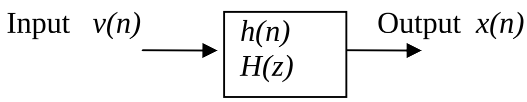
Extra problems with solution

Random Processes, chapter 3

3.1. The input to a filter *h(n)* is *v(n)* and the output is *x(n)*. The power spectrum for *v(n)* is $P_v(e^{j\omega}) = \sigma_v^2$.



a) The filter is given by $H(z) = 1 + z^{-1}$. Determine the power spectra $P_x(z)$ and $P_x(e^{j\omega})$.

b) The filter is given by $H(z) = \dfrac{1}{1 - 0.5\, z^{-1}}$. Determine the power spectra $P_x(z)$ and $P_x(e^{j\omega})$.

c) The filter is given by $h(n) = \{1\ 2\ 1\} = \partial(n) + 2\partial(n-1) + \partial(n-2)$. Determine the power spectra $P_x(z)$ and $P_x(e^{j\omega})$.

d) Determine the cross spectra $P_{xv}(e^{j\omega})$ and $P_{vx}(e^{j\omega})$ using the impulse response $h(n)$ from c).

3.2. The autocorrelation function to a WSS process $x(n)$ is given by

$$r_x(k) = 4 \cdot 0.8^{|k|} .$$

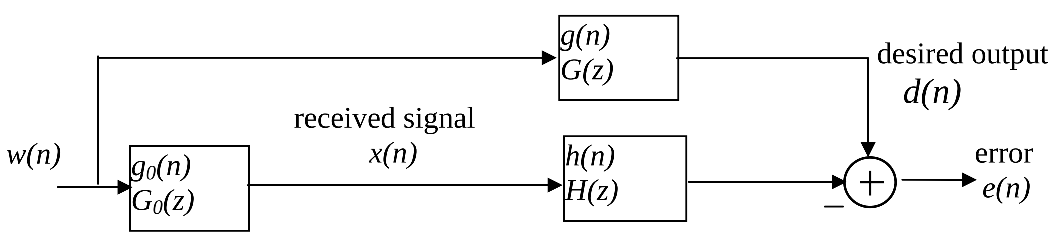
a) Determine the power spectrum $P_x(e^{j\omega})$ for the process $x(n)$.

b) A new process is given by

$$y(n) = h(n) * x(n) + w(n)$$

with $h(n) = \delta(n) + \delta(n-1)$ and $w(n)$ white noise with variance 2. Determine the power spectrum $P_y(e^{j\omega})$ for the process $y(n)$.

3.3. A system is given by the figure below.



The input signal *w(n)* is white noise with variance 1. The filters g(n) and g_0(n) are given by

$$G_0(z) = \frac{1}{1 - 0.5 z^{-1}} \quad \text{and} \quad G(z) = z^{-1} .$$

a) Determine the power spectra $P_x(e^{j\omega})$ and $P_d(e^{j\omega})$.

b) Determine the cross spectra $P_{xw}(e^{j\omega})$ and $P_{dx}(e^{j\omega})$.

Signal modelling, chapter 4:

4.1 We want to model a signal x(n) using an all-pole model

$$H(z) = \frac{1}{1 + a_3(2)z^{-2} + a_3(3)z^{-3}}$$

Derive the normal equations that define the coefficients $a_3(2)$ and $a_3(3)$ that minimize the error $\varepsilon_p = \sum_{n=0}^{\infty} e^2(n)$, $e(n) = x(n) + a_3(2)x(n-2) + a_3(3)x(n-3)$ and derive an expression for the minimum error.

Levinson-Durbin, chapter 5:

5.1 Given the reflection coefficients $\Gamma_1 = 0.75$, $\Gamma_2 = 0.5$, $\Gamma_3 = 0.25$,
Find, the model parameters $a_j(k)$ for *j=1, 2, 3*.

5.2 Given the autocorrelation sequence $r_k(k) = [\;0\;\;1\;\;2\;\;\underset{\uparrow}{3}\;\;2\;\;1\;\;0\;]$

Find the reflection coefficients Γ_j, the model parameters $a_j(k)$ and the model errors ε_j for *j=1, 2.*

Wiener FIR-filtering, chapter 7

7.1 We want to predict $x(n)$ one step and as a predictor we chose

$$\hat{x}(n+1) = x(n) + \alpha\, x(n-1)\ .$$

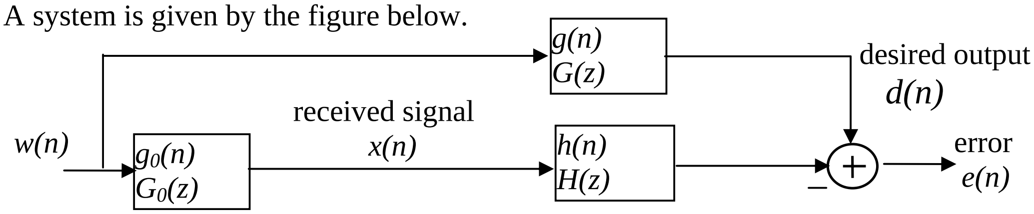
The autocorrelation function is $r_x(k) = [1.0\;\;0.8\;\;0.6\;\;0.5\;\;0.4]$.

a) Determine the Wiener-Hopf equations for the predictor from minimizing $E\{|e(n)|^2\}$ with $e(n) = x(n+1) - \hat{x}(n+1)\}$.

b) Determine the optimal value for the parameter α.

Wiener FIR and IIR filtering, chapter 7

7.2. A system is given by the figure below.



The input signal *w(n)* is white noise with variance 1. The filters g(n) and g_0(n) are given by

$$G_0(z) = \frac{1}{1 - 0.5z^{-1}} \quad \text{and} \quad G(z) = z^{-1}\ .$$

a) Determine the non-causal filter *H(z)* which minimize the error $\xi = E\{e(n)^2\}$

b) Determine the causal filter *H(z)* which minimize the error $\xi = E\{e(n)^2\}$

c) Determine the causal FIR filter of length 3 which minimize the error $\xi = E\{e(n)^2\}$.

Spectrum estimation, chapter 8

8.1. The autocorrelation function to a WSS process $x(n)$ is given by $r_x(k) = 4 \cdot 0.8^{|k|}$.

a) Determine the power spectrum for the process $x(n)$.

b) Estimate the power spectrum from $r_x(k) = 4 \cdot 0.8^{|k|}$ using Blackman-Tukeys method. Chose a rectangular window of length 2M+1=5 (M=2).

OSB, ETTN10 Optimal Signal Processing, 2012

Solutions: Extra problems

3.1 Given this: $x(n) = v(n) * h(n)$ with *v(n)* white noise with $P_v(e^{j\omega}) = \sigma_v^2$

and the filter *H(z)* known.

a). $H(z) = 1 + z^{-1}$, b) $H(z) = \dfrac{1}{1-0.5\,z^{-1}}$,c) $h(n) = \lfloor \underset{\uparrow}{1} \;\; 2 \;\; 1 \rfloor$

Task: a,b,c) Determine $P_x(z)$ and $P_x(e^{j\omega})$.

d) Determine $P_{xv}(e^{j\omega})$ and $P_{vx}(e^{j\omega})$ using $h(n)$ from c).

Solution.

a) $P_x(z) = P_v(z)H(z)H(z^{-1}) = \sigma_v^2(1+z^{-1})(1+z^1) = \sigma_v^2(z+2+z^{-1})$

$P_x(e^{j\omega}) = P_v(e^{j\omega})H(e^{j\omega})H(e^{-j\omega}) = \sigma_v^2(1+e^{-j\omega})(1+e^{j\omega}) =$

$= \sigma_v^2(e^{j\omega}+2+e^{j\omega}) = \sigma_v^2\;(2+2\cos(\omega))$

b)

$$P_x(z) = P_v z)Hz)H(z^{-1}) = \sigma_v^2 \frac{1}{(1-0.5z^{-1})(1-0.5z)} = \sigma_v^2 \frac{1}{-0.5z+5/4-0.5z^{-1}}$$

$$P_x(e^{j\omega}) = P_v(e^{j\omega})H(e^{j\omega})H(e^{-j\omega}) = \sigma_v^2 \frac{1}{(1-0.5e^{-j\omega})(1-0.5e^{j\omega})} = \sigma_v^2 \frac{1}{5/4-\cos(\omega)}$$

c) $h(n) = [1\;2\;1]$; $h(n) * h(-n) = [1\;4\;\underset{\uparrow}{6}\;4\;\;1]$; $H(z)H(z^{-1}) = z^2+4z^1+6+4z^{-1}+z^{-2}$

$P_x(z) = P_v(z)H(z)H(z^{-1}) = \sigma_v^2(z^2+4z^1+6+4z^{-1}+z^{-2})$

$P_x(e^{j\omega}) = P_v(e^{j\omega})H(e^{j\omega})H(e^{-j\omega}) = \sigma_v^2(e^{j\omega}+4e^{j\omega}+6+4e^{-j\omega}+e^{-j\omega 2}) =$

$= \sigma_v^2\,(6+8\cos(\omega)+2\cos(2\omega))$

d) $P_{xv}(e^{j\omega)}) = P_v(e^{j\omega)})H(e^{j\omega)}) = 2\sigma_v^2\,(1+\cos\omega)e^{-j\omega}$

$P_{vx}(e^{j\omega)}) = P_v(e^{j\omega)})H^*(e^{j\omega)}) = 2\sigma_v^2\,(1+\cos\omega)e^{j\omega}$

3.2 Given this: $r_x(k) = 4 \cdot 0.8^{|k|}$

Task: a) Determine the power spectrum $P_x(e^{j\omega})$

b) Determine the power spectrum $P_y(e^{j\omega})$ with $y(n) = h(n) * x(n) + w(n)$

and $h(n) = \delta(n) + \delta(n-1)$, $w(n)$ white noise with variance 2.

Solution: a)

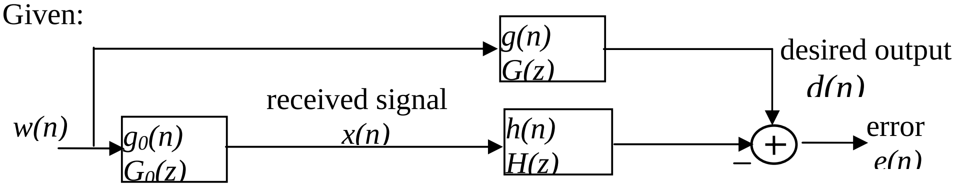
$$P_x(e^{j\omega}) = 4\frac{1-0.8^2}{(1-0.8e^{-j\omega})(1-0.8e^{j\omega})} = \frac{4\cdot 0.36}{1.64-1.6\cos\omega}$$

b)

$$P_y(e^{j\omega}) = P_x(e^{j\omega})H(e^{j\omega})H(e^{-j\omega}) + 2 =$$

$$= \frac{4\cdot 0.36}{1.64-1.6\cos\omega}(2+2\cos(\omega)) + 2$$

3.3. Given:



w(n) white noise with variance 1.

$$G_0(z) = \frac{1}{1-0.5z^{-1}} \quad \text{and} \quad G(z) = z^{-1} .$$

Task: a) Determine the power spectra $P_x(e^{j\omega})$ and $P_d(e^{j\omega})$.

b) Determine the cross spectra $P_{xw}(e^{j\omega})$ and $P_{dx}(e^{j\omega})$.

Solution:

a)

$$P_w(e^{jw}) = 1$$

$$P_x(z) = P_w(e^{j\omega})G_0(e^{j\omega})G_0^*(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}} \cdot \frac{1}{1-0.5e^{j\omega}} = \frac{4}{5-4\cos\omega}$$

$$P_d(z) = P_w(e^{j\omega})G(e^{j\omega})G^*(e^{j\omega}) = e^{-j\omega}e^{j\omega} = 1$$

b)

$$P_{xw}(e^{jw}) = G_0(e^{jw})P_w(e^{jw}) = \frac{1}{1-\frac{1}{2}e^{-jw}}$$

$$P_{dx}(e^{jw}) = G(e^{jw})G_0^*(e^{jw})P_w(z) = (e^{-jw})\frac{1}{1-\frac{1}{2}e^{jw}}$$

Determine the cross correlation from the definition

$$r_{dx}(k) = E\{d(n)x(n-k)\} = E\{\sum_m g(m)w(n-m) \cdot \sum_l g_0(l)w(n-k-l) \cdot\} =$$

$$= \underbrace{\sum_m g(m) \underbrace{\sum_l g_0(l) \underbrace{E\{w(n-m)\ w(n-k-l)\}}_{r_w(k-m+l)}}_{f(k-m)=g_0(-(k-m))*r_w(k-m)}}_{g(k)*g_0(-k)*r_w(k)}$$

$$r_{dx}(k) = g(k)*g_0(-k)*r_w(k) = \lfloor \underset{\uparrow}{0},\ 1 \rfloor * \ (0.5^{-k}\ u(-k)) =$$

$$= \{.\ ...1/8,\ 1/4,\ \underset{\uparrow}{\underbrace{1/2}},\ 1,\ 0,\ 0,\ 0..\} = \{...,1/4,\ \underset{\uparrow}{0.5},\ 1,\ 0,\ 0...\}$$

4.1 Given this: $H(z) = \dfrac{1}{1+a_3(2)z^{-2}+a_3(3)z^{-3}}$

$$\varepsilon_p = \sum_{n=0}^{\infty} e^2(n), \qquad e(n) = x(n) + a_3(2)\,x(n-2) + a_3(3)\,x(n-3)$$

Task Determine the equation for $a_3(2),\ a_3(3)\ and\ \varepsilon_{p,\min}$

Solution:

$$\begin{cases} \dfrac{\partial \varepsilon_p}{\partial a_3(2)} = \dfrac{\partial}{\partial a_3(2)} \sum_n e^2(n) = 2\sum_n e(n) \dfrac{\partial}{\partial a_3(2)} e(n) = 2\sum_n e(n)x(n-2) = 0 \\ \dfrac{\partial \varepsilon_p}{\partial a_3(3)} = \dfrac{\partial}{\partial a_3(3)} \sum_n e^2(n) = 2\sum_n e(n) \dfrac{\partial}{\partial a_3(3)} e(n) = 2\sum_n e(n)x(n-3) = 0 \end{cases}$$

$$\begin{cases} \sum_n [x(n)+a_3(2)\,x(n-2)+a_3(3)\,x(n-3)]x(n-2)=0 \\ \sum_n [x(n)+a_3(2)\,x(n-2)+a_3(3)\,x(n-3)]x(n-3)=0 \end{cases}$$

$$\begin{cases} r_x(2)+a_3(2)\,r_x(0)+a_3(3)\,r_x(1)=0 \\ r_x(3)+a_3(2)\,r_x(1)+a_3(3)\,r_x(0)=0 \end{cases}$$

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}\begin{bmatrix} a_3(2) \\ a_3(3) \end{bmatrix}=-\begin{bmatrix} r_x(2) \\ r_x(3) \end{bmatrix} \qquad \text{Answer}$$

Derive the minimum error

$$\varepsilon_p=\sum_{n=0}^{\infty} e(n)e(n)=\sum_{n=0}^{\infty} e(n)[x(n)+a_3(2)\,x(n-2)+a_3(3)\,x(n-3)]=$$

$$=\sum_{n=0}^{\infty} e(n)x(n)+a_3(2)\underbrace{\sum_{n=0}^{\infty} e(n)x(n-2)}_{0}+a_3(3)\underbrace{\sum_{n=0}^{\infty} e(n)x(n-3)}_{0}$$

$$\varepsilon_p=\sum_{n=0}^{\infty} [x(n)+a_3(2)\,x(n-2)+a_3(3)\,x(n-3)]x(n)=$$

$$=r_x(0)+a_3(2)r_x(2)+a_3(3)\,r_x(3)$$

$$\varepsilon_p=r_x(0)+a_3(2)r_x(2)+a_3(3)\,r_x(3) \qquad \text{Answer}$$

5.1 Given this: $\Gamma_1=0.75,\ \Gamma_2=0.5,\ \Gamma_3=0.25,$

Task: Find, the model parameters $a_j(k)$ for $j=1, 2, 3.$

Solution:

$$a_1=\begin{bmatrix} 1 \\ \Gamma_1 \end{bmatrix}=\begin{bmatrix} 1 \\ 3/4 \end{bmatrix}$$

$$a_2=\begin{bmatrix} 1 \\ 3/4 \\ 0 \end{bmatrix}+1/2\begin{bmatrix} 0 \\ 3/4 \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ 9/8 \\ 1/2 \end{bmatrix}$$

$$a_3=\begin{bmatrix} 1 \\ 9/8 \\ 1/2 \\ 0 \end{bmatrix}+1/4\begin{bmatrix} 0 \\ 1/2 \\ 9/8 \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ 5/4 \\ 25/32 \\ 1/4 \end{bmatrix}$$

5.2

Given this: $r_k(k)=[\ 0\ \ 1\ \ 2\ \ \underset{\uparrow}{3}\ \ 2\ \ 1\ \ 0\ ]$

Task: Determine Γ_j, $a_j(k)$ and ε_j for $j=1, 2.$

Solution: See solution for problem 5.1

Step 1

Init $a_0(0)=1,$ and $\varepsilon_0=r_x(0)=3$

Step 2

j=0

$a) \quad \gamma_0 = r_x(1) = 2$

$b) \quad \Gamma_1 = -\gamma_0 / \varepsilon_0 = -2/3$

$c)$ empty

$d) \quad a_1(1) = \Gamma_1 = -2/3$

$e) \quad \varepsilon_1 = \varepsilon_0(1 - \Gamma_1{}^2) = 3(1 - 4/9) = 5/3$

j=1

$a) \quad \gamma_1 = r_x(2) + a_1(1)\, r_x(1) = 1 - 2/3 \cdot 2 = -1/3$

$b) \quad \Gamma_2 = -\gamma_1 / \varepsilon_1 = -(-\frac{1/3}{5/3}) = 1/5$

$c) \quad a_2(1) = a_1(1) + \Gamma_2 \; a_1(1) = -2/3 - 1/5 \cdot 2/3 = -4/5$

$d) \quad a_2(2) = \Gamma_2 = 1/5$

$e) \quad \varepsilon_2 = \varepsilon_1(1 - \Gamma_2{}^2) = 5/3(1 - 1/25) = 8/5$

Answer:

$a_1 = [1, -2/3]^T, \; a_2 = [1, -4/5, 1/5]^T, \;\; \varepsilon_1 = 5/3, \;\; \varepsilon_2 = 8/5$

$\Gamma = [-2/3, 1/5]^T$

7.1 Given this: Predict $x(n)$ one step.

Input signal: $\hat{x}(n+1) = x(n) + \alpha\, x(n-1)$.

Autocorr. seq. $r_x(k) = [1.0 \;\; 0.8 \;\; 0.6 \;\; 0.5 \;\; 0.4]$.

Task:

a) Determine Wiener-Hopf for $E\{|e(n)|^2\}$ with $e(n) = x(n+1) - \hat{x}(n+1)\}$.

b) Determine α

Solution:

a) $e(n) = x(n+1) - \hat{x}(n+1) = x(n+1) - x(n) - \alpha x(n-1)$

Then, $E\{|e(n)|^2\} = E\{e(n)e(n)\} = E\{e(n)[x(n+1) - x(n) - \alpha x(n-1)]\}$

$= \underbrace{E\{e(n)[x(n+1) - x(n)]\}}_{\text{minimum error}} - \alpha \underbrace{E\{e(n)x(n-1)\}}_{\text{must be 0}}$

$E\{e(n)x(n-1)\} = E\{[x(n+1) - x(n) - \alpha x(n-1)]x(n-1)\} =$

$= r_x(2) - r_x(1) - \alpha\, r_x(0) = 0$

b) $\alpha = \frac{r_x(2) - r_x(1)}{r_x(0)} = \frac{0.6 - 0.8}{1} = -0.2$

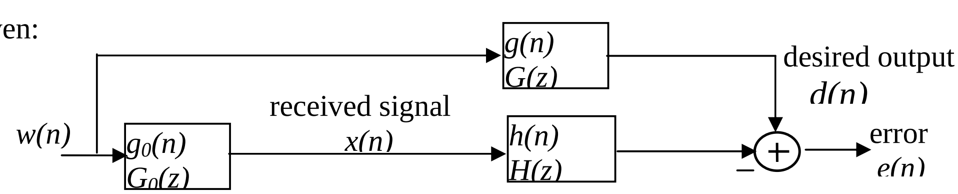
Minimum error is

$E\{(x(n+1) - x(n) - ax(n-1))(x(n+1) - x(n)\} =$

$= 2(r_x(0) - r_x(1)) + a\,(r_x(1) - r_x(2)) =$

$= 2(1 - 0.8) - 0.2(0.8 - 0.6) = 0.36$

7.2 . Given:



w(n) white noise with variance 1.

$G_0(z) = \frac{1}{1 - 0.5z^{-1}}$ and $G(z) = z^{-1}$.

Task: a) Determine non-causal IIR $H(z)$.
b) Determine causal IIR $H(z)$.
c) Determine causal FIR $H(z)$ of length 3.

Solution: a) From homework 1

$$P_x(z) = P_w(z)G_0(z)G_0(z^{-1}) = \frac{1}{1-0.5z^{-1}} \cdot \frac{1}{1-0.5z}$$

$$P_{dx}(z) = G(z)G_0(z^{-1})P_w(z) = z^{-1}G_0(z) = (z^{-1})\frac{1}{1-\frac{1}{2}z}$$

or

$$r_{dx}(k) = g(k) * g_0(-k) * r_w(k) = \lfloor \underset{\uparrow}{0},\ 1 \rfloor * (0.5^{-k}\ u(-k)) =$$

$$= \{.\ ...1/8,\ 1/4,\ \underset{\uparrow}{\underbrace{1/2}},\ 1,\ 0,\ 0,\ 0..\} = \{...,1/4,\ \underset{\uparrow}{0.5},\ 1,\ 0,\ 0...\}$$

$$r_x(k) = \frac{4}{3} \cdot 0.5^{|k|}$$

a) $$H_{noncausal}(z) = \frac{P_{dx}(z)}{P_x(z)}$$

$$H(z) = \frac{P_{dx}(z)}{P_x(z)} = \frac{G(z)G_0(z^{-1})P_w(z)}{G_0(z)G_0(z^{-1})P_w(z)} = \frac{G(z)G_0(z^{-1})P_w(z)}{G_0(z)G_0(z^{-1})P_w(z)} =$$

$$= z^{-1}(1-0.5z^{-1}) = z^{-1} - 0.5z^{-2}$$

b) $$Q(z) = G_0(z); H_{causal} = \frac{1}{\sigma^2 Q(z)}\left[\frac{P_{dx}(z)}{Q(z^{-1})}\right] = \frac{1}{\sigma^2 Q(z)}\left[\frac{z^{-1}Q(z^{-1})}{Q(z^{-1})}\right]_+ = H_{noncausal}(z)$$

c) See also computer exercise 1.

$$\begin{bmatrix} r_x(0) & r_x(1) & r_x(2) \\ r_x(1) & r_x(0) & r_x(1) \\ r_x(2) & r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \\ r_{dx}(2) \end{bmatrix}$$

$$4/3 * \begin{bmatrix} 1 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$$

Inverting a first order correlation matrix, see Hayes or the exercises

$$\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \frac{3}{4}\begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 5/4 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1/2 \end{bmatrix}$$ same as in a) and b)

8.1 Given this: Autocorrelation function $r_x(k) = 4 \cdot 0.8^{|k|}$.

Task: a) Determine the power spectrum.
b) Estimate power spectrum with Blackman-Tukey method. Rectangular window 2M+1=5 (M=2).
c) Is the estimate in b) a true power spectrum? Plot the power spectra

Solution: a) From formula table

$$P_x(e^{j\omega}) = 4\frac{1-0.8^2}{(1-0.8e^{-j\omega})(1-0.8e^{j\omega})} = \frac{1.44}{1.64-1.6\cos(\omega)}$$

b) $$r_x(k) = 4 \cdot 0.8^{|k|} = 4[0.64e^{j\omega 2}\quad 0.8e^{j\omega}\quad \underset{\uparrow}{1}\quad 0.8e^{-j\omega}\quad 0.64e^{j\omega 2}] \quad -2 \le k \le 2$$

$$P_x(e^{j\omega}) = 4[1+1.6\cos(\omega)+1.28\cos(2\omega)]$$