Lund Institute of Technology
Department of Electro and Information Technology

Exam in Optimal Signal Processing - (ETT074)
Date: October 20, 2011
Hour: 14.00 - 19.00
Room: Victoria stadium

Auxiliaries: Tables, calculator.
Write your name on all sheets.
The exam consists of 5 problems, each problem gives 0-1 point
Limits: 2.0-2.9 points give the grade 3
3.0-3.9 points give the grade 4
4.0-5.0 points give the grade 5
The solutions must be easy to follow and a clear answer must be given. Explain your solutions and assumptions.
Hints: Draw figures. Write definitions.

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1 We want to model a signal *x(n)* using the all-pole model below,

$$H(z) = \frac{1}{1 + a_4(3)\,z^{-3} + a_4(4)\,z^{-4}}$$

a) Determine the normal equations for the filter coefficients $a_4(3)$ and $a_4(4)$ which minimize the error $\varepsilon_p = \sum_{n=0}^{\infty} e^2(n)$ with

$$e(n) = x(n) + a_4(3)\,x(n-3) + a_4(4)\,x(n-4)$$

Determine also the minimum error $\varepsilon_{p,\,\min}$

b) Determine the numerical values of $a_4(3)$ and $a_4(4)$ for

$$r_x(k) = 0.5^{|k|}$$

2. A FIR-filter *H(z)* is given by

$$H(z) = 1 + 0.5\,z^{-1} + 0.5\,z^{-2}$$

a) Determine the coefficients in a lattice FIR- filter given by H(z) above.

b) Determine the minimum error e_2 if the Levinson-Durbin algorithm is used for the calculation of the lattice-filter coefficients in an all pole model for $r_x(0) = 1$.

3. The autocorrelation function for a random process is given below.

$$r_x(0) = 3, \ \ r_x(1) = 2, \ \ r_x(2) = 1, \ \ r_x(3) = 0.5, \ \ r_x(4) = 0$$

We want to determine a model for the input signal if the signal is described by filtering white noise through the filter *g(n), G(z)* with

$$G(z) = \frac{B_q(z)}{A_p(z)}.$$

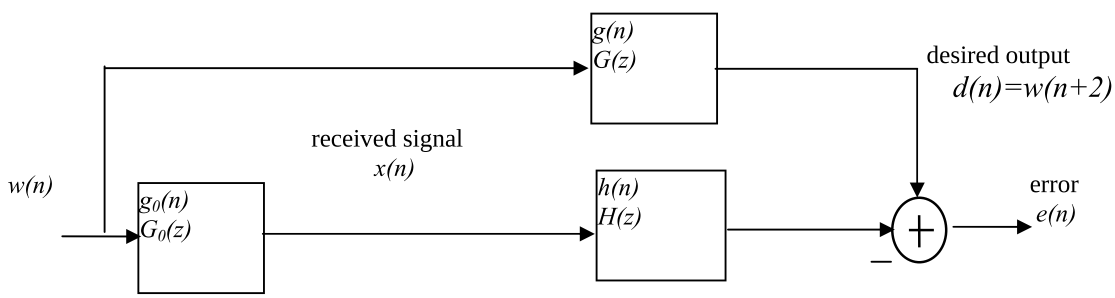
a) Determine a second order pole model for the input signal, i.e. p=2, q=0.
b) Determine a first order pole-zero model for the input signal, i.e. p=1, q=1.

4. The reflection coefficients for the all-pole model of the signal *x(n)* are

$$\Gamma_1 = 0.25, \quad \Gamma_2 = 0.5, \ \Gamma_3 = 0.25,$$

a) Determine the system function $H(z) = \dfrac{1}{A_3(z)}$
b) Determine the corresponding autocorrelation sequence $r_x(0),\ r_x(1),\ r_x(2),\ r_x(3)$ when the model error $\mathrm{e}_3 = (15/16)^2$.

5. A system that predict the input signal two step is given by the figure below.



The input signal *w(n)* is white noise with variance 1. The filters *g(n)* and $g_0(n)$ are given by

$$G_0(z) = 1 - 0.5z^{-1} \quad \text{and} \quad G(z) = z^2$$

a) Determine the non-causal IIR filter *H(z)* which minimize the error $\xi = E\{e(n)^2\}$
b) Determine the causal IIR filter *H(z)* which minimize the error $\xi = E\{e(n)^2\}$
c) Determine the causal FIR filter of length 3 which minimize the error

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Solutions for Exam in Optimal Signal Processing, October 20, 2011 (ETT074)

1. **Given:** $H(z)=\dfrac{1}{1+a_4(3)\,z^{-3}+a_4(4)\,z^{-4}}$

$$\varepsilon_p=\sum_{n=0}^{\infty} e^2(n) \quad \text{with } e(n)=x(n)+a_4(3)\,x(n-3)+a_4(4)\,x(n-4)$$

Task: Determine the equation for $a_4(3)$, $a_4(4)$ *and* $\varepsilon_{p,\min}$

Solution:

a) Orthogonality gives: $\sum_n e(n)\,x(n-3)=0,\qquad \sum_n e(n)\,x(n-4)=0$ gives

$$\begin{cases}\sum_n \,(x(n)+a_4(3)\,x(n-3)+a_4(4)\,x(n-4))\;\; x(n-3)=0\\ \sum_n \,(x(n)+a_4(3)\,x(n-3)+a_4(4)\,x(n-4))\;\; x(n-4)=0\end{cases}$$

$$\begin{cases}r_x(3)+a_4(3)\; r_x(0)+a_4(4)r_x(1)=0\\ r_x(4)+a_4(3)\; r_x(1)+a_4(4)r_x(0)=0\end{cases}$$

Answer: $\begin{bmatrix} r_x(0) & r_x(1)\\ r_x(1) & r_x(0)\end{bmatrix}\cdot\begin{bmatrix} a_4(3)\\ a_4(4)\end{bmatrix}=-\begin{bmatrix} r_x(3)\\ r_x(4)\end{bmatrix}$;

Min error $E_P=r_x(0)+a_4(3)\,r_x(3)+a_4(4)\,r_x(4)$

b) $\begin{bmatrix} 1 & 1/2\\ 1/2 & 1\end{bmatrix}\cdot\begin{bmatrix} a_4(3)\\ a_4(4)\end{bmatrix}=-\begin{bmatrix} 1/8\\ 1/16\end{bmatrix};\qquad \begin{bmatrix} a_4(3)\\ a_4(4)\end{bmatrix}=-\begin{bmatrix} 1/8\\ 0\end{bmatrix}$

$$E_P=r_x(0)+a_4(3)\,r_x(3)+a_4(4)\,r_x(4)=1-0.125^2=0.984$$

2. **Given:** $H(z)=1+0.5\;z^{-1}+0.5\;z^{-2}$

Task: a) Determine Γ_1, Γ_2 b) Minimum error e_2

Solution: a)

$A_2(z)=1+0.5\;z^{-1}+0.5\,z^{-2}$ *gives* $\Gamma_2=0.5$

Then $A_1(z)=1+\dfrac{1}{3}\;z^{-1}$ *gives* $\Gamma_1=\dfrac{1}{3}$

b) $e_2=r_x(0)\,(1-\Gamma_1^2)\;(1-\Gamma_2^2)=1\cdot(1-\dfrac{1}{9})\cdot(1-\dfrac{1}{4})=\dfrac{2}{3}$

3 **Given:** $r_x(0)=3,\; r_x(1)=2,\; r_x(2)=1,\; r_x(3)=0.5,\; r_x(4)=0$

Task: Model for the input signal described by white noise through the filter *g(n), G(z)*

$G(z)=\dfrac{B_q(z)}{A_p(z)}$. a) p=2, q=0 b) p=1, q=1

Solution:

3a) $\begin{bmatrix} 3 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix} a_1\\ a_2\end{bmatrix}=-\begin{bmatrix} 2\\ 1\end{bmatrix}$ gives $\begin{bmatrix} a_1\\ a_2\end{bmatrix}=\begin{bmatrix} -4/5\\ 1/5\end{bmatrix}$ $\quad (b_0)^2=r_x(0)+a_1r_x(1)+a_2\,r_x(2)=$
$=3-0.8\cdot 2+0.2\cdot 1=8/5$

$$G(z)=\frac{\sqrt{8/5}}{1-0.8\,z^{-1}+0.2\,z^{-2}}=\frac{1.26}{1-0.8\,z^{-1}+0.2\,z^{-2}}$$

3b) Problem : Estimate a first order ARMA model from $r_x=\{3,2,1,0.5,0,...\}$

Solution: We have $p=q=1$ which gives the equations

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \end{bmatrix} = \begin{bmatrix} c_1(0) \\ c_1(1) \\ 0 \end{bmatrix}$$ Then, we found $a_1 = -1/2$ from the lower equation and

$$\begin{bmatrix} c_1(0) \\ c_1(1) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix} \quad \text{and} \quad \begin{pmatrix} c_1(k)=0 & k \ge 2 \\ c_1(k) & k<0 \quad not \quad used \end{pmatrix}$$

Now, we have to identify $B_q(z)$. We have

$$C_q(z) A_p^*(1/z^*) = \underbrace{\underbrace{(\ldots + 2 + \frac{1}{2} z^{-1})}_{C_q(z)} \underbrace{(1 - \frac{1}{2} z)}_{A_p^*(1/z^*)}}_{\ldots \frac{7}{4} + \frac{1}{2} z^{-1}}$$

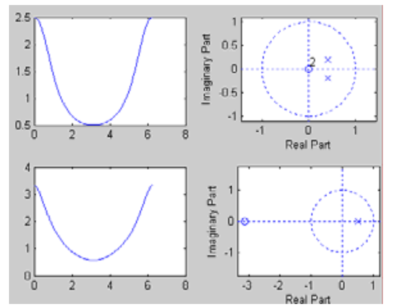
But $B_q(z) B_q^*(1/z^*)$ must be symmetrical. Then

$$B_q(z) B_q^*(1/z^*) = \frac{1}{2} z + \frac{7}{4} + \frac{1}{2} z^{-1} = (c_1 + c_2 z^{-1})(c_1 + c_2 z) =$$

$$= \begin{cases} (1.26 + 0.4 z^{-1})(1.26 + 0.4 z) & minimum \ \ phase \\ (0.4 + 1.26 z^{-1})(0.4 + 1.26 z) & not \ minimum \ phase \end{cases}$$

which gives the filter (choose minimum phase)

$$G(z) = \frac{1.26 + 0.4 z^{-1}}{1 - \frac{1}{2} z^{-1}} = 1.26 \frac{1 + 0.31 z^{-1}}{1 - \frac{1}{2} z^{-1}}$$



Plot of $|G(\omega)|$ in a) and b)

4. **Given:** The reflection coefficients $\Gamma_1 = 0.25$, $\Gamma_2 = 0.5$, $\Gamma_3 = 0.25$,

Task: a) Determine the system function $H(z) = \dfrac{1}{A_3(z)}$

b) Determine the corresponding autocorrelation sequence $r_x(0)$, $r_x(1)$, $r_x(2)$, $r_x(3)$ when the model error $\text{e}_3 = (15/16)^2$.

Solution:

$$a_1 = \begin{bmatrix} 1 \\ \Gamma_1 \end{bmatrix}, = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix}, \ a_2 = \begin{bmatrix} 1 \\ 1/4 \\ 0 \end{bmatrix} + 1/2 \begin{bmatrix} 0 \\ 1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/8 \\ 1/2 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 1 \\ 3/8 \\ 1/2 \\ 0 \end{bmatrix} + 1/4 \begin{bmatrix} 0 \\ 1/2 \\ 3/8 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 19/32 \\ 1/4 \end{bmatrix},$$

$$\text{e}_3 = r_x(0)(1-\Gamma_1^2)(1-\Gamma_2^2)(1-\Gamma_3^2) = 1 \cdot (1-1/16)(1-1/4)(1-1/16) = (15/16)^2$$

gives $r_x(0) = 4/3$

And $r_x(j+1) = -\sum_{i=1}^{j+1} a_{j+1}(i) \ r_x(j+1-i)$

$j = 0 \qquad r_x(1) = -a_1(1) \ r_x(0) = -1/3$

$j = 1 \qquad r_x(2) = -a_2(1) \ r_x(1) - a_2(2) \ r_x(0) = -13/24$

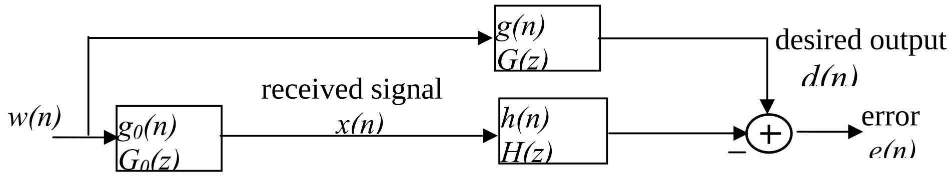
$j = 2 \qquad r_x(3) = -a_3(1) \ r_x(2) - a_3(2) \ r_x(1) - a_3(3) \ r_x(0) = 13/96$

Answer a) $H(z) = \dfrac{1}{1 + 1/2 z^{-1} \ + 19/32 \ z^{-3} \ + 1/4 z^{-3})}$

b) $r_x = [4/3, \ -1/3, \ -13/24, \ \ 13/96]^T =$

$= [1.3333, \ -0.3333, \ -0.5417, \ 0.1354]^T$

5 **Given:**



w(n) white noise with variance 1. $G_0(z) = 1 - 0.5z^{-1}$ and $G(z) = z^2$.

Task: a) Determine non-causal IIR $H(z)$. b) Determine causal IIR $H(z)$

c) Determine causal FIR $H(z)$ of length 3.

Solution: a) $H(z) = \dfrac{P_{dx}(z)}{P_x(z)}$

$$r_{dx}(k) = E\{d(n)\,x(n-k)\} = E\{\sum_m g(m)w(n-m)\cdot\sum_l g_0(l)w(n-k-l)\cdot\} =$$

$$= \underbrace{\sum_m g(m)\ \underbrace{\sum_l\ g_0(l)\ \underbrace{E\{w(n-m)\ w(n-k-l)\}}_{r_w(k-m+l)}}_{f(k-m)=g_0(-(k-m))*r_w(k-m)}}_{g(k)*g_0(-k)*r_w(k)}$$

$$P_{dx}(z)\ = G(z)\,G_0(z^{-1})P_w(z)\ =\ z^2\ \cdot(1-\frac{1}{2}z)\ \ P_w(z)$$

$$P_x(z) = G_0(z)\,G_0(z^{-1})P_w(z)\ = (1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)\ \ P_w(z)$$

$$H(z) = \frac{P_{dx}(z)}{P_x(z)} = \frac{z^2\ \cdot(1-\frac{1}{2}z)}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)} = \frac{z^2}{(1-\frac{1}{2}z^{-1})}$$

b) and c) The memory length of $g_0(n)$ is one so we can't predict 2 step forward so we must have $H_{causal}(z) = H_{FIR}(z) = 0$

b) $Q(z) = G_0(z);\ H_{causal} = \dfrac{1}{\sigma^2 Q(z)}\left[\dfrac{P_{dx}(z)}{Q(z^{-1})}\right]_+ = \dfrac{1}{\sigma^2 Q(z)}\left[\dfrac{z^2 Q(z^{-1})}{Q(z^{-1})}\right]_+ = 0$

c) $r_{dx}(k) = g(k)*g_0(-k)*r_w(k) = \{1\ 0\ \underset{\uparrow}{0}\}*\{-0.5\ \underset{\uparrow}{1}\}*\{1\} = \{-0.5\ 1\ 0\ \underset{\uparrow}{0}\}$

$$\begin{bmatrix} r_x(0) & r_x(1) & r_x(2) \\ r_x(1) & r_x(0) & r_x(1) \\ r_x(2) & r_x(1) & r_x(0) \end{bmatrix}\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \\ r_{dx}(2) \end{bmatrix} = 0;\quad \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$