Problems - Lecture 3

Linear Algebra for Wireless Communications

1. Answer the following questions:

   (a) Is a projection matrix $\mathbf{P}$ invertible or not? Explain!

   (b) If $\mathbf{Q}$ is othogonal, is the same true for $\mathbf{Q}^3$?

   (c) Is it true that the projections $\mathbf{Px}$ and $\mathbf{Py}$ are orthogonal if the original vectors $\mathbf{x}$ and $\mathbf{y}$ are?

   (d) What is the QR factorization of

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & 0 \end{bmatrix}.$$

2. MIMO and QR

   During the lecture it was stated that a QR-factorization can help the decoding process of MIMO systems. Two separate cases were mentioned, where the channel matrix $\mathbf{H}$, or its hermitian transpose $\mathbf{H}^H$, were QR-factored. Let us study these situations in a little more detail!

   The MIMO system model referred to is the complex baseband equivalent of a wireless MIMO system with N_T transmit and N_R receive antennas,

$$\mathbf{y} = \mathbf{Hx} + \mathbf{n},$$

   where $\mathbf{y}$ is the received N_R-vector, $\mathbf{H}$ the $N_R \times N_T$ channel matrix, $\mathbf{x}$ the transmitted N_T-vector, and $\mathbf{n}$ an N_R-vector with independent and identically-distributed (i.i.d) zero-mean circularly-symmetric complex Gaussian (ZMCSCG) noise. Further, we assume that the transmitted vector $\mathbf{x}$ belongs to a discrete set $\mathcal{X}$ of possible transmitted vectors and that the channel matrix is completely known to the receiver.

   Under these conditions the *maximum-likelihood* (ML) detector finds the most probable transmitted vector $\hat{\mathbf{x}}_{ML}$, given the received vector $\mathbf{y}$. Since the noise is i.i.d. ZMCSCG, this is equivalent to solving the discrete least-squares problem

$$\hat{\mathbf{x}}_{ML} = \arg\min_{\mathbf{x}\in\mathcal{X}} \|\mathbf{Hx} - \mathbf{y}\|^2.$$

   This detector has one important problem. Its complexity easily become overwhelming. If we transmit an M-ary constellation point on each transmit antenna, the number of vectors that has to be checked to find the minimum is $|\mathcal{X}| = M^{N_T}$. Other, more efficient algorithms are needed.

For simplicity, we restrict ourselves to the situation where each entry in the transmitted vector $\mathbf{x}$ contains an M-ary constellation point.

(a) Assume that the receiver performs a QR-factorization of the channel matrix, $\mathbf{H} = \mathbf{QR}$, and post-process the received vector to obtain $\tilde{\mathbf{y}} = \mathbf{Q}^H\mathbf{y} = \mathbf{Rx} + \mathbf{Q}^H\mathbf{n}$, which it uses in the detection process instead of $\mathbf{y}$.

i. What are the requirements on N_T and N_R to make successive interference cancellation (successive detection of the symbols in $\mathbf{x}$, from last to first) possible?

ii. Successive interference cancellation can result in error propagation if we make an incorrect decision. We would therefore like to minimize the probability of making wrong decisions early in the process. The probabilty of an incorrect decision increases with decreasing signal-to-noise ratio (SNR), so we would like to maximize the SNR of each stage in the detection. Propose (and motivate) an alternative strategy to the direct QR-factorization of $\mathbf{H}$, which obtains this objective. *Hints: When looking at the SNR of the kth detection stage, assume that the detected symbols in previous stages are error free. A source of inspiration may be the problem with small pivot-elements in Gaussian elimination.*

(b) Now, assume that the receiver performs a QR-factorization of the hermitian transpose of the channel matrix, $\mathbf{H}^H = \mathbf{QR}$, and send $\mathbf{Q}$ to the transmitter. The transmitter uses $\mathbf{Q}$ as a pre-coding of the transmit vector $\mathbf{x}$ to obtain $\tilde{\mathbf{y}} = \mathbf{R^H Q^H Q x} + \mathbf{n} = \mathbf{R^H x} + \mathbf{n}$. The resulting structure of the transmission channel, again, becomes triangular and successive interference cancellation is possible. We, however, seem to have the same problem as above with minimizing error propagation.

i. Is there another solution to this, which (ideally) eliminates the problem with error propagation?
Hint: What can be done if the transmitter knows $\mathbf{R}$ as well?

ii. The solution which eliminates the problem with error propagation (if you find the one I'm looking for) has another, quite serious, drawback! Which?
Hint: The solution to this new problem can be (at least partly) solved with techniques similar to Tomlinson-Harashima precoding.

(c) Now, assume the N_T transmitting antennas are placed on a single base station and that the (same number) $N_R = N_T$ receiving antennas are placed on N_R different terminals (one on each terminal). Further, assume that the data on transmit antenna k is intended for the kth terminal. Make a qualitative analysis of the necessary flow of information between terminals when the different detection strategies above are used. Which one seems the most "feasible"?

3. Let us assume that we have a family of complex $N \times N$ matrices $\mathbf{H}_k$, which all have the property that they are diagonalized by the N-point normalized

DFT (Vandermonde) matrix

$$\mathbf{T}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{(N-1)} \\ 1 & \vdots & \ddots & \vdots \\ 1 & w^{(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix},$$

where $w = \exp(-j2\pi/N)$ is the primitive Nth root of 1. This means that they all satisfy $\mathbf{T}_N \mathbf{H}_k \mathbf{T}_N^H = \mathbf{\Delta}_k$, where the $\mathbf{\Delta}_k$s are diagonal matrices.

(a) First, assume that we have a block matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \\ \vdots \\ \mathbf{H}_P \end{bmatrix}$$

and show, by construction, that it is possible to find unitary matrices $\mathbf{U}_A$ and $\mathbf{V}_A$ such that

$$\mathbf{R}_A = \mathbf{U}_A^H \mathbf{A} \mathbf{V}_A$$

is a rectangular matrix with non-zero elements only on its main diagonal. *Hint: A rectangular complex matrix with orthogonal (normalized) columns can be extended to a unitary matrix by adding a basis for its left nullspace. Assume that you have access to a function* $[\mathbf{n}_1, \ldots, \mathbf{n}_m] = g(\mathbf{C})$ *that gives you an orthogonal basis for the left nullspace a matrix* $\mathbf{C}$.

(b) Secondly, assume that we have a block matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \cdots & \mathbf{H}_Q \\ \mathbf{H}_{Q+1} & \mathbf{H}_{Q+2} & \cdots & \mathbf{H}_{2Q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_{P(Q+1)} & \mathbf{H}_{P(Q+2)} & \cdots & \mathbf{H}_{PQ} \end{bmatrix}.$$

Is it possible to to find unitary matrices $\mathbf{U}_B$ and $\mathbf{V}_B$, in the same manner as above, such that

$$\mathbf{R}_B = \mathbf{U}_B^H \mathbf{B} \mathbf{V}_B$$

is a rectangular (or square, if $P = Q$) matrix with non-zero elements only on its main diagonal?

(This problem may seem a bit odd right now, but a more interesting follow-up will come later in the course.)