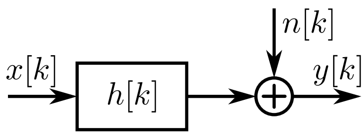
Problems - Lecture 6

Linear Algebra for Wireless Communications

1. Show that the matrix equation $\mathbf{AX} + \mathbf{XB} = -\mathbf{C}$, where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are given, has a unique solution if and only if $\mathbf{A}$ and -$\mathbf{B}$ have no common eigenvalues.

2. Show that if $\mathbf{A}$ is skew-symmetric ($\mathbf{A}^T = -\mathbf{A}$), then $e^{\mathbf{A}t}$ is an orthogonal matrix.

3. Orthogonal frequency-division multiplexing (OFDM) can convert a channel with time dispersion (aka frequency selectivity) to a set of parallel and independent transmission channels. During Lecture 6 the details were, however, quite scarce. In this problem we go through the steps necessary to derive this relationship.



Assume that we have a (not very realistic) discrete-time transmission channel where a transmitted signal $x[k]$ is dispersed by a channel impulse response $h[k]$ and a white Gaussian noise $n[k]$ representing the receiver noise is added to the channel output. The received signal can be written as

$$y[k] = (x * h)[k] + n[k] = \sum_n x[n]h[k-n] + n[k]. \tag{1}$$

We will throughout this problem assume that the time dispersion of the channel is limited to L time steps, *i.e.*, that $h[k] \neq 0$ only for $0 \leq k \leq L$.

(a) Assuming that we have an input signal $x[k]$ to the channel of length N, *i.e.*, that $x[k] \neq 0$ only for $0 \leq k \leq N-1$. Introduce matrix notation where $\mathbf{H}$ is a matrix representing the channel, $\mathbf{x}$, $\mathbf{n}$, and $\mathbf{y}$, are vectors representing the corresponding input, noise and output signals with increasing time index from first to last vector element. Give the input-output relation in matrix form and specify the contents of the channel matrix and the signal vectors, with minimal size, so that all output signal components (affected by the non-zero part of the input signal) can be calculated.

(b) In OFDM systems, we add a cyclic prefix of length L to our transmitted OFDM-symbol, consisting of copies of the last L samples of the symbol. Assuming that the length N input signal from (a) is the main part of the OFDM symbol, we can add a cyclic prefix by assigning $x[-n] = x[N-n]$ for $n = 1, 2, \ldots, L$. Re-write the expressions in (a) to reflect this change in structure.

(c) Further, assume that we only care about the N received samples $y[k]$ for $k = 0, 1, \ldots, N-1$. Re-write the expressions above, exploiting any multiple copies of matrix and vector elements, so that $\mathbf{H}$ becomes an $N \times N$ matrix and all vectors are of size N, and show that the resulting channel matrix is diagonalized by a DFT matrix, independent of the channel impulse response.

(d) Diagonalize the system in (c) and describe the relationship between the channel attenuation on channel m (diagonal element m in the diagonalized channel matrix) and the channel impulse response of the transmission channel.

4. In Lecture 5 we derived the largest obtainable minimum signal-to-interference ratio (SIR), when path gains $G_{i,j}$ of all interfering links were known. All receivers were treated equally and only interference was taken into account. A more realistic scenario would be to also include different receiver requirements on signal quality and acknowledge the fact that receivers will have ambient noise.
Let's re-formulate the problem from Lecture 5 for a situation where we also take into account that receiver m needs a certain SIR level of γ_m. This would, in the notation of Lecture 5, lead to a requirement

$$\mathrm{SIR}_m = \frac{G_{m,m} P_m}{\sum_{n \neq m} G_{m,n} P_n} \geq \gamma_m$$

or

$$P_m \geq \gamma_m \sum_{n \neq m} \frac{G_{m,n}}{G_{m,m}} P_n$$

which in matrix notation can be written

$$\mathbf{DGp} \leq \mathbf{p}.$$

This notation is following Lecture 5, slide 19, with an addition of the diagonal matrix $\mathbf{D}$ containing the SIR requirements γ_m on its diagonal. Under the given conditions, Perron's theorem says that there exists a positive power vector $\mathbf{p}$ and an eigenvalue $\lambda_{PF} > 0$ of the matrix $\mathbf{DG}$ such that we obtain the equality

$$\mathbf{DGp} = \lambda_{PF} \mathbf{p}$$

and all other eigenvalues of $\mathbf{DG}$ have magnitudes lower than λ_{PF}. This solution is, however, only valid if $\lambda_{PF} \leq 1$, since we require that $\mathbf{DGp} \leq \mathbf{p}$.

Use Gershgorin's theorem to derive a sufficient condition on the required SIRs, in relation to the given link gains, so that the above eigenvalue solution provides the desired power distribution solution (up to a certain common scaling factor).