Lesson 6

Chapter 8.
Spectrum estimation

LTH

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Spectrum estimation, Chapter 8

Nonparametric methods:

The periodogram
The modified Periodogram (windowing)
Averaging periodogram
- **Bartlett**
- **Welch**

The Blackman-Tukey method

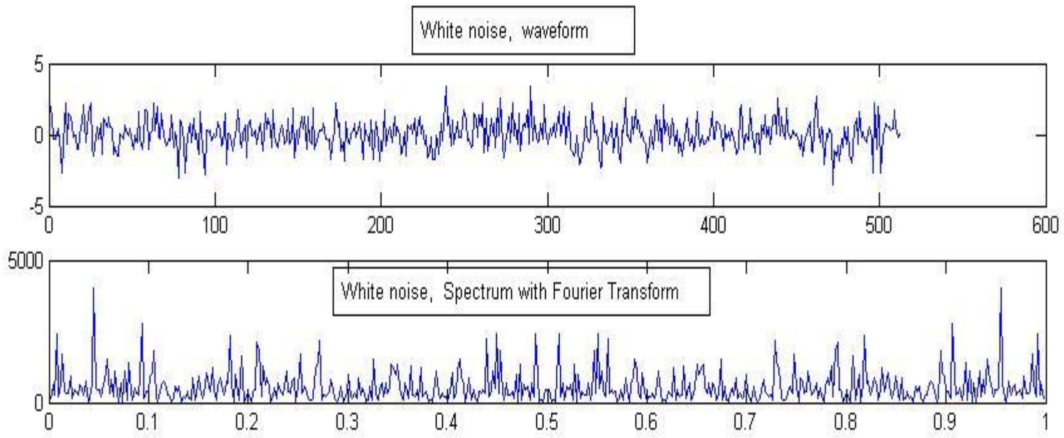
Parametric methods:

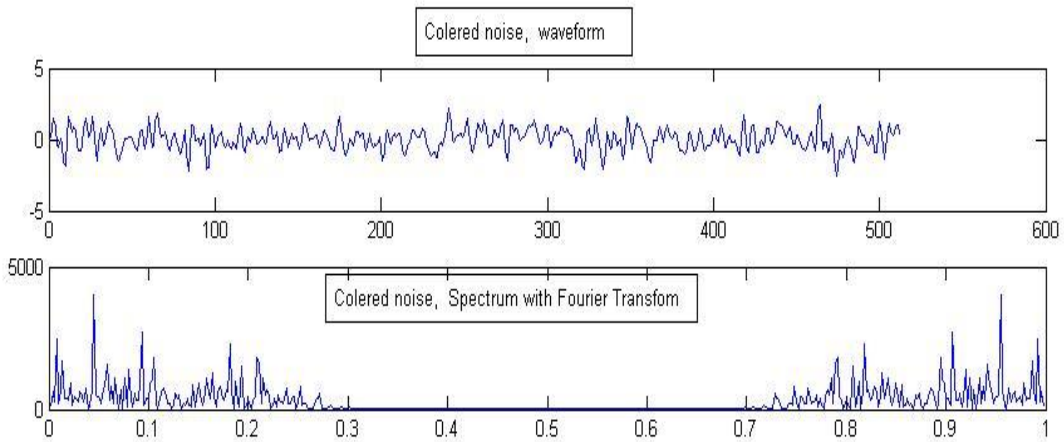
Described in chapter 4

Frequency estimation (Estimation of sinusoids), lesson 7

The well known methods like Pisarenco Harmonic Decomposition and the MUSIC algorithm are presented here. These methods are based on the eigenvectors of the correlation matrix.

Examples of waveforms and Fourier Transforms





Row 1: White noise (N=512 values)
Row 2: Fourier transform of the signal in row 1 (magnitude) (N=512 values)

Row 3: Coloured noise (output from 4th order Butterworth filter)
Row 4: Fourier transform of the signal in row 3 (magnitude) (N=512 values)

Estimation of power spectra – periodogram
(page 393-394)

We want to estimate $r_x(k)$ **from** ***x(n)*** **in the interval** $0 \le n \le N-1$**.**
In chapter 3 we had

$$\hat{r}_x(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\,x(n-k)$$

To ensure that the values that fall outside the interval are excluded, we write

$$\hat{r}_x(k) = \frac{1}{N}\sum_{n=0}^{N-1-k} x(n+k)\,x(n) \qquad 0 \le k \le N-1$$

Using a rectangular window

$$w_R(n) = [\underbrace{1\ 1\ 1\cdots 1}_{N}] \quad \text{rectangular window}$$

this can be written

$$x_N(n) = x(n)\cdot w_R(n)$$

or

$$x_N(n) = \begin{cases} x(n) & 0 \le n \le N-1 \\ 0 & otherwise \end{cases}$$

The estimated autocorrelation can now be written

$$\hat{r}_x(k) = \frac{1}{N} \sum_{n=-\infty}^{\infty} x_N(n+k) x_N(n)$$

$$= \frac{1}{N} \sum_{n=-\infty}^{\infty} x(n+k) w_R(n+k) x(n) w_R(n)$$

$$= \frac{1}{N} x_N(k) * x_N(-k)$$

Then $\hat{r}_x(k)$ **is defined for** $-N+1 \le k \le N-1$

Now, we take the Fourier Transform of $\hat{r}_x(k)$**, and then we get**

$$\hat{P}_{per}(e^{j\omega}) = \sum_{k=-N+1}^{N-1} \hat{r}_x(k)\ e^{-j\omega k}$$

which is called the periodogram.

We see that it also can be written

$$\hat{P}_{per}(e^{j\omega}) = \frac{1}{N} X_N(e^{j\omega}) X_N{}^*(e^{j\omega}) = \frac{1}{N} | X_N(e^{j\omega}) |^2$$

Using DFT (FFT), the periodogram will be

$$\hat{P}_{per}(e^{j2\pi k/N}) = \frac{1}{N} X_N(k) X_N{}^*(k) = \frac{1}{N} | X_N(k) |^2$$

The Performance of the Periodogram **(page 398-399)**

The estimate is unbiased if

$$E\{\hat{P}_x(e^{j\omega})\} = P_x(e^{j\omega})$$

The estimate is consistent if it is (asymptotically) unbiased and if

$$\lim_{N\to\infty} \mathrm{var}\{\hat{P}_x(e^{j\omega})\} = 0$$

Taking the mean of $\hat{r}_x(k)$, we got ($k \geq 0$) (page 398-399)

$$E\{\hat{r}(k)\} = \frac{1}{N}\sum_{n=-\infty}^{\infty} E\{x_N(n+k)x_N(n)\} =$$

$$= \frac{1}{N}\sum_{n=0}^{N-1-k} E\{x(n+k)x(n)\} = \frac{1}{N}\sum_{n=0}^{N-1-k} r_x(k) = \frac{N-k}{N}\; r_x(k)$$

Defining the Bartlett (triangular) window

$$w_B(k) = \begin{cases} \dfrac{N-|k|}{N} & |k| \leq N \\ 0 & |k| > N \end{cases}$$

we can write

$$E\{\hat{r}_x(k)\} = w_B(k)\, r_x(k)$$

Using this, we have (page 399)

$$E\{\hat{P}_{per}(e^{j\omega})\} = E\{\sum_{k=-N+1}^{N-1} \hat{r}_x(k)\; e^{-j\omega k}\} =$$

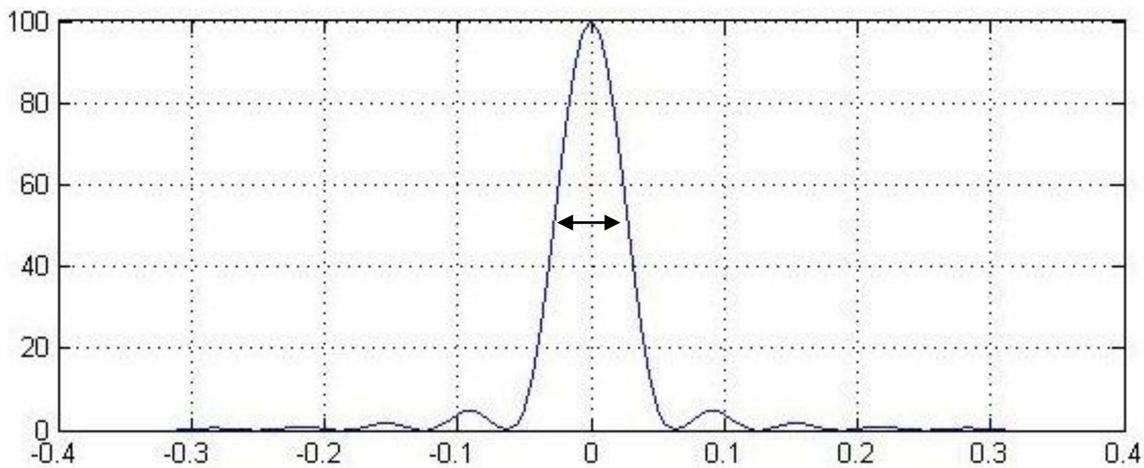
$$= \sum_{k=-N+1}^{N-1} E\{\hat{r}_x(k)\} e^{-j\omega k} = \sum_{k=-\infty}^{\infty} r_x(k)\, w_B(k)\, e^{-j\omega k}$$

or

$$E\{\hat{P}_{per}(e^{j\omega})\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega})$$

The Bartlett (triangular) window can be seen as the convolution of two rectangular windows. The window is

$$W_B(e^{j\omega}) = \frac{1}{N}\left[\frac{\sin(N\omega/2)}{\sin(\omega/2)}\right]^2$$



Plot of $W_B(e^{j\omega})$ **, N=100, bandwidth 0.89*2π/N**

The estimate in asymptotically unbiased due to

$$\lim_{N\to\infty} E\{\hat{P}_{per}(e^{j\omega})\} = P_x(e^{j\omega})$$

The variance is (textbook page 404, 405)

$$\mathrm{var}\{\hat{P}_{per}(e^{j\omega})\} \approx P_x^{\ 2}(e^{j\omega})$$

so the periodogram is not a consistent estimate of the power spectrum.

Table 8.1 ***Properties of the Periodogram***

$$\hat{P}_{per}(e^{j\omega}) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{-jn\omega}\right|^2$$

Bias

$$E\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega})$$

Resolution

$$\Delta\omega = 0.89\frac{2\pi}{N}$$

Variance

$$\mathrm{Var}\left\{\hat{P}_{per}(e^{j\omega})\right\} \approx P_x^2(e^{j\omega})$$

The Modified Periodogram (windowing *x(n)*)

The periodogram use a rectangular window $w_R(n)$

$$\hat{P}_{per}(e^{j\omega}) = \frac{1}{N} \left| X_N(e^{j\omega}) \right|^2 = \frac{1}{N} \left| \sum_{n=-\infty}^{\infty} x(n)\, w_R(n)\; e^{-j\omega n} \right|^2$$

If we use other windows, we got the modified periodogram

$$\hat{P}(e^{j\omega}) = \frac{1}{NU} \left| \sum_{n=-\infty}^{\infty} x(n) w(n) e^{-j\omega n} \right|^2$$

$$U = \frac{1}{N} \sum_{n=0}^{N-1} \left| w(n) \right|^2$$

Table 8.2 ***Properties of a Few Commonly Used Windows. Each Window is Assumed to be of Length N.***

Window	Sidelobe Level (dB)	3 dB BW $(\Delta\omega)_{3dB}$
Rectangular	−13	$0.89(2\pi/N)$
Bartlett	−27	$1.28(2\pi/N)$
Hanning	−32	$1.44(2\pi/N)$
Hamming	−43	$1.30(2\pi/N)$
Blackman	−58	$1.68(2\pi/N)$

Properties of the modified periodogram

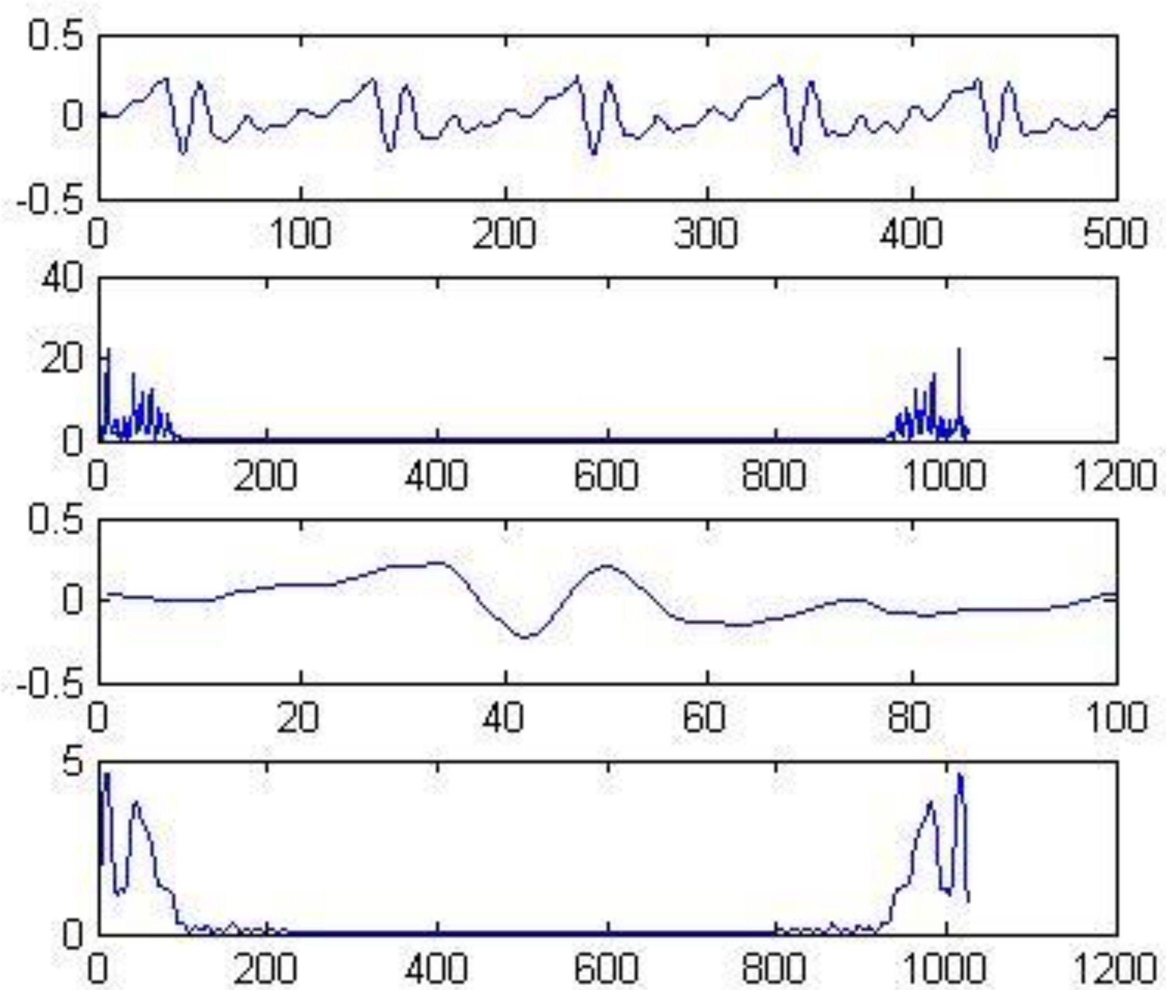
Table 8.3 Properties of the Modified Periodogram

$$\hat{P}_M(e^{j\omega}) = \frac{1}{NU}\left|\sum_{n=-\infty}^{\infty} w(n)x(n)e^{-jn\omega}\right|^2$$

$$U = \frac{1}{N}\sum_{n=0}^{N-1}|w(n)|^2$$

Bias	$E\left\{\hat{P}_M(e^{j\omega})\right\} = \frac{1}{2\pi NU} P_x(e^{j\omega}) * \|W(e^{j\omega})\|^2$
Resolution	Window dependent
Variance	$\text{Var}\left\{\hat{P}_M(e^{j\omega})\right\} \approx P_x^2(e^{j\omega})$

Example of resolution



Row 1: **Waveform of a vowel ‘a’, N=500 (50 ms)-**
Row 2: **Fourier transform of the N=500 values in row 1.**
Row 3: **Part of the waveform in row 1, N=100, (10 ms)**
Row 4: **Fourier transform of the N=100 values in row 3.**

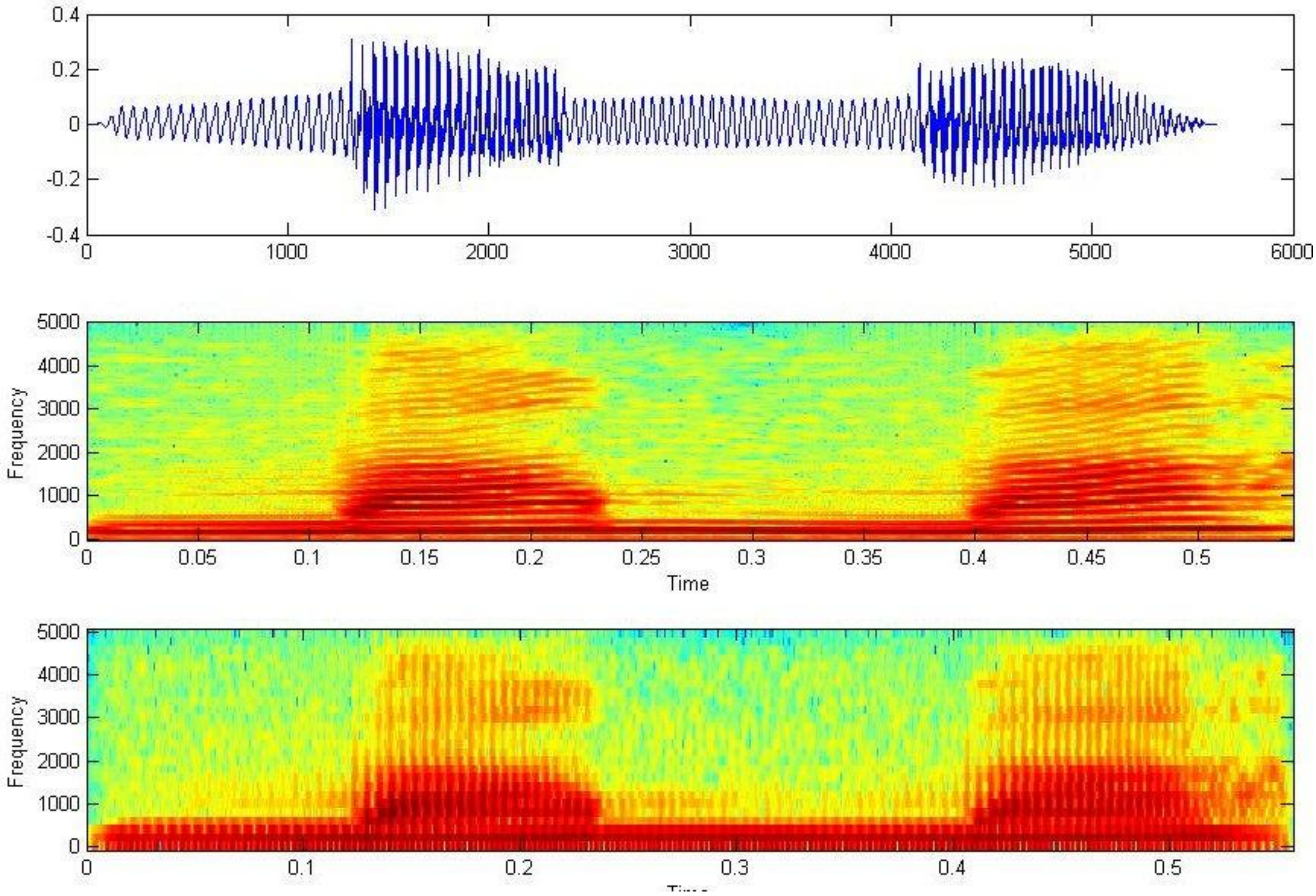
Spectrogram

Spectrogram is a plot of spectrum as function of the time using a sliding window. The command in Matlab is

specgram(x,Nfft,Fs);

Nfft is the length of the time window (length of the fft).
Fs is the sample frequency.

Example of spectrogram of the word ‘mamma’.



Top: **Waveform of the word ‘mamma’.**
Middle: **Specgram with wide time window, N=200 (20 ms)**
Bottom: **Specgram with narrow time window, N=50 (5 ms)**

Averaging periodogram. Bartlett's Method
(page 412.414)

In order to reduce the variance we may use averaging.

We divide the input sequence *x(n)* of length N into K blocks of length L,

$$K = \frac{N}{L}$$

Then, determine the power spectra for each block and take the average. <u>The variance will decrease but at the same time the resolution will decrease</u>.

The variance will be

$$\mathrm{var}\{\hat{P}_B(e^{j\omega})\} \approx \frac{1}{K} P_x^2(e^{j\omega})$$

Table 8.4 ***Properties of Bartlett's Method***

$$\hat{P}_B(e^{j\omega}) = \frac{1}{N}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} x(n+iL)e^{-jn\omega}\right|^2$$

Bias

$$E\left\{\hat{P}_B(e^{j\omega})\right\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega})$$

Resolution

$$\Delta\omega = 0.89K\,\frac{2\pi}{N}$$

Variance

$$\mathrm{Var}\left\{\hat{P}_B(e^{j\omega})\right\} \approx \frac{1}{K} P_x^2(e^{j\omega})$$

Averaging periodogram. Welch's Method
(page 419)

The method of Welch is similar to the Bartlett's method but we allow overlapping of the blocks and using windows *w(n)*.

The estimated properties of Welch's method is found in table 8.5

Table 8.5 *Properties of Welch's Method*

$$\hat{P}_W(e^{j\omega}) = \frac{1}{KLU} \sum_{i=0}^{K-1} \left| \sum_{n=0}^{L-1} w(n)x(n+iD)e^{-jn\omega} \right|^2$$

$$U = \frac{1}{L} \sum_{n=0}^{L-1} |w(n)|^2$$

Bias

$$E\left\{\hat{P}_W(e^{j\omega})\right\} = \frac{1}{2\pi LU} P_x(e^{j\omega}) * |W(e^{j\omega})|^2$$

Resolution Window dependent

Variance†

$$\text{Var}\left\{\hat{P}_W(e^{j\omega})\right\} \approx \frac{9}{16}\frac{L}{N} P_x^2(e^{j\omega})$$

Blackman-Tukey Method
(page 420-423)

With the Blackman-Tukey method we calculate $\hat{r}_x(k)$ from all N data. But for large *k*, the estimate has high variance.

Multiply $\hat{r}_x(k)$ with a window symmetric around *k=0* and take the Fourier Transform. This gives

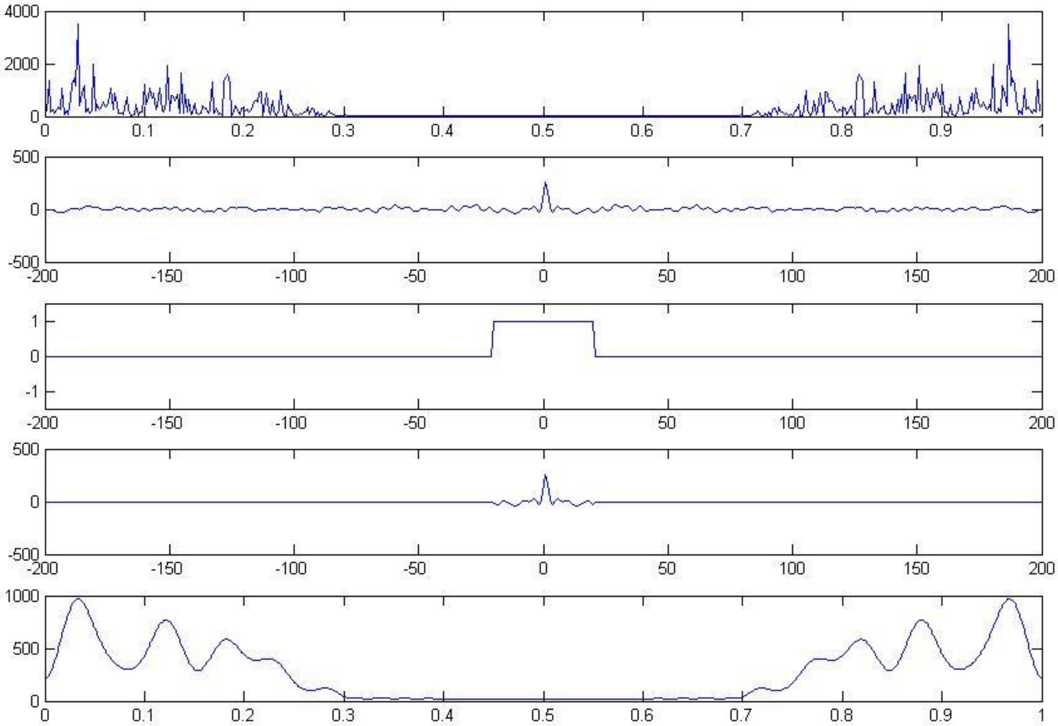
$$\hat{P}_{BT}(e^{j\omega}) = \sum_{k=-N+1}^{N-1} \hat{r}_x(k)\, w(k)\, e^{-j\omega k}$$

The spectrum of the window must be positive for all frequencies, i.e. $W(e^{j\omega}) \geq 0$, to guarantee that $P_{BT}(e^{j\omega}) \geq 0$. This is not true for a rectangular time window.

Table 8.6 ***Properties of the Blackman-Tukey Method***

	$\hat{P}_{BT}(e^{j\omega}) = \sum_{k=-M}^{M} \hat{r}_x(k) w(k) e^{-jk\omega}$
Bias	
	$E\left\{\hat{P}_{BT}(e^{j\omega})\right\} \approx \frac{1}{2\pi} P_x(e^{j\omega}) * W(e^{j\omega})$
Resolution	Window dependent
Variance	
	$\text{Var}\left\{\hat{P}_{BT}(e^{j\omega})\right\} \approx P_x^2(e^{j\omega}) \frac{1}{N} \sum_{k=-M}^{M} w^2(k)$

Example of the Blackman-Tukey Method



Row 1: **Spectrum from FFT.**
Row 2: **Autocorrelation**
Row 3: **Time window**
Row 4: **Windowed autocorrelation**
Row 5: **Blackman-Tukey Spectrum (from windowed autocorr)**

Conclusion

We have always a trade-off between resolution and variance.

Time windows

Rectangular window has the best resolution but also highest leakage (highest side lobes)

Averaging

Averaging decreases the variance but for fix length of data the resolution also will decrease.

Performance comparisons

Definitions see page 424-426

Resolution: $\Delta\omega$

Variability:
$$\nu = \frac{\operatorname{var}\{\hat{P}_x(e^{j\omega}\}}{(E\{\hat{P}_x(e^{j\omega}\})^2}$$

Figure of merit: $M = \nu \cdot \Delta\omega$

Quality factor:
$$Q = \frac{1}{\nu}$$

Filter bank implementation of periodogram

We can interpret the periodogram as the output from of bank of band pass filters.

$$P_x(e^{j\omega}) = \frac{1}{N} \left| \sum_{k=0}^{N-1} x(k) e^{-j\omega k} \right|^2$$

For the frequency ω_i , this can be written

$$P_x(e^{j\omega_i}) = | y(n) |_{n=0}^2 = N \left| \sum_{k=0}^{N-1} x(k) h_i(n-k) \right|_{n=0}^2$$

i.e. the squared of the output from the filter at n=0;

The band pass filters are then

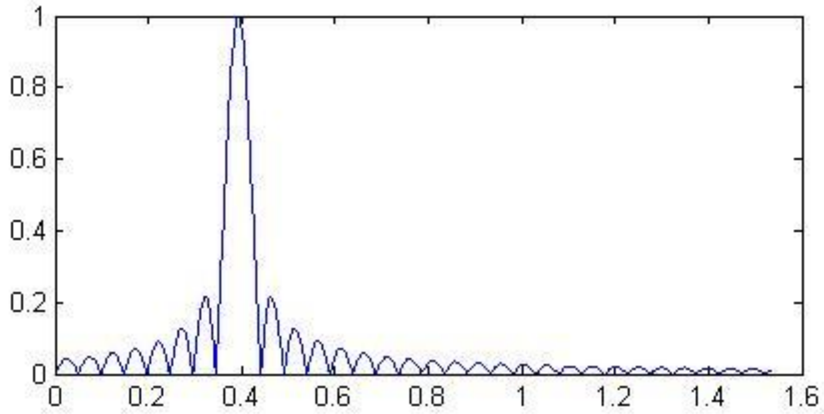
$$h_i(k) = \begin{cases} \frac{1}{N} e^{j\omega_i k} & k = -(N-1),...,0 \\ 0 & otherwise \end{cases}$$

The Fourier transform of the filters

$$h_i(k) = \begin{cases} \frac{1}{N} e^{j\,\omega_i k} & k = -(N-1),\ldots,0 \\ 0 & otherwise \end{cases}$$

are

$$H_i(e^{j\,\omega_i}) = \frac{\sin(N(\omega - \omega_i)/2)}{N\sin((\omega - \omega_i)/2)}\; e^{-j(\omega-\omega_i)(N-1)/2}$$



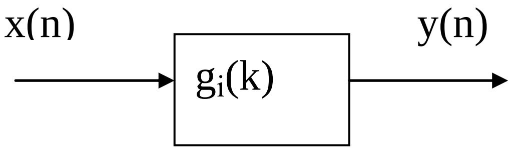
Conclusion: The value of the spectrum at this frequency is the output at n=0 from the band pass filter. The bandwidth is approximately

$$\Delta\omega = 2\pi / N$$

$$\Delta f = 1/N$$

Minimum variance spectral estimation
(page 426-429)

We use the idea of band pass filters



The output y(n) is an narrowband signal out from the band pass filter.

$$g_i(k), \quad G_i(e^{j\omega_i})$$

1. **Design a bank of band pass filters $g_i(k)$ with center frequency ω_i so that each filter rejects the maximum out-of-band power while passing component at ω_i with no distortions.**

2. **Filter x(n) with each filter and estimate the output power.**

3. **Set $\hat{P}_x(e^{j\omega_i})$ equal to the estimated power in step 2 divided by the filter bandwidth.**

The band pass filter depends on the properties of the signal x(n).

Use the vector notation:

Band pas sfilter: $g_i = [g_i(0), g_i(1), ..., g_i(p)]^T$

Sinusoids: $$e_i = [1, e^{j\omega_i}, e^{j\omega_i 2}, ..., e^{j\omega_i p}]^T$$

Output: $$y_i(n) = \sum_{k=0}^{p} g_i(k)\, x(n-k) = g_i^T x$$

Using the definition of e_i **, the Fourier transform of** g **at frequency** ω_i **can be written**

$$G(e^{j\omega_i}) = \sum_{k=0}^{p} g(k) e^{-j\omega_i k} = e_i^H g = (g^H e_i)^H$$

Now, the spectrum estimate can be written (complex signals)

$$P(e^{j\omega_i}) = E\{y_i(n)y_i^*(n)\} = E\{g_i^H\ x\ x^H g_i\} =$$
$$= g_i^H\ R_x g_i$$

We must also normalize the band pass filters so that

$$G(e^{j\omega_i}) = \sum_{k=0}^{p} g_i(k)e^{-j\omega_i k} = e_i^H g_i = (g_i^H e_i)^H = 1$$

Then, we now want to minimize

$$P(e^{j\omega_i}) = \ g_i^H\ R_x g_i$$

due to the linear constraints

$$G_i(e^{j\omega_i}) = g_i^H e_i = 1$$

This can be done using Lagrange multipliers (page 50-52)

Introduce the Lagrange multiplier μ **and minimize (page 50-52)**

$$L(g_i,\mu)=\underbrace{\frac{1}{2}g_i^H R_x g_i}_{\text{minimize this}} + \mu \underbrace{(1-g_i^H e_i)}_{\text{this should be zero}}$$

Differentiate $L(g_i,\mu)$ **with respect to** g_i^H **. Then**

$$\nabla_{g_i^*} L(g_i,\mu)=R_x g_i - \mu e_i = 0$$

and

$$g_i=\mu R_x^{-1} e_i$$

Differentiate $L(g_i,\mu)$ **with respect to** μ gives

$$\frac{\delta}{\delta\mu}L(g_i,\mu)=1-g_i^H e_i = 0$$

Then using

$$g_i = \mu \; R_x^{-1} e_i$$

we have

$$\mu = \frac{1}{e_i^H R_x^{-1} e_i}$$

This gives the filter

$$g_i = \frac{R_x^{-1} e_i}{e_i^H R_x^{-1} e_i}$$

The power at frequency ω_i **is estimated as**

$$P(e^{j\omega_i}) = g_i^H \; R_x g_i = \frac{1}{e_i^H R_x^{-1} e_i}$$

We normalized the band pass filter but we must also normalize for the bandwidth of the band pass filter (length p+1).

A correction factor (p+1) (see page 429) finally gives the minimum variance estimate for any ω

$$P_{MV}(e^{j\omega}) = \frac{p+1}{e^H R_x^{-1} e}$$