{L}+Z_\mathrm{in}\right|^2}$$

we finally obtain

$$P_\mathrm{a} = \frac{1}{2}\operatorname{Re}Z_\mathrm{L}\left|I^{\mathrm{r,sc}}\right|^2\left|\frac{Z_\mathrm{in}}{Z_\mathrm{L}+Z_\mathrm{in}}\right|^2$$

and where the short-circuited current is, see (3.10) on page 10

$$I^{\mathrm{r,sc}} = -\frac{4\pi\mathrm{i}}{kZ}\frac{\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0}{1+S} = -\frac{2\pi\mathrm{i}}{k}\frac{\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0}{Z_\mathrm{in}}$$

Similarly, in the receiving mode, the current I in the transmission line is (use $V = -Z_\mathrm{L}I$, regarding the sign, see Footnote 8 on page 8)

$$I = -\frac{2\pi\mathrm{i}}{k}\frac{\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0}{Z_\mathrm{L}+Z_\mathrm{in}}$$

and the far field amplitude is

$$\boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{r}}) = \frac{2\pi\mathrm{i}}{k}\left\{-Z\frac{\boldsymbol{f}_\mathrm{Z}(\hat{\boldsymbol{r}})\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i})}{Z_\mathrm{L}+Z_\mathrm{in}} + \mathbf{t}_\mathrm{Z}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\right\}\cdot\boldsymbol{E}_0$$

which gives the following expressions for different loads, see Table 2:

$$\begin{cases}
\boldsymbol{F}^{\mathrm{r,oc}}(\hat{\boldsymbol{r}}) = \dfrac{2\pi\mathrm{i}}{k}\mathbf{t}_\mathrm{Z}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0 \\
\boldsymbol{F}^{\mathrm{r,sc}}(\hat{\boldsymbol{r}}) = \dfrac{2\pi\mathrm{i}}{k}\left\{-\dfrac{Z}{Z_\mathrm{in}}\boldsymbol{f}_\mathrm{Z}(\hat{\boldsymbol{r}})\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i}) + \mathbf{t}_\mathrm{Z}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\right\}\cdot\boldsymbol{E}_0 \\
\boldsymbol{F}^{\mathrm{r,m}}(\hat{\boldsymbol{r}}) = \dfrac{2\pi\mathrm{i}}{k}\left\{-Z\dfrac{\boldsymbol{f}_\mathrm{Z}(\hat{\boldsymbol{r}})\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i})}{Z+Z_\mathrm{in}} + \mathbf{t}_\mathrm{Z}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\right\}\cdot\boldsymbol{E}_0 \\
\boldsymbol{F}^{\mathrm{r,cm}}(\hat{\boldsymbol{r}}) = \dfrac{2\pi\mathrm{i}}{k}\left\{-Z\dfrac{\boldsymbol{f}_\mathrm{Z}(\hat{\boldsymbol{r}})\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i})}{Z_\mathrm{in}^*+Z_\mathrm{in}} + \mathbf{t}_\mathrm{Z}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\right\}\cdot\boldsymbol{E}_0
\end{cases}$$

With these expressions, we rewrite the far field amplitude as

$$\boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{r}}) = -\frac{2\pi\mathrm{i}}{k}Z\frac{\boldsymbol{f}_\mathrm{Z}(\hat{\boldsymbol{r}})\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0}{Z_\mathrm{L}+Z_\mathrm{in}} + \boldsymbol{F}^{\mathrm{r,oc}}(\hat{\boldsymbol{r}}) = -\frac{ZV^{\mathrm{r,oc}}}{Z_\mathrm{L}+Z_\mathrm{in}}\boldsymbol{f}_\mathrm{Z}(\hat{\boldsymbol{r}}) + \boldsymbol{F}^{\mathrm{r,oc}}(\hat{\boldsymbol{r}})$$

Load	**Load impedance**
Open-circuit	$Z_\mathrm{L} = \infty$
Short-circuit	$Z_\mathrm{L} = 0$
Matched	$Z_\mathrm{L} = Z$
Conjugate matched	$Z_\mathrm{L} = Z_\mathrm{in}^*$

Table 2: The four canonical impedance loads for the antenna.

where $V^{\mathrm{r,oc}} = 2\pi\mathrm{i}\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0/k$ is the open circuit voltage. Alternatively, we also have

$$\boldsymbol{F}^\mathrm{r}(\hat{\boldsymbol{r}}) = \frac{2\pi\mathrm{i}}{k}\frac{Z}{Z_\mathrm{in}}Z_\mathrm{L}\frac{\boldsymbol{f}_\mathrm{Z}(\hat{\boldsymbol{r}})\boldsymbol{s}_\mathrm{Z}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0}{Z_\mathrm{L} + Z_\mathrm{in}} + \boldsymbol{F}^{\mathrm{r,sc}}(\hat{\boldsymbol{r}}) = -\frac{ZZ_\mathrm{L}I^{\mathrm{r,sc}}}{Z_\mathrm{L} + Z_\mathrm{in}}\boldsymbol{f}_\mathrm{Z}(\hat{\boldsymbol{r}}) + \boldsymbol{F}^{\mathrm{r,sc}}(\hat{\boldsymbol{r}})$$

Finally, the reciprocity relations in (5.6) on page 19 in the new quantities become

$$\begin{cases} 2\boldsymbol{f}_Z(\hat{\boldsymbol{r}}) = \boldsymbol{s}_Z(-\hat{\boldsymbol{r}}) \\ \mathbf{t}_Z(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) = \mathbf{t}_Z^t(-\hat{\boldsymbol{k}}_\mathrm{i}, -\hat{\boldsymbol{r}}) \end{cases}$$

Appendix: The formal characterization of an antenna

The purpose of this appendix is to verify the form of the expression (1.5) on page 5, *i.e.*,

$$\begin{cases} \alpha_- = S\alpha_+ + \dfrac{2\pi \mathrm{i}}{k}\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \\ \boldsymbol{F}(\hat{\boldsymbol{r}}) = \boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+ + \dfrac{2\pi \mathrm{i}}{k}\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0 \end{cases}$$

These expressions can be obtained by general linear system arguments. Such arguments lead to this general form, but it lacks a means to compute the four components S, $\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})$, $\boldsymbol{f}(\hat{\boldsymbol{r}})$, and $\mathbf{t}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_\mathrm{i})$, that characterize the antenna. In this appendix, we pursue in a more constructive way, even though no explicit computations are made.

The antenna consists of two parts — one part characterized by the reference surface S_p, and a second part, which is the boundary surface of the rest of the antenna S_a, see Figure 1. The entire, closed surface of the antenna is $S_\mathrm{a} \cup S_\mathrm{p}$. For simplicity, we assume the antenna is metallic, *i.e.*, S_a is a PEC surface, but generalizations with non-metallic antennas can be made. However, non-metallic antennas make the analysis less transparent.

A The fields in the port S_p

In the reference plane, the electric and the magnetic fields have a prescribed form. There are several ways to realize this condition — the most common way is by a coaxial cable or transmission line, which is adopted in the supplement. Common to all these realizations is that the transverse electric and the magnetic fields at S_p have the form, see (1.1) on page 3

$$\begin{cases} \boldsymbol{E}(\boldsymbol{r}) = (\alpha_+ + \alpha_-)\,\boldsymbol{E}_m(\boldsymbol{r}) \\ Z\boldsymbol{H}(\boldsymbol{r}) = (\alpha_+ - \alpha_-)\,\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m(\boldsymbol{r}) \end{cases} \quad \boldsymbol{r} \in S_\mathrm{p} \tag{A.1}$$

where the function $\boldsymbol{E}_m$ is a single TEM-mode, and Z is the wave impedance of the medium exterior to the antenna, which is a real number. The dimension of the components of the mode is m^{-1}. The assumption of a real-valued impedance guarantees that no energy is lost as the wave propagates in the transmission line. The exact form of the function, $\boldsymbol{E}_m$, is irrelevant for the analysis below, and it is enough to specify that the fields in the port have the particular form in (A.1) and that $\boldsymbol{E}_m \cdot \hat{\boldsymbol{\nu}} = 0$, *i.e.*, the electric mode $\boldsymbol{E}_m$ is tangential to the reference plane S_p. As shown above, the amplitude α_+ quantifies the power propagating in the positive $\hat{\boldsymbol{\nu}}$ direction, while α_- quantifies the power propagating in the negative $\hat{\boldsymbol{\nu}}$ direction. Moreover, the integral over the magnitude of the field $\boldsymbol{E}_m$ is normalized to unity, see (1.2) on page 3, *i.e.*,

$$\iint_{S_\mathrm{p}} |\boldsymbol{E}_m|^2 \,\mathrm{d}S = \iint_{S_\mathrm{p}} (\boldsymbol{E}_m \times (\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)) \cdot \hat{\boldsymbol{\nu}} \,\mathrm{d}S = \iint_{S_\mathrm{p}} |\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m|^2 \,\mathrm{d}S = 1$$

B The exterior Calderón operator

We now introduce an important operator, the exterior Calderón operator C^e, which is instrumental in the analysis presented in this appendix.

For the entire antenna structure, $S_\mathrm{a} \cup S_\mathrm{p}$, the exterior problem — satisfying the radiation condition far away from the antenna — is uniquely solved, if we specify the tangential electric field, $\boldsymbol{E}$, on $S_\mathrm{a} \cup S_\mathrm{p}$ [18, 19]. In particular, the tangential magnetic field $\hat{\boldsymbol{\nu}} \times \boldsymbol{H}$ on $S_\mathrm{a} \cup S_\mathrm{p}$ is determined. The mapping from the tangential electric field on $S_\mathrm{a} \cup S_\mathrm{p}$ to the tangential magnetic field on $S_\mathrm{a} \cup S_\mathrm{p}$ defines the exterior Calderón operator C^e, see also Comment B.1, *i.e.*,

$$\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r}) = C^\mathrm{e}(\hat{\boldsymbol{\nu}} \times \boldsymbol{E})(\boldsymbol{r}), \quad \boldsymbol{r} \in S_\mathrm{a} \cup S_\mathrm{p}$$

The exterior Calderón operator is realized numerically by *e.g.*, the admittance matrix in a Method of Moments (MoM) implementation of the antenna geometry.

Comment B.1

Let V be a finite domain with bounding surface S (outward unit normal vector $\hat{\boldsymbol{\nu}}$), see Figure 3. The domain exterior to V is denoted V_e. Consider the following exterior problem where the trace (limit value) of the scattered electric field on the boundary is given by a fixed vector $\boldsymbol{m} = \hat{\boldsymbol{\nu}} \times \boldsymbol{E}$,

$$1) \begin{cases} \nabla \times \boldsymbol{E}(\boldsymbol{r}) = \mathrm{i}k\boldsymbol{H}(\boldsymbol{r}) \\ \nabla \times \boldsymbol{H}(\boldsymbol{r}) = -\mathrm{i}k\boldsymbol{E}(\boldsymbol{r}) \end{cases} \quad \boldsymbol{r} \in V_\mathrm{e}$$

$$2) \begin{cases} \hat{\boldsymbol{r}} \times \boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{H}(\boldsymbol{r}) = o(1/r) \\ \text{or} \\ \hat{\boldsymbol{r}} \times \boldsymbol{H}(\boldsymbol{r}) + \boldsymbol{E}(\boldsymbol{r}) = o(1/r) \end{cases} \quad \text{as } r \to \infty$$

where $r = |\boldsymbol{r}|$. This problem has a unique solution, which determines the trace of the magnetic field on S. For more details on this problem and the definition of the appropriate Sobolev space of the solution, we refer to the literature, *e.g.*, [3, 15]. This mapping defines the exterior Calderón operator $C^\mathrm{e} : \boldsymbol{m} \mapsto \hat{\boldsymbol{\nu}} \times \boldsymbol{H}$ on S uniquely.

The exterior Calderón operator has the following properties that are useful in many applications [3]:

1. $$\operatorname{Re} \iint_S C^\mathrm{e}(\boldsymbol{m}) \cdot (\hat{\boldsymbol{\nu}} \times \boldsymbol{m}^*) \,\mathrm{d}S \geq 0 \qquad \text{for all } \boldsymbol{m}$$
   where $\mathrm{d}S$ denotes the surface measure of S, and the star denotes the complex conjugate.
2. The exterior Calderón operator satisfies
   $$(C^\mathrm{e})^2 = -I,$$
   where I is the identity operator.

3. The exterior Calderón operator is a bounded invertible linear map in the space $H^{-1/2}(\mathrm{div}, S)$ [3, 15, 19], and consequently there exist constants $0 < c_{\mathrm{C}} \leq C_{\mathrm{C}}$, such that

$$c_{\mathrm{C}} \|\boldsymbol{m}\|_{H^{-1/2}(\mathrm{div},S)} \leq \|C^{\mathrm{e}}(\boldsymbol{m})\|_{H^{-1/2}(\mathrm{div},S)} \leq C_{\mathrm{C}} \|\boldsymbol{m}\|_{H^{-1/2}(\mathrm{div},S)}$$

4. The exterior Calderón operator is independent of the material properties of the volume inside the surface S. The operator is completely specified by the geometry of the antenna.

C Transmitting antenna

The purpose of the transmitting antenna is to generate electromagnetic energy that radiates into the volume outside the antenna. The main difference between a passive scatterer, such as the radome, is the presence of a port, which is manifested by the reference plane S_{p} in Figure 1. The antenna consists of two disjoint parts — the port S_{p} and the specific antenna S_{a}, which we assume is metallic.

C.1 The reflection coefficient

In the transmitting mode, the only excitation of the antenna is specified by the coefficient α_+, and there are two unknowns, the reflected field in the port — quantified by the coefficient α_- — and the electric field $\boldsymbol{E}$ outside the antenna. We now formally solve for these unknown in the known coefficient α_+.

The electric field is decomposed in two disjoint parts, one over S_{p} and one over S_{a}, $\boldsymbol{E}_m$ and $\boldsymbol{E}_{\mathrm{a}}$, respectively, see (A.1), *i.e.*,

$$\begin{aligned}\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}(\boldsymbol{r}) &= \chi_{S_{\mathrm{p}}}(\boldsymbol{r})(\alpha_+ + \alpha_-)\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_m(\boldsymbol{r}) + \chi_{S_{\mathrm{a}}}(\boldsymbol{r})\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_{\mathrm{a}}(\boldsymbol{r}) \\ &= \chi_{S_{\mathrm{p}}}(\boldsymbol{r})(\alpha_+ + \alpha_-)\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_m(\boldsymbol{r}), \quad \boldsymbol{r} \in S_{\mathrm{a}} \cup S_{\mathrm{p}} \end{aligned} \tag{C.1}$$

where χ_S is the characteristic function of the surface S, *i.e.*, $\chi_S(\boldsymbol{r}) = 1$ if $\boldsymbol{r} \in S$, and zero otherwise. We have here used $\chi_{S_{\mathrm{a}}}\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_{\mathrm{a}} = \boldsymbol{0}$, due to the assumption of a metallic antenna. Moreover, the magnetic field is

$$\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r}) = \chi_{S_{\mathrm{p}}}(\boldsymbol{r})\frac{-\alpha_+ + \alpha_-}{Z}\boldsymbol{E}_m(\boldsymbol{r}) + \chi_{S_{\mathrm{a}}}(\boldsymbol{r})\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}_{\mathrm{a}}(\boldsymbol{r}), \quad \boldsymbol{r} \in S_{\mathrm{a}} \cup S_{\mathrm{p}}$$

Due to linearity of the exterior Calderón operator, we obtain

$$\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r}) = (\alpha_+ + \alpha_-)C^{\mathrm{e}}(\chi_{S_{\mathrm{p}}}\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)(\boldsymbol{r}), \quad \boldsymbol{r} \in S_{\mathrm{a}} \cup S_{\mathrm{p}} \tag{C.2}$$

In (C.2), restrict the argument to $\boldsymbol{r} \in S_{\mathrm{p}}$, and we get

$$\frac{-\alpha_+ + \alpha_-}{Z}\boldsymbol{E}_m = \hat{\boldsymbol{\nu}} \times \boldsymbol{H} = (\alpha_+ + \alpha_-)\, C^{\mathrm{e}}(\chi_{S_{\mathrm{p}}}\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)\big|_{S_{\mathrm{p}}}, \quad \boldsymbol{r} \in S_{\mathrm{p}} \tag{C.3}$$

which proves that $\boldsymbol{E}_m(\boldsymbol{r})$ and $C^{\mathrm{e}}(\chi_{S_{\mathrm{p}}}\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)(\boldsymbol{r})$ are linearly dependent when $\boldsymbol{r} \in S_{\mathrm{p}}$. Note that the function $C^{\mathrm{e}}(\chi_{S_{\mathrm{p}}}\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)(\boldsymbol{r})$ contains information of the entire antenna geometry as well as the port configuration.

Take the scalar product of (C.3) with $\boldsymbol{E}_m^*$ and integrate over S_p. We get (use the normalization of $\boldsymbol{E}_m$)

$$\frac{-\alpha_+ + \alpha_-}{Z} = (\alpha_+ + \alpha_-)\iint_{S_\mathrm{p}} C^\mathrm{e}(\chi_{S_\mathrm{p}}\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)\cdot \boldsymbol{E}_m^*\ \mathrm{d}S$$

and solve for α_-. The result is

$$\alpha_- = \frac{1+N_m}{1-N_m}\alpha_+ = S\alpha_+ \tag{C.4}$$

where

$$S = \frac{1+N_m}{1-N_m}$$

and

$$N_m = Z\iint_{S_\mathrm{p}} C^\mathrm{e}(\chi_{S_\mathrm{p}}\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)\cdot \boldsymbol{E}_m^*\ \mathrm{d}S$$

Notice that the reflection coefficient S is completely characterized by the geometry of the antenna and the feeding transmission line. In particular, S is a real number if N_m is real. A real-valued reflection coefficient S is the criterion for resonance. The antenna is matched to the transmission line if $S = 0$, *i.e.*, $N_m = -1$.

For $\boldsymbol{m} = \chi_{S_\mathrm{p}}\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m$, Item 1 in Comment B.1 implies

$$\operatorname{Re} N_m = -Z \operatorname{Re}\iint_{S_\mathrm{p}} C^\mathrm{e}(\boldsymbol{m})\cdot(\hat{\boldsymbol{\nu}} \times \boldsymbol{m}^*)\ \mathrm{d}S \le 0$$

since $\hat{\boldsymbol{\nu}} \times \boldsymbol{m} = \hat{\boldsymbol{\nu}} \times (\chi_{S_\mathrm{p}}\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m) = -\chi_{S_\mathrm{p}}\boldsymbol{E}_m$. This inequality then implies that $|S| \le 1$. In fact,

$$1 - \left|\frac{1+N_m}{1-N_m}\right|^2 = -4\frac{\operatorname{Re} N_m}{|1-N_m|^2} \ge 0$$

This result is related to preservation of energy.

C.2 The far field amplitude

The far field amplitude $\boldsymbol{F}(\boldsymbol{r})$ generated by the antenna is expressed by the integral [17]

$$\boldsymbol{F}(\hat{\boldsymbol{r}}) = \frac{\mathrm{i}k}{4\pi}\hat{\boldsymbol{r}} \times \iint_{S_\mathrm{a}\cup S_\mathrm{p}} \Big[\hat{\boldsymbol{\nu}}(\boldsymbol{r}') \times \boldsymbol{E}(\boldsymbol{r}') - Z\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{\nu}}(\boldsymbol{r}') \times \boldsymbol{H}(\boldsymbol{r}'))\Big]\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\ \mathrm{d}S'$$

or if the tangential magnetic field in (C.2) and the tangential electric field in (C.1) on the antenna are inserted

$$\begin{aligned}\boldsymbol{F}(\hat{\boldsymbol{r}}) = \alpha_+\frac{\mathrm{i}k}{4\pi}\hat{\boldsymbol{r}} \times \left\{(1+S)\iint_{S_\mathrm{p}} \hat{\boldsymbol{\nu}}(\boldsymbol{r}') \times \boldsymbol{E}_m(\boldsymbol{r}')\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\ \mathrm{d}S'\right\}\\ - \alpha_+\frac{\mathrm{i}kZ}{4\pi}(1+S)\hat{\boldsymbol{r}} \times \left\{\hat{\boldsymbol{r}} \times \iint_{S_\mathrm{a}\cup S_\mathrm{p}} C^\mathrm{e}(\chi_{S_\mathrm{p}}\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)(\boldsymbol{r}')\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\ \mathrm{d}S'\right\}\end{aligned}$$

where we also used $\alpha_- = S\alpha_+$. Now use (C.3), and we get

$$\begin{aligned}\boldsymbol{F}(\hat{\boldsymbol{r}}) = -\alpha_+\frac{\mathrm{i}kZ}{4\pi}(1+S)\hat{\boldsymbol{r}}\times\left\{\hat{\boldsymbol{r}}\times\iint\limits_{S_\mathrm{a}} C^\mathrm{e}(\chi_{S_\mathrm{p}}\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_m)(\boldsymbol{r}')\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\,\mathrm{d}S'\right\}\\ +\alpha_+\frac{\mathrm{i}k}{4\pi}\hat{\boldsymbol{r}}\times\left\{(1+S)\iint\limits_{S_\mathrm{p}}\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\boldsymbol{E}_m(\boldsymbol{r}')\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\right\}\\ +\alpha_+\frac{\mathrm{i}k}{4\pi}(1-S)\hat{\boldsymbol{r}}\times\left\{\hat{\boldsymbol{r}}\times\iint\limits_{S_\mathrm{p}}\boldsymbol{E}_m(\boldsymbol{r}')\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\,\mathrm{d}S'\right\} = \boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+\end{aligned}\tag{C.5}$$

where

$$\begin{aligned}\boldsymbol{f}(\hat{\boldsymbol{r}}) = -\frac{\mathrm{i}kZ}{4\pi}(1+S)\hat{\boldsymbol{r}}\times\left\{\hat{\boldsymbol{r}}\times\iint\limits_{S_\mathrm{a}} C^\mathrm{e}(\chi_{S_\mathrm{p}}\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_m)(\boldsymbol{r}')\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\,\mathrm{d}S'\right\}\\ +\frac{\mathrm{i}k}{4\pi}\hat{\boldsymbol{r}}\times\left\{(1+S)\iint\limits_{S_\mathrm{p}}\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\boldsymbol{E}_m(\boldsymbol{r}')\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\right\}\\ +\frac{\mathrm{i}k}{4\pi}(1-S)\hat{\boldsymbol{r}}\times\left\{\hat{\boldsymbol{r}}\times\iint\limits_{S_\mathrm{p}}\boldsymbol{E}_m(\boldsymbol{r}')\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\,\mathrm{d}S'\right\}\end{aligned}\tag{C.6}$$

Finally, we can conclude that the transmitting antenna is characterized by, see (C.4) and (C.5)

$$\begin{cases}\alpha_- = S\alpha_+\\ \boldsymbol{F}(\hat{\boldsymbol{r}}) = \boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+\end{cases}$$

where the complex constant (reflection) coefficient S and the complex-valued vector $\boldsymbol{f}(\hat{\boldsymbol{r}})$ are quantities that characterize the antenna. These relations quantify the transmitting antenna, and is identical to (1.5) on page 5 with $\boldsymbol{E}_0 = \boldsymbol{0}$.

D Receiving antenna

If the antenna is used as a receiving antenna, the antenna is illuminated by an electromagnetic field, which now acts as the source of the problem. We assume the sources of this electromagnetic field is far away from the antenna, and locally at the antenna, it is a plane wave with a general expression

$$\boldsymbol{E}_\mathrm{i}(\boldsymbol{r}) = \boldsymbol{E}_0\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}$$

The field $\boldsymbol{E}_0$ is the amplitude of this excitation at the origin, which is assumed to be located in the vicinity of the antenna. The total electric field consists of the sum of two parts — the incident plane wave, $\boldsymbol{E}_\mathrm{i}$ and the scattered field $\boldsymbol{E}_\mathrm{s}$. The scattered field satisfies the radiation condition at infinity, and consequently we can apply the exterior Calderón operator to this field to obtain the magnetic counterpart.

D.1 The reflection coefficient

As in the transmitting case, the tangential fields in the port are

$$\begin{cases} \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}(\boldsymbol{r}) = \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_{\rm i}(\boldsymbol{r}) + \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_{\rm s}(\boldsymbol{r}) = (\alpha_+ + \alpha_-)\, \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_m(\boldsymbol{r}) \\ \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r}) = \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}_{\rm i}(\boldsymbol{r}) + \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}_{\rm s}(\boldsymbol{r}) = \dfrac{-\alpha_+ + \alpha_-}{Z} \boldsymbol{E}_m(\boldsymbol{r}) \end{cases} \qquad \boldsymbol{r} \in S_{\rm p}$$

Again, if the antenna structure is perfectly conducting, the tangential fields on the antenna are

$$\begin{cases} \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}(\boldsymbol{r}) = \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_{\rm i}(\boldsymbol{r}) + \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_{\rm s}(\boldsymbol{r}) = \boldsymbol{0} \\ \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r}) = \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}_{\rm i}(\boldsymbol{r}) + \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}_{\rm s}(\boldsymbol{r}) \\ \qquad\qquad\quad = \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}_{\rm i}(\boldsymbol{r}) + C^{\rm e}(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_{\rm s})(\boldsymbol{r}) \end{cases} \qquad \boldsymbol{r} \in S_{\rm a}$$

Note that the argument of the exterior Calderón operator is the tangential scattered electric field. On the entire surface $S_{\rm a} \cup S_{\rm p}$, the tangential scattered electric field is

$$\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_{\rm s}(\boldsymbol{r}) = -\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_{\rm i}(\boldsymbol{r}) + \chi_{S_{\rm p}}(\boldsymbol{r})\, (\alpha_+ + \alpha_-)\, \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_m(\boldsymbol{r}), \quad \boldsymbol{r} \in S_{\rm a} \cup S_{\rm p} \tag{D.1}$$

Due to linearity of the exterior Calderón operator

$$\begin{aligned} \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}_{\rm s}(\boldsymbol{r}) = C^{\rm e}(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_{\rm s})(\boldsymbol{r}) &= -C^{\rm e}(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_{\rm i})(\boldsymbol{r}) \\ &+ (\alpha_+ + \alpha_-)\, C^{\rm e}(\chi_{S_{\rm p}} \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)(\boldsymbol{r}), \quad \boldsymbol{r} \in S_{\rm a} \cup S_{\rm p} \end{aligned} \tag{D.2}$$

In this relation for the scattered magnetic field, restrict the argument to the port, $\boldsymbol{r} \in S_{\rm p}$. We get

$$\begin{aligned} \frac{-\alpha_+ + \alpha_-}{Z} \boldsymbol{E}_m(\boldsymbol{r}) - \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}_{\rm i}(\boldsymbol{r})\big|_{S_{\rm p}} &= -\, C^{\rm e}(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_{\rm i})\big|_{S_{\rm p}} \\ &+ (\alpha_+ + \alpha_-)\, C^{\rm e}(\chi_{S_{\rm p}} \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)\big|_{S_{\rm p}}, \quad \boldsymbol{r} \in S_{\rm p} \end{aligned}$$

We insert the explicit form of the incident electric field $\boldsymbol{E}_{\rm i}(\boldsymbol{r}) = \boldsymbol{E}_0 {\rm e}^{{\rm i}k\hat{\boldsymbol{k}}_{\rm i} \cdot \boldsymbol{r}}$ and the incident magnetic field $Z\boldsymbol{H}_{\rm i}(\boldsymbol{r}) = \hat{\boldsymbol{k}}_{\rm i} \times \boldsymbol{E}_0 {\rm e}^{{\rm i}k\hat{\boldsymbol{k}}_{\rm i} \cdot \boldsymbol{r}}$

$$\begin{aligned} &(-\alpha_+ + \alpha_-)\, \boldsymbol{E}_m(\boldsymbol{r}) - \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \left(\hat{\boldsymbol{k}}_{\rm i} \times \boldsymbol{E}_0\right) {\rm e}^{{\rm i}k\hat{\boldsymbol{k}}_{\rm i} \cdot \boldsymbol{r}}\Big|_{S_{\rm p}} \\ &\quad = -Z\, C^{\rm e}(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_0 {\rm e}^{{\rm i}k\hat{\boldsymbol{k}}_{\rm i} \cdot \boldsymbol{r}})\Big|_{S_{\rm p}} + (\alpha_+ + \alpha_-)\, Z\, C^{\rm e}(\chi_{S_{\rm p}} \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)\big|_{S_{\rm p}}, \quad \boldsymbol{r} \in S_{\rm p} \end{aligned}$$

Take the scalar product with $\boldsymbol{E}_m^*$ in this equation and integrate over $S_{\rm p}$. We get (use the normalization of $\boldsymbol{E}_m$)

$$(-\alpha_+ + \alpha_-) - F_m = -G_m + (\alpha_+ + \alpha_-)\, N_m \tag{D.3}$$

where

$$\begin{aligned} F_m &= \iint_{S_{\rm p}} \boldsymbol{E}_m^*(\boldsymbol{r}) \cdot \left(\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \left(\hat{\boldsymbol{k}}_{\rm i} \times \boldsymbol{E}_0\right)\right) {\rm e}^{{\rm i}k\hat{\boldsymbol{k}}_{\rm i} \cdot \boldsymbol{r}}\, {\rm d}S \\ &= \iint_{S_{\rm p}} \left(\left(\boldsymbol{E}_m^*(\boldsymbol{r}) \cdot \hat{\boldsymbol{k}}_{\rm i}\right) \hat{\boldsymbol{\nu}}(\boldsymbol{r}) - \left(\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \cdot \hat{\boldsymbol{k}}_{\rm i}\right) \boldsymbol{E}_m^*(\boldsymbol{r})\right) {\rm e}^{{\rm i}k\hat{\boldsymbol{k}}_{\rm i} \cdot \boldsymbol{r}}\, {\rm d}S \cdot \boldsymbol{E}_0 \end{aligned}$$

and

$$G_m = Z \iint\limits_{S_\mathrm{p}} \boldsymbol{E}_m^*(\boldsymbol{r}) \cdot C^\mathrm{e}\left(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}\right) \,\mathrm{d}S = Z \iint\limits_{S_\mathrm{p}} \boldsymbol{E}_m^*(\boldsymbol{r}) \cdot C^\mathrm{e}\left(\hat{\boldsymbol{\nu}} \times \mathbf{I}_3 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}\right) \,\mathrm{d}S \cdot \boldsymbol{E}_0$$

and, as above

$$N_m = Z \iint\limits_{S_\mathrm{p}} C^\mathrm{e}(\chi_{S_\mathrm{p}} \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m) \cdot \boldsymbol{E}_m^* \,\mathrm{d}S$$

The dyadic-valued operator $C^\mathrm{e}\left(\hat{\boldsymbol{\nu}} \times \mathbf{I}_3 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}\right)$ is defined as

$$C^\mathrm{e}\left(\hat{\boldsymbol{\nu}} \times \mathbf{I}_3 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}\right) = C^\mathrm{e}\left(\hat{\boldsymbol{\nu}} \times \hat{\boldsymbol{x}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}\right)\hat{\boldsymbol{x}} + C^\mathrm{e}\left(\hat{\boldsymbol{\nu}} \times \hat{\boldsymbol{y}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}\right)\hat{\boldsymbol{y}} + C^\mathrm{e}\left(\hat{\boldsymbol{\nu}} \times \hat{\boldsymbol{z}} \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}\right)\hat{\boldsymbol{z}}$$

In (D.3), solve for α_-. The result is

$$\alpha_- = \frac{1+N_m}{1-N_m}\alpha_+ + \frac{F_m - G_m}{1-N_m} = S\alpha_+ + \frac{2\pi\mathrm{i}}{k}\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) \cdot \boldsymbol{E}_0$$

where

$$S = \frac{1+N_m}{1-N_m}$$

and

$$\begin{aligned}\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i}) = \frac{k}{2\pi\mathrm{i}} \iint\limits_{S_\mathrm{p}} \left(\left(\boldsymbol{E}_m^*(\boldsymbol{r}) \cdot \hat{\boldsymbol{k}}_\mathrm{i}\right)\hat{\boldsymbol{\nu}}(\boldsymbol{r}) - \left(\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \cdot \hat{\boldsymbol{k}}_\mathrm{i}\right)\boldsymbol{E}_m^*(\boldsymbol{r})\right) \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}} \,\mathrm{d}S \\ - \frac{kZ}{2\pi\mathrm{i}} \iint\limits_{S_\mathrm{p}} \boldsymbol{E}_m^*(\boldsymbol{r}) \cdot C^\mathrm{e}\left(\hat{\boldsymbol{\nu}} \times \mathbf{I}_3 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}\right) \,\mathrm{d}S\end{aligned} \quad \text{(D.4)}$$

D.2 The far field amplitude

The receiving antenna also gives rise to a scattered field and a re-radiated field with an amplitude $\boldsymbol{F}(\hat{\boldsymbol{r}})$.

$$\boldsymbol{F}(\hat{\boldsymbol{r}}) = \frac{\mathrm{i}k}{4\pi}\hat{\boldsymbol{r}} \times \iint\limits_{S_\mathrm{a}\cup S_\mathrm{p}} \left[\hat{\boldsymbol{\nu}}(\boldsymbol{r}') \times \boldsymbol{E}(\boldsymbol{r}') - Z\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{\nu}}(\boldsymbol{r}') \times \boldsymbol{H}(\boldsymbol{r}'))\right] \mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'} \,\mathrm{d}S'$$

The fields are, see (D.1)

$$\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}(\boldsymbol{r}) = \chi_{S_\mathrm{p}}(\boldsymbol{r})\,(\alpha_+ + \alpha_-)\,\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{E}_m(\boldsymbol{r}), \quad \boldsymbol{r} \in S_\mathrm{a} \cup S_\mathrm{p}$$

and, see (D.2)

$$\begin{aligned}\hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r}) = \hat{\boldsymbol{\nu}}(\boldsymbol{r}) \times \boldsymbol{H}_\mathrm{i}(\boldsymbol{r}) - C^\mathrm{e}(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_\mathrm{i})(\boldsymbol{r}) \\ + (\alpha_+ + \alpha_-)\,C^\mathrm{e}(\chi_{S_\mathrm{p}}\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_m)(\boldsymbol{r}), \quad \boldsymbol{r} \in S_\mathrm{a} \cup S_\mathrm{p}\end{aligned}$$

In particular, in the port S_p

$$\hat{\boldsymbol{\nu}}(\boldsymbol{r})\times\boldsymbol{H}(\boldsymbol{r})=\frac{-\alpha_+ +\alpha_-}{Z}\boldsymbol{E}_m(\boldsymbol{r}),\quad \boldsymbol{r}\in S_\mathrm{p}$$

The explicit form of the incident electric field $\boldsymbol{E}_\mathrm{i}(\boldsymbol{r})=\boldsymbol{E}_0\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}$ and the incident magnetic field $Z\boldsymbol{H}_\mathrm{i}(\boldsymbol{r})=\hat{\boldsymbol{k}}_\mathrm{i}\times\boldsymbol{E}_0\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}$ implies

$$\begin{aligned}
\boldsymbol{F}(\hat{\boldsymbol{r}})&=\frac{\mathrm{i}k}{4\pi}(\alpha_+ +\alpha_-)\,\hat{\boldsymbol{r}}\times\iint_{S_\mathrm{p}}\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\boldsymbol{E}_m(\boldsymbol{r}')\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\,\mathrm{d}S'\\
&\quad-\frac{\mathrm{i}k}{4\pi}(-\alpha_+ +\alpha_-)\,\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\iint_{S_\mathrm{p}}\boldsymbol{E}_m(\boldsymbol{r}')\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\,\mathrm{d}S'\right)\\
&\quad-\frac{\mathrm{i}k}{4\pi}\hat{\boldsymbol{r}}\times\Bigg[\hat{\boldsymbol{r}}\times\iint_{S_\mathrm{a}}\Big(\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\Big(\hat{\boldsymbol{k}}_\mathrm{i}\times\boldsymbol{E}_0\Big)\,\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}'}-ZC^\mathrm{e}(\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_0\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}})(\boldsymbol{r}')\\
&\qquad+(\alpha_+ +\alpha_-)\,ZC^\mathrm{e}(\chi_{S_\mathrm{p}}\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_m)(\boldsymbol{r}')\Big)\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\,\mathrm{d}S'\Bigg]\\
&=\boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+ -\frac{\mathrm{i}k}{4\pi}\hat{\boldsymbol{r}}\times\Bigg[\hat{\boldsymbol{r}}\times\iint_{S_\mathrm{a}}\Big(\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\Big(\hat{\boldsymbol{k}}_\mathrm{i}\times\boldsymbol{E}_0\Big)\,\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}'}\\
&\qquad-ZC^\mathrm{e}(\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_0\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}})(\boldsymbol{r}')\,\mathrm{d}S'\Bigg]=\boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+ +\frac{2\pi\mathrm{i}}{k}\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0
\end{aligned}$$

where $\boldsymbol{f}(\hat{\boldsymbol{r}})$ is defined in (C.6) and where

$$\begin{aligned}
\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})=-\frac{\mathrm{i}k}{4\pi}\hat{\boldsymbol{r}}\times\Bigg[\hat{\boldsymbol{r}}\times\iint_{S_\mathrm{a}}\Big(\hat{\boldsymbol{k}}_\mathrm{i}\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}'}-\Big(\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\cdot\hat{\boldsymbol{k}}_\mathrm{i}\Big)\,\mathbf{I}_3\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}'}\\
-ZC^\mathrm{e}(\hat{\boldsymbol{\nu}}\times\mathbf{I}_3\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}})(\boldsymbol{r}')\Big)\mathrm{e}^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}\,\mathrm{d}S'\Bigg]
\end{aligned}\tag{D.5}$$

We collect the results for the receiving antenna, see (D.4) and (D.5)

$$\begin{cases}\alpha_- =S\alpha_+ +\dfrac{2\pi\mathrm{i}}{k}\boldsymbol{s}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\\ \boldsymbol{F}(\hat{\boldsymbol{r}})=\boldsymbol{f}(\hat{\boldsymbol{r}})\alpha_+ +\dfrac{2\pi\mathrm{i}}{k}\mathbf{t}(\hat{\boldsymbol{r}},\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0\end{cases}$$

These relations quantify the receiving antenna and they are identical to the relations given in (1.5) on page 5.

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Index