Lund Institute of Technology
Department of Electro and Information Technology

Exam in Optimal Signal Processing (ETTN10)
Date: October 25, 2012
Hour: 14.00-19.00
Room: Sparta

Auxiliaries: Tables, calculator.
Write your name on all sheets.
The exam consists of 5 problems, each problem gives 0-1 point
Limits: 2.0-2.9 points give the grade 3
3.0-3.9 points give the grade 4
4.0-5.0 points give the grade 5
The solutions must be easy to follow and a clear answer must be given. Explain your solutions and assumptions.
Hints: Draw figures. Write definitions.

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1. The autocorrelation function to a WSS process $x(n)$ is given by

$$r_x(k) = 4 \cdot 0.8^{|k|}.$$

a) Determine the power spectrum for the process $x(n)$.

b) Estimate the power spectrum from $r_x(k) = 4 \cdot 0.8^{|k|}$ using Blackman-Tukeys method.
Chose a rectangular window of length 2M+1=5 (M=2).

c) A new process is given by
$$y(n) = h(n) * x(n) + w(n)$$
with $h(n) = \delta(n) + \delta(n-1)$ and $w(n)$ white noise with variance 2.
Determine the power spectrum for the process $y(n)$.

2. The system function for the prediction error filter $A_3(z)$ is

$$A_3(z) = 1 - 0.5z^{-1} + 0.25z^{-2} + 0.5z^{-3}$$

a) Determine the reflection coefficients.

b) Determine the errors ε_k, $k = 0,1,2,3$ for $r_x(0) = 4$.

c) Determine the autocorrelation function for $r_x(0) = 4$.

3 A signal $s(n)$ is generated according to

$$s(n) = v(n) + 0.5v(n-1)$$

with $v(n)$ white noise with variance 1.

a) Determine the impulse response to the FIR filter of length 2 which estimates the signal $s(n-2)$.

b) Determine the impulse response to the non-causal IIR filter which estimates the signal $s(n-2)$.

c) Determine the impulse response to the causal IIR filter which estimates the signal $s(n-2)$.

d) Determine also $\mathcal{E}_{\min}$ for the filters in a), b) and c). Make comments of the size of $\mathcal{E}_{\min}$. Which filter is the best? Explain.

4 A signal $s(n)$ is generated according to

$$s(n) = v(n) + v(n-1)$$

with $v(n)$ white noise with variance 1.
The received signal is $x(n) = s(n) + w(n)$
with $w(n)$ white noise with variance 1.

a) Determine the system function $H(z)$ for the FIR filter of length 2 which estimates the signal $s(n+1)$ from $x(n)$.

b) Determine the system function $H(z)$ for the causal IIR filter which estimates the signal $s(n+1)$ from $x(n)$.

c) Determine also $\mathcal{E}_{\min}$ for the filters in a) and b). Make comments of the size of $\mathcal{E}_{\min}$.

5. The signal $x(n)$ is $x(n) = n \cdot 0.5^n \quad n \geq 0$.

a) Determine a second order pole-zero model $H(z)$ (p=2 and q=2) with Pade's method for the signal $x(n)$.

b) Determine a second order pole-zero model $H(z)$ (p=2 and q=2) with Prony's method for the signal $x(n)$ using the first 6 values of $x(n)$ $0 \leq n \leq 5$.

Lund Institute of Technology, Department of Electro and Information Technology

Solutions for Exam in Optimal Signal Processing, October 25, 2012 (ETTN10)

1.

Given this: $r_x(k) = 4 \cdot 0.8^{|k|}$

Task: Determine the power spectrum a) $P_x(e^{j\omega})$, b) $\hat{P}_{BT}(e^{j\omega})$, c) $P_y(e^{j\omega})$,

Solution: a) From formula table

$$P_x(e^{j\omega}) = 4\frac{1-0.8^2}{(1-0.8e^{-j\omega})(1-0.8e^{j\omega})} = \frac{1.44}{1.64-1.6\cos\omega}$$

b) $r_x(k) = 4 \cdot 0.8^{|k|} = 4\ \{...0.8^2,\ 0.8,\ \underset{\uparrow}{1},\ 0.8,\ 0.8^2,...\}$

$$\hat{P}_{BT}(e^{j\omega}) = 4\{0.64e^{j2\omega} + 0.8e^{j\omega} + 1 + 0.8e^{-j\omega} + 0.64e^{j2\omega}\} =$$
$$= 4(1+1.6\cos\omega + 1.28\cos 2\omega$$

c)
$$P_y(e^{j\omega}) = P_x(e^{j\omega})H(e^{j\omega})H(e^{-j\omega}) + P_w(e^{j\omega}) =$$
$$= 1.44\frac{(1+e^{-j\omega})(1+e^{j\omega})}{(1-0.8e^{-j\omega})(1-0.8e^{j\omega})} + 2 = \frac{1.44(2+2\cos\omega)}{1.64-1.6\cos\omega} + 2$$

2.

Given this: $A_3(z) = 1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{2}z^{-3}$

Task: Determine a) $\Gamma_1, \Gamma_2, \Gamma_3$

b) $r_x(0),\quad r_x(1),\quad r_x(2),\quad r_x(3)$

c) $\varepsilon_k \quad k = 0,1,2,3$

Solution: a) $\Gamma_1 = -\frac{1}{2},\ \Gamma_2 = \frac{2}{3},\ \Gamma_3 = \frac{1}{2},\quad A_2(z) = 1 - \frac{5}{6}z^{-1} + \frac{2}{3}z^{-2}\quad A_1(z) = 1 - \frac{1}{2}z^{-1}$

b)

$$\varepsilon_0 = r_x(0) = 4$$

$$\varepsilon_1 = \varepsilon_0\ (1-\Gamma_1^{\ 2}) = 4 \cdot \frac{3}{4} = 3$$

$$\varepsilon_2 = \varepsilon_1\ (1-\Gamma_2^{\ 2}) = 3 \cdot \frac{15}{16} = \frac{45}{16} = 2.8$$

$$\varepsilon_3 = \varepsilon_2\ (1-\Gamma_3^{\ 2}) = \frac{46}{16} \cdot \frac{3}{4} = 2.1$$

c)

$$r_x(0) = 4$$
$$r_x(1) = -r_x(0)\ \Gamma_1 = 2$$
$$r_x(2) = -a_2(1)\ r_x(1) - a_2(2)\ r_x(0) = \frac{5}{6} \cdot 2 - \frac{2}{3} \cdot 4 = -1$$
$$r_x(3) = -a_3(1)\ r_x(2) - a_3(2)\ r_x(1) - a_3(3)\ r_x(0) = \frac{1}{2} \cdot (-1) - \frac{1}{4} \cdot 2 - \frac{1}{2} \cdot 4 = -3$$

3

Given this $s(n)=v(n)+0.5\,v(n-1)$ with $v(n)$ white noise with variance 1 and

Task: Estimate the signal $s(n-2)$ using

a) FIR filter of length 2

b) non-causal IIR filter

c) causal IIR filter

d) Determine mimimum errors.

Solution: a) Wiener-Hopf equation $x(n)=s(n)$

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}\begin{bmatrix} h(0) \\ h(1) \end{bmatrix}=\begin{bmatrix} r_x(-2) \\ r_x(-1) \end{bmatrix}$$

$$P_x(z)=(1+0.5\,z^{-1})(1+0.5\,z)=0.5\,z+1.25+0.5\,z^{-1}$$

$$r_x(k)=[\ 0.5\ \ \underset{\uparrow}{1.25}\ \ \ 0.5\ ]$$

$$\begin{bmatrix} 1.25 & 0.5 \\ 0.5 & 1.25 \end{bmatrix}\begin{bmatrix} h(0) \\ h(1) \end{bmatrix}=\begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \Rightarrow \begin{bmatrix} h(0) \\ h(1) \end{bmatrix}=\begin{bmatrix} -0.1905 \\ 0.4762 \end{bmatrix}$$

b) $$H(z)=\frac{P_{dx}(z)}{P_x(z)}=\frac{z^{-2}\ P_x(z)}{P_x(z)}=z^{-2}; \quad h(n)=\delta(n-2)$$

c)

$$H(z)=\frac{1}{\sigma^2 Q(z)}\left[\frac{P_{dx}(z)}{Q_x(z^{-1})}\right]_+ = \frac{1}{1+0.5\,z^{-1}}\left[\frac{z^{-2}(1+0.5\,z^{-1})(1+0.5\,z)}{(1+0.5\,z)}\right]_+ = z^{-2}$$

$$h(n)=\delta(n-2)$$

d) $$\xi_{\min}=r_d(0)-\sum_{l=-\infty}^{\infty} h(l)r_{dx}(l)=r_d(0)-\sum_{l=-\infty}^{\infty} h(l)r_x(l-2)$$

a) FIR $$\xi_{\min}=r_d(0)-h(0)r_x(-2)\ -h(1)r_x(1)\ = =1.25-(-0.1905)\cdot 0-0.4762\cdot 0.5=1.0119$$

b) and c) noncausal and causal IIR has the same $h(n)=\delta(n-2)$ which estimates $s(n-2)$ without errors. Then, of course

$$\xi_{\min}=r_d(0)-\sum_{l=0}^{\infty} h(l)r_x(l-2)=r_d(0)-h(2)\,r_x(0)= =1.25-1\cdot 1.25=0$$

4

Given this $s(n) = v(n) + v(n-1)$ with $v(n)$ white noise with variance 1 and

$x(n) = s(n) + w(n)$ with $w(n)$ white noise with variance 1.

Task: Estimate the signal $s(n+1)$ using

a) FIR filter of length 2

b) causal IIR filter

c) Derive minimum errors.

Solution: a) Wiener-Hopf equation for prediction

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} r_x(1) \\ r_x(2) \end{bmatrix}$$

$$P_s(z) = (1+z^{-1})(1+z) = z + 2 + z^{-1}, \qquad P_x(z) = P_s(z) + 1 \qquad r_{dx}(k) = r_s(k+1)$$

$$r_s(k) = [\ 1\ \underset{\uparrow}{2}\ \ 1] \qquad r_x(k) = [\ 1\ \underset{\uparrow}{3}\ \ 1] \qquad P_{dx}(z) = zP_s(z)$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} 3/8 \\ -1/8 \end{bmatrix}$$

b) $P_x(z) = z + 3 + z^{-1} = \sigma^2(1+az^{-1})(1+az)$

gives $a = 0.382,\ \sigma^2 = 2.618,\ \ Q(z) = (1+0.382z^{-1})$

$$H(z) = \frac{1}{\sigma^2 Q(z)} \left[\frac{P_{dx}(z)}{Q_x(z^{-1})} \right]_+ = \frac{0.382}{1+0.382z^{-1}} \left[\frac{z(1+z^{-1})(1+z)}{(1+0.382z)} \right]_+ =$$

$$= \frac{0.382}{1+0.382z^{-1}} \underbrace{\left[\frac{1+2z+z^2}{(1+0.382z)} \right]_+}_{=1} = \frac{0.382}{1+0.382z^{-1}}$$

c) $$\xi_{\min} = r_d(0) - \sum_{l=-\infty}^{\infty} h(l) r_{dx}(l) = r_s(0) - \sum_{l=-\infty}^{\infty} h(l) r_s(l+1) \qquad r_d(0) = r_s(0)$$

FIR $\xi_{\min} = r_d(0) - h(0)r_s(1) - h(1)r_s(2) = 2 - 3/8 \cdot 1 = 1.625$

causal IIR

$$\xi_{\min} = r_d(0) - \sum_{l=0}^{\infty} h(l) r_x(l+1) = r_s(0) - \sum_{l=0}^{\infty} 0.38(-0.38)^l r_s(l+1) =$$

$$= r_d(0) - 0.382(-0.382)^0 r_x(1) = 2 - 0.382 \cdot 1 = 1.618$$

5. Given this: $x(n) = n\ 0.5^n\ u(n)$

Problem:
a) Use Padé to determine $H(z)$ for $p=2,\ q=2$
b) Use Padé to determine $H(z)$ for $p=2,\ q=2$ from $x(n),\ n = 0,1,2...,5$

Solution: $x(n) = [0\ \ 1/2\ \ 1/2\ \ 3/8\ \ 1/4\ \ 5/32\ \ 3/32\ \ 7/128\ \ ...]$

a) This gives the matrix equation

$$\begin{bmatrix} x(0) & 0 & 0 \\ x(1) & x(0) & 0 \\ x(2) & x(1) & x(0) \\ \cdots & \cdots & \cdots \\ x(3) & x(2) & x(1) \\ x(4) & x(3) & x(2) \\ x(5) & x(4) & x(3) \\ x(6) & x(5) & x(4) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ a(1) \\ a(2) \end{bmatrix} = \begin{bmatrix} b(0) \\ b(1) \\ b(2) \\ \cdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Pade: Use the rows 4 and 5 to solve a(1),a(2), Then rows 1,2,3 to solve b(0),b(1),b(2)

$$\begin{bmatrix} x(3) & x(2) & x(1) \\ x(4) & x(3) & x(2) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ a(1) \\ a(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} x(2) & x(1) \\ x(3) & x(2) \end{bmatrix}}_{X_q} \cdot \underbrace{\begin{bmatrix} a(1) \\ a(2) \end{bmatrix}}_{\bar{a}} = -\begin{bmatrix} x(3) \\ x(4) \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 \\ 3/8 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = -\begin{bmatrix} 3/8 \\ 1/4 \end{bmatrix} \qquad \begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = \begin{bmatrix} -1 \\ 0.25 \end{bmatrix}$$

Then, use row 1,2 and 3 to determine b(n)

$$b(0) = x(0) = 0$$
$$b(1) = x(1) + a(1)x(0) = 0.5$$
$$b(2) = x(2) + a(1)x(1) + a(2)x(0) = 0$$

which gives the filter $H(z) = \dfrac{0.5\,z^{-1}}{1 - z^{-1} + 0.25\,z^{-2}}$

Prony: We is allowed to use x(0), x(1)...x(5).
Then use the row 4,5,6 and solve it as an overdetermined equation system

$$X_q = \begin{bmatrix} x(2) & x(1) \\ x(3) & x(2) \\ x(4) & x(3) \end{bmatrix} \quad \textbf{and} \quad X_q^T\, X_q = \begin{bmatrix} x(2) & x(3) & x(4) \\ x(1) & x(2) & x(3) \end{bmatrix} \begin{bmatrix} x(2) & x(1) \\ x(3) & x(2) \\ x(4) & x(3) \end{bmatrix} = \begin{bmatrix} 0.45 & 0.53 \\ 0.53 & 0.64 \end{bmatrix}$$

$$\begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = (X_q^T\ X_q)^{-1}\ X_q^T\ x_{q+1} = ... = \begin{bmatrix} -1 \\ 0.25 \end{bmatrix}$$

Then b(n) the same as in a), which gives $H(z) = \dfrac{0.5\,z^{-1}}{1 - z^{-1} + 0.25\,z^{-2}}$ **.**

Comment: The z-transform of *x(n)* can be found in a formula table to be just
$H(z) = \dfrac{0.5\,z^{-1}}{1 - z^{-1} + 0.25\,z^{-2}}$ **and because of no noise , both methods gives the exact solution.**