Lesson 7

Chapter 8.
Frequency estimation

LTH

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Chapter 8, Spectrum estimation

Nonparametric methods: lesson 6

The periodogram
The modified Periodogram (windowing)
Averaging periodogram
- **Bartlett**
- **Welch**

The Minimum variance method
The Blackman-Tukey method

Parametric methods:

Described in chapter 4
- **Pade**
- **Prony**
- **All-pole model**

Lattice structures in chapter 6

Frequency estimation (Estimation of sinusoids), lesson 7

The well known methods like Pisarenco Harmonic Decomposition and the MUSIC algorithm are presented here. These methods are based on the eigenvectors of the correlation matrix.

Pisarenco Harmonic Decomposition
The MUSIC algorithm
The Eigenvector method (EV)
(Minimum norm)
Principal components Blackman-Tukey frequency estimation
Minimum variance Frequency estimation

Frequency estimation

The model is that we have sinusoids in white noise.

$$x(n) = \sum_{i=1}^{p} A_i \, e^{j\omega_i n} + w(n)$$

with the complex amplitude

$$A_i = |A_i| e^{j\phi_i}$$

The phase is randomly distributed in the interval $-\pi \leq \phi_i \leq \pi$

We want to estimate

I: **The amplitudes** $|A_i|;\qquad A_i = |A_i| e^{j\phi_i}$

II: **The frequency** $\omega_i\,,\ \ f_i$

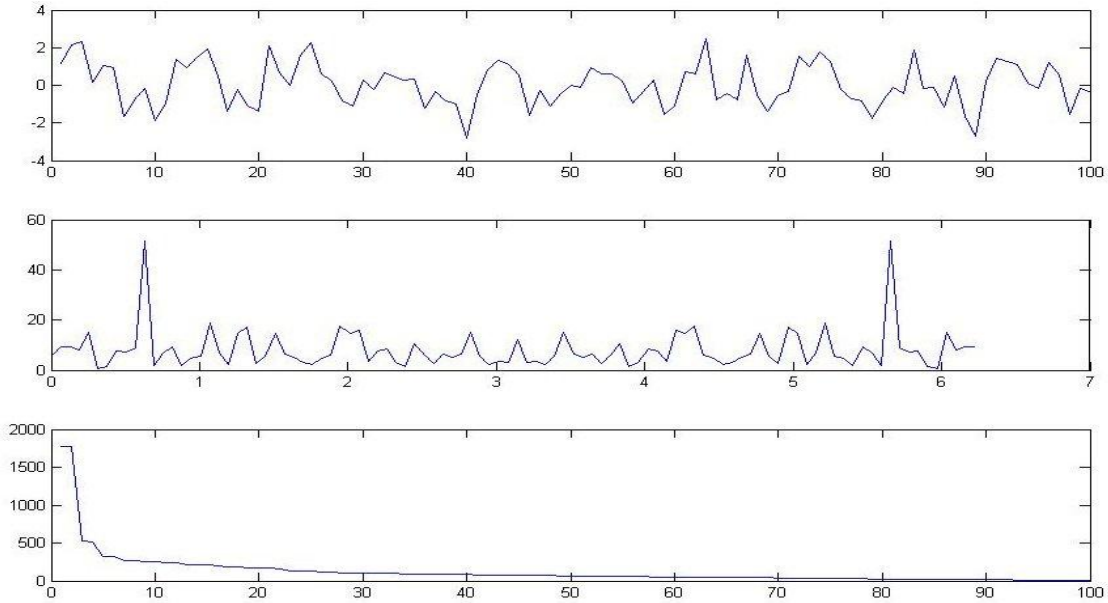
III: **Number of sinusoids p**

Frequency estimation,

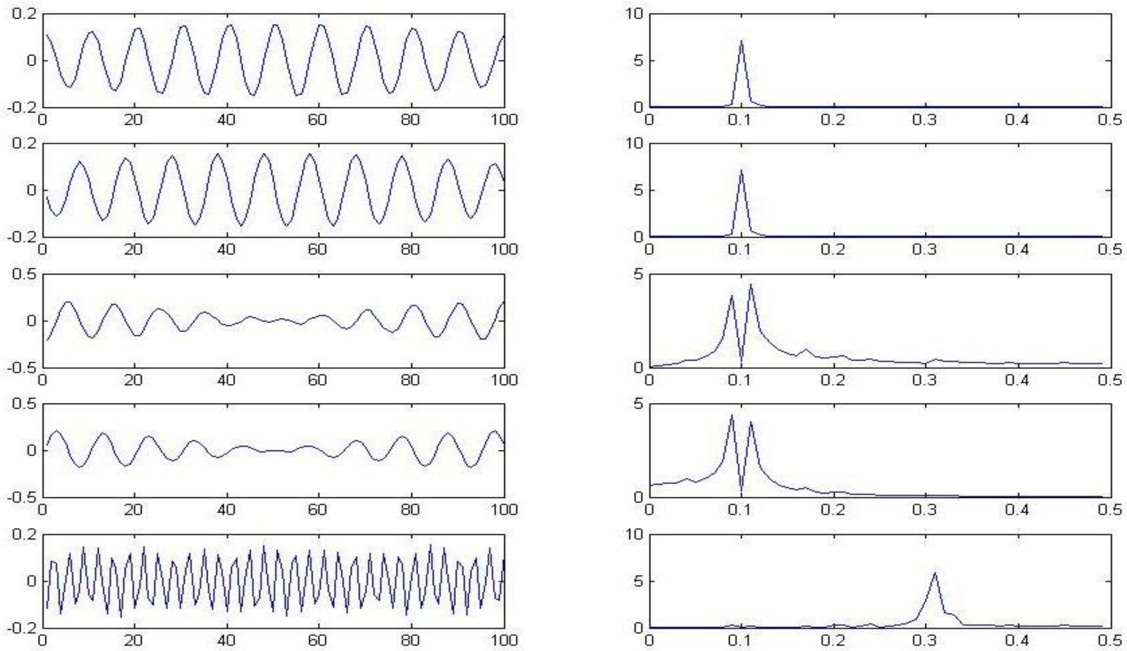
Examples on eigenvectors and eigenvalues of the correlation matrix.

Sinusoid in white noise

$$x(n) = \sin(2*\pi*0.1*n) \ + \ w(n)$$



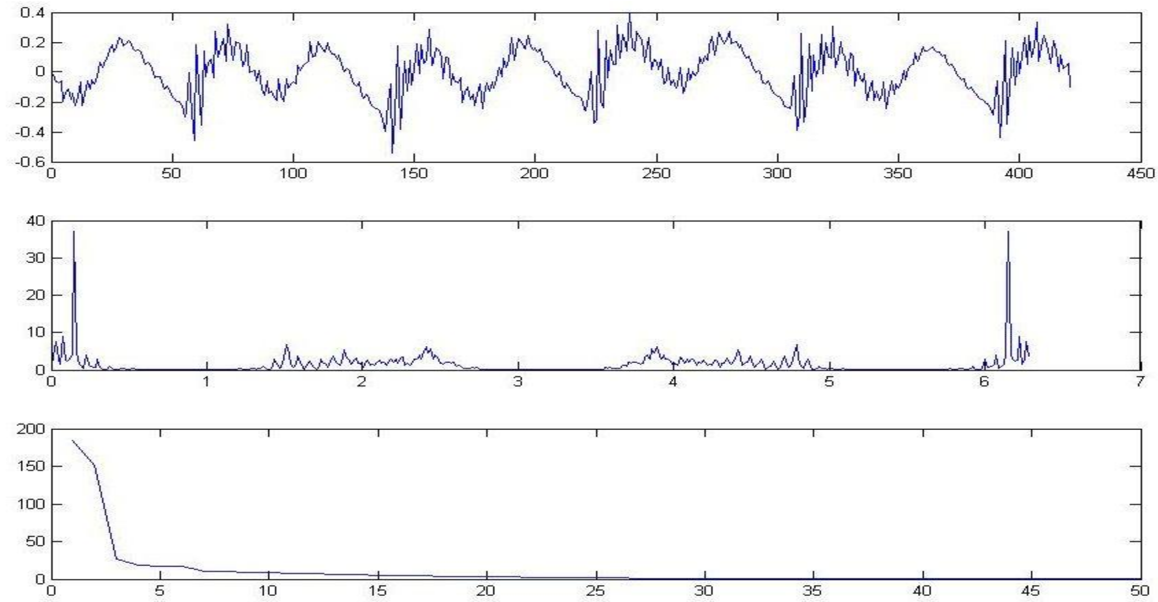
Upper: Waveform of a sinusoid in white noise
Middle: Spectrum from DFT
Lower: The eigenvalues of the correlation matrix R_x.



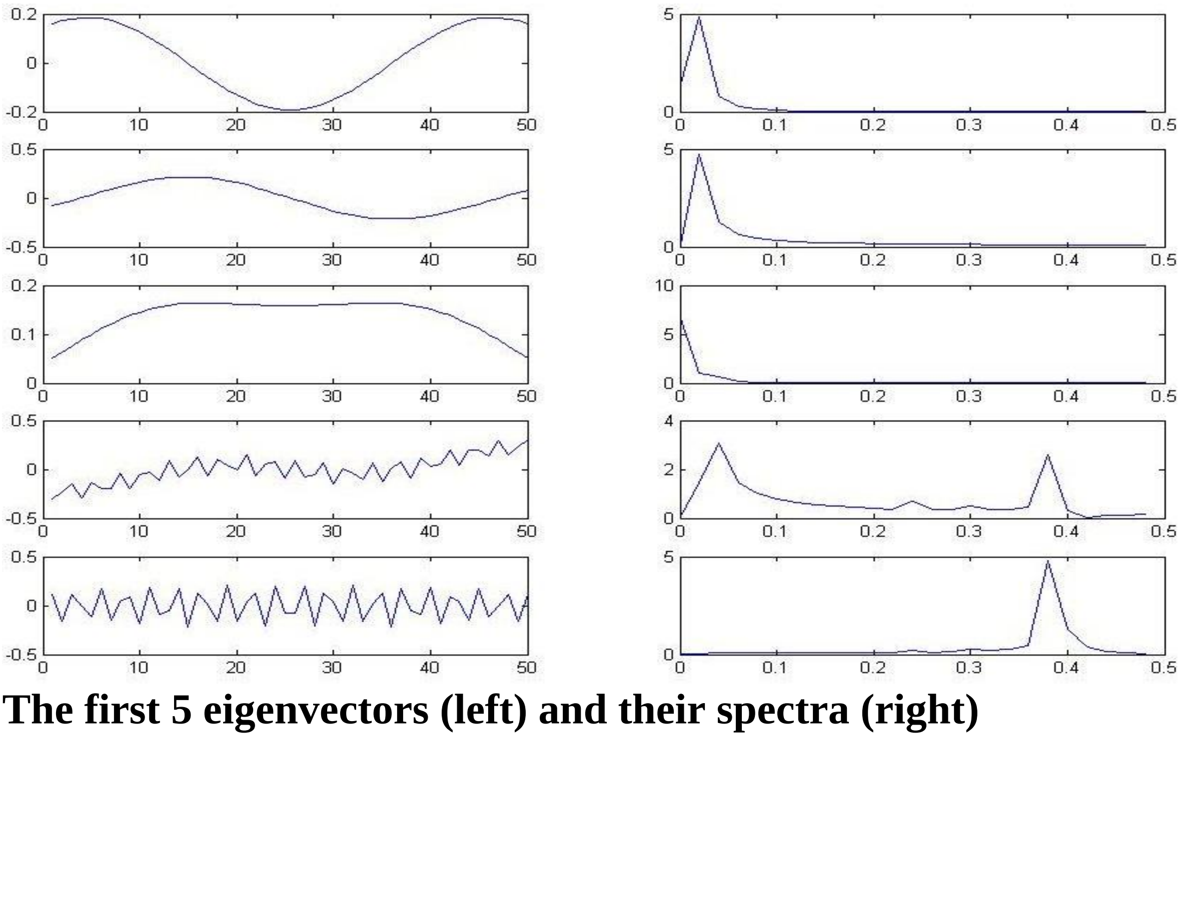
The first 5 eigenvectors (left) and their spectra (right)

Frequency estimation,

Examples on eigenvectors and eigenvalues of the correlation matrix. Vowel 'i'.



Upper: Waveform of a vowel 'i'.
Middle: Spectrum from DFT
Lower: The eigenvalues of the correlation matrix R_x.



The first 5 eigenvectors (left) and their spectra (right)

Frequency estimation, correlation matrix.

We assume first that p=1,

$$x(n) = A_1 e^{j\omega_1 n} + w(n) \quad n = 0,\ldots,N-1$$

or

$$x = A_1 e_1 + w$$

with

$$x = \left[x(0),\ x(1),\ \ldots\ ,x(N-1)\right]^T$$

$$e_1 = \left[1, e^{j\omega_1}, e^{j\omega_1 2}, \ldots , e^{j\omega_1 (N-1)}\right]^T$$

$$w = \left[w(0),\ w(1),\ \ldots\ ,w(N-1)\right]^T$$

The correlation matrix is

$$R_x = E\{x\,x^H\} =$$

$$= E\{(A_1 e_1 + w)(A_1 e_1 + w)^H\} =$$

$$= P_1 e_1 e_1^H + \sigma_w^2 I$$

The power of the sinusoids is $P_1 = |A_1|^2$.

Frequency estimation, eigenvalues and eigenvectors.

Eigenvalues and eigenvectors for sinusoids in white noise.

Multiply R_x **with** e_1 ,

$$R_x e_1 = (P_1 e_1 e_1^H + \sigma_w^2 I) e_1 =$$

$$= (P_1 e_1 e_1^H e_1 + \sigma_w^2 e_1) =$$

$$= (P_1 N + \sigma_w^2) e_1$$

We now identify one eigenvalue and corresponding eigenvector

$$R_x e_1 = (P_1 N + \sigma_w^2) e_1$$

$$\lambda_1 = P_1 N + \sigma_w^2 \qquad (\lambda_{\max})$$

$$v_1 = e_1 \qquad (only\ if \quad p = 1)$$

Frequency estimation, eigenvalues and eigenvectors.

The other eigenvectors must be orthogonal to eigenvector 1.

$$R_x v_i = (P_1 e_1 e_1^H + \sigma_w^2 I) v_i =$$

$$= \sigma_w^2 v_i \qquad\qquad i = 2,3,\ldots,N$$

$$\lambda_2 = \lambda_3 = \ldots = \lambda_N = \sigma_w^2 \qquad (\lambda_{\min})$$

The signal subspace is determined by v_1

The noise subspace is determined by v_i ,i=2,...,N

Frequency estimation.

A: Estimate R_x and determine the eigenvalues and eigenvectors.

B: Estimate the variance of the noise as $\sigma_w^2 = \lambda_{\min}$ **.**

C: Estimate the signal power as

$$\hat{P}_1 = \frac{\lambda_{\max} - \lambda_{\min}}{N}$$

Note that

$$\lambda_1 = P_1 N + \sigma_w^2$$
$$v_1 = e_1$$

D: Estimate the frequency from the eigenvector 1.

$$\omega_1 = \arg\{v_1(1)\}$$ **(second index)**

Frequency estimation, Frequency estimation function

The eigenvectors v_2 **to** v_N **are orthogonal to** $v_1 = e_1$.

$$e_1^H v_i = 0, \quad i = 2, \ldots, N$$

But

$$V_i(e^{j\omega}) = e^H v_i = \sum_{k=0}^{N-1} v_i(k) e^{-j\omega k}$$

For $\omega = \omega_1$ **we have**

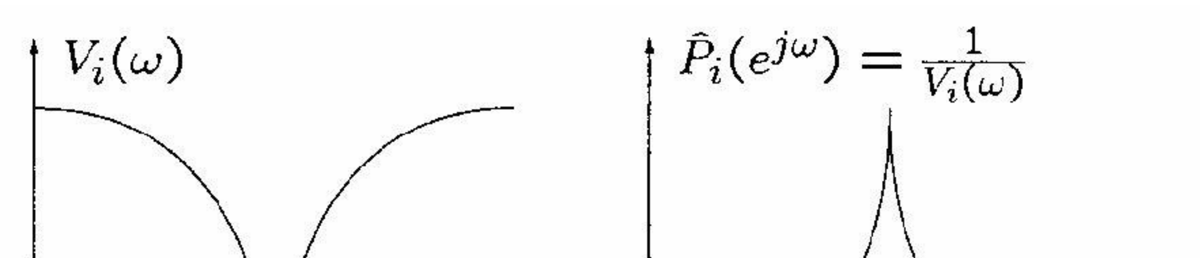
$$V_i(e^{j\omega_1}) = e_1^H v_i = \sum_{k=0}^{N-1} v_i(k) e^{-j\omega_1 k} = 0$$

This is valid for all eigenvectors v_2 **to** v_N.

We define the frequency estimation function as

$$\hat{P}_i(e^{j\omega}) = \left| \frac{1}{V_i(e^{j\omega k})} \right|^2 = \frac{1}{\left| \sum_{k=0}^{N-1} v_i(k) e^{-j\omega k} \right|^2} =$$

$$= \frac{1}{|e^H v_i|^2}$$



We can also compute the Z-transform

$$V_i(z) = \sum_{k=0}^{N-1} v_i(k) z^{-k}$$

and determine the zeroes of $V_i(z)$

Frequency estimation.

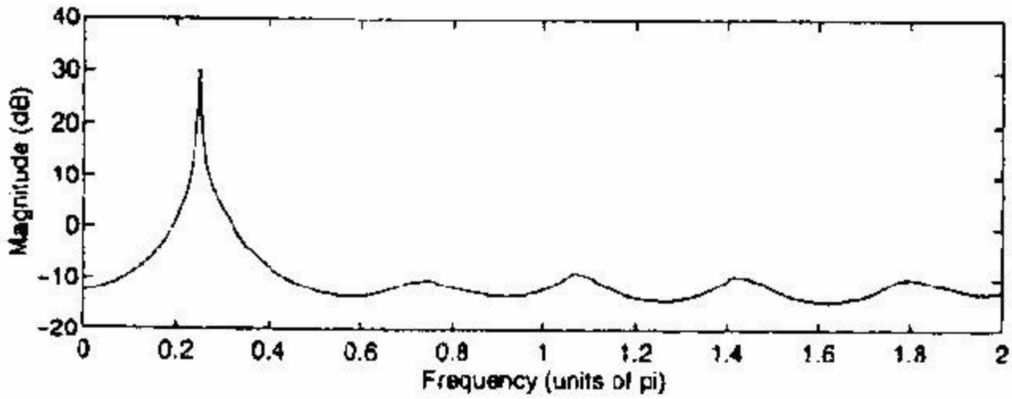
Averaging over all noise eigenvectors yield

$$\hat{P}(e^{j\omega}) = \frac{1}{\sum_{i=2}^{N} \alpha_i \,|e^H v_i|^2}$$

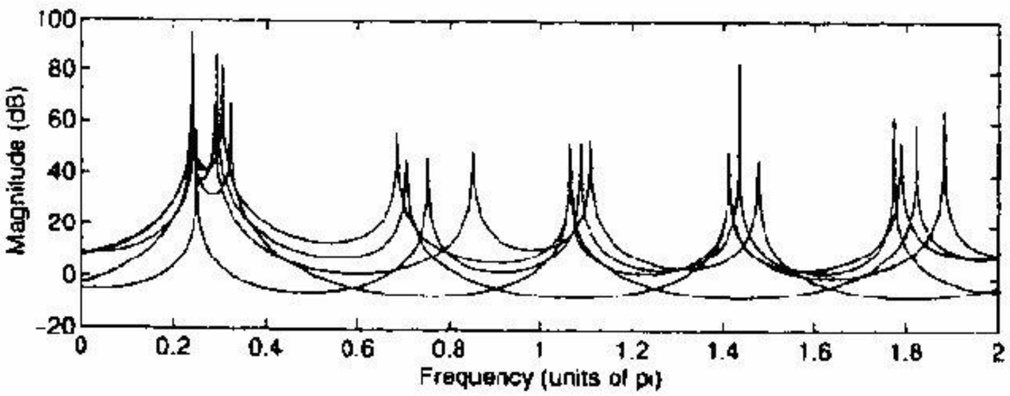
Example from the textbook page 455

Upper figure: **Averaging over the noise eigenvectors with the weight equal to one.**

Lower figure: **Overlay plot over the frequency estimation function from each of the noise eigenvectors**



(a)



Frequency estimation.
Several sinusoids in white noise

$$x(n) = \sum_{i=1}^{p} A_i \, e^{j\omega_i n} + w(n)$$

and for p=2

$$R_x = E\{x\,x^H\} =$$

$$= E\{(A_1 e_1 + A_2 e_2 + w)(A_1 e_1 + A_2 e_2 + w)^H\} =$$

$$= P_1 e_1 e_1^H + P_2 e_2 e_2^H + \sigma_w^2 \, I$$

and

Eigenvaules

$$\lambda_i \approx \begin{cases} \upsilon_i + \sigma_w^2 & signal\ \ subspace \\ \sigma_w^2 & noise\ \ subspace \end{cases}$$

$$\upsilon_i \ are\ the\ eigenvalues\ in\ the\ signal\ \ subspace$$

The frequency estimation function is now

$$\hat{P}(e^{j\omega}) = \frac{1}{\sum_{i=p+1}^{N} \alpha_i \, |e^H v_i|^2}$$

We will now look at some methods using the frequency estimation function above.

The first is called the Pisarenco Decomposition method. This method is very sensitive to the noise but describe the principle for the methods.

A well known method is the MUSIC algorithm.

The frequency estimation function is sometimes called the pseudospectrum or eigenspectrum.

Frequency estimation: Pisarenko Harmonic Decomposition (PHD)

1: Assume *p* complex sinusoids in white noise

2: Assume the dimension of R_x (p+1)*(p+1), i.e. only one noise eigenvector.

This assumptions means that only one eigenvector corresponds to the noise subspace.

Then $$\lambda_{\min} = \lambda_{p+1} = \sigma_w^2$$

and the frequency estimation function (pseudospectrum) is defined

$$\hat{P}_{PHD}(e^{j\omega}) = \frac{1}{|\sum_{k=0}^{p} v_{\min}(k) e^{-j\omega k}|^2} =$$

$$= \frac{1}{|e^H v_{\min}|^2}$$

$$V_{\min}(z) = \sum_{k=0}^{p} v_{\min}(k)\, z^{-k}$$

The power of the p complex sinusoids are given by the solution of;

$$\begin{bmatrix} \left|V_1(e^{jw_1})\right|^2 & \left|V_1(e^{jw_2})\right|^2 & \cdots & \left|V_1(e^{jw_p})\right|^2 \\ \left|V_2(e^{jw_1})\right|^2 & \left|V_2(e^{jw_2})\right|^2 & \cdots & \left|V_2(e^{jw_p})\right|^2 \\ \vdots & \vdots & & \vdots \\ \left|V_p(e^{jw_1})\right|^2 & \left|V_p(e^{jw_2})\right|^2 & \cdots & \left|V_p(e^{jw_p})\right|^2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_p \end{bmatrix} = \begin{bmatrix} \lambda_1 - \sigma_w^2 \\ \lambda_2 - \sigma_w^2 \\ \vdots \\ \lambda_p - \sigma_w^2 \end{bmatrix}$$

(8.160)

Frequency estimation: MUSIC

Page 464, 465

MUSIC: MUltiple **SI**gnal **C**haracterization

The frequency estimation is achieved by averaging the pseudospectra over the noise eigenvectors.

$$\hat{P}_{MU}(e^{j\omega}) = \frac{1}{\sum_{i=p+1}^{M} |e^H v_i|^2}$$

Then estimate the position of the peaks in $\hat{P}_{MU}(e^{j\omega})$

Principal Components Spectrum Estimation.

These methods use the signal subspace. (page 470-471)

The Blackman-Tukey power spectrum was determined from a windowed autocorrelation sequence

$$\hat{P}_{BT}(e^{j\omega}) = \sum_{k=-N+1}^{N-1} \hat{r}_x(k)\, w(k)\, e^{-j\omega k}$$

If w(k) is a Bartlett window, the Blackman-Tukey estimate can be written in terms of the autocorrelation matrix

$$\hat{P}_{BT}(e^{j\omega}) = \frac{1}{M} \sum_{k=-M}^{M} (M - |k|)\, \hat{r}_x(k) e^{j\omega k} = \frac{1}{M} e^H R_x e$$

In terms of eigenvectors (eigendecomposition) this is

$$\hat{P}_{BT}(e^{j\omega}) = \frac{1}{M} \sum_{i=1}^{M} \lambda_i \, | e^H v_i |^2$$

Now, use only the eigenvectors corresponding to the sinusoids. Then the Blackman-Tukey principal frequency estimation is defined by

$$\hat{P}_{PC-BT}(e^{j\omega}) = \frac{1}{M}\sum_{i=1}^{p} \lambda_i |e^H v_i|^2$$

The minimum variance power spectrum estimate was defined by

$$P_{MV}(e^{j\omega}) = \frac{M}{e^H R_x^{-1} e}$$

Rewrite this in terms of eigenvectors and only use eigenvectors corresponding to the sinusoids gives the minimum variance frequency estimation

$$P_{PC-MV}(e^{j\omega}) = \frac{M}{\sum_{i=1}^{p} \frac{1}{\lambda_i} e^H v_i}$$

Table 8.10 Noise Subspace Methods for Frequency Estimation

Pisarenko	$\hat{P}_{PHD}(e^{j\omega}) = \dfrac{1}{\lvert \mathbf{e}^H \mathbf{v}_{\min} \rvert^2}$
MUSIC	$\hat{P}_{MU}(e^{j\omega}) = \dfrac{1}{\sum_{i=p+1}^{M} \lvert \mathbf{e}^H \mathbf{v}_i \rvert^2}$
Eigenvector Method	$\hat{P}_{EV}(e^{j\omega}) = \dfrac{1}{\sum_{i=p+1}^{M} \frac{1}{\lambda_i} \lvert \mathbf{e}^H \mathbf{v}_i \rvert^2}$
Minimum Norm	$\hat{P}_{MN}(e^{j\omega}) = \dfrac{1}{\lvert \mathbf{e}^H \mathbf{a} \rvert^2}$; $\mathbf{a} = \lambda \mathbf{P}_n \mathbf{u}_1$

$\mathbf{P}_n = \mathbf{V}_n \mathbf{V}_n^H$ **where** $\mathbf{V}_n$ **is a matrix spanning the noise subspace.**

$\mathbf{u}_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ **and** $\lambda = \dfrac{1}{\mathbf{u}_1^H \mathbf{P}_n \mathbf{u}_1}$

Table 8.11 *Signal Subspace Methods*

Blackman-Tukey	$\hat{P}_{PC-BT}(e^{j\omega}) = \frac{1}{M}\sum_{i=1}^{p} \lambda_i \lvert \mathbf{e}^H \mathbf{v}_i \rvert^2$
Minimum variance	$\hat{P}_{PC-MV}(e^{j\omega}) = \frac{M}{\sum_{i=1}^{p} \frac{1}{\lambda_i} \lvert \mathbf{e}^H \mathbf{v}_i \rvert^2}$
Autoregressive	$\hat{P}_{PC-AR}(e^{j\omega}) = \frac{1}{\left\lvert \sum_{i=1}^{p} \alpha_i \mathbf{e}^H \mathbf{v}_i \right\rvert^2}$

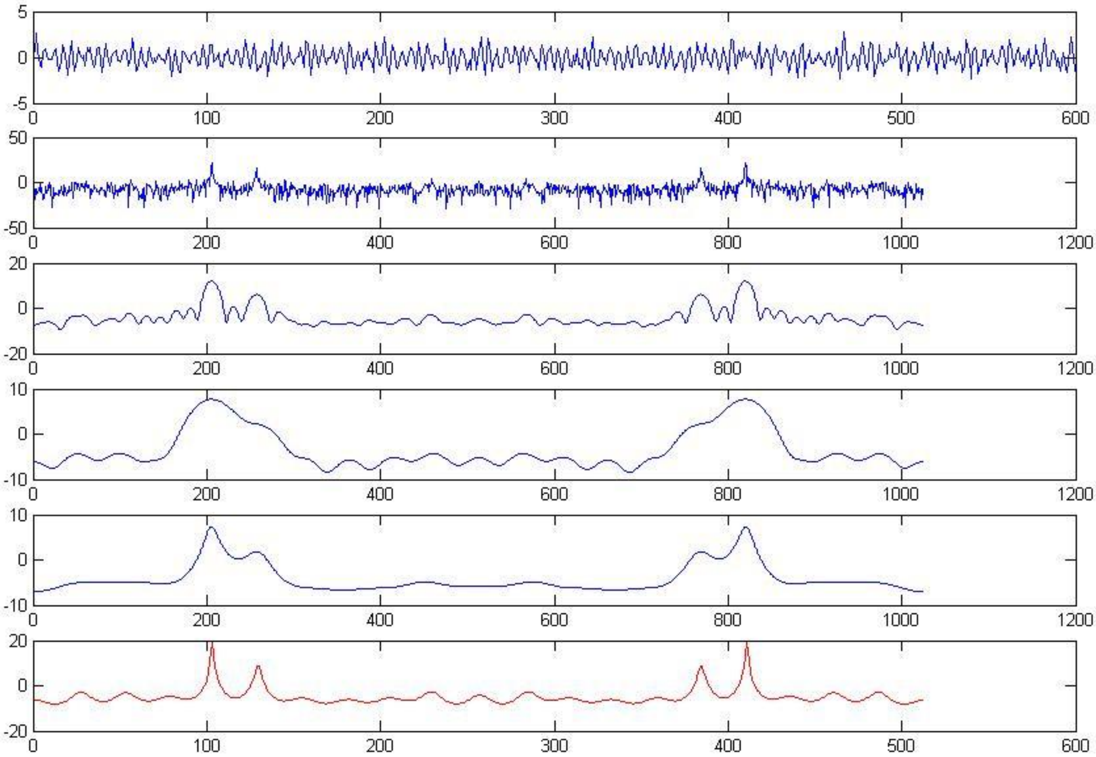
$$\alpha_i = v_i^*(0) / \lambda_i$$

Example of sinusoids in white noise

Power spectrum estimation

$$x(n) = \sin(2\pi \cdot 0.20 \cdot n) + 0.5\sin(2\pi \cdot 0.25 \cdot n) + v(n)$$

$$\text{with} \quad v(n) = \text{white noise with varaiance } 1$$



Row 1: Waveform of input signal x(n)
Row 2 FFT of x(n), N=1024 (Periodogram)

Row 3 Averaging with the Welch method (10 subintervals, rectangular time window)
Row 4 Blackman-Tukey estimate with M=20 (hamming window)
Row 5 Minimum variance method with M=20,
Row 6 All-pole model of order M=20 (Levinson Durbin algorithm).

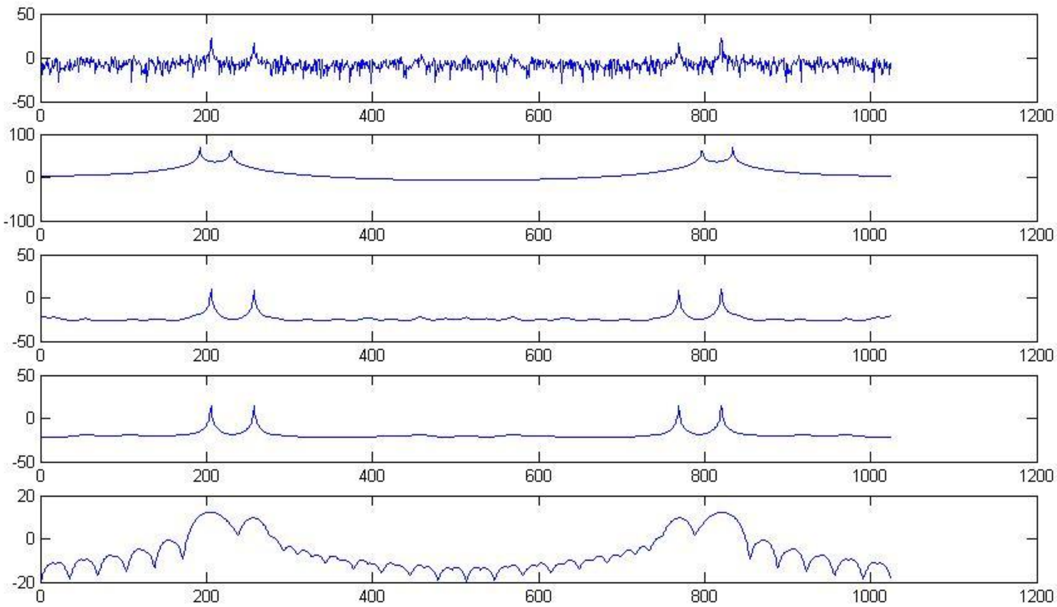
(All spectra in 1024 frequency points. y axis in dB)

Example of sinusoids in white noise

Frequency estimation methods

$$x(n) = \sin(2\pi \cdot 0.20 \cdot n) + 0.5\sin(2\pi \cdot 0.25 \cdot n) + v(n)$$

$$\text{with} \quad v(n) = \text{white noise with varaiance } 1$$



Row 1 FFT of x(n), N=1024 (Periodogram).
Row 2 Pisarenco Harmonic Decomposition p=4, M=5.
Row 3 The MUSIC algorithm p=4, M=30.
Row 4 The Eigenvector method (EV) p=4, M=30.
Row 5 Principal components Blackman-Tukey frequency estimation (PC-BT) p=4, M=30.

(All spectra in 1024 frequency points, y axis in dB)