Lund Institute of Technology
Department of Electro and Information Technology

Exam in Optimal Signal Processing - (ETT074)
Date: October 21, 2010
Hour: 14.00 - 19.00
Room: MA:10 IJ

Auxiliaries: Tables, calculator.
Write your name on all sheets.
The exam consists of 5 problems, each problem gives 0-1 point
Limits: 2.0-2.9 points give the grade 3
3.0-3.9 points give the grade 4
4.0-5.0 points give the grade 5
The solutions must be easy to follow and a clear answer must be given. Explain your solutions and assumptions.
Hints: Draw figures. Write definitions.

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1 Given the reflection coefficients

$$\Gamma_1 = 0.75,\ \Gamma_2 = 0.5,\ \Gamma_3 = 0.25,$$

Find, the model parameters $a_j(k)$ for *j=1, 2, 3*.

2 We want to model a signal x(n) using an all-pole model

$$H(z) = \frac{1}{1 + a_3(2)z^{-2} + a_3(3)z^{-3}}$$

Derive the normal equations that define the coefficients $a_3(2)$ and $a_3(3)$ that minimize the error

$$\varepsilon_p = \sum_{n=0}^{\infty} e^2(n), \qquad e(n) = x(n) + a_3(2)\,x(n-2) + a_3(3)\,x(n-3)$$

and derive an expression for the minimum error.

3. We want to predict $x(n)$ one step and as a predictor we chose

$$\hat{x}(n+1) = x(n) + \alpha\, x(n-1)\,.$$

The autocorrelation function is

$$r_x(k) = [1.0\ \ 0.8\ \ 0.6\ \ 0.5\ \ 0.4].$$

a) Determine the Wiener-Hopf equations for the predictor from minimizing $E\{|e(n)|^2\}$ with $e(n) = x(n+1) - \hat{x}(n+1)\}$.

b) Determine the optimal value for the parameter α.

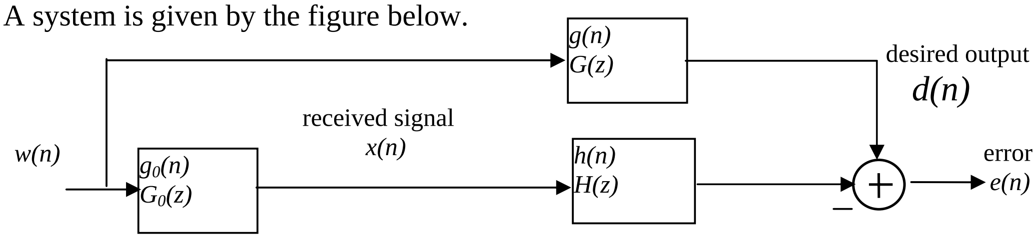
4. The autocorrelation function to a WSS process $x(n)$ is given by

$$r_x(k) = 4 \cdot 0.8^{|k|}.$$

a) Determine the power spectrum for the process $x(n)$.

b) Estimate the power spectrum from $r_x(k) = 4 \cdot 0.8^{|k|}$ using Blackman-Tukeys method. Chose a rectangular window of length 2M+1=5 (M=2).

5. A system is given by the figure below.



The input signal *w(n)* is white noise with variance 1 The filters g(n) and $g_0(n)$ are given by

$$G_0(z) = \frac{1}{1 - 0.5 z^{-1}} \quad \text{and} \quad G(z) = z^{-1}\,.$$

a) Determine the non-causal filter *H(z)* which minimize the error $\xi = E\{e(n)^2\}$

b) Determine the causal filter *H(z)* which minimize the error $\xi = E\{e(n)^2\}$

c) Determine the causal FIR filter of length 3 which minimize the error

Lund Institute of Technology. Department of Electroscience
Solutions

Exam in Optimal Signal Processing, October 21, 2010 (ETT074)

1. **Given this:** $\Gamma_1 = 0.75,\ \Gamma_2 = 0.5,\ \Gamma_3 = 0.25,$

   Task: Find, the model parameters $a_j(k)$ for $j=1, 2, 3$.

   Solution:

$$a_1 = \begin{bmatrix} 1 \\ \Gamma_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/4 \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 1 \\ 3/4 \\ 0 \end{bmatrix} + 1/2 \begin{bmatrix} 0 \\ 3/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9/8 \\ 1/2 \end{bmatrix}$$

$$a_3 = \begin{bmatrix} 1 \\ 9/8 \\ 1/2 \\ 0 \end{bmatrix} + 1/4 \begin{bmatrix} 0 \\ 1/2 \\ 9/8 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5/4 \\ 25/32 \\ 1/4 \end{bmatrix}$$

2 Given this: $H(z) = \dfrac{1}{1 + a_3(2)z^{-2} + a_3(3)z^{-3}}$

$$\varepsilon_p = \sum_{n=0}^{\infty} e^2(n), \qquad e(n) = x(n) + a_3(2)\,x(n-2) + a_3(3)\,x(n-3)$$

Task Determine the equation for $a_3(2),\ a_3(3)\ and\ \varepsilon_{p,\min}$

Solution:

$$\begin{cases} \dfrac{\partial \varepsilon_p}{\partial a_3(2)} = \dfrac{\partial}{\partial a_3(2)} \sum_n e^2(n) = 2\sum_n e(n) \dfrac{\partial}{\partial a_3(2)} e(n) = 2\sum_n e(n)x(n-2) = 0 \\ \dfrac{\partial \varepsilon_p}{\partial a_3(3)} = \dfrac{\partial}{\partial a_3(3)} \sum_n e^2(n) = 2\sum_n e(n) \dfrac{\partial}{\partial a_3(3)} e(n) = 2\sum_n e(n)x(n-3) = 0 \end{cases}$$

$$\begin{cases} \sum_n [x(n) + a_3(2)\,x(n-2) + a_3(3)\,x(n-3)]x(n-2) = 0 \\ \sum_n [x(n) + a_3(2)\,x(n-2) + a_3(3)\,x(n-3)]x(n-3) = 0 \end{cases}$$

$$\begin{cases} r_x(2) + a_3(2)\,r_x(0) + a_3(3)\,r_x(1) = 0 \\ r_x(3) + a_3(2)\,r_x(1) + a_3(3)\,r_x(0) = 0 \end{cases}$$

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} a_3(2) \\ a_3(3) \end{bmatrix} = -\begin{bmatrix} r_x(2) \\ r_x(3) \end{bmatrix}$$ Answer

Derive the minimum error

$$\varepsilon_p = \sum_{n=0}^{\infty} e(n)e(n) = \sum_{n=0}^{\infty} e(n)[x(n) + a_3(2)\,x(n-2) + a_3(3)\,x(n-3)] =$$

$$= \sum_{n=0}^{\infty} e(n)x(n) + a_3(2)\underbrace{\sum_{n=0}^{\infty} e(n)x(n-2)}_{0} + a_3(3)\underbrace{\sum_{n=0}^{\infty} e(n)x(n-3)}_{0}$$

$$\varepsilon_p = \sum_{n=0}^{\infty} [x(n) + a_3(2)\,x(n-2) + a_3(3)\,x(n-3)]x(n) =$$

$$= r_x(0) + a_3(2)r_x(2) + a_3(3)\,r_x(3)$$

$$\varepsilon_p = r_x(0) + a_3(2)r_x(2) + a_3(3)\,r_x(3)$$ Answer

3.

Given this: Predict $x(n)$ one step.

Input signal: $\hat{x}(n+1) = x(n) + \alpha\, x(n-1)$.

Autocorr. seq. $r_x(k) = [1.0 \;\; 0.8 \;\; 0.6 \;\; 0.5 \;\; 0.4]$.

Task: a) Determine Wiener-Hopf for $E\{|e(n)|^2\}$ with $e(n) = x(n+1) - \hat{x}(n+1)\}$.

b) Determine α

Solution:

a) $e(n) = x(n+1) - \hat{x}(n+1) = x(n+1) - x(n) - \alpha x(n-1)$

Then, $$E\{|e(n)|^2\} = E\{e(n)e(n)\} = E\{e(n)[x(n+1) - x(n) - \alpha x(n-1)]\}$$
$$= \underbrace{E\{e(n)[x(n+1) - x(n)]\}}_{\text{minimum error}} - \alpha \underbrace{E\{e(n)x(n-1)\}}_{\text{must be 0}}$$

$$E\{e(n)x(n-1)\} = E\{[x(n+1) - x(n) - \alpha x(n-1)]x(n-1)\} =$$
$$= r_x(2) - r_x(1) - \alpha\, r_x(0) = 0$$

b) $$\alpha = \frac{r_x(2) - r_x(1)}{r_x(0)} = \frac{0.6 - 0.8}{1} = -0.2$$

Minimum error is

$$E\{(x(n+1) - x(n) - ax(n-1))(x(n+1) - x(n)\} =$$
$$= 2(r_x(0) - r_x(1)) + a\,(r_x(1) - r_x(2)) =$$
$$= 2\,(1 - 0.8) - 0.2\,(0.8 - 0.6) = 0.36$$

4

Given this: Autocorrelation function $r_x(k) = 4 \cdot 0.8^{|k|}$.

Task: a) Determine the power spectrum.

b) Estimate power spectrum with Blackman-Tukey method. Rectangular window 2M+1=5 (M=2).

c) Is the estimate in b) a true power spectrum? Plot the power spectra

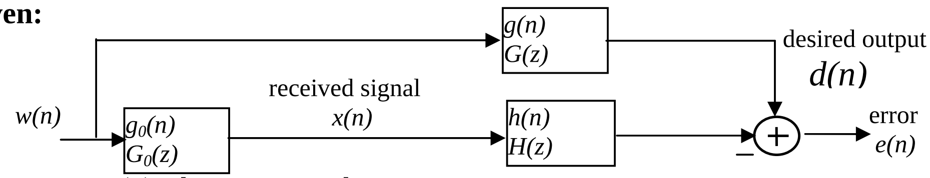
Solution: a) From formula table (see also homework 1)

$$P_x(e^{j\omega}) = 4\frac{1 - 0.8^2}{(1 - 0.8e^{-j\omega})(1 - 0.8e^{j\omega})} = \frac{1.44}{1.64 - 1.6\,\cos(\omega)}$$

b) $$r_x(k) = 4 \cdot 0.8^{|k|} = 4[0.64e^{j\omega 2} \quad 0.8e^{j\omega} \quad \underset{\uparrow}{1} \quad 0.8e^{-j\omega} \quad 0.64e^{j\omega 2}] \quad -2 \le k \le 2$$

$$P_x(e^{j\omega}) = 4[1 + 1.6\cos(\omega) + 1.28\cos(2\omega)]$$

5 **Given:**



w(n) white noise with variance 1.

$G_0(z) = \frac{1}{1-0.5z^{-1}}$ and $G(z) = z^{-1}$.

Task: a) Determine non-causal IIR $H(z)$.

b) Determine causal IIR $H(z)$.

c) Determine causal FIR $H(z)$ of length 3.

Solution:

$$P_x(z) = P_w(z)G_0(z)G_0(z^{-1}) = \frac{1}{1-0.5z^{-1}} \cdot \frac{1}{1-0.5z}$$

$$P_{dx}(z) = G(z)G_0(z^{-1})P_w(z) = z^{-1}G_0(z) = (z^{-1})\frac{1}{1-\frac{1}{2}z}$$

or

$$r_{dx}(k) = g(k) * g_0(-k) * r_w(k) = \left[\underset{\uparrow}{0},\ 1\right] * (0.5^{-k}\ u(-k)) =$$

$$= \{. ...1/8,\ 1/4,\ \underset{\uparrow}{\underbrace{1/2}},\ 1,\ 0,\ 0,\ 0..\} = \{...,1/4,\ \underset{\uparrow}{0.5},\ 1,\ 0,\ 0...\}$$

$$r_x(k) = \frac{4}{3} \cdot 0.5^{|k|}$$

a) $$H_{noncausal}(z) = \frac{P_{dx}(z)}{P_x(z)}$$

$$H(z) = \frac{P_{dx}(z)}{P_x(z)} = \frac{G(z)G_0(z^{-1})P_w(z)}{G_0(z)G_0(z^{-1})P_w(z)} = \frac{G(z)G_0(z^{-1})P_w(z)}{G_0(z)G_0(z^{-1})P_w(z)} =$$

$$= z^{-1}(1-0.5z^{-1}) = z^{-1} - 0.5z^{-2}$$

b) $$Q(z) = G_0(z); H_{causal} = \frac{1}{\sigma^2 Q(z)}\left[\frac{P_{dx}(z)}{Q(z^{-1})}\right] = \frac{1}{\sigma^2 Q(z)}\left[\frac{z^{-1}Q(z^{-1})}{Q(z^{-1})}\right]_+ = H_{noncausal}(z)$$

c) See also computer exercise 1.

$$\begin{bmatrix} r_x(0) & r_x(1) & r_x(2) \\ r_x(1) & r_x(0) & r_x(1) \\ r_x(2) & r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \\ r_{dx}(2) \end{bmatrix}$$

$$4/3 * \begin{bmatrix} 1 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$$

Inverting a first order correlation matrix, see Hayes or the exercises

$$\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \frac{3}{4}\begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 5/4 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1/2 \end{bmatrix}$$ same as in a) and b)