Problems - Lectures 7 & 8

Linear Algebra for Wireless Communications

1. Assume that we have a real symmetric $N \times N$ matrix $\mathbf{A}$.

   (a) Show that the largest eigenvalue of $\mathbf{A}$ is given by the maximum of the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ evaluated on the N-dimensional unit sphere $\mathcal{U}_N = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x}^T\mathbf{x} = 1\}$.

   (b) Show that the second largest eigenvalue of $\mathbf{A}$ is given by the maximum of the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ evaluated on the subset of the N-dimensional unit sphere $\mathcal{U}_N$ consisting of vectors orthogonal to an eigenvector associated with the largest eigenvalue.

2. Assume that the matrix $\mathbf{A}$ has orthogonal columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$, with lengths $\alpha_1, \alpha_2, \ldots, \alpha_n$. Express the SVD of $\mathbf{A}$ in terms of its columns and their lengths.

3. Given that the SVD of a matrix $\mathbf{A}$ is $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$. What is the SVD of

   (a) $\alpha\mathbf{A}$, where α is a real number,

   (b) $\mathbf{A}^H$, and

   (c) $\mathbf{A}^{-1}$?

4. Assume that we have a vector $\mathbf{b}$ and want to find the closest vector $\mathbf{p}$ (Euclidean distance) in the column space $\mathcal{C}(\mathbf{A})$ of a matrix $\mathbf{A}$. Since $\mathbf{A}\mathbf{x} = \mathbf{b}$ may not have a solution, we have solved the normal equations $\mathbf{A}^H\mathbf{A}\mathbf{x}_{LS} = \mathbf{A}^H\mathbf{b}$ and use the least squares solution $\mathbf{x}_{LS} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{b}$ (when $\mathbf{A}^H\mathbf{A}$ is invertible) to build the projection which gives our solution $\mathbf{p} = \mathbf{A}\mathbf{x}_{LS} = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{b}$. By generalizing this situation, we can study a number of interesting cases.

   (a) Show that $(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$ is a left-inverse of $\mathbf{A}$, whenever the columns of $\mathbf{A}$ are linearly independent. (Corresponds to "solving" $\mathbf{A}\mathbf{x} = \mathbf{b}$ above by multiplying with a left-inverse.)

   (b) Show that $\mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}$ is a right-inverse of $\mathbf{A}$, whenever the rows of $\mathbf{A}$ are linearly independent.

   (c) For a *diagonal* (possibly rectangular) matrix $\mathbf{D}$, with elements d_k on its main diagonal and zeros elsewhere, define $\mathbf{D}^+$ as the matrix with elements $1/d_k$ on its main diagonal, for non-zero d_ks, and zeros elsewhere. Show that the left-inverse in (a) and the right-inverse in (b) can both be expressed as a (Moore-Penrose) *pseudo-inverse* $\mathbf{A}^+ = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}^H$, using the SVDs $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$ of the respective $\mathbf{A}$s.

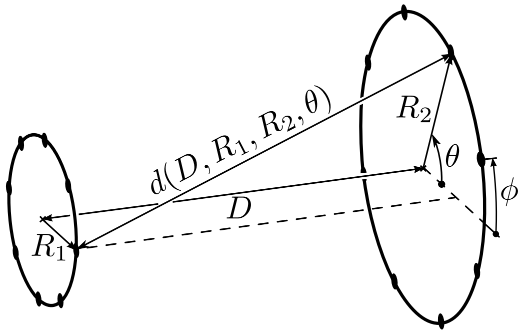

Figure 1: Two uniform circular arrays in free space with radii R_1 and R_2, at a distance D, facing eachother with a common beam axis. Antenna elements indicated by dots on the two circles.

5. Here we will return to the somewhat esoteric Problem 3.3, where we assumed that we had a family of complex $N \times N$ matrices $\mathbf{H}_k$, which all have the property that they are diagonalized by the N-point normalized DFT (Vandermonde) matrix

$$\mathbf{T}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{(N-1)} \\ 1 & \vdots & \ddots & \vdots \\ 1 & w^{(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix},$$

where $w = \exp(-j2\pi/N)$ is the primitive Nth root of 1.

(a) Now, assume that we have two uniform circular arrays in free space, as displayed in Figure 1. Further assume that both arrays have N antenna elements each. Given that the influence of the propagation only depends on the distance-attenuation (amplitude) and the distance traveled (phase), the channel attenuation $h(d)$ between the input of one transmit and the output of one receive antenna element at distance d becomes

$$h(d) = \beta \frac{\lambda}{4\pi d} \exp\left(-j2\pi \frac{d}{\lambda}\right), \tag{1}$$

where the free space (amplitude) loss is given by $\lambda/(4\pi d)$, the additional phase rotation due to propagation distance is introduced by the complex exponential term, λ denotes the wavelength of the used carrier frequency, and complex constant β contains all relevant constants attenuations and phase rotations caused by antenna electronics and antenna patterns on both sides.

Show that the resulting MIMO channel matrix $\mathbf{H}$, between the two uniform circular arrays in Figure 1 fulfil the requirements for being diagonalized by $\mathbf{T}_N$, independent of antenna radii R_1 and R_2, array distance D and relative array rotation ϕ.

Hint: Start by finding an expression for the antenna element distances in terms of array distance D, array radii R_1, R_2, and angle θ between antenna elements, i.e., find $d(D, R_1, R_2, \theta)$. Use this expression in (1) and relate this to properties of $\mathbf{H}$.

(b) Now assume a case where the receiver has TWO concentric circles of N antenna elements with (possibly) different radii R_{21} and R_{22}. The transmitter is still a single uniform circular array with radius R_1. Show that the structure of the resulting $2N \times N$ MIMO channel matrix $\mathbf{H}$ matches the structure studied in Problem 3.3(a).

(c) We have seen that the singular-value decomposition (SVD) is useful when analyzing the performance of MIMO systems. Now we compare the result obtained in Problem 3.3 (a), to the SVD of the MIMO channel matrix in (b) above. Show that the "diagonalization" derived in Problem 3.3 (a), can be used to find an analytical expression for the SVD of our channel matrix, and hence an analytical expression for the channel capacity.

Hint: If you solved Problem 3.3(a) you probably obtained (in the terminology of that problem) a diagonalization of the given matrix $\mathbf{A}$ as (rectangular matrix with nonxero elements only on its main diagonal) $\mathbf{R}_A = \mathbf{U}_A^H \mathbf{A} \mathbf{V}_A$, where $\mathbf{V}_A = \mathbf{T}_N^H$ and the first N rows of $\mathbf{U}_A^H$ are given by

$$\left(\sum_{k=1}^{P} \boldsymbol{\Delta}_k^H \boldsymbol{\Delta}_k \right)^{-1/2} \left[\boldsymbol{\Delta}_1^H \mathbf{T}_N | \cdots | \boldsymbol{\Delta}_P^H \mathbf{T}_N \right] \tag{2}$$

(d) What are the "ideal beamformers" on transmitter and receiver side (derived from the SVD) when we have the particular array constructions discussed above? No explicit expression needed. A discussion is enough.