LUND INSTITUTE of TECHNOLOGY
Dept. of Elektro- och Information technology

Exam in OPTIMAL SIGNAL PROCESSING - ETTN10
Date: October 25, 2013
Time: 14.00–19.00
Room: Kår:Gasq

Allowed items — Course literature (”Statistical Digital Signal Processing and Modeling”, by Monson H. Hayes), Calculator and table of formulas.

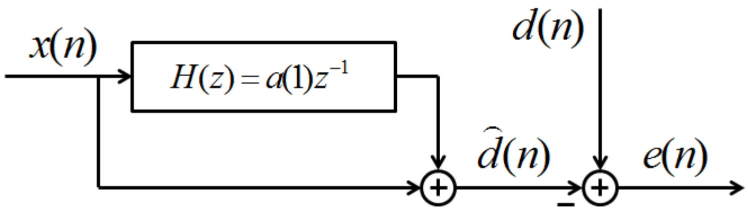
Observandum — In order to simplify the correction process:
-Solve only one task per paper sheet.
-Write your name on all papers.
Motivations and/or equations are necessary.
The exam consists of 5 questions and each gives a maximum of 1 point.
Grading: 3 (≥2.0p), 4 (≥3.0p), 5 (≥4.0p).

1. The auto-correlation sequence of a random process, $x(n)$, is given by,

$$r_x(0) = 1,\ r_x(1) = 0.8,\ r_x(2) = 0.5,\ r_x(3) = 0.1$$

   a) Determine the reflection coefficients, $\Gamma_j,\ j = 1, 2, 3$
   b) Determine the AR-parameters, $a_j(k),\ j = 1, 2, 3$
   c) Determine the model errors, $\epsilon_j,\ j = 1, 2, 3$

2. We are given the system below, to estimate a process $d(n)$ from $x(n)$, where $\sigma_d^2 = 4$,



Determine the value of $a(1)$ which minimizes the error $\xi = E\left\{|e(n)|^2\right\}$ and also calculate the minimum error ξ. The auto-correlation and the cross-correlation functions are given by,

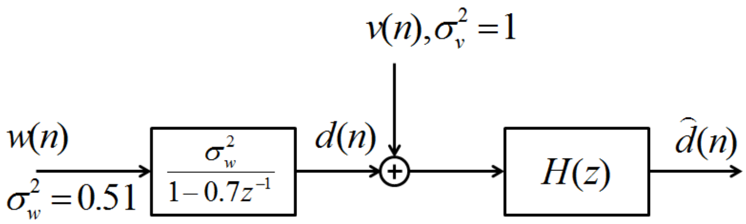
$$\mathbf{r}_x = [1.0 \quad 0.5 \quad 0.25]^T, \qquad \mathbf{r}_{dx} = [-1.0 \quad 1.0]^T.$$

3. The result of a third-order Padés approximation is,

$$H(z) = \frac{1 - z^{-1} + 2z^{-2}}{1 + 2z^{-1} + z^{-2} + 3z^{-3}}$$

Provide the four first samples of $x(n)$ which gives the above approximation.

4. We have the following system, and we wish to find optimal filters $H(z)$ to remove the influence of the noise, $v(n)$, on the signal $d(n)$. The white noise $v(n)$ is uncorrelated with the white noise $w(n)$.



a) Find the first-order FIR-filter, $H(z)$, (i.e. $L = 2$) that minimizes $E\{|e(n)|^2\} = E\left\{|d(n) - \hat{d}(n)|^2\right\}$.

b) Find the non-causal IIR filter, $H(z)$, that minimizes $E\{|e(n)|^2\}$.

c) Find the causal IIR filter, $H(z)$, that minimizes $E\{|e(n)|^2\}$.

d) Calculate the corresponding minimum errors from a), b) and c)!

5. We wish to model a signal $x(n)$ which is the result of filtering $\delta(n)$ through a sparse all-pole model of the form

$$H(z) = \frac{b(0)}{1 + \sum_{k=1}^{p} a_p(k) z^{-ck}}$$

where c is an arbitrary integer constant. Derive the Prony-model based normal equations which finds the parameters $a_p(k)$ from a given input signal $x(n)$. Specifically, provide the matrix normal equation when $c = 2$ and $p = 2$.

Good Luck

Answers/Solutions EXAMINATION 2013-10-25

- **Ans: 1**

  Levinson-Durbins recursion in Table 5.1 gives

$$
\begin{aligned}
\text{Initiate:}\quad & a_0 = 1 \\
& \epsilon_0 = 1 \\
j = 0: \quad & \gamma_0 = 0.8 \\
& \Gamma_1 = -0.8 \\
& a_1(1) = \Gamma_1 = -0.8 \\
& \epsilon_1 = 0.36 \\
j = 1: \quad & \gamma_1 = -0.14 \\
& \Gamma_2 = 0.3889 \\
& \qquad a_2(1) = -1.111 \\
& a_2(2) = \Gamma_2 = 0.3889 \\
& \epsilon_2 = 0.3056 \\
j = 2: \quad & \gamma_2 = -0.1444 \\
& \Gamma_3 = 0.4727 \\
& \qquad a_3(1) = -0.9273 \\
& \qquad a_3(2) = -0.1364 \\
& a_3(3) = \Gamma_3 = 0.4727 \\
& \epsilon_3 = 0.2373
\end{aligned}
$$

  The answer is: $\Gamma_1 = -0.8$, $\Gamma_2 = 0.3889$, $\Gamma_3 = 0.4727$,

$$
\mathbf{a_1} = \begin{bmatrix} 1 \\ -0.8 \end{bmatrix} \quad \mathbf{a_2} = \begin{bmatrix} 1 \\ -1.11 \\ 0.3889 \end{bmatrix} \quad \mathbf{a_3} = \begin{bmatrix} 1 \\ -0.9273 \\ -0.1364 \\ 0.4727 \end{bmatrix}
$$

  and $\epsilon_0 = 1$, $\epsilon_1 = 0.36$, $\epsilon_2 = 0.3056$ and $\epsilon_3 = 0.2373$.

- **Ans: 2** The orthogonality principle gives that $e(n) \perp x(n-1)$:

$$
E[e(n)x(n-1)] = E[(d(n)-x(n)-a(1)x(n-1))x(n-1)] = r_{dx}(1)-r_x(1)-a(1)r_x(0) = 0
$$

$$
a(1) = \frac{r_{dx}(1) - r_x(1)}{r_x(0)} = \frac{1}{2}
$$

$$
\begin{aligned}
\epsilon_{min} &= E(e(n)e(n)) \\
&= E(e(n)[d(n) - x(n) - a(1)x(n-1)]) \\
&= E(e(n)(d(n) - x(n))) \\
&= r_d(0) - r_{dx}(0) - a(1)r_{dx}(1) - r_{dx}(0) + r_x(0) + a(1)r_x(1) \\
&= 6.75
\end{aligned}
$$

- **Ans: 3** We have the following;

$$H(z) = \frac{1 - z^{-1} + 2z^{-2}}{1 + 2z^{-1} + z^{-2} + 3z^{-3}}$$

According to Padés approximation, given in table 4.1 we have four unknown parameters and four equations, according to

$$\begin{bmatrix} x(0) & 0 & 0 & 0 \\ x(1) & x(0) & 0 & 0 \\ x(2) & x(1) & x(0) & 0 \\ x(3) & x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_p(1) \\ a_p(2) \\ a_p(3) \end{bmatrix} = \begin{bmatrix} b(0) \\ b(1) \\ b(2) \\ 0 \end{bmatrix}$$

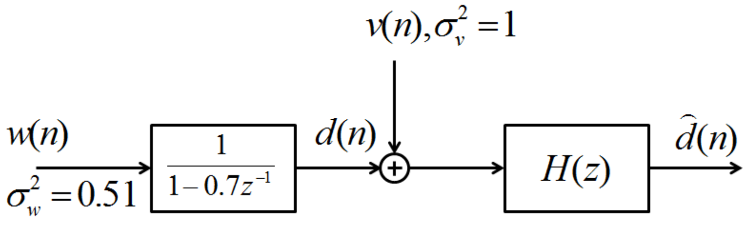
This can be rewritten as,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_p(1) & 1 & 0 & 0 \\ a_p(2) & a_p(1) & 1 & 0 \\ a_p(3) & a_p(2) & a_p(1) & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} b(0) \\ b(1) \\ b(2) \\ 0 \end{bmatrix}$$

and the unique solution is given by

$$[x(0), x(1), x(2), x(3)] = [1, -3, 7, -14]$$

- **Ans: 4** (See example in lecture notes for details)
The following system illustration was intended, (the solution to the system given in the exam can easily be found from the solution below by multiplying the filters by the constant σ_w^2)



a) We wish to find the optimal Wiener FIR filter in the form $H(z) = h(0) + h(1)z^{-1}$. The FIR Wiener-Hopf equations are given by,

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \end{bmatrix}$$

where

$$\begin{aligned} r_d(k) &= 0.7^{|k|} \\ r_x(k) &= 0.7^{|k|} + \sigma_v^2 \delta(k) = 0.7^{|k|} + 1 \\ r_{dx}(k) &= r_d(k) = 0.7^{|k|} \end{aligned}$$

The solution is given by,

$$\begin{bmatrix} 2 & 0.7 \\ 0.7 & 2 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0.7 \end{bmatrix}$$

The solution is given by,

$$H(z) = 0.4302 + 0.1994z^{-1}$$

b) The non-causal IIR-Wiener filter is given by,

$$H(z) = \frac{P_{dx}(z)}{P_x(z)}$$

As in a), $P_{dx}(z) = P_d(z)$, i.e.

$$P_{dx}(z) = P_d(z) = \frac{0.51}{(1 - 0.7z^{-1})(1 - 0.7z)}$$

and

$$P_x(z) = P_d(z) + 1$$

The solution is then given by,

$$H(z) = \frac{0.255}{(1 - 0.4084z^{-1})(1 - 0.4084z^{-1})} => h(n) = 0.3060 \cdot 0.4084^{|n|}$$

c) The causal Wiener IIR filter is given by,

$$H(z) = \frac{1}{\sigma_0^2 Q(z)} \left[\frac{P_{dx}(z)}{Q(z^{-1})} \right]_+$$

We need to write $P_x(z)$ in factorized form according to

$$P_x(z) = \sigma_0^2 Q(z) Q(z^{-1}) = 1.8192 \frac{(1 - 0.4084z^{-1})}{(1 - 0.7z^{-1})} \frac{(1 - 0.4084z)}{(1 - 0.7z)}$$

which gives the causal IIR filter as,

$$H(z) = \frac{1}{1.8192} \frac{(1 - 0.7z^{-1})}{(1 - 0.4084z^{-1})} \frac{0.7577}{(1 - 0.7z^{-1})} = 0.417 \frac{1}{1 - 0.4084z^{-1}}$$

and $h(n) = 0.417 \cdot (0.4084)^n u(n)$

d) The mean-square errors are given by

$$\epsilon_{min} = r_d(0) - \sum_{l=-\infty}^{\infty} h(l) r_{dx}(l)$$

which gives the errors $\epsilon_{min} = 0.4285$ in a), 0.3518 in b) and 0.417 in c)

- **Ans: 5** We want to minimize the error,

$$\mathcal{E}_p = \sum_{n=0}^{\infty} e^2(n)$$

where

$$e(n) = x(n) + \sum_{l=1}^{p} a_p(l) x(n - cl)$$

Taking the derivative, with respect to the parameters $a_p(k)$, gives,

$$\begin{aligned}
\frac{\partial \mathcal{E}_p}{\partial a_p(k)} &= 2 \sum_{n=0}^{\infty} e(n) x(n - ck) \\
&= 2 \sum_{n=0}^{\infty} \left[x(n) + \sum_{l=1}^{p} a_p(l) x(n - cl) \right] x(n - ck) \\
&= 2 \left(r_x(ck) + \sum_{l=1}^{p} a_p(l) r_x(ck - cl) \right) = 0
\end{aligned}$$

for $k = 1 \ldots p$, and the normal equations becomes,

$$\sum_{l=1}^{p} a_p(l) r_x(ck - cl) = -r_x(ck).$$

When $c = 2$ and $p = 2$ the matrix normal equations becomes matrisform

$$\begin{bmatrix} r_x(0) & r_x(2) \\ r_x(2) & r_x(0) \end{bmatrix} \begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = \begin{bmatrix} -r_x(2) \\ -r_x(4) \end{bmatrix}$$