Detection Theory

Detection of random (Gaussian) signals in Gaussian noise

Chapter 5: Rationale

$$\begin{array}{llr} \mathcal{H}_0 : & x[n] = w[n] & n = 0, 1, \ldots, N-1 \\ \mathcal{H}_1 : & x[n] = s[n] + w[n] & n = 0, 1, \ldots, N-1 \end{array}$$

If signal is s[n] = 1 we have $T(\mathbf{x}) = (1/N) \sum_{n=0}^{N-1} x[n]$

Chapter 5: Rationale

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so that $\mathbf{x} \sim \begin{cases} \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mathbf{0}, \mathbf{I} + \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_1 \end{cases}$

Chapter 5: Rationale

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Is $T(\mathbf{x}) = (1/N) \sum_{n=0}^{N-1} x[n]$

still a good choice?

Chapter 5: Rationale

With $T(\mathbf{x}) = (1/N)\sum_{n=0}^{N-1} x[n]$

We have $$T(\mathbf{x}) \sim \begin{cases} \mathcal{N}(0, \sigma^2/N) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(0, \sigma^2/N + 1/N) & \text{under } \mathcal{H}_1 \end{cases}$$

Chapter 5: Rationale

With $T(\mathbf{x}) = \sqrt{(1/N)}\sum_{n=0}^{N-1} x[n]$

Equivalently

We have $T(\mathbf{x}) \sim \begin{cases} \mathcal{N}(0, \sigma^2) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(0, \sigma^2 + 1) & \text{under } \mathcal{H}_1 \end{cases}$

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Thus, $\boxed{T(\mathbf{x}) = (1/N) \sum_{n=0}^{N-1} x[n]}$ gives no gain of having multiple observations

Correlator (or equivalently, matched filtering) does not work with random signals

Chapter 5: Overview

In its most general setting, Chapter 5 deals with the case

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{x} = \mathbf{w} \\ \mathcal{H}_1 &: \mathbf{x} = \mathbf{s} + \mathbf{w} \end{aligned} \qquad \mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w), \mathbf{s} \sim \mathcal{N}(\boldsymbol{\mu}_s, \mathbf{C}_s)$$

- **Signal comprises a deterministic and a random part**
- **Noise has equal statistical properties in both cases**

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With that, we get

$$\begin{aligned}\mathcal{H}_0 &: \mathbf{x} = \mathbf{w}_1 \\ \mathcal{H}_1 &: \mathbf{x} = \mu_s + \mathbf{w}_2 \\ \mathbf{w}_1 &\sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w) \\ \mathbf{w}_2 &\sim \mathcal{N}(\mathbf{0}, \mathbf{C}_s + \mathbf{C}_w)\end{aligned}$$

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This is (almost) equivalent to deterministic signals (chapter 4) but with different noise distributions

(The difference is that the difference of the two covariances must be positive semi-definite)

Chapter 5: Estimator-Correlator

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We have $L(\mathbf{x}) = \dfrac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \gamma$

$$\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}) \text{ under } \mathcal{H}_0$$
$$\mathbf{x} \sim \mathcal{N}(\mathbf{0}, (\sigma_s^2 + \sigma^2) \mathbf{I}) \text{ under } \mathcal{H}_1$$

We start with zero-mean signals, and identity covariance matrices

Chapter 5: Estimator-Correlator

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$$L(\mathbf{x}) = \frac{\dfrac{1}{[2\pi(\sigma_s^2 + \sigma^2)]^{\frac{N}{2}}} \exp\left[-\dfrac{1}{2(\sigma_s^2 + \sigma^2)} \sum_{n=0}^{N-1} x^2[n]\right]}{\dfrac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\dfrac{1}{2\sigma^2} \sum_{n=0}^{N-1} x^2[n]\right]}$$

Chapter 5: Estimator-Correlator

Taking logs and straightforward manipulations give

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x^2[n] > \gamma'$$

For some data-independent γ'

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Chapter 5: Estimator-Correlator

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Scaling with 1/N shows that *T*(***x***) is a **variance estimator**

Chapter 5: Estimator-Correlator

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$T(\boldsymbol{x})$ is the sum of squared independent Gaussian variables, not with unit variance

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Standard distribution: If x[k] are N(0,1) variables, $\sum_{k=1}^{N} |x[k]|^2$ is a chi-2 variable with N DoFs

χ_N^2

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Chapter 5: Estimator-Correlator

Standard formulas….

$$P_{FA} =$$

$$=$$

$$P_D =$$

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Chapter 5: Estimator-Correlator

Standard formulas….

$$P_{FA} = \Pr\{T(\mathbf{x}) > \gamma'; \mathcal{H}_0\}$$

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Chapter 5: Estimator-Correlator

Standard formulas….

$$\begin{aligned} P_{FA} &= \Pr\{T(\mathbf{x}) > \gamma'; \mathcal{H}_0\} \\ &= \Pr\left\{\frac{T(\mathbf{x})}{\sigma^2} > \frac{\gamma'}{\sigma^2}; \mathcal{H}_0\right\} \end{aligned}$$

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Chapter 5: Estimator-Correlator

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Standard function for chi-2
Comparable to Q(x)

$$P_D = \Pr\{T(\mathbf{x}) > \gamma'; \mathcal{H}_1\}$$

under $\mathcal{H}_0$ $\quad E[x^2[n]] = \sigma^2$ $\quad$ Thus, $\dfrac{T(\mathbf{x})}{\sigma^2} \sim \chi^2_N$

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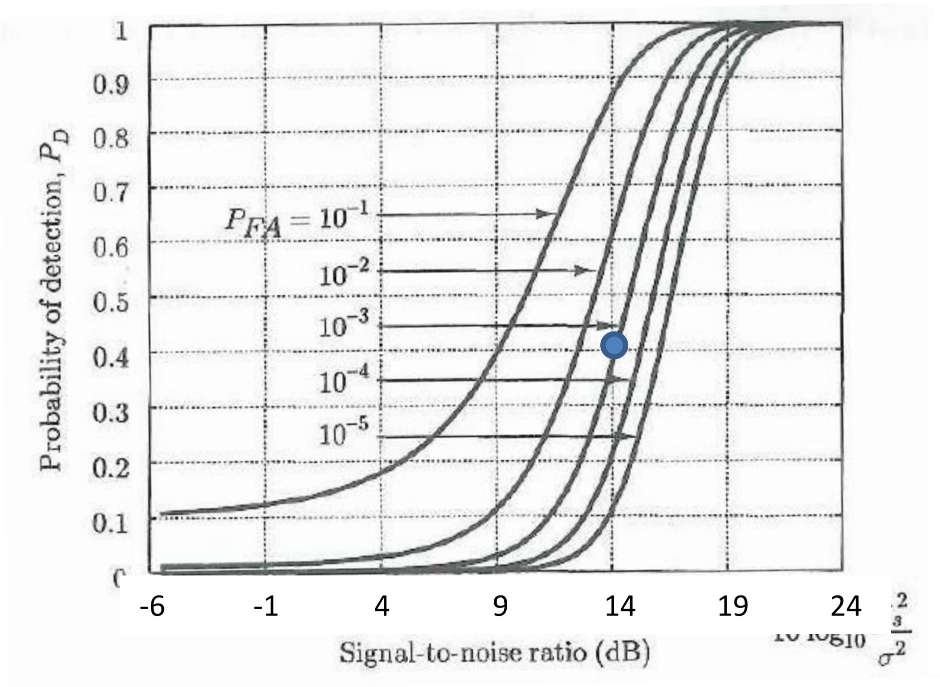
Standard function for chi-2
Comparable to Q(x)

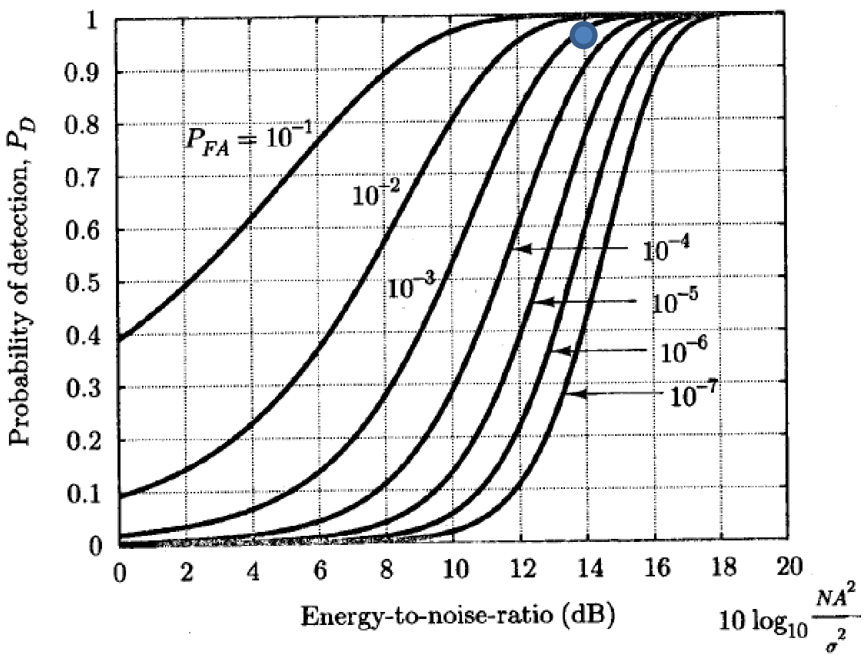
$$\begin{aligned} P_D &= \Pr\{T(\mathbf{x}) > \gamma'; \mathcal{H}_1\} \\ &= Q_{\chi^2_N}\left(\frac{\gamma'}{\sigma_s^2 + \sigma^2}\right) = Q_{\chi^2_N}\left(\frac{\gamma'/\sigma^2}{\sigma_s^2/\sigma^2 + 1}\right) \end{aligned}$$

under $\mathcal{H}_0$ $E[x^2[n]] = \sigma^2$ Thus, $\frac{T(\mathbf{x})}{\sigma^2} \sim \chi^2_N$

under $\mathcal{H}_1$ $E[x^2[n]] = {\sigma_s}^2 + \sigma^2$ Thus, $\frac{T(\mathbf{x})}{\sigma_s^2 + \sigma^2} \sim \chi^2_N$

Comparison, random vs. deterministic





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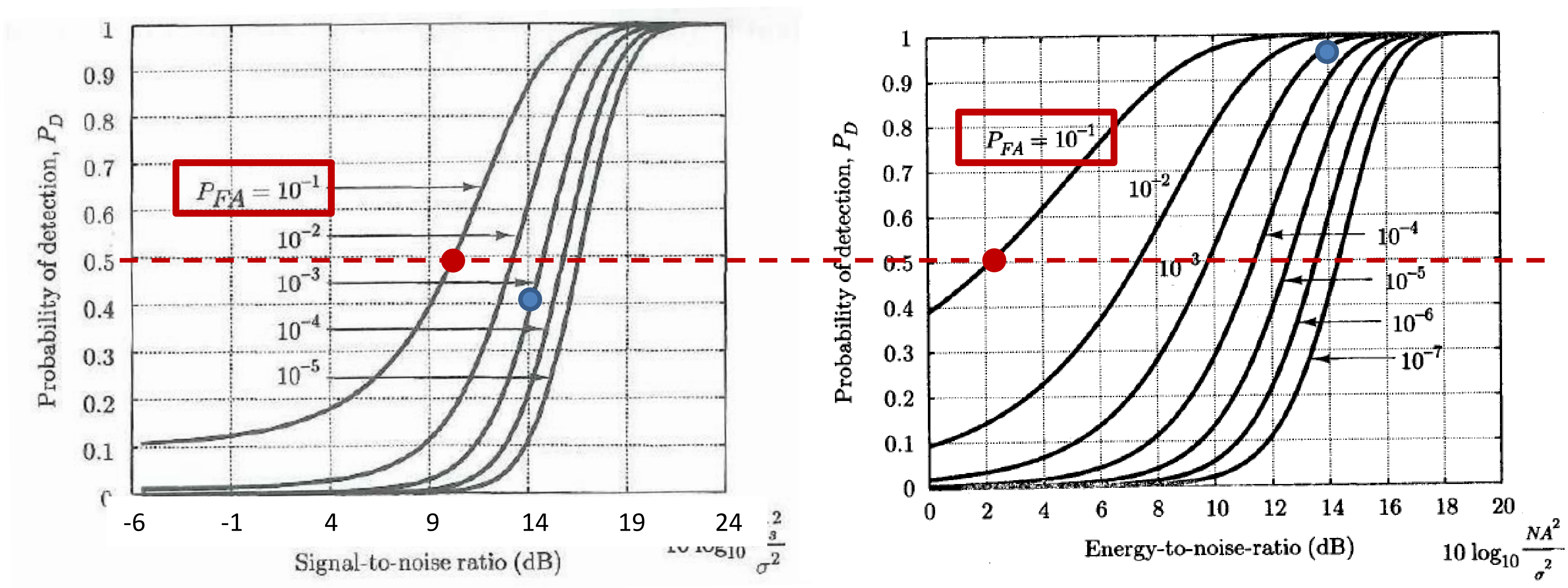

Chapter 5: Estimator-Correlator

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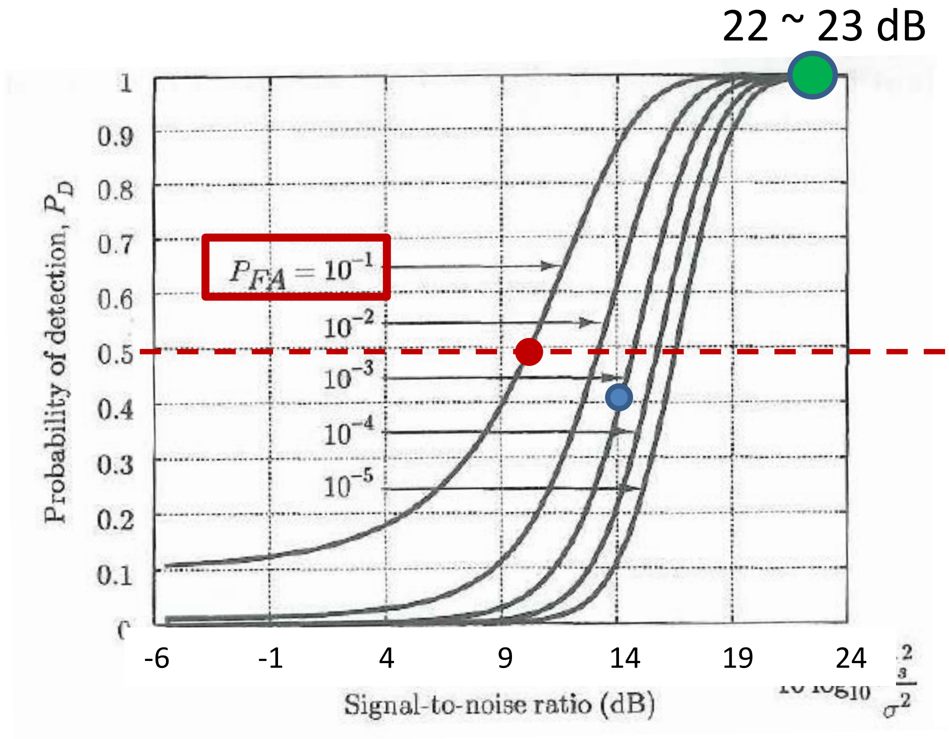

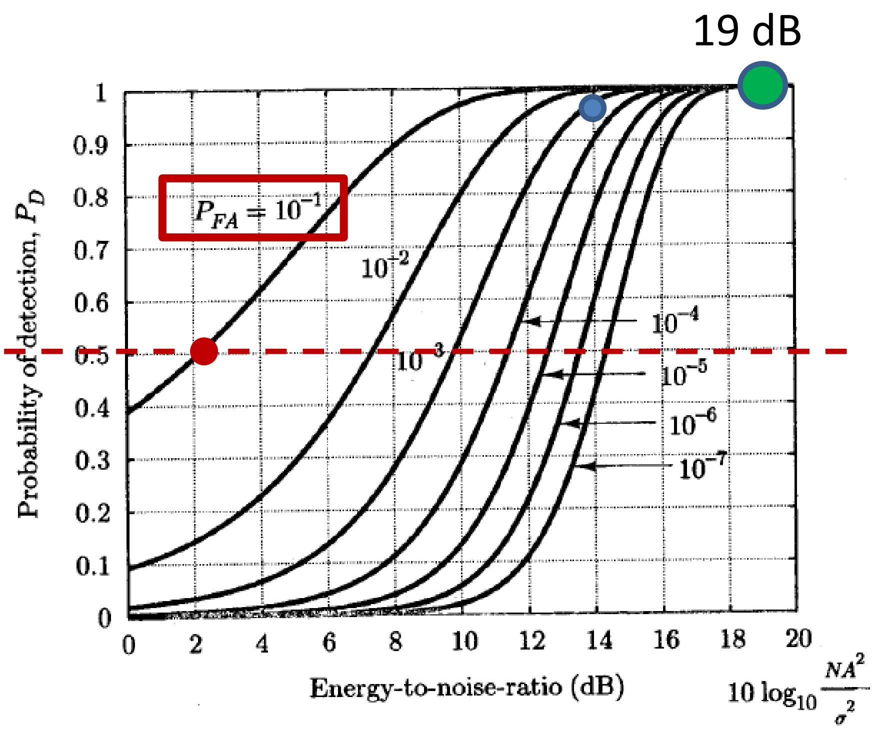

Chapter 5: Estimator-Correlator

We now move on to

$$\mathbf{x} \sim \begin{cases} \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mathbf{0}, \mathbf{C}_s + \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_1 \end{cases}$$

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} =$$

Chapter 5: Estimator-Correlator

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$$L(\mathbf{x}) = \frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} = \frac{\dfrac{1}{(2\pi)^{\frac{N}{2}}\det^{\frac{1}{2}}(\mathbf{C}_s+\sigma^2\mathbf{I})}\exp\left[-\dfrac{1}{2}\mathbf{x}^T(\mathbf{C}_s+\sigma^2\mathbf{I})^{-1}\mathbf{x}\right]}{\dfrac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left(-\dfrac{1}{2\sigma^2}\mathbf{x}^T\mathbf{x}\right)}$$

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Multiplicative constant -> absorb into the threshhold

Chapter 5: Estimator-Correlator

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Take logartihm of ***L*(*x*)**

Chapter 5: Estimator-Correlator

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Multiplicative constant -> absorb into the threshhold
Take logartihm of *L*(*x*)

$$-\frac{1}{2}\mathbf{x}^T\left[(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1} - \frac{1}{\sigma^2}\mathbf{I}\right]\mathbf{x} > \gamma'$$

Chapter 5: Estimator-Correlator

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Multiplicative constant -> absorb into the threshhold

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Manipulate a bit, for later use

$$\boxed{T(\mathbf{x}) = \sigma^2\mathbf{x}^T\left[\frac{1}{\sigma^2}\mathbf{I} - (\mathbf{C}_s+\sigma^2\mathbf{I})^{-1}\right]\mathbf{x} > 2\gamma'\sigma^2}$$

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Take logartihm of *L*(*x*)

$$-\frac{1}{2}\mathbf{x}^T\left[(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1} - \frac{1}{\sigma^2}\mathbf{I}\right]\mathbf{x} > \gamma'$$

(Always!) Deal with this using the matrix inversion Lemma

Manipulate a bit, for later use

$$\boxed{T(\mathbf{x}) = \sigma^2\mathbf{x}^T\left[\frac{1}{\sigma^2}\mathbf{I} - (\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\right]\mathbf{x} > 2\gamma'\sigma^2}$$

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Straightforward application of matrix inversion lemma yields

$$\underline{(\sigma^2\mathbf{I} + \mathbf{C}_s)^{-1}} = \frac{1}{\sigma^2}\mathbf{I} - \frac{1}{\sigma^4}\left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1}\right)^{-1}$$

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$$(\sigma^2\mathbf{I} + \mathbf{C}_s)^{-1} = \frac{1}{\sigma^2}\mathbf{I} - \frac{1}{\sigma^4}\left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1}\right)^{-1}$$

This cancels that

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Chapter 5: Estimator-Correlator

We now move on to

$$\mathbf{x} \sim \begin{cases} \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mathbf{0}, \mathbf{C}_s + \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_1 \end{cases}$$

Straightforward application of matrix inversion lemma yields

$$(\sigma^2\mathbf{I} + \mathbf{C}_s)^{-1} = \frac{1}{\sigma^2}\mathbf{I} - \frac{1}{\sigma^4}\left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1}\right)^{-1}$$

$$T(\mathbf{x}) = \sigma^2\mathbf{x}^T\left[\frac{1}{\sigma^4}\left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1}\right)^{-1}\right]\mathbf{x}$$

$$T(\mathbf{x}) = \sigma^2\mathbf{x}^T\left[\frac{1}{\sigma^2}\mathbf{I} - (\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\right]\mathbf{x} > 2\gamma'\sigma^2$$

Chapter 5: Estimator-Correlator

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Chapter 5: Estimator-Correlator

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$$T(\mathbf{x}) = \sigma^2 \mathbf{x}^T \left[\frac{1}{\sigma^4} \left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1} \right)^{-1} \right] \mathbf{x}$$

$$T(\mathbf{x}) = \mathbf{x}^T \left[\frac{1}{\sigma^2} \left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1} \right)^{-1} \right] \mathbf{x}$$

But,

$$\begin{aligned} & \frac{1}{\sigma^2} \left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1} \right)^{-1} \mathbf{x} \\ = \quad & \frac{1}{\sigma^2} \left[\frac{1}{\sigma^2} (\mathbf{C}_s + \sigma^2\mathbf{I}) \mathbf{C}_s^{-1} \right]^{-1} \mathbf{x} \\ = \quad & \mathbf{C}_s (\mathbf{C}_s + \sigma^2\mathbf{I})^{-1} \mathbf{x} \end{aligned}$$

Chapter 5: Estimator-Correlator

We now move on to

$$\boxed{\mathbf{x} \sim \begin{cases} \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mathbf{0}, \mathbf{C}_s + \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_1 \end{cases}}$$

$$T(\mathbf{x}) = \mathbf{x}^T \left[\frac{1}{\sigma^2}\left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1}\right)^{-1}\right]\mathbf{x}$$

But,

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Chapter 5: Estimator-Correlator

We now move on to

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But,

$$\frac{1}{\sigma^2}\left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1}\right)^{-1}\mathbf{x}$$

$$= \frac{1}{\sigma^2}\left[\frac{1}{\sigma^2}(\mathbf{C}_s + \sigma^2\mathbf{I})\mathbf{C}_s^{-1}\right]^{-1}\mathbf{x}$$

$$= \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x}$$

$$= \hat{\mathbf{s}}$$

$$T(\mathbf{x}) = \mathbf{x}^T\left[\frac{1}{\sigma^2}\left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1}\right)^{-1}\right]\mathbf{x}$$

$$\boxed{T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}}}$$

Chapter 5: Estimator-Correlator

We now move on to

$$\mathbf{x} \sim \begin{cases} \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mathbf{0}, \mathbf{C}_s + \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_1 \end{cases}$$

But where is the estimator ?

$$T(\mathbf{x}) = \mathbf{x}^T \left[\frac{1}{\sigma^2} \left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1} \right)^{-1} \right] \mathbf{x}$$

$$\boxed{T(\mathbf{x}) = \mathbf{x}^T \hat{\mathbf{s}}}$$

But,

$$\begin{aligned} & \frac{1}{\sigma^2} \left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1} \right)^{-1} \mathbf{x} \\ = & \frac{1}{\sigma^2} \left[\frac{1}{\sigma^2}(\mathbf{C}_s + \sigma^2\mathbf{I})\mathbf{C}_s^{-1} \right]^{-1} \mathbf{x} \\ = & \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x} \\ = & \hat{\mathbf{s}} \end{aligned}$$

Chapter 5: Estimator-Correlator

We now move on to

$$\mathbf{x} \sim \begin{cases} \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mathbf{0}, \mathbf{C}_s + \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_1 \end{cases}$$

$$\boxed{\hat{\mathbf{s}} = \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x}}$$

$\hat{\mathbf{s}}$ **is the Wiener estimator of s**

But,

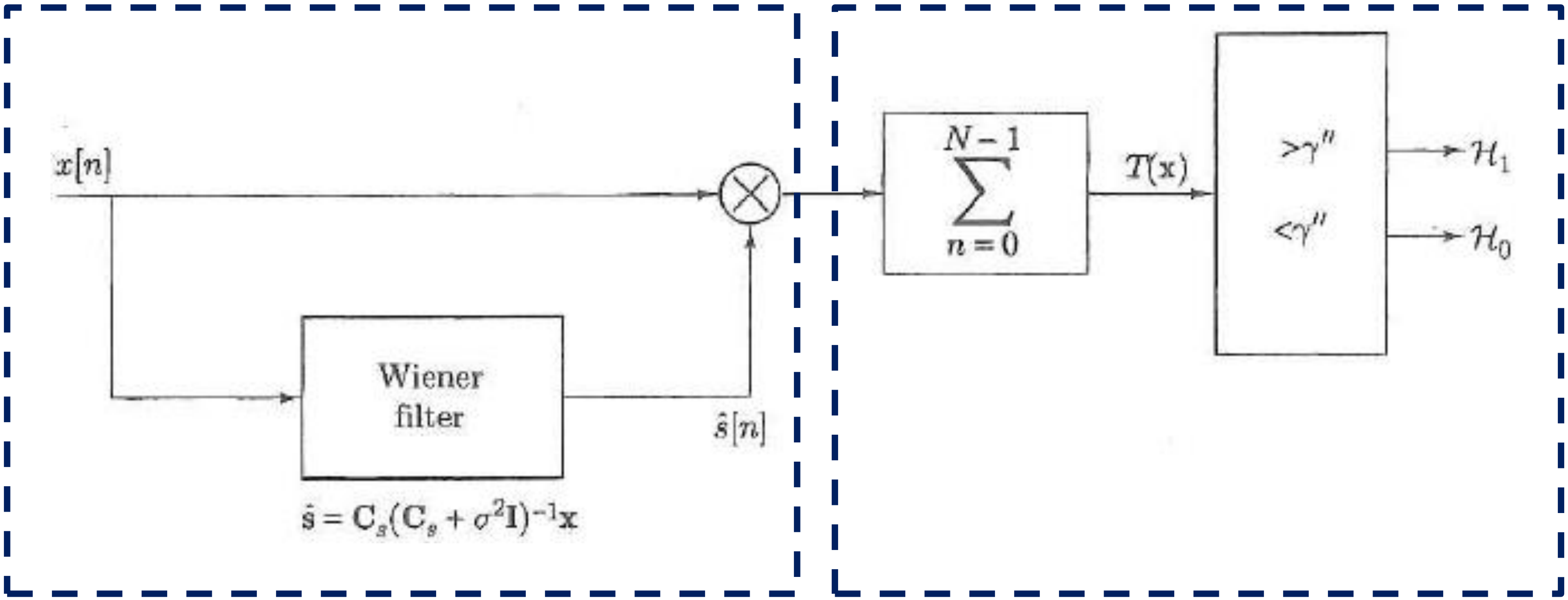
$$T(\mathbf{x}) = \mathbf{x}^T\left[\frac{1}{\sigma^2}\left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1}\right)^{-1}\right]\mathbf{x}$$

$$\begin{aligned} &\frac{1}{\sigma^2}\left(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{C}_s^{-1}\right)^{-1}\mathbf{x} \\ =\ & \frac{1}{\sigma^2}\left[\frac{1}{\sigma^2}(\mathbf{C}_s + \sigma^2\mathbf{I})\mathbf{C}_s^{-1}\right]^{-1}\mathbf{x} \\ =\ & \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x} \\ =\ & \hat{\mathbf{s}} \end{aligned}$$

$$\boxed{T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}}}$$

Chapter 5: Estimator-Correlator

Wiener estimation of the unknown signal   Correlator as if the signal was known



Chapter 5: Estimator-Correlator

It will be instrumental for understanding later to note the following generalization to non-white noise

White noise

$$\boxed{T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}}}$$

Not White noise

$$\boxed{T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_w^{-1}\hat{\mathbf{s}}}$$

Chapter 5: Estimator-Correlator

Make an eigendecomposition of the signal covariance matrix

$$\mathbf{C}_s = \mathbf{V}\mathbf{\Lambda}_s\mathbf{V}^T$$

This gives

$$T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x}$$

Chapter 5: Estimator-Correlator

Make an eigendecomposition of the signal covariance matrix

$$\mathbf{C}_s = \mathbf{V}\mathbf{\Lambda}_s\mathbf{V}^T$$

This gives

$$\begin{aligned} T(\mathbf{x}) &= \mathbf{x}^T\mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x} \\ &= \mathbf{y}^T\mathbf{\Lambda}_s(\mathbf{\Lambda}_s + \sigma^2\mathbf{I})^{-1}\mathbf{y} \end{aligned} \qquad \mathbf{y} = \mathbf{V}^T\mathbf{x}$$

Chapter 5: Estimator-Correlator

Make an eigendecomposition of the signal covariance matrix

$$\mathbf{C}_s = \mathbf{V}\mathbf{\Lambda}_s\mathbf{V}^T$$

This gives

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$$\mathbf{y} = \mathbf{V}^T\mathbf{x}$$

Due to diagonal form

Make an eigendecomposition of the signal covariance matrix

$$\mathbf{C}_s = \mathbf{V}\mathbf{\Lambda}_s\mathbf{V}^T$$

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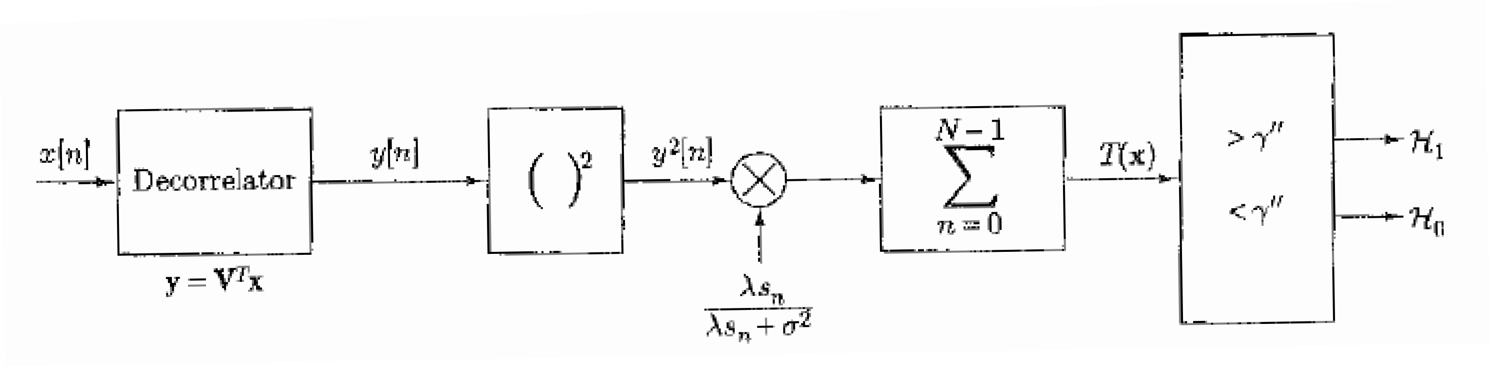
Due to diagonal form



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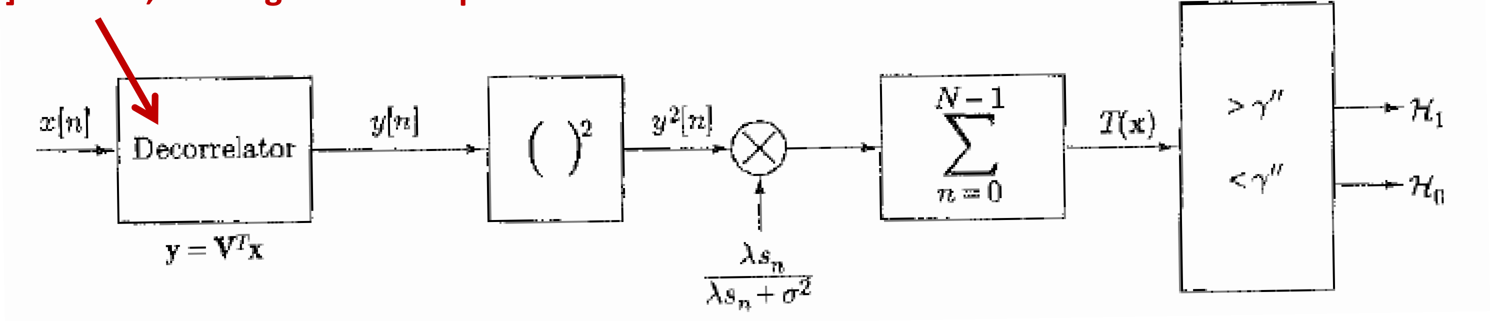
$$\mathbf{y} = \mathbf{V}^T\mathbf{x}$$

Due to diagonal form

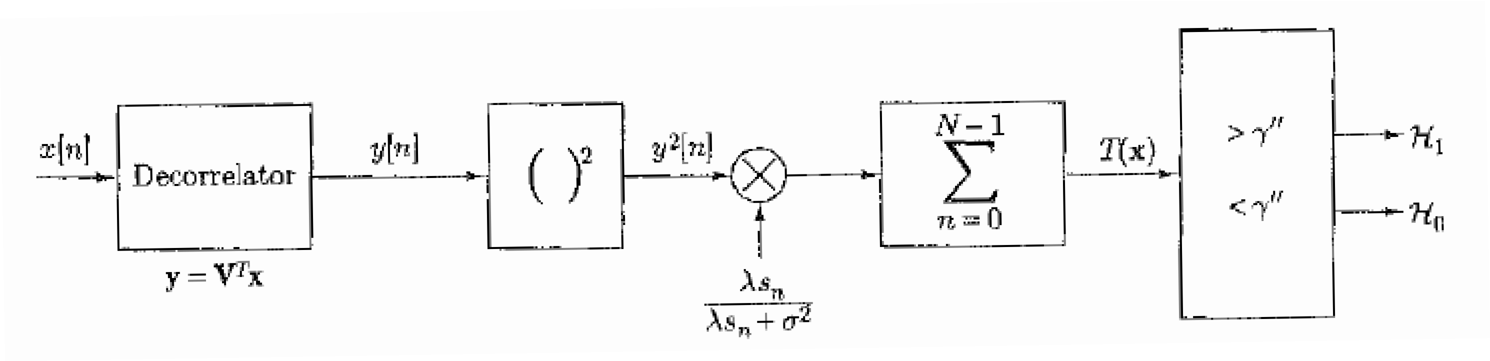
Signal y[n] is white, although with unequal variances



Chapter 5: Estimator-Correlator

Questions:

1. **Where is now the boundary between the estimator and the correlator?**
2. **What is the signal y[n] in the case of a large data record with WSS signals?**



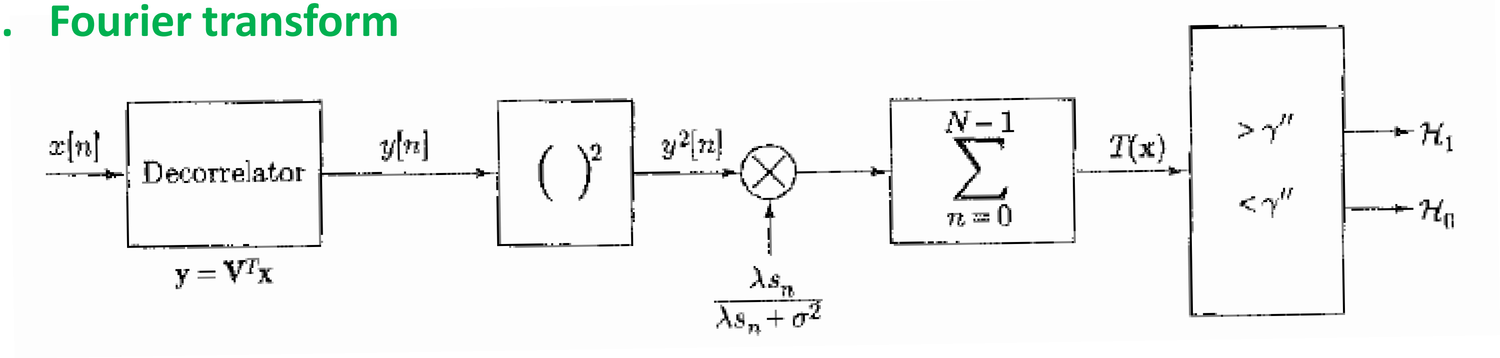
Chapter 5: Estimator-Correlator

Questions:

1. **Where is now the boundary between the estimator and the correlator?**
2. **What is the signal y[n] in the case of a large data record with WSS signals?**

Answers:

1. **Sort of trick question…This structure is not visible in this picture**
2. **Fourier transform**



Performance

Check your understanding: If I say performance, what is it that we should compute?

Performance

We already did performance before for the case of white input signals, is this any harder?

From before

$$\begin{aligned} P_{FA} &= \Pr\{T(\mathbf{x}) > \gamma'; \mathcal{H}_0\} \\ &= \Pr\left\{\frac{T(\mathbf{x})}{\sigma^2} > \frac{\gamma'}{\sigma^2}; \mathcal{H}_0\right\} \\ &= Q_{\chi^2_N}\left(\frac{\gamma'}{\sigma^2}\right) \end{aligned}$$

Standard function for chi-2

Comparable to Q(x)

$$\begin{aligned} P_D &= \Pr\{T(\mathbf{x}) > \gamma'; \mathcal{H}_1\} \\ &= Q_{\chi^2_N}\left(\frac{\gamma'}{\sigma_s^2 + \sigma^2}\right) \cdot = Q_{\chi^2_N}\left(\frac{\gamma'/\sigma^2}{\sigma_s^2/\sigma^2 + 1}\right) \end{aligned}$$

Chapter 5: Estimator-Correlator

Performance

We already did performance before for the case of white input signals, is this any harder?

YES!

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} \frac{\lambda_{s_n}}{\lambda_{s_n} + \sigma^2} y^2[n]$$

This holds (used before)

Standard distribution: If *x*[k] are *N*(0,1) variables, $\sum_{k=1}^{N} |x[k]|^2$ is a chi-2 variable with N DoFs

Chapter 5: Estimator-Correlator

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This holds (used before)

Standard distribution: If *x*[k] are *N*(0,1) variables, $\sum_{k=1}^{N} |x[k]|^2$ is a chi-2 variable with N DoFs

But sadly this also holds

If *x*[k] are *N*(0,**δ[k]**) variables, $\sum_{k=1}^{N} |x[k]|^2$ is **not** a chi-2 variable

Chapter 5: Estimator-Correlator

Another, more advanced, technique is used

Start from $\mathbf{y} = \mathbf{V}^T\mathbf{x}$

$$\mathbf{y} \sim \begin{cases} \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mathbf{0}, \mathbf{\Lambda}_s + \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_1 \end{cases}$$

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Chapter 5: Estimator-Correlator

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$$T(\mathbf{x}) = \sum_{n=0}^{N-1} \frac{\lambda_{s_n}}{\lambda_{s_n} + \sigma^2} y^2[n]$$

$$P_{FA} = \Pr\{T(\mathbf{x}) > \gamma''; \mathcal{H}_0\} = \Pr\left\{\sum_{n=0}^{N-1} \frac{\lambda_{s_n}\sigma^2}{\lambda_{s_n} + \sigma^2} z^2[n] > \gamma''; \mathcal{H}_0\right\}$$

$$z[n] = y[n]/\sigma$$

IID under H_0

Chapter 5: Estimator-Correlator

Another, more advanced, technique is used

Start from $\mathbf{y} = \mathbf{V}^T\mathbf{x}$ $\quad \mathbf{y} \sim \begin{cases} \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mathbf{0}, \mathbf{\Lambda}_s + \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_1 \end{cases}$ $\quad T(\mathbf{x}) = \sum_{n=0}^{N-1} \frac{\lambda_{s_n}}{\lambda_{s_n} + \sigma^2} y^2[n]$

$$P_{FA} = \Pr\{T(\mathbf{x}) > \gamma''; \mathcal{H}_0\} = \Pr\left\{\sum_{n=0}^{N-1} \underbrace{\frac{\lambda_{s_n}\sigma^2}{\lambda_{s_n} + \sigma^2}}_{\alpha_n} z^2[n] > \gamma''; \mathcal{H}_0\right\}$$

$z[n] = y[n]/\sigma$

IID under H_0

$$\alpha_n = \frac{\lambda_{s_n}\sigma^2}{\lambda_{s_n} + \sigma^2}$$

$$T(\mathbf{x}) = \underbrace{\sum_{n=0}^{N-1} \alpha_n z^2[n]}_{\text{We need pdf of this}}$$

Chapter 5: Estimator-Correlator

Another, more advanced, technique is used

Start from $\mathbf{y} = \mathbf{V}^T\mathbf{x}$

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$$T(\mathbf{x}) = \sum_{n=0}^{N-1} \frac{\lambda_{s_n}}{\lambda_{s_n} + \sigma^2} y^2[n]$$

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$$\alpha_n = \frac{\lambda_{s_n}\sigma^2}{\lambda_{s_n} + \sigma^2}$$

<u>Fundamental and useful fact:</u>

Characteristic function of a sum of independent variables is the product of the individual characteristic functions

Chapter 5: Estimator-Correlator

$$\boxed{\phi_T(\omega) = E[\exp(j\omega T)] = \prod_{n=0}^{N-1} \phi_{z^2}(\alpha_n \omega)}$$

Definition of char.-func.

The property below

Property of char.-func.s

$$T(\mathbf{x}) = \underbrace{\sum_{n=0}^{N-1} \alpha_n z^2[n]}_{\text{We need pdf of this}}$$

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Chapter 5: Estimator-Correlator

$$\phi_T(\omega) = E[\exp(j\omega T)] = \prod_{n=0}^{N-1} \phi_{z^2}(\alpha_n \omega)$$

But z[n] are IID chi-2 variables,

$$\phi_{z^2}(\omega) = \phi_{\chi_1^2}(\omega) = \frac{1}{\sqrt{1-2j\omega}}$$

Chapter 5: Estimator-Correlator

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Chapter 5: Estimator-Correlator

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So, $$p_T(t) = \int_{-\infty}^{\infty} \phi_T(\omega) \exp(-j\omega t) \frac{d\omega}{2\pi}$$ **By definition of the char. func.**

Chapter 5: Estimator-Correlator

Recall: $P_{FA} = \Pr\{T(\mathbf{x}) > \gamma''; \mathcal{H}_0\}$

So, $$p_T(t) = \int_{-\infty}^{\infty} \phi_T(\omega) \exp(-j\omega t) \frac{d\omega}{2\pi}$$ **By definition of the char. func.**

Chapter 5: Estimator-Correlator

Recall: $$\begin{aligned} P_{FA} &= \Pr\{T(\mathbf{x}) > \gamma''; \mathcal{H}_0\} \\ &= \int_{\gamma''}^{\infty} p_T(t)\, dt \end{aligned}$$

So, $$p_T(t) = \int_{-\infty}^{\infty} \phi_T(\omega) \exp(-j\omega t) \frac{d\omega}{2\pi}$$ **By definition of the char. func.**

Chapter 5: Estimator-Correlator

Recall: $P_{FA} = \Pr\{T(\mathbf{x}) > \gamma''; \mathcal{H}_0\}$

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Chapter 5: Estimator-Correlator

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In general, the outer integral is tough.

Chapter 5: Estimator-Correlator

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This can be done in closed form. Occurs if α_n occurs in pairs.

Chapter 5: Estimator-Correlator

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In general, the outer integral is tough.

This can be done in closed form. Occurs if α_n occurs in pairs.

This happens if we have a complex-valued signal, and then model it in the real domain

Chapter 5: Linear model

Customary to write the hypotheses as

$$
\begin{aligned}
\mathcal{H}_0 &: \mathbf{x} = \mathbf{w} \\
\mathcal{H}_1 &: \mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}
\end{aligned}
$$

Chapter 5: Linear model

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$$\mathcal{H}_0 : \mathbf{x} = \mathbf{w}$$
$$\mathcal{H}_1 : \mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$$

Applying verbatim arguments to this we get

$$T(\mathbf{x}) = \mathbf{x}^T\mathbf{H}\mathbf{C}_\theta\mathbf{H}^T(\mathbf{H}\mathbf{C}_\theta\mathbf{H}^T + \sigma^2\mathbf{I})^{-1}\mathbf{x}$$

which leads to $$T(\mathbf{x}) = \mathbf{x}^T\mathbf{H}\hat{\boldsymbol{\theta}}$$

$\hat{\theta}$ is the Wiener estimator (MMSE) for the parameter vector

Chapter 5: Linear model

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{H} \hat{\boldsymbol{\theta}}$$

The astute listener will make a connection to the situation with deterministic signals:

$$T(\mathbf{x}) = \hat{\boldsymbol{\theta}}^T \mathbf{C}_{\hat{\theta}}^{-1} \boldsymbol{\theta}_1$$

Where $\boldsymbol{\theta}_1$ is the parameter value under H_1

Chapter 5: Linear model

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Random signals:
- MMSE estimate is used as signal
- Observation is original data record
- There is no whitening

Deterministic signals:
- MMSE estimate is used as observation
- There is whitening
- Signal is the known parameter

Chapter 5: Linear model

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Random signals:

- MMSE estimate is used as signal
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- **<u>There is no whitening</u>**

Deterministic signals:

- MMSE estimate is used as observation
- There is whitening
- Signal is the known parameter

No whitening needed since estimate is orthogonal to recevied signal

Chapter 5: Large data records

Again, when the data record is large, one goes over to the Fourier plane (assuming WSS)

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It is **very** useful to be able to quickly understand a formula of the following type

Log-pdf of long signal x[n] with PSD $P_{xx}(f)$

$$\ln p(\mathbf{x}) \approx -\frac{N}{2}\ln 2\pi - \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \ln P_{xx}(f)df - \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{I(f)}{P_{xx}(f)}df$$

$$I(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]\exp(-j2\pi fn)\right|^2$$

Periodogram

make sure that you can, it saves lots of time in many cases – not limited to this course

Chapter 5: Large data records

Under H_0, we have $P_{xx}(f) = \sigma^2$ Under H_1, we have $P_{xx}(f) = P_{ss}(f) + \sigma^2$

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Chapter 5: Large data records

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$$\begin{aligned} l(\mathbf{x}) &= \ln p(\mathbf{x}; \mathcal{H}_1) - \ln p(\mathbf{x}; \mathcal{H}_0) \\ &= -\frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \ln(P_{ss}(f)+\sigma^2)df - \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{I(f)}{P_{ss}(f)+\sigma^2}df \\ &\quad + \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \ln\sigma^2 df + \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{I(f)}{\sigma^2}df \end{aligned}$$

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Chapter 5: Large data records

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&= -\frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \ln\left(\frac{P_{ss}(f)}{\sigma^2}+1\right)df + \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} I(f)\frac{P_{ss}(f)}{(P_{ss}(f)+\sigma^2)\sigma^2}df.
\end{aligned}
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\end{aligned}
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$$
\boxed{
\begin{aligned}
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Chapter 5: Large data records

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$$\boxed{T(\mathbf{x}) = N \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{P_{ss}(f)}{P_{ss}(f) + \sigma^2} I(f) df}$$

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Chapter 5: Large data records

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Thus,

$$\begin{aligned} T(\mathbf{x}) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} H(f)X(f)X^*(f)df \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f)(H(f)X(f))^* df \end{aligned}$$

H(f) is real valued

Chapter 5: Large data records

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Chapter 5: General case

$$\mathcal{H}_0 : \mathbf{x} = \mathbf{w}$$
$$\mathcal{H}_1 : \mathbf{x} = \mathbf{s} + \mathbf{w}$$

$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w), \mathbf{s} \sim \mathcal{N}(\boldsymbol{\mu}_s, \mathbf{C}_s)$$

Think over why this is the most general case although w is the same in both cases.

Chapter 5: General case

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Fairly short and standard derivation gives

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Deterministic part vanishes if μ_s is 0

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With random signals with means, we can write

$H_1 : \mathbf{x} + \boldsymbol{\mu}_s + \mathbf{w}_s + \mathbf{w}$

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This has that covariance

Chapter 5: General case

Non-trivial interpretation of result

$$T'(\mathbf{x}) = \mathbf{x}^T(\mathbf{C}_s + \mathbf{C}_w)^{-1}\boldsymbol{\mu}_s + \frac{1}{2}\mathbf{x}^T\mathbf{C}_w^{-1}\mathbf{C}_s(\mathbf{C}_s + \mathbf{C}_w)^{-1}\mathbf{x}.$$

Recall the estimator-correlator for the zero-mean case

$$\hat{\mathbf{s}} = \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x}$$

$$T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_w^{-1}\hat{\mathbf{s}}$$

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Intuition:

This holds

$$T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_w^{-1}\hat{\mathbf{s}}$$

But with modified (accounting for mean) MMSE estimator

$$\hat{\mathbf{s}} = \boldsymbol{\mu}_s + \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}(\mathbf{x} - \boldsymbol{\mu}_s)$$

Chapter 5: General case

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Chapter 5: *correlator + (estimator-correlator)*

Very surprisingly, we get the following situation in the general case

$$\boxed{T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}_w^{-1} \hat{\mathbf{s}} + \mathbf{x}^T \mathbf{C}_w^{-1} \boldsymbol{\mu}_s}$$

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MMSE estimate of signal

Chapter 5: correlator + (estimator-correlator)

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Correlation with that estimate

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MMSE estimate of signal

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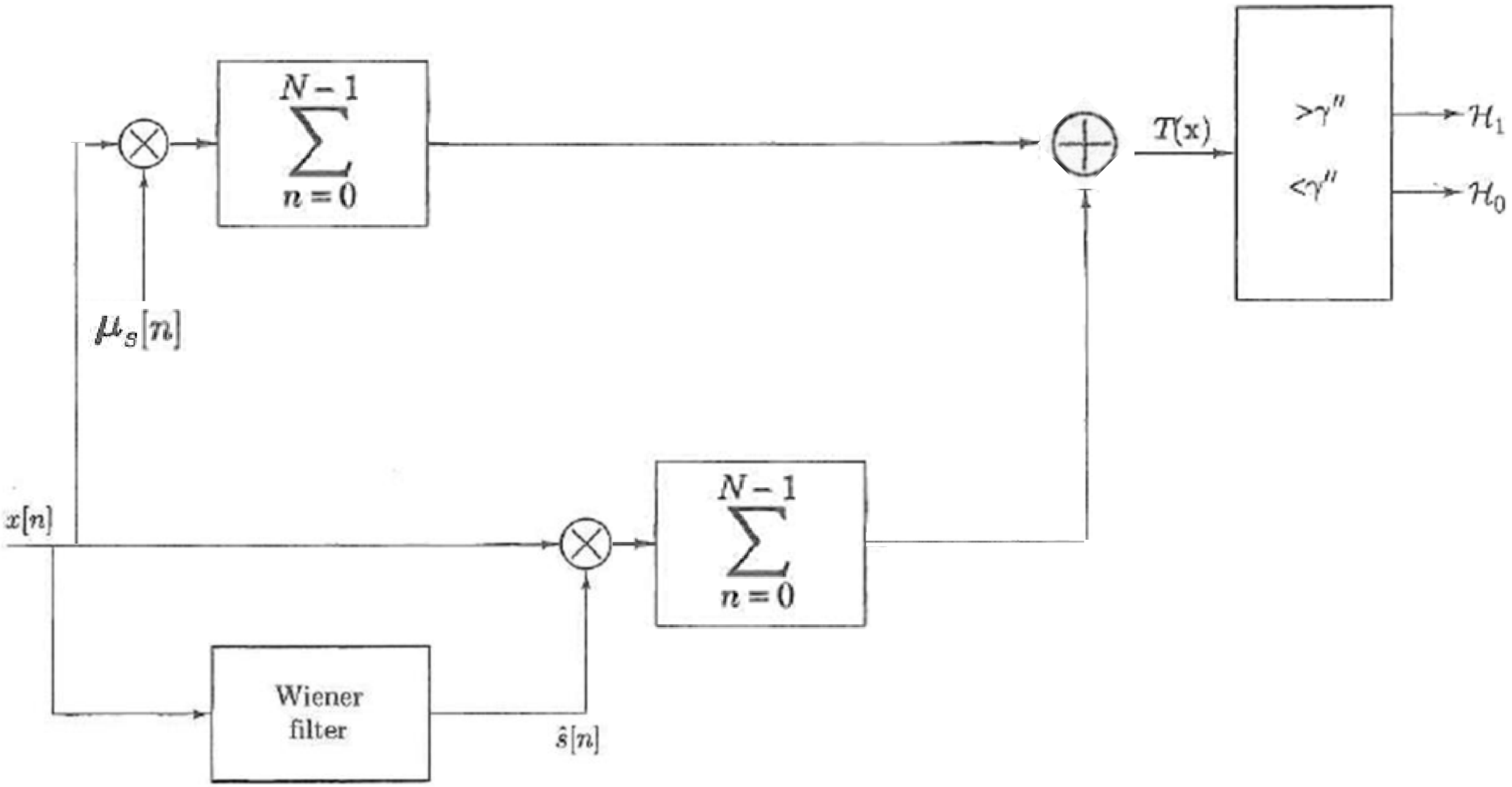
<u>Note:</u> The estimator-correlator contains the deterministic part
Thus, it is counted twice, likely since it is of superior quality

Chapter 5: *correlator + (estimator-correlator)*

(In white noise)



I have never seen this structure in any book