LUND INSTITUTE
OF TECHNOLOGY
Lund University
Department of Electrical and Information Technology

Formula Table

Digital Signal Processing

Bengt Mandersson

Lund 2011

Department of Electrical and Information Technology, Lund University, Sweden

Innehåll

1 Basic trigonometrical formula

1.1 Trigonometry

$\sin\alpha = \cos(\alpha - \pi/2)$	$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\cos\alpha = \sin(\alpha + \pi/2)$	$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos^2\alpha + \sin^2\alpha = 1$	$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$
$\cos^2\alpha - \sin^2\alpha = \cos 2\alpha$	$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$
$2\sin\alpha\cos\alpha = \sin 2\alpha$	$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$
$\sin(-\alpha) = -\sin\alpha$	$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
$\cos(-\alpha) = \cos\alpha$	$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$	

$$\cos\alpha = \tfrac{1}{2}\left(e^{j\alpha} + e^{-j\alpha}\right), \quad \sin\alpha = \tfrac{1}{2j}\left(e^{j\alpha} - e^{-j\alpha}\right), \quad e^{j\alpha} = \cos\alpha + j\sin\alpha$$

$$A\cos\alpha + B\sin\alpha = \sqrt{A^2+B^2}\cos(\alpha-\beta)$$

$$\text{with } \cos\beta = \tfrac{A}{\sqrt{A^2+B^2}}, \ \sin\beta = \tfrac{B}{\sqrt{A^2+B^2}}$$

$$\text{och } \beta = \begin{cases} \arctan\frac{B}{A} & \text{if } A \geq 0 \\ \arctan\frac{B}{A} + \pi & \text{if } A < 0 \end{cases}$$

$$A\cos\alpha + B\sin\alpha = \sqrt{A^2+B^2}\sin(\alpha+\beta)$$

$$\text{with } \cos\beta = \tfrac{B}{\sqrt{A^2+B^2}}, \ \sin\beta = \tfrac{A}{\sqrt{A^2+B^2}}$$

$$\text{och } \beta = \begin{cases} \arctan\frac{A}{B} & \text{if } B \geq 0 \\ \arctan\frac{A}{B} + \pi & \text{if } B < 0 \end{cases}$$

Degree	Rad	sin	cos	tan	cot
0	0	0	1	0	$\pm\infty$
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
90	$\frac{\pi}{2}$	1	0	$\pm\infty$	0

1.2 Matrix theory

Notation of matrix A and vector x

A matrix $\mathbf{A}$ of order mxn and a vector $\mathbf{x}$ with dimension n are defined by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The matrix $\mathbf{A}$ is symmetrical if $a_{ij} = a_{ji} \; \forall \; ij$.
$\mathbf{I}$ denotes the unit matrix.

Transpose of a matrix A

$$\mathbf{B} = \mathbf{A}^T \text{ där } b_{ij} = a_{ji}$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

Determinant of a matrix A

$$det\mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix} = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} det\mathbf{M}_{ij}$$

there $\mathbf{M}_{ij}$ is the resulting matrix if row i and column j in the matrix $\mathbf{A}$ are deleted.

$$det\mathbf{AB} = det\mathbf{A} \cdot det\mathbf{B}$$

Specially for a 2x2 matrix:

$$det\mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$$

Inverse of the matrix A

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \quad (\text{om } det\mathbf{A}\#0)$$

$$\mathbf{A}^{-1} = \frac{1}{det\mathbf{A}} \cdot \mathbf{C}^T$$

with $\mathbf{C}$ defined by

$$c_{ij} = (-1)^{i+j} \cdot det\mathbf{M}_{ij}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Specially for a 2x2 matrix:

$$\mathbf{A}^{-1} = \frac{1}{det\mathbf{A}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Eigenvalues and eigenvectors

Eigenvalues ($\lambda_i,\ i = 1, 2, \ldots, n$) and eigenvectors ($\mathbf{q}_i,\ i = 1, 2, \ldots, n$) are the solution to the equation system

$$\mathbf{A}\mathbf{q} = \lambda\mathbf{q} \text{ eller } (\mathbf{A} - \lambda\mathbf{I})\mathbf{q} = 0$$

The eigenvalues can be determined as the solution to t6he characteristic equation to $\mathbf{A}$

$$det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0 = 0$$

$det(\lambda\mathbf{I} - \mathbf{A})$ is called the characteristic polynomial to $\mathbf{A}$.

1.3 Notation of some basic signals

Unit step function $$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The impuls function $$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \# 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

Rectangular function $$p(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$

Sinc-function $$sinc\ x = \frac{\sin \pi x}{\pi x}$$

Periodical sinc-function $$diric(x, N) = \frac{\sin\left(\frac{Nx}{2}\right)}{N \sin\left(\frac{x}{2}\right)}$$

Complex sinusoids $$e^{st} = e^{\sigma t} e^{j\Omega t}$$

Complex undamped sinusoids $$e^{j\Omega t} = \cos \Omega t + j \sin \Omega t$$

1.4 Often used relations

Sum of a geometrical series.

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N & \text{if } a = 1 \\ \frac{1-a^N}{1-a} & \text{if } a \neq 1 \end{cases}$$

Sum of a sinusoids over a full periods.

$$\sum_{n=0}^{N-1} e^{j2\pi\ kn/N} = \begin{cases} N & \text{if } k = 0, \pm N, \ldots \\ 0 & \text{f.ö.} \end{cases}$$

1.5 Correlation

Correlation, cross correlation, spectrum, cross spectrum and coherence between input and out signals.

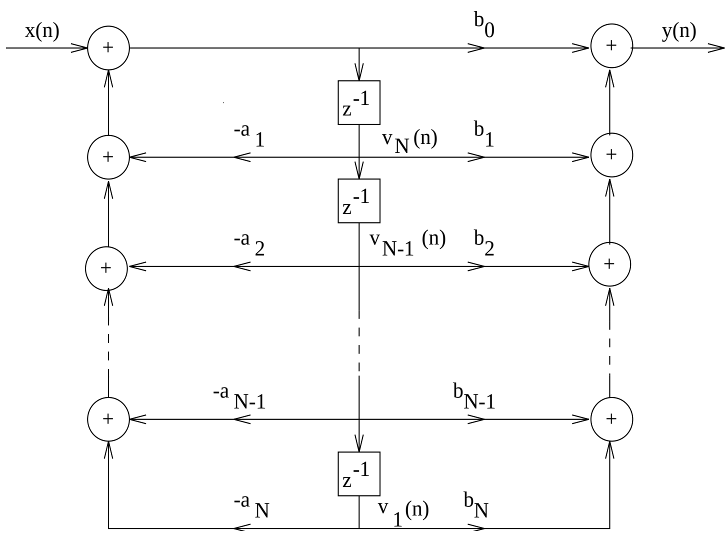
$$
\begin{array}{lcllcl}
y(t) & = & h(t) * x(t) & y(n) & = & h(n) * x(n) \\
Y(F) & = & H(F) \cdot X(F) & Y(f) & = & H(f) \cdot X(f) \\
r_{yy}(\tau) & = & r_{hh}(\tau) * r_{xx}(\tau) & r_{yy}(n) & = & r_{hh}(n) * r_{xx}(n) \\
R_{yy}(F) & = & |H(F)|^2 R_{xx}(F) & R_{yy}(f) & = & |H(f)|^2 \cdot R_{xx}(f) \\
r_{yx}(\tau) & = & h(\tau) * r_{xx}(\tau) & r_{yx}(n) & = & h(n) * r_{xx}(n) \\
R_{yx}(F) & = & H(F) \cdot R_{xx}(F) & R_{yx}(f) & = & H(f) \cdot R_{xx}(f) \\
r_{xx}(\tau) & = & \int_t x(t)x(t-\tau)dt & r_{xx}(n) & = & \sum_\ell x(\ell)x(\ell-n) \\
r_{yx}(\tau) & = & \int_t y(t)x(t-\tau)dt & r_{yx}(n) & = & \sum_\ell y(\ell)x(\ell-n) \\
\gamma_{xx}(\tau) & = & E\{x(t)x(t-\tau)\} & \gamma_{xx}(n) & = & E\{x(\ell)x(\ell-n)\} \\
\gamma_{yx}(\tau) & = & E\{y(t)x(t-\tau)\} & \gamma_{yx}(n) & = & E\{y(\ell)x(\ell-n)\}
\end{array}
$$

Gaussian random variables. $X_i \in N(m_i, \sigma_i)$

$$
\begin{array}{rcl}
E\{X_1X_2X_3X_4\} & = & E\{X_1X_2\}\ E\{X_3X_4\} + E\{X_1X_3\}\ E\{X_2X_4\} + \\
 & + & E\{X_1X_4\}\ E\{X_2X_3\} - 2m_1m_2m_3m_4
\end{array}
$$

1.6 Circuit model (single input, single output)

1) Canonical form (direct form II)



2) The difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

3) Steady state notation

$$\begin{cases} \mathbf{v}(n+1) &= \mathbf{F}\mathbf{v}(n) + \mathbf{q}\cdot x(n) \\ y(n) &= \mathbf{g^T}\mathbf{v}(n) + d\cdot x(n) \end{cases}$$

with

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 1 \\ -a_k & -a_{k-1} & \dots & -a_2 & -a_1 \end{pmatrix} \quad ; \quad \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\mathbf{g^T} = (b_k, \dots, b_2, b_1) - b_0(a_k, \dots, a_2, a_1) \quad ; \quad d = b_0$$

4) The system function

$$\mathcal{H}(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

1.7 Input output relations

1) Convolution

$$y(n) = h * x = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$$

2) Steady state

a) Direct solution

$$y(n) = \mathbf{g^T}\cdot\mathbf{F}^n\mathbf{v}(0) + \sum_{k=0}^{n-1}\mathbf{g^T}\cdot\mathbf{F}^{n-1-k}\mathbf{q}x(k)u(n-1) + dx(n)$$

b) Impulse response

$$h(n) = \mathbf{g^T}\cdot\mathbf{F}^{n-1}\mathbf{q}u(n-1) + d\delta(n)$$

c) System function

$$\mathcal{H}(z) = \mathbf{g^T}[z\mathbf{I} - \mathbf{F}]^{-1}\mathbf{q} + d$$

1.8 Analogous sinusoids through a linear, causal filter

1) Complex, non-causal input signal

$$x(t) = e^{j\Omega_0 t} = (\cos(\Omega_0 t) + j\ \sin(\Omega_0 t)) \quad -\infty < t < \infty$$

$$y(t) = \int_{\tau=0}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{\tau=0}^{\infty} h(\tau)e^{j\Omega_0(t-\tau)}d\tau = \underbrace{H(s)|_{s=j\Omega_0}\ e^{j\Omega_0 t}}_{\text{stationary}}$$

2) Complex, causal input signal

$$x(t) = e^{j\Omega_0 t}u(t) = (\cos(\Omega_0 t) + j\ \sin(\Omega_0 t))\ u(t); \qquad X(s) = \frac{1}{s - j\Omega_0}$$

$$Y(s) = H(s)X(s) = \frac{T(s)}{N(s)}\ \frac{1}{s-j\Omega_0} = \underbrace{\frac{T_1(s)}{N(s)}}_{\text{transient}} + \underbrace{H(s)|_{s=j\Omega_0}\ \frac{1}{s-j\Omega_0}}_{\text{stationary}}$$

$$y(t) = \text{transient} + \underbrace{H(s)|_{s=j\Omega_0}\ e^{j\Omega_0 t}}_{\text{stationary}}$$

3) Real, non-causal input signal

$$x(t) = Re\{e^{j\Omega_0 t}\} = \cos(\Omega_0 t) \quad -\infty < t < \infty$$

$$y(t) = \int_{\tau=0}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{\tau=0}^{\infty} h(\tau)\frac{1}{2}(e^{j\Omega_0(t-\tau)} + e^{-j\Omega_0(t-\tau)})d\tau =$$

$$= \underbrace{|H(s)|_{s=j\Omega_0}\ \cos(\Omega_0 t + arg\{H(s)|_{s=j\Omega_0}\})}_{\text{stationary}}$$

4) Complex, causal input signal

$$x(t) = Re\{e^{j\Omega_0 t}\}\ u(t) = \cos(\Omega_0 t)\ u(t); \qquad X(s) = \frac{s}{s^2 + \Omega_0^2}$$

$$Y(s) = H(s)X(s) = \frac{T(s)}{N(s)}\ \frac{s}{s^2+\Omega_0^2} = \underbrace{\frac{T_1(s)}{N(s)}}_{\text{transient}} + \underbrace{\frac{C_1 s + C_0}{s^2+\Omega_0^2}}_{\text{stationary}}$$

$$H(s)|_{s=j\Omega_0} = A\ e^{j\theta};\ C_1 = A\cos(\theta);\ C_0 = -A\Omega_0 \sin\theta$$

$$y(t) = \text{transient} + \underbrace{C_1\cos(\Omega_0 t) + \frac{C_0}{\Omega_0}\sin(\Omega_0 t)}_{\text{stationary}} =$$

$$= \text{transient} + \underbrace{|H(s)|_{s=j\Omega_0}\ \cos(\Omega_0 t + arg\{H(s)|_{s=j\Omega_0}\})}_{\text{stationary}}$$

1.9 Time discrete sinusoids through a linear, causal filter

1) Complex, non-causal input signal

$$x(n) = e^{j\omega_0 n} = (\cos(\omega_0 n) + j\ \sin(\omega_0 n)) \quad -\infty < n < \infty$$

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} h(k)e^{j\omega_0(n-k)} = \underbrace{H(z)|_{z=e^{j\omega_0}}\ e^{j\omega_0 n}}_{\text{stationary}}$$

2) Complex, causal input signal

$$x(n) = e^{j\omega_0 n}u(n) = (\cos(\omega_0 n) + j\ \sin(\omega_0 n))\ u(n); \qquad X(z) = \frac{1}{1 - e^{j\omega_0}z^{-1}}$$

$$Y(z) = H(z)X(z) = \frac{T(z)}{N(z)}\ \frac{1}{1 - e^{j\omega_0}z^{-1}} = \underbrace{\frac{T_1(z)}{N(z)}}_{\text{transient}} + \underbrace{H(z)|_{z=e^{j\omega_0}}\ \frac{1}{1 - e^{j\omega_0}z^{-1}}}_{\text{stationary}}$$

$$y(n) = \text{transient} + \underbrace{H(z)|_{z=e^{j\omega_0}}\ e^{j\omega_0 n}}_{\text{stationary}}$$

3) Real, non-causal input signal

$$x(n) = Re\{e^{j\omega_0 n}\} = \cos(\omega_0 n) \quad -\infty < n < \infty$$

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} h(k)\frac{1}{2}(e^{j\omega_0(n-k)} + e^{-j\omega_0(n-k)}) =$$

$$= \underbrace{|H(z)|_{z=e^{j\omega_0}}\ \cos(\omega_0 n + arg\{H(z)|_{z=e^{j\omega_0}}\})}_{\text{stationary}}$$

4) Real, causal input signal

$$x(n) = Re\{e^{j\omega_0 n}\}\ u(n) = \cos(\omega_0 n)\ u(n); \qquad X(z) = \frac{1 - \cos\omega_0 z^{-1}}{1 - 2\cos\omega_0 z^{-1} + z^{-2}}$$

$$Y(z) = H(z)X(z) = \frac{T(z)}{N(z)}\ \frac{1 - \cos\omega_0 z^{-1}}{1 - 2\cos\omega_0 z^{-1} + z^{-2}} = \underbrace{\frac{T_1(z)}{N(z)}}_{\text{transient}} + \underbrace{\frac{C_0 + C_1 z^{-1}}{1 - 2\cos\omega_0 z^{-1} + z^{-2}}}_{\text{stationary}}$$

$$H(z)|_{z=e^{j\omega_0}} = A\ e^{j\theta};\ C_0 = A\cos(\theta);\ C_1 = -A(\sin\omega_0\sin\theta + \cos\omega_0\cos\theta)$$

$$y(n) = \text{transient} + \underbrace{C_0\cos(\omega_0 n) + \frac{C_1 + C_0\cos(\omega_0)}{\sin(\omega_0)}\sin(\omega_0 n)}_{\text{stationary}} =$$

$$= \text{transient} + \underbrace{|H(z)|_{z=e^{j\omega_0}}\ \cos(\omega_0 n + arg\{H(z)|_{z=e^{j\omega_0}}\})}_{\text{stationary}}$$

2 Transforms

2.1 Laplace transform

2.1.1 Laplace transform of causal signals

In the table below, $f(t) = 0$ for $t < 0$ (i.e. $f(t) \cdot u(t) = f(t)$).

1.	$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \mathcal{F}(s) e^{st} ds$	$\longleftrightarrow$	$\mathcal{F}(s) = \int_{0-}^{\infty} f(t) e^{-st} dt$
2.	$\sum_\nu a_\nu f_\nu(t)$	$\longleftrightarrow$	$\sum_\nu a_\nu \mathcal{F}_\nu(s)$ Linearity
3.	$f(at)$	$\longleftrightarrow$	$\frac{1}{a}\ \mathcal{F}(\frac{s}{a})$ Scaling
4.	$\frac{1}{a}\ f(\frac{t}{a})$	$\longleftrightarrow$	$\mathcal{F}(as)$ $a > 0$ Scalning
5.	$f(t - t_0);\ t \geq t_0$	$\longleftrightarrow$	$\mathcal{F}(s)\ e^{-st_0}$ Time shift
6.	$f(t) \cdot e^{-at}$	$\longleftrightarrow$	$\mathcal{F}(s + a)$ Frequency shift
7.	$\frac{d^n f}{dt^n}$	$\longleftrightarrow$	$s^n \mathcal{F}(s)$ Derivate
8.	$\int_{0-}^{t} f(\tau) d\tau$	$\longleftrightarrow$	$\frac{1}{s}\ \mathcal{F}(s)$ Integrate
9.	$(-t)^n f(t)$	$\longleftrightarrow$	$\frac{d^n \mathcal{F}(s)}{ds^n}$ Derivation in frequency
10.	$\frac{f(t)}{t}$	$\longleftrightarrow$	$\int_s^\infty \mathcal{F}(z) dz$ Integration in frequency
11.	$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s \cdot \mathcal{F}(s)$		Initial value-theorem
12.	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s \cdot \mathcal{F}(s)$		
13.	$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t - \tau) d\tau = \int_0^t f_1(t - \tau) f_2(\tau) d\tau$	$\longleftrightarrow$	$\mathcal{F}_1(s) \cdot \mathcal{F}_2(s)$ Convolution in time domain
14.	$f_1(t) \cdot f_2(t)$	$\longleftrightarrow$	$\frac{1}{2\pi j}\ \mathcal{F}_1(s) * \mathcal{F}_2(s) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \mathcal{F}_1(z) \cdot \mathcal{F}_2(s - z) \cdot dz$ Convolution in frequency domain
15.	$\int_{0-}^{\infty} f_1(t) \cdot f_2(t) dt = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \mathcal{F}_1(s) \cdot \mathcal{F}_2(-s) ds$		Parseval's relation

16. $\delta(t) \longleftrightarrow 1$

17. $\delta^n(t) \longleftrightarrow s^n$

18. $1 \longleftrightarrow \frac{1}{s}$

19. $\frac{1}{n!}\, t^n \longleftrightarrow \frac{1}{s^{n+1}}$

20. $e^{-\sigma_0 t} \longleftrightarrow \frac{1}{s+\sigma_0}$

21. $\frac{1}{(n-1)!}\, t^{n-1}e^{-\sigma_0 t} \longleftrightarrow \frac{1}{(s+\sigma_0)^n}$

22. $\sin\Omega_0 t \longleftrightarrow \frac{\Omega_0}{s^2+\Omega_0^2}$

23. $\cos\Omega_0 t \longleftrightarrow \frac{s}{s^2+\Omega_0^2}$

24. $t\cdot\sin\Omega_0 t \longleftrightarrow \frac{2\Omega_0 s}{(s^2+\Omega_0^2)^2}$

25. $t\cdot\cos\Omega_0 t \longleftrightarrow \frac{s^2-\Omega_0^2}{(s^2+\Omega_0^2)^2}$

26. $e^{-\sigma_0 t}\sin\Omega_0 t \longleftrightarrow \frac{\Omega_0}{(s+\sigma_0)^2+\Omega_0^2}$

27. $e^{-\sigma_0 t}\cos\Omega_0 t \longleftrightarrow \frac{s+\sigma_0}{(s+\sigma_0)^2+\Omega_0^2}$

28. $e^{-\sigma_0 t}\sin(\Omega_0 t+\phi) \longleftrightarrow \frac{(s+\sigma_0)\sin\phi+\Omega_0\cos\phi}{(s+\sigma_0)^2+\Omega_0^2}$

2.1.2 One-side Laplace transform of non-causal signals

Notations

$\mathcal{F}^+(s)=\int_{0-}^{\infty} f(t)e^{-st}dt$	One-side Laplacetransform, $f(t)$ not nessesary causal.
$\mathcal{F}(s)=\mathcal{F}^+(s)$	For causal signals

Taking the derivative of $f(t)$ yelds

$$\frac{d}{dt}\, f(t) \longleftrightarrow s\cdot\mathcal{F}^+(s)-f(0-) \qquad \text{First deivative}$$

$$\frac{d^n}{dt}\, f(t) \longleftrightarrow s^n\mathcal{F}^+(s)-s^{n-1}f(0-)$$
$$-s^{n-2}f^{(1)}(0-)-\ldots f^{(n-1)}(0-) \text{ The } n:th \text{ derivative}$$

2.2 Fourier transform of time continous signals

$$\Omega = 2\pi F$$

1.	$w(t) = \mathcal{F}^{-1}\{W(F)\} = \int_{-\infty}^{\infty} W(F) e^{j2\pi Ft} dF$	$\longleftrightarrow$	$W(F) = \mathcal{F}\{w(t)\} = \int_{-\infty}^{\infty} w(t) e^{-j2\pi Ft} dt$
2.	$\sum_\nu a_\nu w_\nu(t)$	$\longleftrightarrow$	$\sum_\nu a_\nu W_\nu(F)$
3.	$w^*(-t)$	$\longleftrightarrow$	$W^*(F)$
4.	$W(t)$	$\longleftrightarrow$	$w(-F)$
5.	$w(at)$	$\longleftrightarrow$	$\frac{1}{\lvert a\rvert}\ W\left(\frac{F}{a}\right)$
6.	$w(t - t_0)$	$\longleftrightarrow$	$W(F) \cdot e^{-j2\pi F t_0}$
7.	$w(t) \cdot e^{j2\pi F_0 t}$	$\longleftrightarrow$	$W(F - F_0)$
8.	$w^*(t)$	$\longleftrightarrow$	$W^*(-F)$
9.	$\frac{d^n w(t)}{dt^n}$	$\longleftrightarrow$	$(j2\pi F)^n\ W(F)$
10.	$\int_{-\infty}^{t} w(\tau) d\tau$	$\longleftrightarrow$	$\frac{1}{j2\pi F}\ W(F)$ om $W(F) = 0$ för $F = 0$
11.	$-j2\pi t\ w(t)$	$\longleftrightarrow$	$\frac{dw}{dF}$
12.	$w_1(t) * w_2(t)$	$\longleftrightarrow$	$W_1(F) \cdot W_2(F)$
13.	$w_1(t) \cdot w_2(t)$	$\longleftrightarrow$	$W_1(F) * W_2(F)$
14.	$\int_{-\infty}^{\infty} \lvert w(t)\rvert^2 dt = \int_{-\infty}^{\infty} \lvert W(F)\rvert^2 dF$		Parseval's relation
15.	$\int_{-\infty}^{\infty} w_1(t) \cdot w_2(t) dt = \int_{-\infty}^{\infty} W_1(F) \cdot W_2^*(F) dF$		$w_1(t),\ w_2(t)$ real
16.	$\delta(t)$	$\longleftrightarrow$	1
17.	1	$\longleftrightarrow$	$\delta(F)$
18.	$u(t)$	$\longleftrightarrow$	$\frac{1}{j2\pi F} + \frac{1}{2}\ \delta(F)$
19.	$e^{-at} u(t)$	$\longleftrightarrow$	$\frac{1}{a + j\Omega}$

20.	$e^{-a\|t\|}$	$\longleftrightarrow$	$\frac{2a}{a^2+\Omega^2}$
21.	$e^{j2\pi F_0 t}$	$\longleftrightarrow$	$\delta(F-F_0)$
22.	$sin 2\pi F_0 t$	$\longleftrightarrow$	$j\ \frac{1}{2}\ \{\delta(F+F_0)-\delta(F-F_0)\}$
23.	$sin 2\pi F_0 t \cdot u(t)$	$\longleftrightarrow$	$\frac{\Omega_0}{\Omega_0^2-\Omega^2} + j\ \frac{1}{4}\ \{\delta(F+F_0)-\delta(F-F_0)\}$
24.	$cos 2\pi F_0 t$	$\longleftrightarrow$	$\frac{1}{2}\ \{\delta(F+F_0)+\delta(F-F_0)\}$
25.	$cos 2\pi F_0 t \cdot u(t)$	$\longleftrightarrow$	$\frac{j\Omega}{\Omega_0^2-\Omega^2} + \frac{1}{4}\{\delta(F+F_0)+\delta(F-F_0)\}$
26.	$\frac{1}{\sqrt{2\pi\sigma^2}}\ e^{-t^2/2\sigma^2}$	$\longleftrightarrow$	$e^{-(\Omega\sigma)^2/2}$
27.	$e^{-at} sin 2\pi F_0 t \cdot u(t)$	$\longleftrightarrow$	$\dfrac{\Omega_0}{(j\Omega+a)^2+(\Omega_0)^2}$
28.	$e^{-a\|t\|} sin 2\pi F_0 \|t\|$	$\longleftrightarrow$	$\dfrac{2\Omega_0(\Omega_0^2+a^2-\Omega^2)}{(\Omega^2+a^2-\Omega_0^2)^2+4a^2\Omega_0^2}$
29.	$e^{-at} cos 2\pi F_0 t \cdot u(t)$	$\longleftrightarrow$	$\dfrac{j\Omega+a}{(j\Omega+a)^2+(\Omega_0)^2}$
30.	$e^{-a\|t\|} cos 2\pi F_0 t$	$\longleftrightarrow$	$\dfrac{2a(\Omega_0^2+a^2+\Omega^2)}{(\Omega^2+a^2-\Omega_0^2)^2+4a^2\Omega_0^2}$
31.	$rect(at) = \begin{cases} 1 \text{ for } \|t\| < \frac{1}{2a} \\ 0 \text{ elsewhere} \end{cases}$	$\longleftrightarrow$	$\frac{1}{a}\ sinc(\frac{F}{a})\quad a>0$
32.	$sinc(at) = \frac{sin(\pi at)}{\pi at}$	$\longleftrightarrow$	$\frac{1}{a}\ rect(\frac{F}{a})\quad a>0$
33.	$rep_T(w(t)) = \sum_{m=-\infty}^{\infty} w(t-mT)$	$\longleftrightarrow$	$\frac{1}{\|T\|}\ comb_{1/T}(W(F))$
34.	$\|T\| comb_T(w(t)) =$ $\|T\| \sum_{m=-\infty}^{\infty} w(mT)\delta(t-mT)$	$\longleftrightarrow$	$rep_{1/T}(W(F))$
35.	$\sum_{n=-\infty}^{\infty} c_n \delta(t-nT)$	$\longleftrightarrow$	$\sum_{n=-\infty}^{\infty} \frac{1}{T}\ c_n \delta(F-\frac{n}{T}) = \sum c_n e^{-j2\pi nTF}$

2.3 Z-transform

2.3.1 The z-transform of causal signals

1.	$\mathcal{X}(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$	Transform
2.	$x(n) = Z^{-1}[\mathcal{X}(z)] = \frac{1}{2\pi j}\int_\Gamma \mathcal{X}(z)z^{n-1}dz$	Inverse transform
3.	$\sum_\nu a_\nu x_\nu(n) \longleftrightarrow \sum_\nu a_\nu \mathcal{X}_\nu(z)$	Linearity
4.	$x(n-n_0) \longleftrightarrow z^{-n_0}\mathcal{X}(z)$	Skift (n_0 positive or negativte integer)
5.	$nx(n) \longleftrightarrow -z\,\frac{d}{dz}\,\mathcal{X}(z)$	Multiplication with n
6.	$a^n x(n) \longleftrightarrow \mathcal{X}\left(\frac{z}{a}\right)$	Scaling
7.	$x(-n) \longleftrightarrow \mathcal{X}\left(\frac{1}{z}\right)$	Folding in time
8.	$\left[\sum_{\ell=-\infty}^{n} x(\ell)\right] \longleftrightarrow \frac{z}{z-1}\,\mathcal{X}(z)$	Sum
9.	$x * y \longleftrightarrow \mathcal{X}(z)\cdot\mathcal{Y}(z)$	Convolution
10.	$x(n)\cdot y(n) \longleftrightarrow \frac{1}{2\pi j}\int_\Gamma \mathcal{Y}(\xi)\mathcal{X}\left(\frac{z}{\xi}\right)\xi^{-1}d\xi$	Multiplication
11.	$x(0) = \lim_{z\to\infty}\mathcal{X}(z)$ (om gränsvärdet existerar)	Initial value theorem
12.	$\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)\mathcal{X}(z)$ (if ROC includes th unit circle)	Final value theorem
13.	$\sum_{\ell=-\infty}^{\infty} x(\ell)y(\ell) = \frac{1}{2\pi j}\int_\Gamma x(z)y\left(\frac{1}{z}\right)z^{-1}dz$	Parseval's theorem for real sequencies
14.	$\sum_{\ell=-\infty}^{\infty} x^2(\ell) = \frac{1}{2\pi j}\int_\Gamma \mathcal{X}(z)\mathcal{X}(z^{-1})z^{-1}dz$	– --

	Time sequency	$\longleftrightarrow$	Transform
	$x(n)$	$\longleftrightarrow$	$\mathcal{X}(z)$
15.	$\delta(n)$	$\longleftrightarrow$	1
16.	$u(n)$	$\longleftrightarrow$	$\frac{1}{1-z^{-1}}$
17.	$nu(n)$	$\longleftrightarrow$	$\frac{z^{-1}}{(1-z^{-1})^2}$
18.	$\alpha^n u(n)$	$\longleftrightarrow$	$\frac{1}{1-\alpha z^{-1}}$
19.	$(n+1)\alpha^n u(n)$	$\longleftrightarrow$	$\frac{1}{(1-\alpha z^{-1})^2}$
20.	$\frac{(n+1)(n+2)\dots(n+r-1)}{(r-1)!}\alpha^n u(n)$	$\longleftrightarrow$	$\frac{1}{(1-\alpha z^{-1})^r}$
21.	$\alpha^n \cos\beta n\ u(n)$	$\longleftrightarrow$	$\frac{1-z^{-1}\alpha\cos\beta}{1-z^{-1}2\alpha\cos\beta+\alpha^2 z^{-2}}$
22.	$\alpha^n \sin\beta n\ u(n)$	$\longleftrightarrow$	$\frac{z^{-1}\alpha\sin\beta}{1-z^{-1}2\alpha\cos\beta+\alpha^2 z^{-2}}$
23.	$\mathbf{F}^n u(n)$	$\longleftrightarrow$	$(\mathbf{I}-z^{-1}\mathbf{F})^{-1}$

2.3.2 One-side Z-transform of non-causal signals

Notations

$\mathcal{X}^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$	One-side Z-transform, $x(n)$ not nessesary causal
$\mathcal{X}(z) = \mathcal{X}^+(z)$	For causal signals

Shift of $x(n)$ yields:
i) shift one step

$$x(n-1) \longleftrightarrow z^{-1}\mathcal{X}^+(z) + x(-1)$$
$$x(n+1) \longleftrightarrow z\mathcal{X}^+(z) - x(0)\cdot z$$

ii) shift n_0 step ($n_0 \geq 0$)

$$\begin{aligned} x(n-n_0) &\longleftrightarrow z^{-n_0}\mathcal{X}^+(z) + x(-1)z^{-n_0+1} + \\ &\quad +x(-2)z^{-n_0+2} + \ldots + x(-n_0) \\ x(n+n_0) &\longleftrightarrow z^{n_0}\mathcal{X}^+(z) - x(0)z^{n_0} - x(1)z^{n_0-1} - \ldots - x(n_0-1)z \end{aligned}$$

2.4 Fourier transform of time discrete signals

1.	$X(f) = \mathcal{F}(x(n)) =$ $= \sum_{\ell=-\infty}^{\infty} x(\ell) e^{-j2\pi f \ell} \quad \omega = 2\pi f$		Transform
2.	$x(n) = \int_{-1/2}^{1/2} X(f) e^{j2\pi f n} df =$ $= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(f) e^{j\omega n} d\omega$		Inverse transform
3.	$\sum a_\nu x_\nu(n)$	$\longleftrightarrow$	$\sum_\nu a_\nu X_\nu(f)$ Linearity
4.	$x(n-n_0)$	$\longleftrightarrow$	$X(f) \cdot e^{-j2\pi f n_0}$ Shift
5.	$x(n) e^{j2\pi f_0 n}$	$\longleftrightarrow$	$X(f-f_0)$ Frequency translation
6.	$x(n) \cdot \cos 2\pi f_0 n$	$\longleftrightarrow$	$\frac{1}{2} \left[X(f-f_0) + X(f+f_0) \right]$ Modulation
7.	$x(n) \cdot \sin 2\pi f_0 n$	$\longleftrightarrow$	$\frac{1}{2j} \left[X(f-f_0) - X(f+f_0) \right]$ Modulation
8.	$x * y$	$\longleftrightarrow$	$X(f) \cdot Y(f)$ Convoloution
9.	$x \cdot y$	$\longleftrightarrow$	$\int_{-1/2}^{1/2} X(\lambda) \cdot Y(f-\lambda) d\lambda$ Multiplication
10.	$\sum_{\ell=-\infty}^{\infty} x(\ell) y(\ell) =$ $= \int_{-1/2}^{1/2} X(f) Y^*(f) df$		Parseval's theorem for real valued sequencies
11.	$X(f) = \mathcal{X}(e^{j\omega})$		If $x(n) = 0$ for $n < n_0$ and $\sum_{\ell=-\infty}^{\infty} \lvert x(\ell) \rvert^2 < \infty$ (In eq.: 18,19,20,21 and 22 in the Z-transform table for $\lvert \alpha \rvert < 1$)
12.	$\delta(n)$	$\longleftrightarrow$	1
13.	$\delta(n-n_0)$	$\longleftrightarrow$	$e^{-j\omega n_0}$
14.	$1 \ \forall n$	$\longleftrightarrow$	$\sum_{p=-\infty}^{\infty} \delta(f-p)$
15.	$u(n)$	$\longleftrightarrow$	$\frac{1}{2} \sum_{p=-\infty}^{\infty} \delta(f-p) + \frac{1}{2} + \frac{1}{j \cdot 2 \cdot \tan(\pi f)}$

16. $2f_1 \cdot sinc(2f_1 \cdot n) = 2f_1 \; \dfrac{\sin(2\pi f_1 n)}{2\pi f_1 n}$

$$\longleftrightarrow rect_p\left(\frac{f}{2f_1}\right) = \begin{cases} 1 & |f-n| < f_1 < 1/2 \ , \ n \text{ integer} \\ \\ 0 & \text{f.ö.} \end{cases}$$

Ideal LP-filter

17. $4f_1 sinc(2f_1 n)\cos(2\pi f_0 n)$

$\longleftrightarrow rect_p\left(\dfrac{f-f_0}{2f_1}\right) + rect_p\left(\dfrac{f+f_0}{2f_1}\right)$ Ideal BP-filter

18. $\dfrac{2\pi f_1 n \cos 2\pi f_1 n - \sin 2\pi f_1 n}{\pi n^2}$

$$\longleftrightarrow (j2\pi f)_p = \begin{cases} j2\pi(f-n) & |f-n| < f_1 < 1/2 \ , \ n \text{ integer} \\ \\ 0 & \text{f.ö.} \end{cases}$$

Derivation

19. $\cos(2\pi f_0 n) \longleftrightarrow \dfrac{1}{2}\sum_{p=-\infty}^{\infty}[\delta(f-f_0-p)+\delta(f+f_0-p)]$

20. $\alpha^{|n|} \longleftrightarrow \dfrac{1-\alpha^2}{1+\alpha^2-2\alpha\cos 2\pi f}$

21. $\alpha^{|n|}\cos(2\pi f_0 n)$

$$\longleftrightarrow \frac{1-\alpha^2}{2}\left[\frac{1}{1+\alpha^2-2\alpha\cos 2\pi(f+f_0)} + \frac{1}{1+\alpha^2-2\alpha\cos 2\pi(f-f_0)}\right]$$

22. $p_r(n) = \begin{cases} 1 & |n| \leq \dfrac{M-1}{2} \\ \\ 0 & \text{f.ö.} \end{cases}$ M udda

$\longleftrightarrow P_r(f) = \dfrac{\sin(\pi f M)}{\sin(\pi f)}$ Rekctangular window

2.5 Fourier serial expansion

2.5.1 Continuous in time

A periodical function with the period T_0 , i.e. $f(t) = f(t - T_0)$, can be expressed in a serial expansion

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{j2\pi k F_0 t}$$

with

$$c_k = \frac{1}{T_0} \int_{T_0} f(t) e^{-j2\pi k F_0 t} dt \; ; F_0 = \frac{1}{T_0}$$

If $f(t)$ real, this can be written

$$\begin{aligned} f(t) &= c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(2\pi k F_0 t + \theta_k) = \\ &= a_0 + \sum_{k=1}^{\infty} a_k \cos 2\pi k F_0 t - b_k \sin 2\pi k F_0 t \end{aligned}$$

with

$$\begin{aligned} a_0 &= c_0 = \frac{1}{T_0} \int_{T_0} f(t) dt \\ a_k &= 2|c_k| \cos\theta_k = \frac{2}{T_0} \int_{T_0} f(t) \cos(2\pi k F_0 t) dt \\ b_k &= 2|c_k| \sin\theta_k = \frac{-2}{T_0} \int_{T_0} f(t) \sin(2\pi k F_0 t) dt \end{aligned}$$

The power is given by (Parseval's relation)

$$P = \frac{1}{T_0} \int_{T_0} |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

For real signals also yields

$$P = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2 = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

2.5.2 Discrete time

A periodical function with the period N, i.e. $f(n) = f(n - N)$, can be expressed in a serial expansion

$$f(n) = \sum_{k=0}^{N-1} c_k \, e^{j2\pi k \; n/N}$$

with

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} f(n) \; e^{-j2\pi k \; n/N}, \quad k = 0, \ldots, N-1$$

The serial expansion is often denoted DTFS (Discrete-Time Fourier Series).
If $f(n)$ real, this can be written

$$\begin{aligned} f(n) &= c_0 + 2\sum_{k=1}^{L} |c_k| \cos\left(2\pi \, \frac{kn}{N} + \theta_k\right) = \\ &= a_0 + \sum_{k=1}^{L} \left(a_k \cos\left(2\pi \, \frac{kn}{N}\right) - b_k \sin\left(2\pi \, \frac{kn}{N}\right)\right) \end{aligned}$$

there

$$
\begin{aligned}
a_0 &= c_0 \\
a_k &= 2|c_k|\cos(\theta_k) \\
b_k &= 2|c_k|\sin(\theta_k) \\
L &= \begin{cases} \frac{N}{2} & \text{if } N \text{ even} \\ \\ \frac{N-1}{2} & \text{if } N \text{ odd} \end{cases}
\end{aligned}
$$

The power is

$$P = \frac{1}{N}\sum_{n=0}^{N-1}|f(n)|^2 = \sum_{k=0}^{N-1}|c_k|^2$$

and the energy over one period is

$$E_N = \sum_{n=0}^{N-1}|f(n)|^2 = N\sum_{k=0}^{N-1}|c_k|^2$$

2.6 Discrete Fourier Transform (DFT)

2.6.1 Definition

$$
\begin{aligned}
X_k = DFT\ (x_n) &= \sum_{n=0}^{N-1} x_n e^{-j2\pi nk/N} \quad k = 0,1,\ldots,N-1 \quad \text{Transform} \\
x_n = IDFT\ (X_k) &= \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi nk/N} \quad n = 0,1,\ldots,N-1 \quad \text{Inversion}
\end{aligned}
$$

Note:

$$\sum_{n=0}^{N-1} e^{j2\pi\ \frac{k-k_0}{N}\cdot n} = N\cdot\delta(k-k_0, (\text{modulo } N))$$

2.6.2 Circular convolution

$$x_n \ \text{Ⓝ}\ y_n = \sum_{\ell=0}^{N-1} x_\ell y_{n-\ell, moduloN} \stackrel{\text{DFT}}{\longleftrightarrow} X_k Y_k \quad \text{Circular convolution}$$

i.e. index is determined modulo N.
Circular convolution is also denoted $x(n) \circledast y(n)$.

2.6.3 Non-circular convolution using the DFT

If $x(n) = 0$ for $n \neq [0, L-1]$ and $y(n) = 0$ for $n \neq [0, M-1]$ yields $x * y = 0$ for $n \neq [0, N-1]$ with $N \geq L + M - 1$.
The convolution can also be determined from

$$x * y = \begin{cases} x \ \text{Ⓝ}\ y = \ \text{IDFT}(X_k Y_k) & n = 0,1,\ldots,N-1 \\ \\ 0 & \text{f.ö.} \end{cases}$$

with

$$
\begin{aligned}
X_k &= \text{DFT}(x(n)) \\
Y_k &= \text{DFT}(y(n))
\end{aligned}
$$

2.6.4 Relation to the Fourier transform $X(f)$:

$$\begin{aligned}X(k/N) &= X_k = DFT(x(n)) \text{ if } x(n) = 0 \text{ for } n \neq [0, N-1] \\ X(k/N) &= X_k = DFT(x_p(n)) \text{ in general } x(n) \text{ there } x_p(n) = \sum_{\ell=-\infty}^{\infty} x(n - \ell N)\end{aligned}$$

2.6.5 Relation to Fourier series

$$X\left(\frac{k}{N}\right) = X_k = DFT(x(n)) = N \cdot c_k$$

if

$$x(n) = x_p(n), \quad 0 \leq n \leq N-1$$

there

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi \frac{nk}{N}} \qquad -\infty < n < \infty$$

and

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi \frac{nk}{N}} \quad k = 0, 1, \ldots, N-1$$

2.6.6 Parseval's theorem

$$\sum_{n=0}^{N-1} x(n) y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_k Y^*(k)$$

2.6.7 Some properties of the DFT

Time	Frequency
$x(n),\ y(n)$	$X(k),\ Y(k)$
$x(n) = x(n+N)$	$X(k) = X(k+N)$
$x(N-1)$	$X(N-k)$
$x((n-1))_N$	$X(k)e^{-j2\pi k1/N}$
$x(n)e^{j2\pi 1n/N}$	$X((k-1))_N$
$x^*(n)$	$X^*(N-k)$
$x_1(n) \ \textcircled{N} \ x_2(n)$	$X_1(k)X_2(k)$
$x(n) \ \textcircled{N} \ y^*(-n)$	$X(k)Y^*(k)$
$x_1(n)x_2(n)$	$\frac{1}{N} X_1(k) \ \textcircled{N} \ X_2(k)$
$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

2.7 Some window functions and their Fourier transform

i) Window functions symmetrically around origin (M odd) i.e. the functions are non-zero only for $-(M-1)/2 \leq n \leq (M-1)/2$

Rectangular window:

$$\begin{aligned} w_{rect}(n) &= 1 \\ W_{rect}(f) &= M \cdot \frac{sin(\pi f M)}{M sin(\pi f)} \end{aligned}$$

Hanning window:

$$\begin{aligned} w_{hanning}(n) &= 0.5 + 0.5cos\left(\frac{2\pi n}{M-1}\right) \\ W_{hanning}(f) &= 0.5\ W_{rect}(f) + \\ &+0.25\ W_{rect}\left(f - \frac{1}{M-1}\right) + \\ &+0.25\ W_{rect}\left(f + \frac{1}{M-1}\right) \end{aligned}$$

Hamming window:

$$\begin{aligned} w_{hamming}(n) &= 0.54 + 0.46cos\left(\frac{2\pi n}{M-1}\right) \\ W_{hamming}(n) &= 0.54\ W_{rect}(f) + \\ &+0.23\ W_{rect}\left(f - \frac{1}{M-1}\right) + \\ &+0.23\ W_{rect}\left(f + \frac{1}{M-1}\right) \end{aligned}$$

Blackman window:

$$w_{blackman}(n) = 0.42 + 0.5cos\frac{2\pi n}{M-1} + 0.08cos\frac{4\pi n}{M-1}$$

$$
\begin{aligned}
W_{blackman}(f) &= 0.42\ W_{rect}(f) + \\
&+0.25\ W_{rect}\left(f - \frac{1}{M-1}\right) + \\
&+0.25\ W_{rect}\left(f + \frac{1}{M-1}\right) + \\
&+0.04\ W_{rect}\left(f - \frac{2}{M-1}\right) + \\
&+0.04\ W_{rect}\left(f + \frac{2}{M-1}\right)
\end{aligned}
$$

Bartlett window (triangular window):

$$
\begin{aligned}
w_{triangel}(n) &= 1 - \frac{|n|}{(M-1)/2} \\
W_{triangel}(f) &= \frac{M}{2}\left(\frac{sin\frac{\pi f M}{2}}{\frac{M}{2}\ sin(\pi f)}\right)^2 \approx \frac{2}{M}\ W_{rect}^2\left(\frac{f}{2}\right) \text{ for small } f
\end{aligned}
$$

ii) Window functions defined for the interval $0 \leq n \leq M-1$ (M odd)
Hanning window

$$
\begin{aligned}
w_{hanning}(n) &= 0.5\ \left(1 + cos\frac{2\pi\left(n - \frac{M-1}{2}\right)}{M-1}\right) = \\
&= 0.5\ \left(1 - cos\left(2\pi\frac{n}{M-1}\right)\right)
\end{aligned}
$$

Hamming window

$$
\begin{aligned}
w_{hamming}(n) &= 0.54 + 0.46\ cos\ \frac{2\pi\left(n - \frac{M-1}{2}\right)}{M-1} = \\
&= 0.54 - 0.46\ cos\ \frac{2\pi n}{M-1}
\end{aligned}
$$

Blackman window

$$
\begin{aligned}
w_{blackman}(n) &= 0.42 + 0.5\ cos\ \frac{2\pi\left(n - \frac{M-1}{2}\right)}{M-1} + \\
&\quad +0.08\ cos\ \frac{4\pi\left(n - \frac{M-1}{2}\right)}{M-1} = \\
&= 0.42 - 0.5\ cos\ \frac{2\pi n}{M-1} + 0.08\ cos\ \frac{4\pi n}{M-1}
\end{aligned}
$$

Bartlett (triangular) window

$$
w_{triangel}(n) = 1 - \frac{\left(n - \frac{M-1}{2}\right)}{\frac{M-1}{2}}
$$

3 Sampling of analogous signals

3.1 Sampling and reconstruction

Fourier transforms

Time continuous signals:

$$\begin{cases} X_a(F) & = & \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi F t} dt \\ \\ x_a(t) & = & \int_{-\infty}^{\infty} X_a(F) e^{j2\pi F t} dF \end{cases}$$

Time discrete signal:

$$\begin{cases} X(f) & = & \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \\ \\ x(n) & = & \int_{-1/2}^{1/2} X(f) e^{j2\pi f n} df \end{cases}$$

The sampling theorem

For a bandlimited signal $x_a(t)$, that is $X_a(F) = 0$ for $|F| \geq 1/2T$ yields

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin \frac{\pi}{T}(t - nT)}{\frac{\pi}{T}(t - nT)}$$

Sampling frequency $F_s = 1/T$.

Sampling

$$\begin{aligned} x(n) &= x_a(nT) \; ; \quad T = \frac{1}{F_s} \\ X(f) &= X\left(\frac{F}{F_s}\right) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \\ \Gamma(f) &= \Gamma\left(\frac{F}{F_s}\right) = F_s \sum_{k=-\infty}^{\infty} \Gamma_a(F - kF_s) \end{aligned}$$

Reconstruction (ideal)

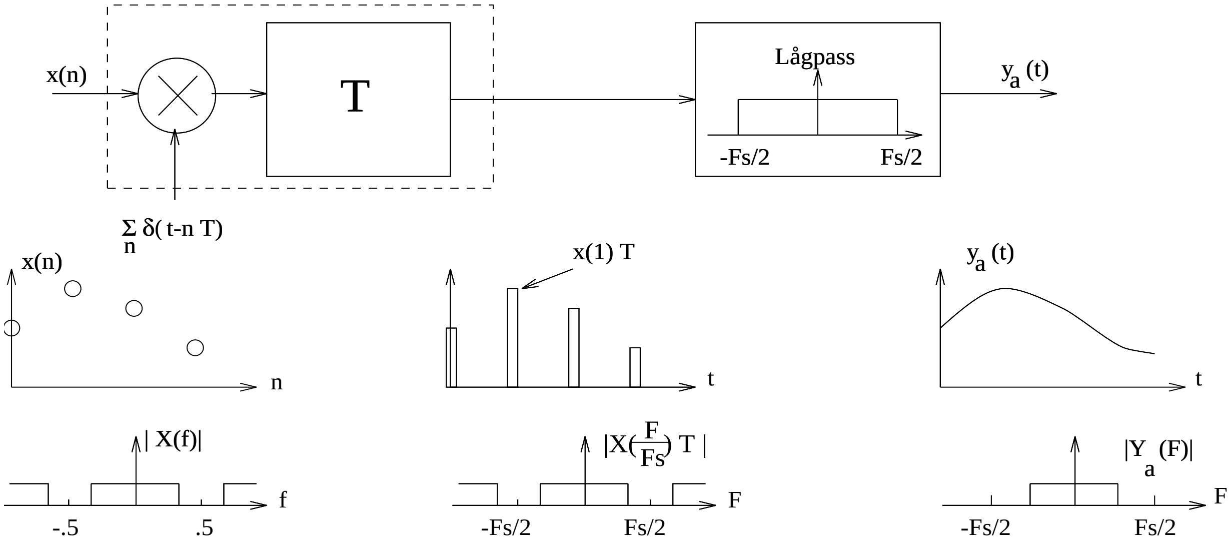
$$\begin{aligned} x_a(t) &= \sum_{n=-\infty}^{\infty} x(n) \frac{sin \frac{\pi}{T}(t - nT)}{\frac{\pi}{T}(t - nT)} \\ X_a(F) &= \frac{1}{F_s} X\left(\frac{F}{F_s}\right) \quad |F| \leq \frac{F_s}{2} \\ \Gamma_a(F) &= \frac{1}{F_s} \Gamma\left(\frac{F}{F_s}\right) \quad |F| \leq \frac{F_s}{2} \end{aligned}$$

Reconstruction using a sample-and-hold circuit

$$\begin{aligned} X_a(F) &= \frac{1}{F_s} X\left(\frac{F}{F_s}\right) \cdot \frac{sin(\pi F T)}{\pi F T} \, e^{-j2\pi F \, \frac{T}{2}} \cdot H_{LP}(F) \\ \Gamma_a(F) &= \frac{1}{F_s} \Gamma\left(\frac{F}{F_s}\right) \left|\frac{\sin(\pi F T)}{\pi F T}\right|^2 \cdot |H_{LP}(F)|^2 \end{aligned}$$

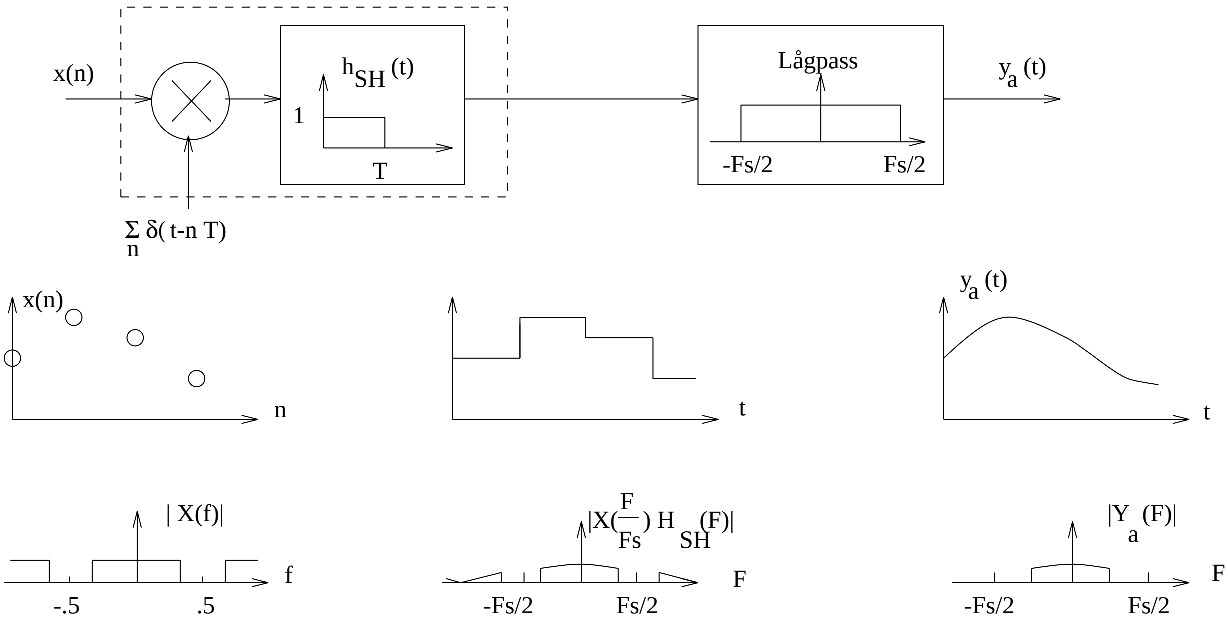
Block diagram over D/A-conversion

Ideal reconstruction



$$
\begin{aligned}
y_a(t) &= \sum_{n=-\infty}^{\infty} x(n) \, \frac{sin \, \frac{\pi}{T} \, (t-nT)}{\frac{\pi}{T} \, (t-nT)} \\
Y_a(F) &= \frac{1}{F_s} \, X\left(\frac{F}{F_s}\right) \quad |F| \leq \frac{F_s}{2}
\end{aligned}
$$

Reconstruction using a sample-and-hold curcuit



$$
Y_a(F) = \frac{1}{F_s} \, X\left(\frac{F}{F_s}\right) \cdot \frac{sin(\pi F T)}{\pi F T} \, e^{-j2\pi F \, \frac{T}{2}} \cdot H_{LP}(F)
$$

3.2 Distortion measurements

3.2.1 Aliasing distortion from sampling

Spectrum after anti-aliasing filter:

$$\Gamma_{in}(F)$$

Aliasing distortion:

$$D_A = 2 \cdot \int_{F_s - F_p}^{\infty} \Gamma_{in}(F) dF$$

Signal power:

$$D_s = 2 \int_0^{F_p} \Gamma_{in}(F) dF$$

there $0 \leq F_p \leq F_s/2$

Aliasing distortion ratio:

$$\text{A:} \quad SDR_A = \frac{D_S}{D_A} = \frac{\int_0^{F_p} \Gamma_{in}(F) dF}{\int_{F_s - F_p}^{\infty} \Gamma_{in}(F) dF}$$

$$\text{B:} \quad SDR_A^0 = \min_{|F| \leq F_p} \frac{\Gamma_{in}(F)}{\Gamma_{in}(F_s - F)}$$

Monotonic decreasing spectrum yields

$$SDR_A^0 = \frac{\Gamma_{in}(F_p)}{\Gamma_{in}(F_s - F_p)}$$

3.2.2 Signal-to-noise ratio during reconstruction

Signal-to-quantization noise ratio:

$$D_P = 2 \cdot \int_{F_s/2}^{\infty} \Gamma_{out}(F) dF$$

Signal power:

$$D_S = 2 \cdot \int_0^{F_s/2} \Gamma_{out}(F) dF$$

Signal-to-quantization noise ratio:

$$\text{A:} \quad SDR_P = \frac{D_S}{D_P} = \frac{\int_0^{F_s/2} \Gamma_{out}(F) dF}{\int_{F_s/2}^{\infty} \Gamma_{out}(F) dF}$$

$$\text{B:} \quad SDR_P^0 = \min_{|F| < F_s/2} \frac{\Gamma_{out}(F)}{\Gamma_{out}(F_s - F)}$$

A useful measurement is

$$SDR_P^0 = \frac{\Gamma_{out}(F_p)}{\Gamma_{out}(F_s - F_p)}$$

there F_p is the highest frequency component in the digital signal.

3.3 Quantization noise

$$D_Q \simeq \frac{\Delta^2}{12} \text{ linear quantization, } \Delta \text{ small}$$

$$SDR_Q = \frac{\text{Signal power}}{D_q}$$

Quantization noise for sinusoids, maximal amplitude, r bits

$$SDR_Q = 1.76 + 6 \cdot r[dB]$$

Quantization noise for sinusoids, amplitude expressed in peak- and RMS, r bits

$$SDR_Q = 6 \cdot r + 1.76 - 10^{10} \log \left(\frac{A_{peak}}{A_{RMS} \cdot \sqrt{2}} \right)^2 - 10^{10} \log \left(\frac{V}{A_{peak}} \right)^2$$

there $[-V, V]$ is the maximal range of the signal.

3.4 Sampling rate conversion, decimation and interpolation

Down-sampling a factor M

$$\downarrow M \quad y(n) = \{\ldots u(0), u(M), u(2M) \ldots\}$$

$$Y(f) = \tfrac{1}{M} \ \textstyle\sum_{i=0}^{M-1} \ U\left(\tfrac{f-i}{M}\right)$$

Up-sampling a factor L

$$\uparrow L \quad w(n) = \{\ldots x(0), \underbrace{0, 0, \ldots}_{\text{L-1 st}}, x(1), \underbrace{0, 0, \ldots}_{\text{L-1 st}}, x(2) \ldots\}$$

$$W(f) = X(fL)$$

4 Analogous filters

4.1 Filter approximations of ideal LP-filter

General form of the approximated amplitude function

$$|H(\Omega)| = \frac{K}{\sqrt{1 + g_N\left(\left(\frac{\Omega}{\Omega_p}\right)^2\right)}} \qquad \Omega = 2\pi F$$

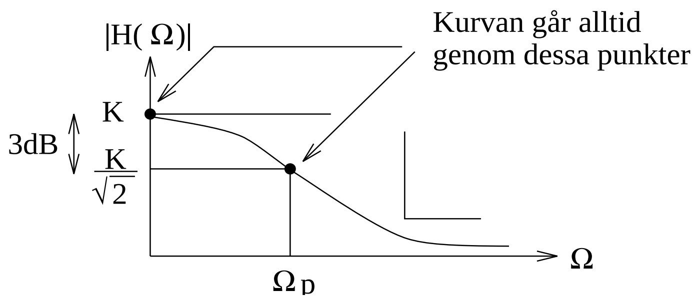
there

$$g_N\left(\left(\frac{\Omega}{\Omega_p}\right)^2\right) \begin{cases} \ll 1 & \left|\frac{\Omega}{\Omega_p}\right| < 1 \\ \gg 1 & \left|\frac{\Omega}{\Omega_p}\right| > 1 \end{cases}$$

and Ω_p is the cutoff frequency of the filter.
Sometimes it is convenient to normalize the angular frequency with Ω_p.
This corresponds to setting $\Omega_p = 1$ below .

4.1.1 The Butterworth filter

$$|H(\Omega)| = \frac{K}{\sqrt{1 + \left(\frac{\Omega}{\Omega_p}\right)^{2N}}}$$



K = Maximum of the amplitude function.
K = the value of the amplitude function for $\Omega = 0$.

The denominator to the system function are the Butterworth polynomial if $\Omega_p = 1$.
These polynomials are showed in table 2.1. Generally, Ω_p yields

$$\mathcal{H}(s) = \frac{K}{\left(\frac{s}{\Omega_p}\right)^N + a_{N-1}\left(\frac{s}{\Omega_p}\right)^{N-1} + \cdots + a_1\left(\frac{s}{\Omega_p}\right) + 1}$$

there $a_1, \ldots, a_{N-1}$ are find in table 4.1.

Table 4.1

Coefficients a_ν in the Butterworth polynomial $s^N + a_{N-1}s^{N-1} + \ldots + a_1 s + 1$

N	a_1	a_2	a_3	a_4	a_5	a_6	a_7
1							
2	$\sqrt{2}$						
3	2	2					
4	2.613	3.414	2.613				
5	3.236	5.236	5.236	3.236			
6	3.864	7.464	9.141	7.464	3.864		
7	4.494	10.103	14.606	14.606	10.103	4.494	
8	5.126	13.138	21.848	25.691	21.848	13.138	5.126

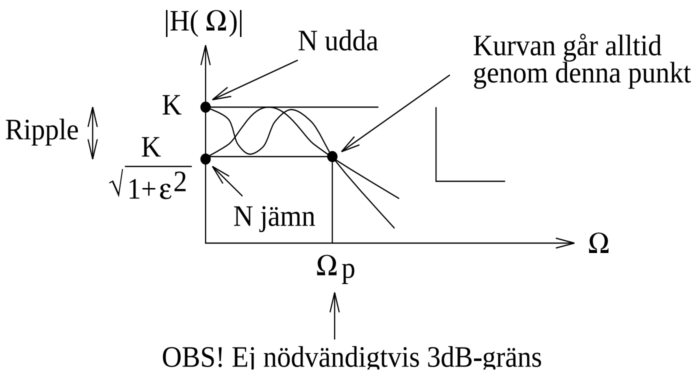
Table 4.2

Factorized Butterworth polynomial for $\Omega_p = 1$. For $\Omega_p \neq 1$ let $s \rightarrow s/\Omega_p$.

N	
1	$(s+1)$
2	$(s^2+\sqrt{2}s+1)$
3	$(s^2+s+1)(s+1)$
4	$(s^2+0.76536s+1)(s^2+1.84776s+1)$
5	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$
6	$(s^2+0.5176s+1)(s^2+\sqrt{2}s+1)(s^2+1.9318s+1)$
7	$(s+1)(s^2+0.4450s+1)(s^2+1.2465s+1)(s^2+1.8022s+1)$
8	$(s^2+0.3896s+1)(s^2+1.1110s+1)(s^2+1.6630s+1)(s^2+1.9622s+1)$

4.1.2 The Chebyshev filters

$$|H(\Omega)| = \frac{K}{\sqrt{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}}$$



Ripple $= 10 \cdot \log(1+\varepsilon^2)$ dB.
K = Maximum of the amplitude function.
$K \neq$ the value of the amplitude function for $\Omega = 0$ for N is even.

$T_N(\frac{\Omega}{\Omega_p})$ is the Chebyshev polynomial. (Denoted with $C_N\left(\frac{\Omega}{\Omega_p}\right)$). These are found in Table 4.3 for $\Omega_p = 1$. For $\Omega_p \neq 1$ let $\Omega \to \frac{\Omega}{\Omega_p}$ i Table 4.3.

System function

$$\mathcal{H}(s) = \frac{K \cdot a_0 \cdot \begin{cases} 1 & N \text{ udda} \\ \frac{1}{\sqrt{1+\varepsilon^2}} & N \text{ jämn} \end{cases}}{\left(\frac{s}{\Omega_p}\right)^N + a_{N-1}\left(\frac{s}{\Omega_p}\right)^{N-1} + \cdots + a_0}$$

there $\varepsilon, a_0, \ldots, a_{N-1}$ are found in table 4.4.
The poles for $\mathcal{H}(s)$ is found in table 4.5 for $\Omega_p = 1$. For $\Omega_p \neq 1$ the poles are multiplied by Ω_p.

Table 4.3

Chebyshev polynomial.

$$T_N(\Omega) = \begin{cases} \cos(N\ arccos\Omega) & |\Omega| \leq 1 \\ \\ \cosh(N\ arccosh\Omega) & |\Omega| \geq 1 \end{cases} \qquad \Omega = 2\pi F$$

or

$$T_N(\Omega) = \frac{\left(\Omega + \sqrt{\Omega^2 - 1}\right)^N + \left(\Omega + \sqrt{\Omega^2 - 1}\right)^{-N}}{2} \qquad |\Omega| \geq 1$$

Recursive determination

$$T_{N+1}(\Omega) = 2\Omega T_N(\Omega) - T_{N-1}(\Omega)$$

N	$T_N(\Omega)$
0	1
1	Ω
2	$2\Omega^2 - 1$
3	$4\Omega^3 - 3\Omega$
4	$8\Omega^4 - 8\Omega^2 + 1$
5	$16\Omega^5 - 20\Omega^3 + 5\Omega$
6	$32\Omega^6 - 48\Omega^4 + 18\Omega^2 - 1$
7	$64\Omega^7 - 112\Omega^5 + 56\Omega^3 - 7\Omega$
8	$128\Omega^8 - 256\Omega^6 + 160\Omega^4 - 32\Omega^2 + 1$
9	$256\Omega^9 - 576\Omega^7 + 432\Omega^5 - 120\Omega^3 + 9\Omega$
10	$512\Omega^{10} - 1280\Omega^8 + 1120\Omega^6 - 400\Omega^4 + 50\Omega^2 - 1$

Table 4.4. Coefficients a_ν in the Chebyshev filter.

0.5dB ripple ($\varepsilon = 0.349,\ \varepsilon^2 = 0.122$).

N	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0
1								2.863
2							1.426	1.516
3						1.253	1.535	0.716
4					1.197	1.717	1.025	0.379
5				1.172	1.937	1.309	0.752	0.179
6			1.159	2.172	1.589	1.172	0.432	0.095
7		1.151	2.413	1.869	1.648	0.756	0.282	0.045
8	1.146	2.657	2.149	2.184	1.148	0.573	0.152	0.024

1-dB ripple ($\varepsilon = 0.509,\ \varepsilon^2 = 0.259$).

N	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0
1								1.965
2							1.098	1.102
3						0.989	1.238	0.491
4					0.953	1.454	0.743	0.276
5				0.937	1.689	0.974	0.580	0.123
6			0.928	1.931	1.202	0.939	0.307	0.069
7		0.923	2.176	1.429	1.357	0.549	0.214	0.031
8	0.920	2.423	1.655	1.837	0.447	0.448	0.107	0.017

2-dB ripple ($\varepsilon = 0.765,\ \varepsilon^2 = 0.585$).

N	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0
1								1.307
2							0.804	0.823
3						0.738	1.022	0.327
4					0.716	1.256	0.517	0.206
5				0.705	1.499	0.693	0.459	0.082
6			0.701	1.745	0.867	0.771	0.210	0.051
7		0.698	1.994	1.039	1.144	0.383	0.166	0.020
8	0.696	2.242	1.212	1.579	0.598	0.359	0.073	0.013

3-dB*) ripple ($\varepsilon = 0.998,\ \varepsilon^2 = 0.995$).

N	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0
1								1.002
2							0.645	0.708
3						0.597	0.928	0.251
4					0.581	1.169	0.405	0.177
5				0.575	1.415	0.549	0.408	0.063
6			0.571	1.663	0.691	0.699	0.163	0.044
7		0.568	1.911	0.831	1.052	0.300	0.146	0.016
8	0.567	2.161	0.972	1.467	0.472	0.321	0.056	0.011

*) The table is determined for "exact"3dB, not for $20 \cdot \log \sqrt{2} \approx 3.01$dB.
Means $\varepsilon \neq 1$ and $a_0 \neq 1$ for $N = 1$.

Table 4.5. Poles for Chebyshev filters.

0.5dB ripple ($\varepsilon = 0.349,\ \varepsilon^2 = 0.122$).

$N = 1$	2	3	4	5	6	7	8
-2.863	-0.713	-0.626	-0.175	-0.362	-0.078	-0.256	-0.044
	±j1.004		±j1.016		±j1.008		±j1.005
		-0.313	-0.423	-0.112	-0.212	-0.057	-0.124
		±j1.022	±j0.421	±j1.011	±j0.738	±j1.006	±j0.852
				-0.293	-0.290	±0.160	-0.186
				±j0.625	±j0.270	±j0.807	±j0.570
						-0.231	-0.220
						±j0.448	±j0.200

1-dB ripple ($\varepsilon = 0.509,\ \varepsilon^2 = 0.259$).

$N = 1$	2	3	4	5	6	7	8
-1.965	-0.549	-0.494	-0.139	-0.289	-0.062	-0.205	-0.035
	±j0.895		±j0.983		±j0.993		±j0.996
		-0.247	-0.337	-0.089	-0.170	-0.046	-0.100
		±j0.966	±j0.407	±j0.990	±j0.727	±j0.995	±j0.845
				-0.234	-0.232	-0.128	-0.149
				±j0.612	±j0.266	±j0.798	±j0.564
						-0.185	-0.176
						±j0.443	±j0.198

2-dB ripple ($\varepsilon = 0.765,\ \varepsilon^2 = 0.585$).

$N = 1$	2	3	4	5	6	7	8
-1.307	-0.402	-0.369	-0.105	-0.218	-0.047	-0.155	-0.026
	±j0.813		±j0.958		±j0.982		±j0.990
		-0.184	-0.253	-0.067	-0.128	-0.034	-0.075
		±j0.923	±0.397	±j0.973	±0.719	±j0.987	±j0.839
				-0.177	-0.175	-0.097	-0.113
				±j0.602	±j0.263	±j0.791	±j0.561
						-0.140	-0.133
						±j0.439	±j0.197

3-dB*) ripple ($\varepsilon = 0.998,\ \varepsilon^2 = 0.995$).

$N = 1$	2	3	4	5	6	7	8
-1.002	-0.322	-0.299	-0.085	-0.177	-0.038	-0.126	-0.021
	±j0.777		±j0.946		±j0.976		±0.987
		-0.1493	-0.206	-0.055	-0.104	-0.028	-0.061
		±j0.904	±j0.392	±j0.966	±0.715	±j0.983	±j0.836
				-0.144	-0.143	-0.079	-0.092
				±j0.597	±j0.262	±j0.789	±j0.559
						-0.114	-0.108
						±j0.437	±j0.196

*) See note for Table 4.4.

4.1.3 The Bessel filter

The Bessel filter gives a maximum flat group delay.
Coefficients to the Bessel polynomial.

n	a_0	a_1	a_2	a_3	a_4	a_5
1	1					
2	3	3				
3	15	15	6			
4	105	105	45	10		
5	945	945	420	105	15	
6	10395	10395	4725	1260	210	21

Rotes to the Bessel polynomial.

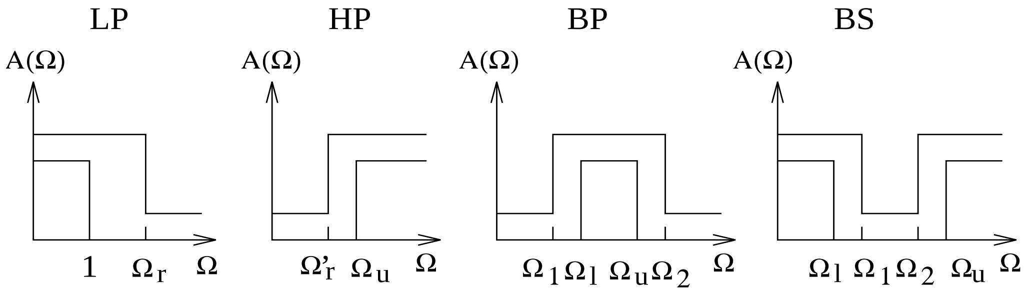
n						
1	-1.0000					
2	-1.5000	$\pm$j0.8660				
3	-2.3222	-1.8389	$\pm$j1.7544			
4	-2.8962	$\pm$j0.8672	-2.1038	$\pm$j2.6574		
5	-3.6467	-3.3520	$\pm$j1.7427	-2.3247	$\pm$j3.5710	
6	-4.2484	$\pm$j0.8675	-3.7357	$\pm$j2.6263	-2.5159	$\pm$j4.4927

Factorized Bessel polynomial

n		
1	$s+1$	1
2	s^2+3s+3	3
3	$(s^2+3.67782s+6.45944)(s+2.32219)$	15
4	$(s^2+5.79242s+9.14013)(s^2+4.20758s+11.4878)$	105
5	$(s^2+6.70391s+14.2725)(s^2+4.64934s+18.15631)(s+3.64674)$	945
6	$(s^2+8.49672s+18.80113)(s^2+7.47142s+20.85282)$ $(s^2+5.03186s+26.51402)$	10395

4.2 Frequency transformation of analogous filters

1. Start with the frequencies from the specification in the analogous high-pass, bandpass or band-stop filter4. In the final filter, this are $\Omega_1\Omega_2 = \Omega_l\Omega_u$.
2. Transform to the LP-filter frequencies $\Omega_p = 1,\ \Omega_r$.
3. Determine the LP-filter coefficients.
4. Transform back to (to HP, BP, BS) by replacing s in $H(s)$ below. For BP, BS it is convenient to transform the poles directly if $H(s)$ should be in a factorized form in 2:a-order polynomial. Determine a new value of Ω_1 or Ω_2 (if $A \neq B$) if needed.



$S \quad \rightarrow$	$\dfrac{\Omega_u}{s}$	$\dfrac{(s^2+\Omega_l\Omega_u)}{s(\Omega_u-\Omega_l)}$	$\dfrac{s(\Omega_u-\Omega_l)}{(s^2+\Omega_l\Omega_u)}$

	Forward	Backward
LP-HP	$\Omega_r' = \Omega_u/\Omega_r$	$\Omega_r = \Omega_u/\Omega_r'$
LP-BP	$\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = \sqrt{\Omega_r^2\Omega_{av}^2 + \Omega_l\Omega_u} - \Omega_{av}\Omega_r$ $\Omega_2 = \sqrt{\Omega_r^2\Omega_{av}^2 + \Omega_l\Omega_u} + \Omega_{av}\Omega_r$ $s_{BP} = S_{LP}\Omega_{av} \pm \sqrt{(S_{LP}\Omega_{av})^2 - \Omega_u\Omega_l)}$	$\Omega_r = min(\lvert A\rvert, \lvert B\rvert)$ $A = (-\Omega_1^2 + \Omega_l\Omega_u)/[\Omega_1(\Omega_u - \Omega_l)]$ $B = (+\Omega_2^2 - \Omega_l\Omega_u)/[\Omega_2(\Omega_u - \Omega_l)]$
LP-BS	$\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = \sqrt{\Omega_{av}^2/\Omega_r^2 + \Omega_l\Omega_u} - \Omega_{av}/\Omega_r$ $\Omega_2 = \sqrt{\Omega_{av}^2/\Omega_r^2 + \Omega_l\Omega_u} + \Omega_{av}/\Omega_r$ $s_{BP} = \Omega_{av}/S_{LP} \pm \sqrt{(\Omega_{av}/S_{LP})^2 - \Omega_u\Omega_l)}$	$\Omega_r = min(\lvert A\rvert, \lvert B\rvert)$ $A = \Omega_1(\Omega_u - \Omega_l)/(-\Omega_1^2 + \Omega_l\Omega_u)$ $B = \Omega_2(\Omega_u - \Omega_l)/(-\Omega_2^2 + \Omega_l\Omega_u)$

5 Time discrete filter

5.1 FIR filters and IIR filters

FIR-filter

$$\begin{aligned} \mathcal{H}(z) &= b_0 + b_1 z^{-1} + \ldots + b_M z^{-M} \\ h(n) &= \begin{cases} b_n & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

IIR-filter

$$\begin{aligned} \mathcal{H}(z) &= \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} \\ h(n) &= Z^{-1}\{\mathcal{H}(z)\} \end{aligned}$$

5.2 FIR filters using the window method

Impulse response

$$h(n) = h_d(n) \cdot w(n)$$

with desired impulse response $h_d(n)$ and spectrum $H_d(\omega)$ (i $0 \leq \omega \leq \pi$) and time window $w(n)$

Low pass:

$$\begin{aligned} h_d(n) &= \frac{\omega_c}{\pi} \frac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\omega_c \left(n - \frac{M-1}{2}\right)} \\ H_d(\omega) &= \begin{cases} e^{-j\omega\ (M-1)/2} & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Band pass:

$$\begin{aligned} h_d(n) &= 2\cos\left(\omega_0 \left(n - \frac{M-1}{2}\right)\right) \cdot \frac{\omega_c}{\pi} \frac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\omega_c \left(n - \frac{M-1}{2}\right)} \\ H_d(\omega) &= \begin{cases} e^{-j\omega\ (M-1)/2} & \omega_0 - \omega_c < |\omega| < \omega_0 + \omega_c \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

High pass:

$$\begin{aligned} h_d(n) &= \delta\left(n - \frac{M-1}{2}\right) - \frac{\omega_c}{\pi} \frac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\omega_c \left(n - \frac{M-1}{2}\right)} \\ H_d(\omega) &= \begin{cases} e^{-j\omega\ (M-1)/2} & |\omega| > \omega_c \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Filter spectrum $H(\omega) = H_d(\omega) * W(\omega)$ and at the cutoff frequency ω_c is the attenuation 6dB.
For dimension of filters, the approximation below gives an approximation of the length M.

Table 5.1

Size of main lobe and side lobes for same useful window.

Window	Approximative size of the main lobe	Highest side lobe (dB)
Rectangular	$4\pi/M$	-13
Bartlett	$8\pi/M$	-27
Hanning	$8\pi/M$	-32
Hamming	$8\pi/M$	-43
Blackman	$12\pi/M$	-58

Table 5.2

Size of width between pass- and stop band main lobe and side lobes for some useful window filters.

Window	Width between pass- and stop band (Hz)	Highest side lobe (dB)
Rektangular	$0.6/M$	-21
Hamming	$1.7/M$	-55
Blackman	$3/M$	-75

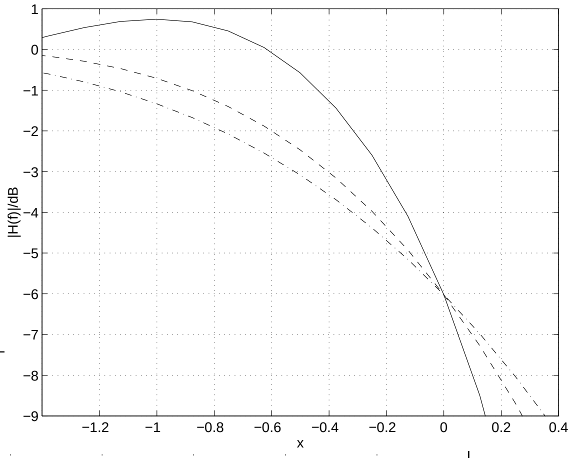
A better approximation is to use (f small, M large)

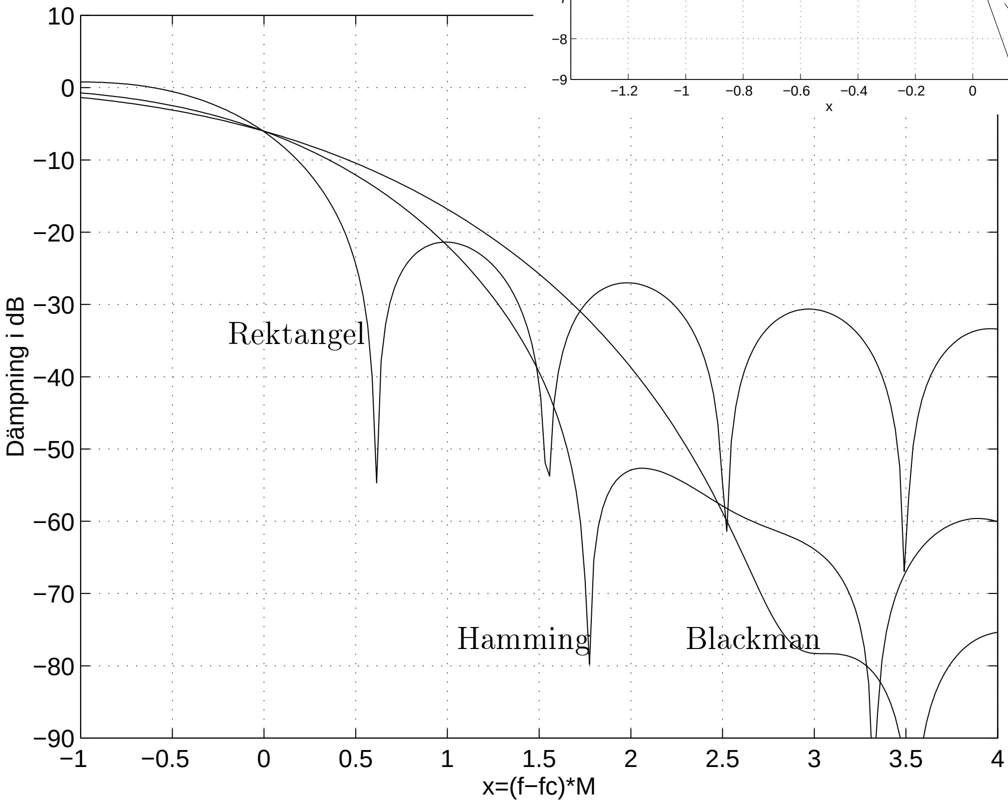
$$\frac{\sin(\pi f M)}{M \sin(\pi f)} \approx \frac{\sin(\pi f M)}{\pi f M}$$

(f small, M large.)
$H(f)$ as a function of $x = (f - f_c) \cdot M$ with $M = 99$, $f_c = 0.1$ for rectangular window, hamming window and blackman window are found in figure below.

Magnitude spectra for some filters using the window method

$$x = \begin{cases} (f - f_c) \cdot M \text{ lågpassfilter} \\ -(f - f_c) \cdot M \text{ högpassfilter} \end{cases}$$



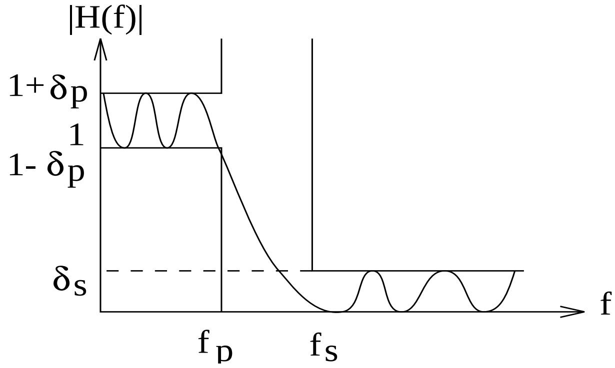


FIR filters using the window method,, $M = 99$ and $f_c = 0.1$. Scale on the x-axis $x = (f - f_c)M$. Higpassfilter, use F $x = -(f - f_c)M$.

Window function used are Rectangular, Hamming and Blackmann.

5.3 Ekviripple FIR filters

Dimensioning of ekviripple filter using the Remez algorithm. Approximatively from Kaiser.



$$
\begin{aligned}
N &= \frac{D_\infty(\delta_p, \delta_s)}{\Delta f} + 1 \\
\Delta f &= f_s - f_p \\
D_\infty(\delta_p, \delta_s) &= \frac{-20 \log \sqrt{\delta_p \delta_s} - 13}{14.6}
\end{aligned}
$$

5.4 FIR filters using Least Squares method

Minimizing

$$\mathcal{E} = \sum_n [x(n) * h(n) - d(n)]^2$$

gives

$$\sum_{n=0}^{M-1} h(n) r_{xx}(n - \ell) = r_{dx}(\ell) \quad \ell = 0, \ldots, M-1$$

and

$$\mathcal{E}_{\min} = r_{dd}(0) - \sum_{k=0}^{M-1} h(k) r_{dx}(k)$$

there $r_{xx}(\ell)$ is the correlation function for $x(n)$ and $r_{dx}(\ell)$ is the cross correlation between $d(n)$ and $x(n)$.
I matrix form this can be written

$$
\begin{aligned}
&\mathbf{R}_{xx} \cdot \mathbf{h} = \mathbf{r}_{dx} \\
&\mathbf{h} = \mathbf{R}_{xx}^{-1} \cdot \mathbf{r}_{dx} \\
&\mathcal{E}_{\min} = r_{dd}(0) - \mathbf{h}^T \cdot \mathbf{r}_{dx}
\end{aligned}
$$

5.5 IIR-filter

Design of IIR-filter from analogous filters.

5.5.1 The impulse invariance method

$$h(n) = h_a(nT)$$

1.

$$h_a(t) = e^{-\sigma_0 t} \longleftrightarrow \mathcal{H}_a(s) = \frac{1}{s+\sigma_0}$$

$$\Rightarrow \mathcal{H}(z) = \frac{1}{1 - e^{-\sigma_0 T} z^{-1}}$$

2.

$$h_a(t) = e^{-\sigma_0 t} \cos \Omega_0 t \longleftrightarrow \mathcal{H}_a(s) = \frac{s+\sigma_0}{(s+\sigma_0)^2 + \Omega_0^2}$$

$$\Rightarrow \mathcal{H}(z) = \frac{1 - z^{-1} e^{-\sigma_0 T} \cos \Omega_0 T}{1 - 2z^{-1} e^{-\sigma_0 T} \cos \Omega_0 T + z^{-2} e^{-2\sigma_0 T}}$$

3.

$$h_a(t) = e^{-\sigma_0 t} \sin \Omega_0 t \longleftrightarrow \mathcal{H}_a(s) = \frac{\Omega_0}{(s+\sigma_0)^2 + \Omega_0^2}$$

$$\Rightarrow \mathcal{H}(z) = \frac{z^{-1} e^{-\sigma_0 T} \sin \Omega_0 T}{1 - 2z^{-1} e^{-\sigma_0 T} \cos \Omega_0 T + z^{-2} e^{-2\sigma_0 T}}$$

5.5.2 Bilinear transformation

Frequency transformation (prewarp)

$$F_{prewarp} = \frac{1}{T} \frac{\tan(\pi f)}{\pi}$$

Analogous filter design in the variable $\Omega_{prewarp}$.

$$\mathcal{H}(z) = \mathcal{H}_a(s) \text{ där } s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

T is a normalizing factor (can often be chosen $=1$).

5.5.3 Quantification of the filter coefficients

Pole movements when the coefficients $a_1, \ldots, a_k$ varies $\Delta a_1, \ldots, \Delta a_k$

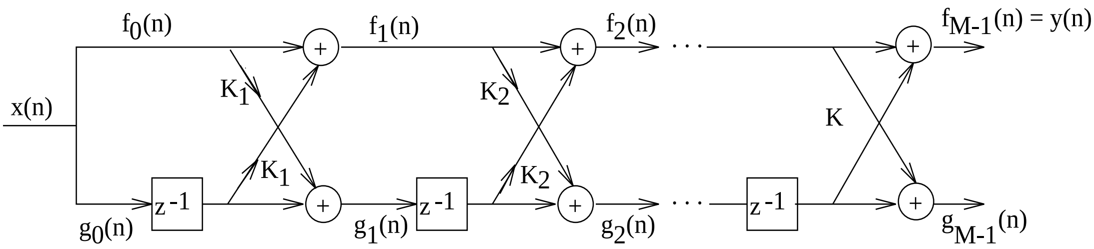
$$\Delta p_i \approx \frac{\partial p_i}{\partial a_1}\ \Delta a_1 + \cdots + \frac{\partial p_i}{\partial a_k}\ \Delta a_k$$

For the normal description (direct form II) yields

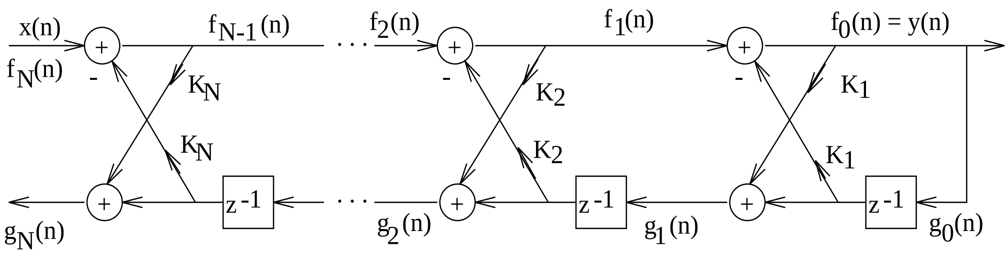
$$\frac{\partial p_i}{\partial a_j} = \frac{-p_i^{k-j}}{\underbrace{(p_i - p_1)(p_i - p_2)\ldots(p_i - p_k)}_{k-1 \text{ factors}}}$$

$(p_i - p_i)$ not included

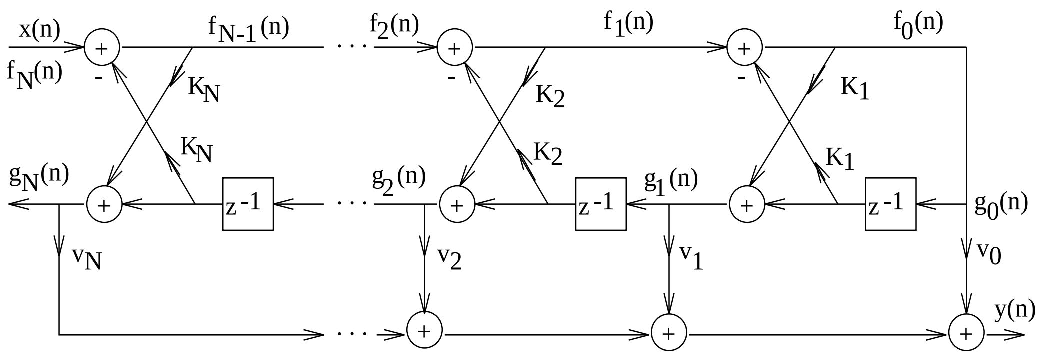
5.6 Lattice filter



Lattice FIR



Lattice all-pole IIR



Lattice-ladder

$$A_0(z) = B_0(z) = 1$$

$$\begin{cases} A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z) \\ B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z) \end{cases}$$

$$A_{m-1}(z) = \frac{1}{1 - K_m^2} \left(A_m(z) - K_m B_m(z)\right)$$

there

$$\begin{aligned} A_m(z) &= Z\{\alpha_m(n)\} \text{ med } K_m = \alpha_m(m) \\ B_m(z) &= Z\{\beta_m(n)\} \end{aligned}$$

Relations between $A_m(z)$ and $B_m(z)$

$$\begin{aligned} B_m(z) &= z^{-m} A_m(z^{-1}) \text{ and} \\ \beta_m(k) &= \alpha_m(m-k) \end{aligned}$$

Lattice-FIR

$$H(z) = A_{M-1}(z)$$

Lattice-all pole IIR

$$H(z) = \frac{1}{A_N(z)}$$

Lattice-ladder

$$H(z) = \frac{C_N(z)}{A_N(z)} = \frac{c_0 + c_1 z^{-1} \ldots c_N z^{-N}}{A_N(z)}$$

there

$$C_m(z) = C_{m-1}(z) + v_m B_m(z)$$

and

$$c_m(m) = v_m \quad m = 0, 1, \ldots, N$$

6 Spectral estimation

Spectral estimation

$$\gamma_{xx}(m) = E\{x(n)x(n+m)\} \text{ autocorrelation}$$

$$\Gamma_{xx}(f) = \sum_{m=-\infty}^{\infty} \gamma_{xx} e^{-j2\pi fm} \text{ power spectrum}$$

The Periodogram

$$r_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-m-1} x(n)x(n+m) \quad 0 \leq m \leq N-1 \quad \text{autocorrelation (estimate)}$$

$$P_{xx}(f) = \sum_{m=-N+1}^{N-1} r_{xx}(m) e^{-j2\pi fm} = \frac{1}{N} \left| \sum_{m=0}^{N-1} x(m) e^{-j2\pi fm} \right|^2$$

power spectrum (estimate)

$$E\{r_{xx}(m)\} = \left(1 - \frac{|m|}{N}\right) \gamma_{xx}(m) \to \gamma_{xx}(m) \text{ when } N \to \infty$$

$$var(r_{xx}(m)) \approx \frac{1}{N} \sum_{n=-\infty}^{\infty} [\gamma_{xx}^2(n) + \gamma_{xx}(n-m)\gamma_{xx}(n+m)] \to 0 \text{ when } N \to \infty$$

$$E\{P_{xx}(f)\} = \int_{-1/2}^{1/2} \Gamma_{xx}(\alpha) W_B(f-\alpha) d\alpha$$

there $W_B(f)$ is the Fourier transform of the Bartlett window $\left(1 - \frac{|m|}{N}\right)$

$$var(P_{xx}(f)) = \Gamma_{xx}^2(f) \left[1 + \left(\frac{sin 2\pi f N}{N sin 2\pi f}\right)^2\right] \to \Gamma_{xx}^2(f) \text{ då } N \to \infty$$

if $x(n)$ Gaussian.

The Periodogram using the DFT:

$$P_{xx}\left(\frac{k}{N}\right) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{nk}{N}} \right|^2 \quad k = 0, \ldots, N-1$$

Mean of periodogram

Quality factor

$$Q = \frac{[E\{P_{xx}(f)\}]^2}{var(P_{xx}(f))}$$

Relative variance $\frac{1}{Q}$
$Q \approx$ time-bandwidth product.

Periodogram	$\Delta f = \frac{0.9}{M}$	$Q = 1$	
Bartlett $(N = K \cdot M)$	$\Delta f = \frac{0.9}{M}$	$Q_B = \frac{N}{M}$	Rectangular window No overlapping
Welch $(N = L \cdot M)$	$\Delta f = \frac{1.28}{M}$	$Q_B = \frac{16}{9} \cdot \frac{N}{M}$	Triangular window 50% overlapping
Blackman/Tukey	$\Delta f = \frac{0.6}{M}$	$Q_B = \frac{1}{2} \cdot \frac{N}{M}$	Rectangular window
	$\Delta f = \frac{0.9}{M}$	$Q_B = \frac{3}{2} \cdot \frac{N}{M}$	Triangular window

Resolution Δf determined in the -3dB points (main lobe).