Lesson 5

Chapter 7.
Wiener Filters

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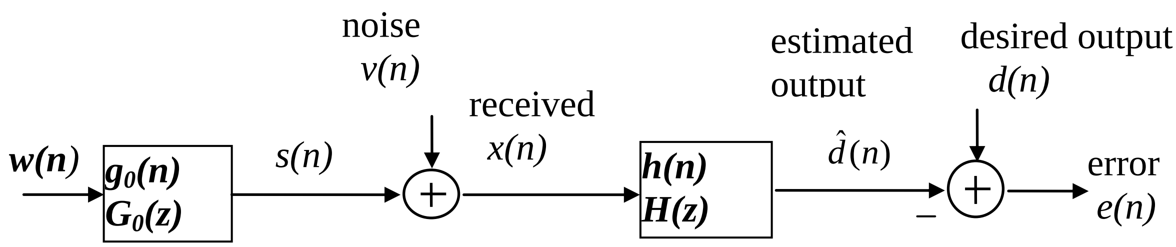
Chapter 7 Wiener Filters

In this chapter we will use the model shown below.

The signal into the receiver is *x(n)* (received signal). Normally, this signal is disturbed by additive white noise *v(n)*. The information is in *s(n)*. Also, we often used the approach that the information signal is modeled as white noise w(n) filtered in a filter *g(n)*.



We will minimize the output error *e(n),* which we describe as the difference between the desired output *d(n)* and the estimated output.

$$\text{minimize} \quad \xi = E[e^2(n)] = E[(d(n) - \hat{d}(n))^2]$$

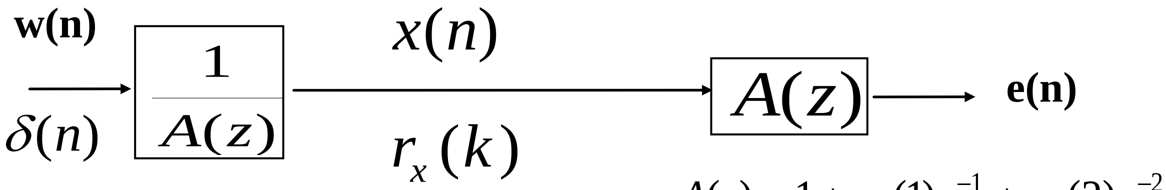
Applications.

Filtering s(n): **The desired signal is *s(n)* and we will determine the optimum filter for noise reduction.**
Smoothing: **Like filtering but we allow an extra delay in the output signal (specially image processing).**
Prediction: **The output is a prediction of future values of *s(n)*. One step predictor. predict next value *s(n+1)*.**
Equalization: **The desired signal is *w(n)* and we will determine the optimum filter for whitening the output spectrum (inverse filtering, deconvolution).**

Other applications: **Echo cancellation. Noise cancellation. Pulse shaping.**

Prediction Error Filter PEF (second order) from chapter 4

Model of the signal x(n).
Input: white noise $w(n)$ **or impulse** $\delta(n)$



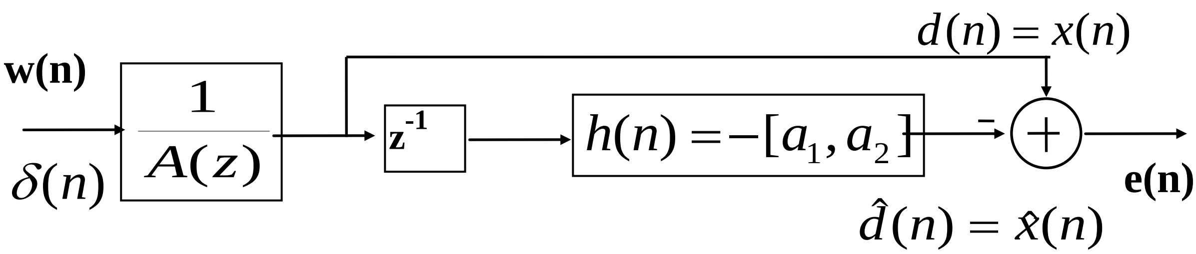
$$A(z) = 1 + a_2(1)z^{-1} + a_2(2)z^{-2}$$

$$e(n) = x(n) + \sum_{l=1}^{2} a_2(l)x(n-l) =$$

$$= x(n) - \hat{x}(n) \qquad with$$

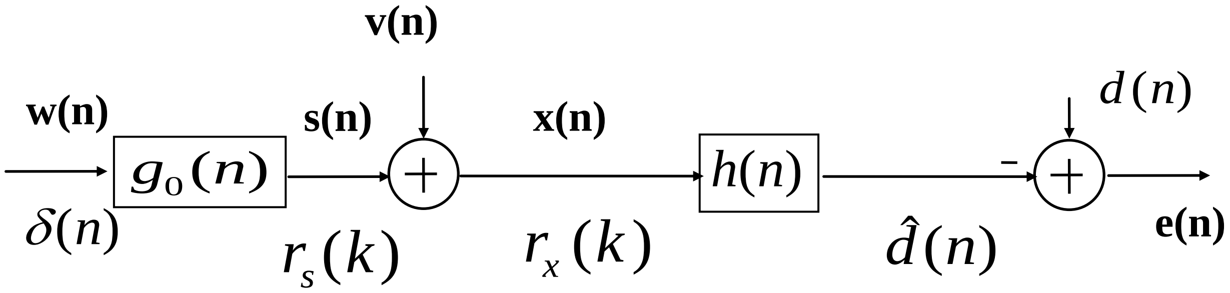
$$\hat{x}(n) = -\sum_{l=1}^{2} a_2(l)x(n-l)$$

We can rewrite the figure



$d(n)$	*desired signal (önskad)*
$\hat{d}(n) = \hat{x}(n)$	*estimated signal*
$e(n) = x(n) - \hat{x}(n)$	*error signal*

Optimum Filters (process the received signal x(n))



We assume uncorrelated noise v(n).

$d(n)$ **could be:**	$s(n)$,	**filtered noisy signal** $x(n)$
	$s(n-n_0)$,	**smoothing (allow delay)**
	$s(n+n_0)$,	**predict future values**
	$\delta(n-n_0)$,	**inverse filtering, deconvolution**

h(n) causal FIR filter: easy, useful (chap. 7.2)
h(n) noncausal IIR filter: easy, less useful (chap. 7.3.1)
h(n) causal IIR filter more difficult, useful (chap. 7.3.2)

We assume that correlation functions $r_x(k)$, $r_{dx}(k)$ **and** $r_d(k)$ **are known or could be estimated.**

Derivation of the optimal solution (Wiener filter).

Real-valued random signals.

We start with

$$\xi = E[e^2(n)] = E[(d(n) - \hat{d}(n))^2]$$

with

$$\hat{d}(n) = \sum_{l=-\infty}^{\infty} h(l)x(n-l)$$ **(in general noncausal filter)**

and

$$e(n) = d(n) - \hat{d}(n) = d(n) - \sum_{l=-\infty}^{\infty} h(l)x(n-l)$$

Set the derivative of ξ **with respect to** h(k) **equal to zero for all k.**

$$\xi = \frac{\partial}{\partial h(k)} E[e^2(n)] = 2E[e(n)\frac{\partial}{\partial h(k)} e(n)] = 2E[e(n)(-x(n-k))] = 0$$

which gives

$$E[e(n)\,x(n-k)] = 0$$ **(The orthogonality principle)**

Replace *e(n)* and then

$$E[(d(n) - \sum_{l=-\infty}^{\infty} h(l)x(n-l))x(n-k)] = 0$$

and we get the Wiener-Hopf equations

$$\sum_{l=-\infty}^{\infty} h(l)r_x(k-l) = r_{dx}(k)$$

Derivation of the minimum error

Writing

$$\xi = E[e^2(n)] = E[e(n)\,e(n)] = E\{e(n)[d(n) - \sum_{l=-\infty}^{\infty} h(l)x(n-l)]\} =$$

$$= \underbrace{E\{e(n)\,d(n)}_{\xi_{\min}} + \sum_{l=-\infty}^{\infty} h(l)\,\underbrace{E\{e(n)\,x(n-l)\}}_{=0}$$

e(n) and given data orthogonal

This gives the minimum error

$$\xi_{\min} = E[e(n)d(n)] = E\{[d(n) - \sum_{l=-\infty}^{\infty} h(l)x(n-l)]d(n)\} =$$

$$= r_d(0) - \sum_{l=\infty}^{\infty} h(l)r_{xd}(-l) = r_d(0) - \sum_{l=\infty}^{\infty} h(l)r_{dx}(l)$$

and

$$\xi_{\min} = r_d(0) - \sum_{l=-\infty}^{\infty} h(l)r_{dx}(l)$$

The Wiener filter was derived from random signals.
For a deterministic approach we have to use the definition of auto-correlation and cross-correlation

$$r_x(k) = \sum_{n=-\infty}^{\infty} x(n)x(n-k)$$

$$r_{dx}(k) = \sum_{n=-\infty}^{\infty} d(n)x(n-k)$$

Then, minimize

$$\varepsilon = \sum_{n=0}^{\infty} e^2(n) = \sum_{n=0}^{\infty} (d(n) - \hat{d}(n))^2$$

The Wiener-Hopf equations will be the same.

Now, we will look at the three types of filters *H(z)*

FIR Wiener filter (in the textbook denoted *W(z))*

Noncausal Wiener IIR filter

Causal Wiener IIR filter (at the end of this chapter)

FIR Wiener filter (pp. 337-339, table 7.1 page 339)

The Wiener-Hopf equations are now

$$\sum_{l=0}^{p-1} h(l) r_x(k-l) = r_{dx}(k) \qquad k = 0,1,\ldots,p-1$$

or in matrix form

$$\underbrace{\begin{bmatrix} r_x(0) & r_x(1) & r_x(2) & \cdots & r(p-1) \\ r_x(1) & r_x(0) & r_x(1) & \cdots & r_x(p-2) \\ r_x(2) & r_x(1) & r_x(0) & \cdots & r_x(p-3) \\ & & \cdots & & \\ r_x(p-1) & r_x(p-2) & r_x(p-3) & \cdots & r_x(0) \end{bmatrix}}_{R_x} \cdot \underbrace{\begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ \cdot \\ h(p-1) \end{bmatrix}}_{h} = \underbrace{\begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \\ r_{dx}(2) \\ \cdot \\ r_{dx}(p-1) \end{bmatrix}}_{r_{dx}}$$

$$R_x \; h = r_{dx}$$

The solution is

$$h = R_x^{-1} \, r_{dx}$$

and the minimum error

$$\xi_{\min} = r_d(0) - \sum_{l=0}^{p-1} h(l) r_{dx}(l)$$

which also can be written

$$\xi_{\min} = r_d(0) - \sum_{l=0}^{p-1} h(l) \; r_{dx}(l) = r_d(0) - r_{dx}^T \; h_{opt} = r_d(0) - r_{dx}^T \; R_x^{-1} r_{dx}$$

Noncausal IIR Wiener filter (pp. 353-356, table 7.2)

The Wiener-Hopf equation are here

$$\sum_{l=-\infty}^{\infty} h(l) r_x(k-l) = r_{dx}(k) \quad all \;\; k$$

Here we have a complete convolution and it can be solved using the Z-transform (and/or the Fourier transform)

$$H(z)P_x(z) = P_{dx}(z)$$

$$H(z) = \frac{P_{dx}(z)}{P_x(z)};$$

$$H(e^{j\omega}) = \frac{P_{dx}(e^{j\omega})}{P_x(e^{j\omega})}$$

The minimum error is

$$\xi_{\min} = r_d(0) - \sum_{l=-\infty}^{\infty} h(l)\; r_{dx}(l)$$

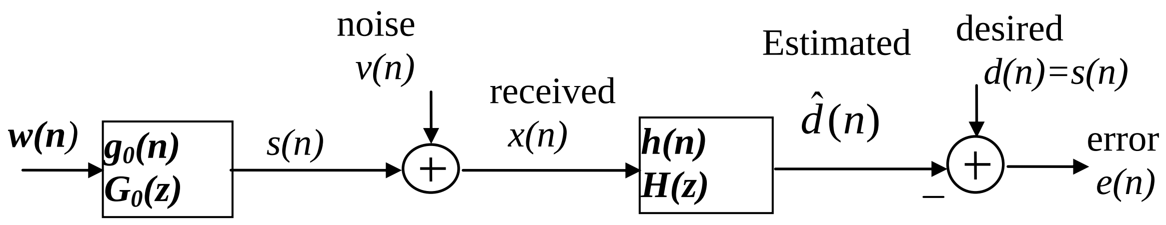
We can use Parseval's relation and also write this in the frequency domain. Then, (see properties of the Fourier transform, see page 356, Table 7.2)

$$\xi_{\min} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[P_d(e^{j\omega}) - H(e^{j\omega}) P_{dx}^{*}(e^{j\omega}) \right] d\omega$$

Filtering received signal for noise reduction

The received signal is disturbed by additive zero mean white noise

$$x(n) = s(n) + v(n)$$



Desired signal is now *s(n)*. Then

$$r_x(k) = r_s(k) + r_v(k)$$

$$r_{dx}(k) = E[d(n)x(n-k)] = E[s(n)(s(n-k) + v(n-k))] = r_s(k)$$

$$P_x(z) = P_s(z) + P_v(z)$$

$$P_{dx}(z) = P_s(z)$$

Causal FIR-filter for noise reduction

The FIR-filter equations are

$$\sum_{l=0}^{p-1} h(l) r_x(k-l) = r_{dx}(k) \quad k = 0,1,\ldots, p-1$$

Now, they will be

$$\sum_{l=0}^{p-1} h(l)(r_s(k-l) + r_v(k-l)) = r_s(k)$$

or

$$(R_s + R_v)\ h = r_s$$

and

$$h_{opt} = (R_s + R_v)^{-1}\ r_s$$

We find the spectrum using the Fourier Transform

$$H(e^{j\omega}) = Fourier\{h_{opt}(n)\}$$

Noncausal IIR-filter for noise reduction

For non-causal IIR filter, we have

$$H(z)P_x(z) = P_{dx}(z)$$

$$H(z) = \frac{P_{dx}(z)}{P_x(z)};$$

$$H(e^{j\omega}) = \frac{P_{dx}(e^{j\omega})}{P_x(e^{j\omega})}$$

In the filtering problem the power spectra are

$$P_x(z) = P_s(z) + P_v(z)$$

$$P_{dx}(z) = P_s(z)$$

which gives the Wiener filter

$$H(z) = \frac{P_s(z)}{P_s(z) + P_v(z)};$$

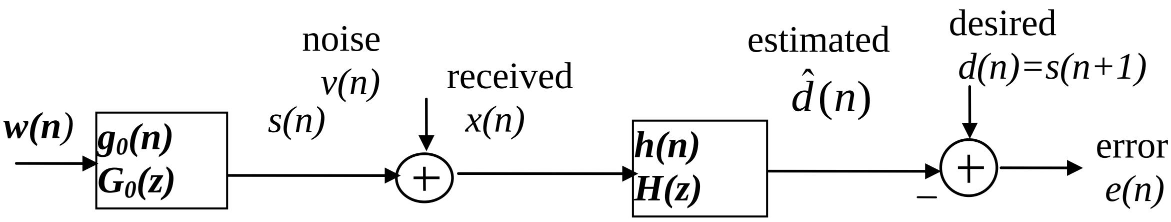
$$H(e^{j\omega}) = \frac{P_s(e^{j\omega})}{P_s(e^{j\omega}) + P_v(e^{j\omega})};$$

We see that for frequencies with low noise,

$$|H(e^{j\omega})| \approx 1$$

Prediction

In a one-step predictor, the desired signal is *s(n+1)*.



Desired signal is now *s(n+1)*. Then

$$r_{dx}(k) = E[d(n)x(n-k)] = E[s(n+1)(s(n-k)] = r_s(k+1)$$
$$P_{dx}(z) = z\,P_s(z)$$

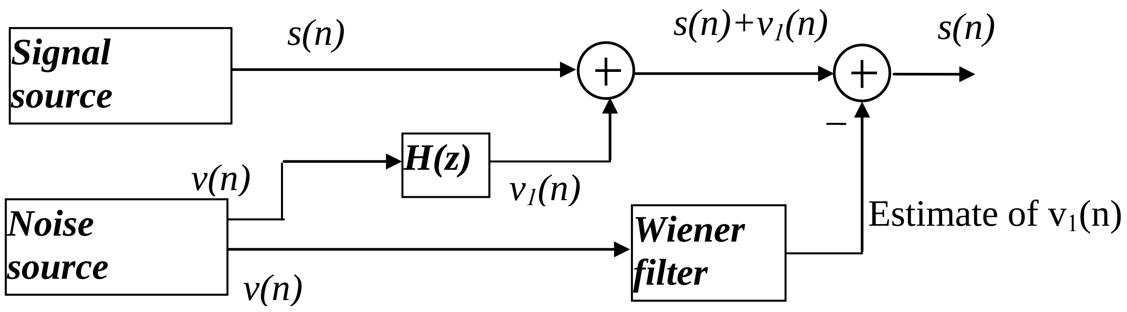
This gives the Wiener-Hopf equation

$$\sum_{l=0}^{p-1} h(l) r_s(k-l) = r_s(k+1) \quad k = 0,1,\ldots, p-1$$

Noise cancellation (page 349)

A signal is disturbed by additive noise $v_1(n)$.

Try to measure the noise v(n) from the source and estimate the noise $v_1(n)$ added to the signal. Then subtract the noise $v_1(n)$ from the received signal.

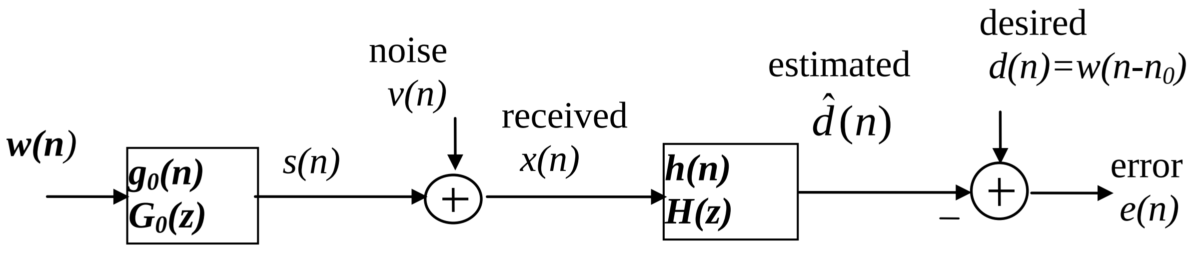


Deconvolution (equalizing, inverse filtering)

Desired signal here is *w(n)* (or allow delay, $w(n-n_0)$).
(see also problem 4.19)

This means that

$$g(n) * h(n) \approx \delta(n - n_0)$$

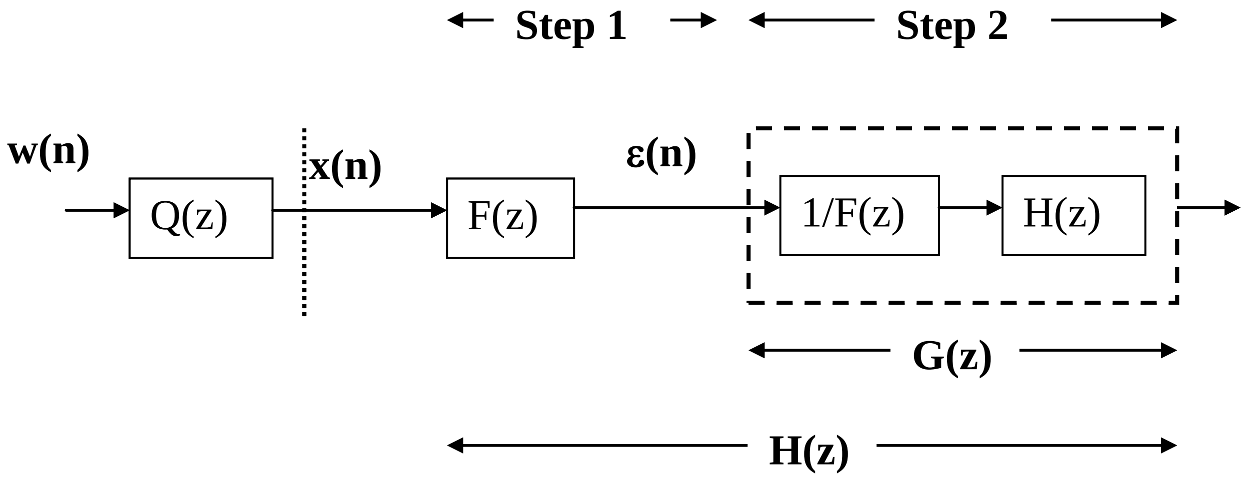


Causal IIR Wiener filter –1 (page 358-362)

Derivation of the causal IIR filter is more difficult.
The Wiener solution is

$$\sum_{l=0}^{\infty} h(l) r_x(k-l) = r_{dx}(k)$$

We divide the solution into two steps.



In step 1, we whitening the input signal *x(n)*. From chapter 3, we have spectral factorization

$$P_x(z) = \sigma_0^2 \; Q(z) \; Q(z^{-1})$$

If we choose $F(z) = \dfrac{1}{\sigma Q(z)}$ **the signal *ε(n)* will be white with variance equal to 1.**

IIR, causal filter – 2

Step 2

In step 2 we now have (Wiener-Hopf equation)

$$\sum_{l=0}^{\infty} g(l) r_{\varepsilon}(k-l) = r_{d\varepsilon}(k)$$

with $$r_{\varepsilon}(k) = \delta(k)$$

The optimal filter (the causal filter, k≥0) is then

$$g(k) = r_{d\varepsilon}(k)\ u(k)$$

with the z-transform

$$G(z) = \left[P_{d\varepsilon}(z)\ \right]_{+}$$

The notation $[\ldots]_{+}$ means the causal part of the argument.

IIR, causal filter – 3

We have to determine $P_{d\varepsilon}(z)$**. Then**

$$r_{d\varepsilon}(k) = E\{d(n)\,\varepsilon(n-k)\} =$$

$$= E\{d(n)(\sum_l f(l)\,x(n-k-l))\} =$$

$$= \sum_l f(l)\,r_{dx}(k+l)$$

where

$$f(n) = Z^{-1}\{F(z)\}$$

Then

$$P_{d\varepsilon}(z) = P_{dx}(z)\,F(z^{-1}) = \frac{P_{dx}(z)}{\sigma_0\,Q(z^{-1})}$$

To find G(z) we take the causal part

$$G(z) = \left[\frac{P_{dx}(z)}{\sigma_0\,Q(z^{-1})}\right]_+$$

Combining step 1 and step 2 gives finally

$$H(z) = F(z)G(z) = \frac{1}{\sigma_0^2\,Q(z)}\left[\frac{P_{dx}(z)}{Q(z^{-1})}\right]_+$$

Relation between causal and non causal IIR Wiener filter

Non causal IIR Wiener filter

$$H(z) = \frac{P_{dx}(z)}{P_x(z)} = \frac{1}{\sigma_0^2\, Q(z)} \; \frac{P_{dx}(z)}{Q(z^{-1})}$$

Causal IIR Wiener filter

$$H(z) = \frac{1}{\sigma_0^2\, Q(z)} \left[\frac{P_{dx}(z)}{Q(z^{-1})} \right]_+$$

We can see both filters as a cascade of two filters where the first is a whitening filter.

The minimum error is as before

$$\xi_{\min} = r_d(0) - \sum_{l=-0}^{\infty} h(l)\; r_{dx}(l)$$

Adaptive filtering.
Chapter 9 or the course ´Adaptive Signal Processing´.

We want to minimize the error

$$\xi = E[e^2(n)] = E[(d(n) - \hat{d}(n))^2]$$

Iterative solution

We can solve this iteratively using the update equation

$$h_{n+1}(k) = h_n(k) - \mu' \frac{\delta \xi}{\delta h_n(k)} =$$

$$= h_n(k) + 2\mu' E\{e(n)\, x(n-k)\}$$

there μ' **is the step size.**

Adaptive solution (Least Mean Square, LMS)

Use the approximation

$$E\{e(n)\, x(n-k)\} \approx e(n)\, x(n-k)$$

which gives

$$h_{n+1}(k) = h_n(k) + 2\mu' e(n)\, x(n-k)$$

How to chose step size μ' **?**
Does the algorithm converge? How fast?