Lecture 4: Capacity

Project info

1. Each project group consists of two students.
2. Each project group should as soon as possible, send an email to fredrik.rusek@eit.lth.se containing Names of each project member.
3. The project group should contact Fredrik Rusek to decide about project and articles.
4. Each group should write a project report.
5. The structure of the project report should follow journal articles published by IEEE. However, two columns are not needed.
6. The project report should be written in English *with your own words, tables and figures*, and contain 4-5 pages.
7. The report should be clearly written, and written to the other students in this course!
8. Observe copyright rules: ”copy & paste” is in general strictly forbidden!

Lecture 4: Capacity

Project info

9. The project report should be sent in .pdf format to Fredrik before Wednesday 9 December, 17.00
10. Feedback on the reports will be provided via zoom.
11. Oral presentations in the week starting with Monday December 14
12. Each group should have relevant comments and questions on the project report and on the oral presentation of another group. NOTE! The project presentation should be clear and aimed to the other students in this course! After the oral presentation the project report and the presentation will be discussed (5 min).
13. Final report should be sent to Fredrik at latest January 11, 2021.

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that

$$P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$$

To meet a specific error probability target, this implies

$$\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$$

Lecture 4: Capacity

Power efficiency

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To meet a specific error probability target, this implies $$\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $$\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$$ or, equivalently, $$R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$$

Lecture 4: Capacity

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Now, divide both sides with the bandwidth W $$\frac{R_b}{W} \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_s \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

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We have seen this before, it is defined as bandwidth efficiency

Now, divide both sides with the bandwidth W $\rho = \frac{R_b}{W} \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$

Lecture 4: Capacity

Power efficiency

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Lecture 4: Capacity

Power efficiency

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Received signal-to-noise-power-ratio

Now, divide both sides with the bandwidth W $$\rho \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$$

Power efficiency

We know from before (e.g., union bound) that $$P_s \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$$

To meet a specific error probability target, this implies $$\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$$

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Definition

Now, divide both sides with the bandwidth W $$\rho \leq \mathcal{SNR}_r \frac{d_{\min}^2}{\mathcal{X}}$$

Lecture 4: Capacity

Power efficiency

"BW efficiency" = "Signal-to-noise-power-ratio" x "Power efficiency"

$$\rho \leq \mathcal{SNR}_r \frac{d_{\min}^2}{\mathcal{X}}$$

Lecture 4: Capacity

Shannon Capacity

Before going on, we go through what the term capacity means

Given a scalar channel of form $y = \sqrt{A}x + n, \; n \sim CN(0, N_0)$

We know that the capacity is $C = \log_2\left(1 + \frac{A}{N_0}\right)$

But what does this mean?

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Build a codebook of all information sequences possible to send of length K

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Build a codebook of all information sequences possible to send of length K

000000	.....	00
000000	.....	01
000000	.....	10
1111111	.....	10
1111111	.....	11

K

Sending K bits of information means:

pick one of the rows, and tell the receiver which row it is

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Build a codebook of codewords to send for each information word, length N

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$

$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
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$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

If this is my data

1111111 10
1111111 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

$x_{2^K 1}x_{2^K 2}x_{2^K 3}x_{2^K 4}$ ..... $x_{2^K (N-1)}x_{2^K N}$

N

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \ n \sim CN(0, N_0)$$

$$C = \log_2 \left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

If this is my data

1111111 10
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K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

I send this one

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Shannon Capacity

As x over this channel used N times

$$y = \sqrt{A}x + n, \quad n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
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If this is my data

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K

Codebook

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N

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Clearly, bit rate is K/N bits/channel use

Information book

000000 00
000000 01
000000 10

1111111 10
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K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
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$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \ldots\ldots y_{(N-1)} y_N$

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
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K

Codebook

$x_{11} x_{12} x_{13} x_{14} \ldots\ldots x_{1(N-1)} x_{1N}$

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N

Lecture 4: Capacity

$$y = \sqrt{A}x + n,\ \ n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \ \ldots\ y_{(N-1)} y_N$

Compare with this one

$$d_1 = \sum_{n=1}^{N} |y_n - x_{1n}|^2$$

Information book

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000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \ \ldots\ x_{1(N-1)} x_{1N}$

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N

Lecture 4: Capacity

$$y = \sqrt{A}x + n, \; n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \; ..... \; y_{(N-1)} y_N$

Compare with this one

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \; ..... \; x_{1(N-1)} x_{1N}$

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N

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \; ..... \; y_{(N-1)} y_N$

Compare with this one

$$d_{2^K} = \sum_{n=1}^{N} |y_n - x_{2^K n}|^2$$

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

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000000 01
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Codebook

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Lecture 4: Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \; ..... \; y_{(N-1)} y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \; ..... \; x_{1(N-1)} x_{1N}$

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Lecture 4: Capacity

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Receiver observes

$y_1 y_2 y_3 y_4 \; ..... \; y_{(N-1)} y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

Information book

000000 00
000000 01
000000 10

So data is this one

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \; ..... \; x_{1(N-1)} x_{1N}$

$x_{21} x_{22} x_{23} x_{24} \; ..... \; x_{2(N-1)} x_{2N}$

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Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \; ..... \; y_{(N-1)} y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

This is ML decoding and is optimal

Capacity means the following

Information book

000000 00
000000 01
000000 10

So data is this one

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \; ..... \; x_{1(N-1)} x_{1N}$

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N

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \, ... \, y_{(N-1)} y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

This is ML decoding and is optimal

Capacity means the following

1. If $K/N \leq C$, and $K \to \infty$ then
   Prob(Correct detection)=1

Information book

000000 00
000000 01
000000 10

So data is this one

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \, ... \, x_{1(N-1)} x_{1N}$

$x_{21} x_{22} x_{23} x_{24} \, ... \, x_{2(N-1)} x_{2N}$

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N

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \; \ldots \; y_{(N-1)} y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

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K

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N

This is ML decoding and is optimal

Capacity means the following

1. If $K/N \leq C$, and $K \to \infty$ then
   Prob(Correct detection)=1
2. If $K/N > C$, then
   Prob(Incorrect detection)=1

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \ldots\ldots y_{(N-1)} y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

To reach C, code-symbols must be
Random complex Gaussian variables
That is, generate codebook randomly

If it is generated with, say, 16QAM
C cannot be reached

Information book

000000 00
000000 01
000000 10

So data is this one

1111111 10
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K

Codebook

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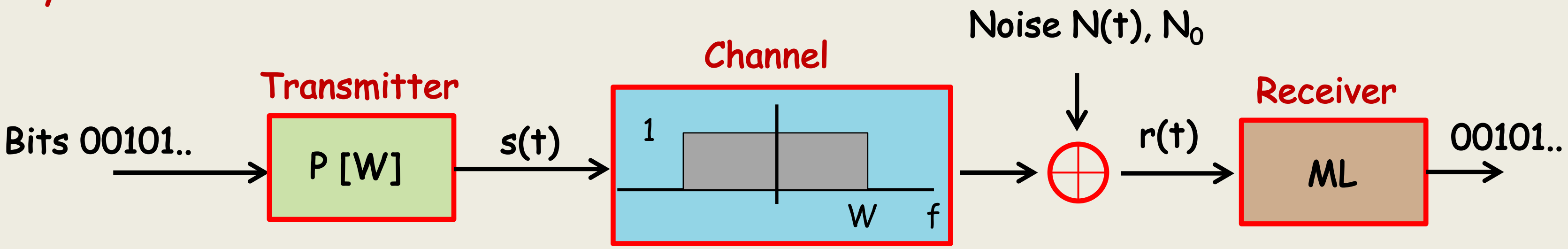
N

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

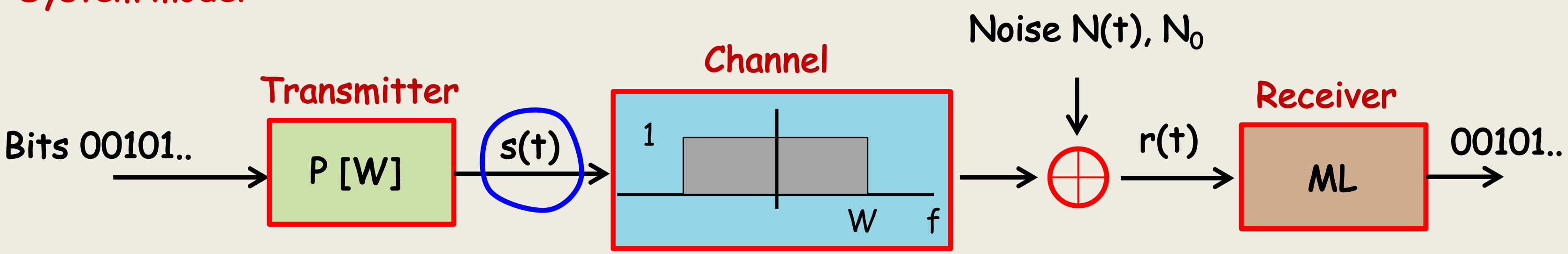
System model:



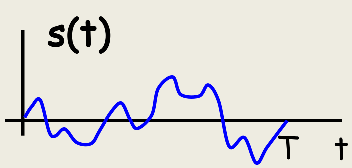
Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



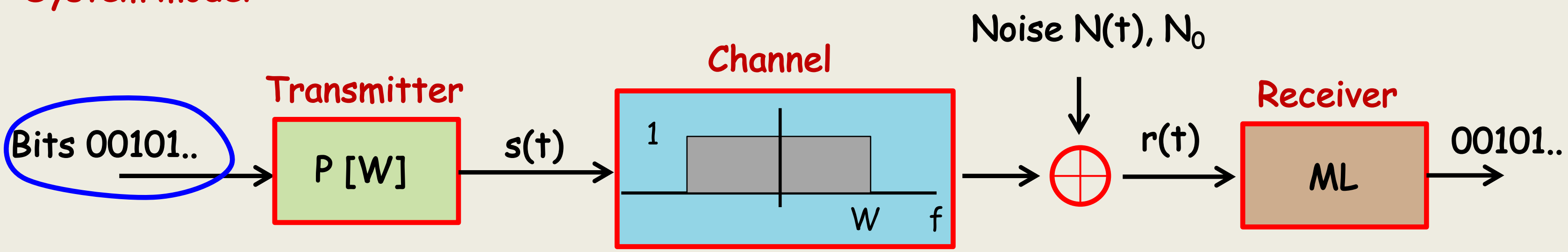
Interpretation of capacity:
Given a transmission of length T (seconds)



Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

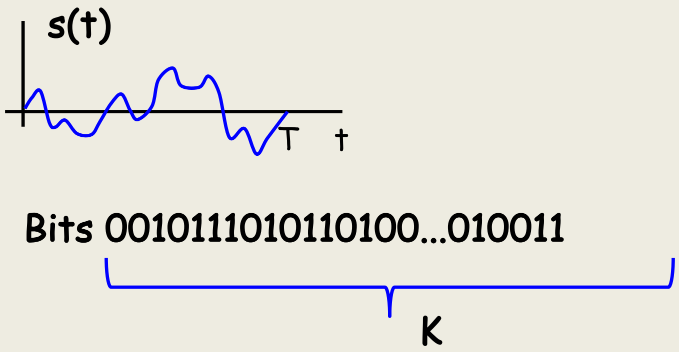
System model:



Interpretation of capacity:
Given a transmission of length T (seconds)

And a number of bits K

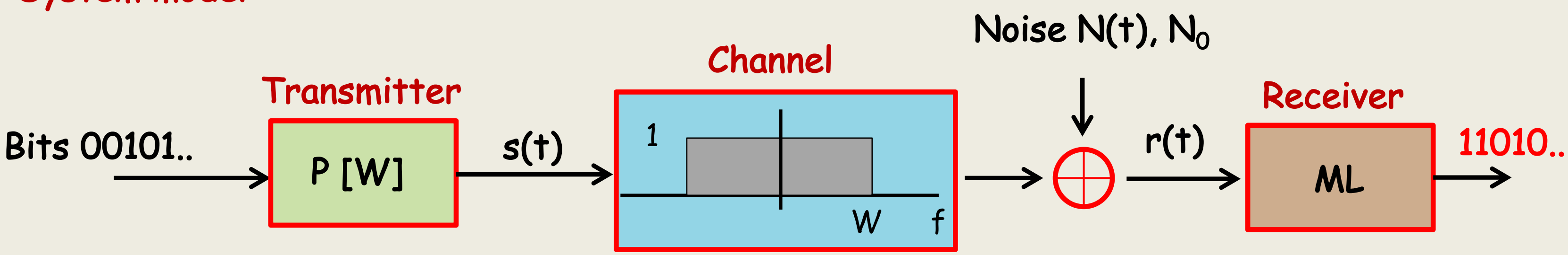
The bitrate is: K/T [bit/sec]



Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:

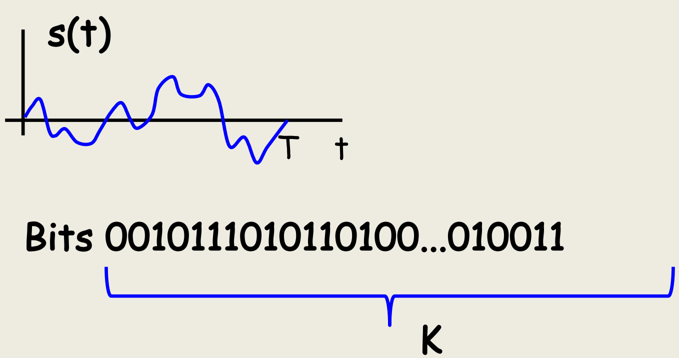


Interpretation of capacity:
Given a transmission of length T (seconds)

And a number of bits K

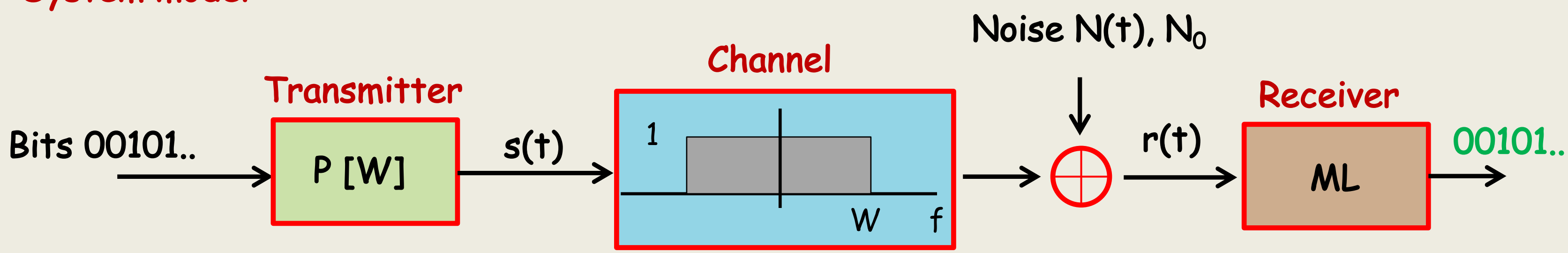
The bitrate is: K/T [bit/sec]

If K/T is too high, then many errors

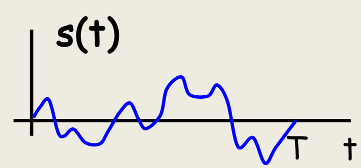


Extension to continuous channel (Shannon '48)

System model:



Interpretation of capacity:
Given a transmission of length T (seconds)



And a number of bits K

Bits 0010111010110100…010011 (K bits)

The bitrate is: K/T [bit/sec]

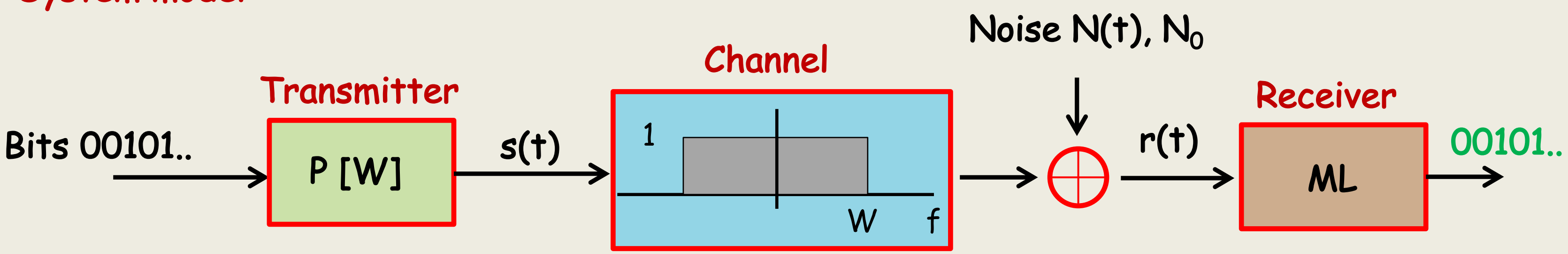
If K/T is too high, then many errors
Shannon proved: Possible to have NO ERRORS if,

1) $T \to \infty$ 2) $\lim_{T\to\infty} \frac{K}{T} = C = W \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

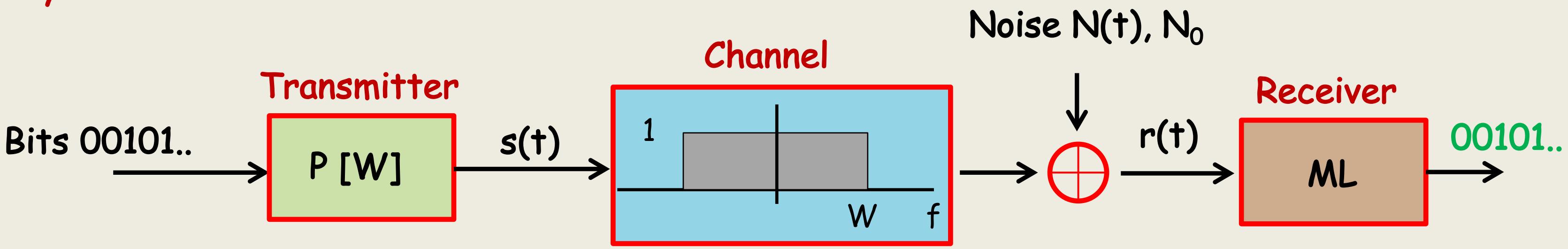
Facts:

1. C is not power-, nor bandwidth efficiency
   (C is not dimensionless)
2. Not easy to reach C
   (i.e., to find a set of s(t) signals)
3. There is no parameter called $d^2_{\min}$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

Facts:

1. C is not power-, nor bandwidth efficiency
   (C is not dimensionless)
2. Not easy to reach C
   (i.e., to find a set of s(t) signals)
3. There is no parameter called $d^2_{\min}$

4. When W grows:

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

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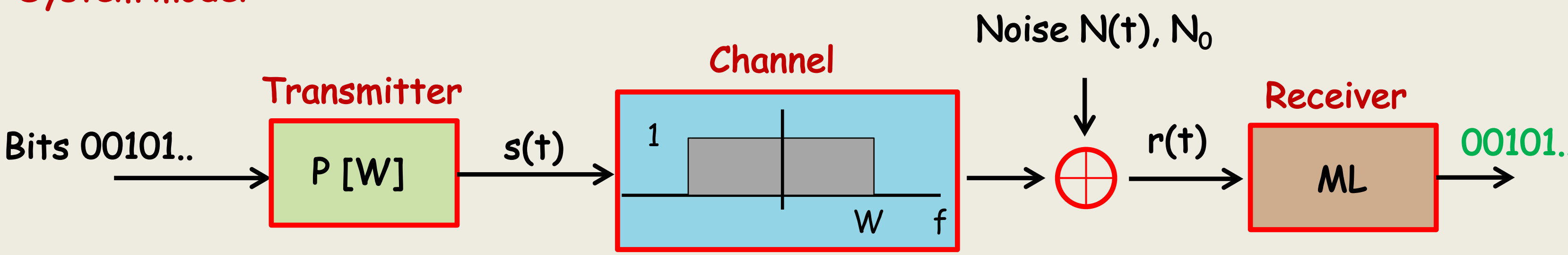

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Extension to continuous channel (Shannon '48)

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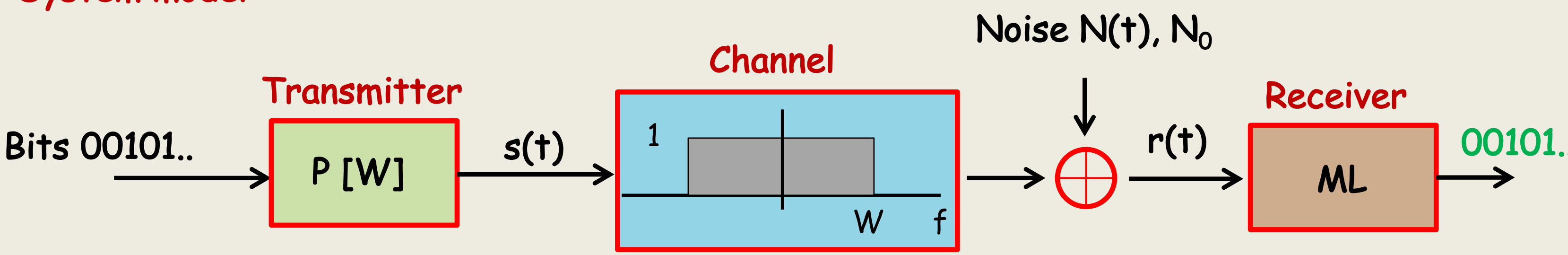
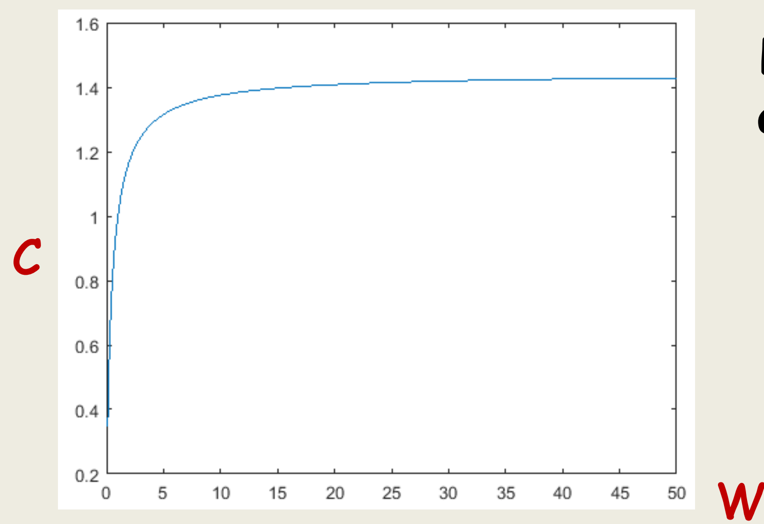


$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

Example, $P/N_0 = 1$

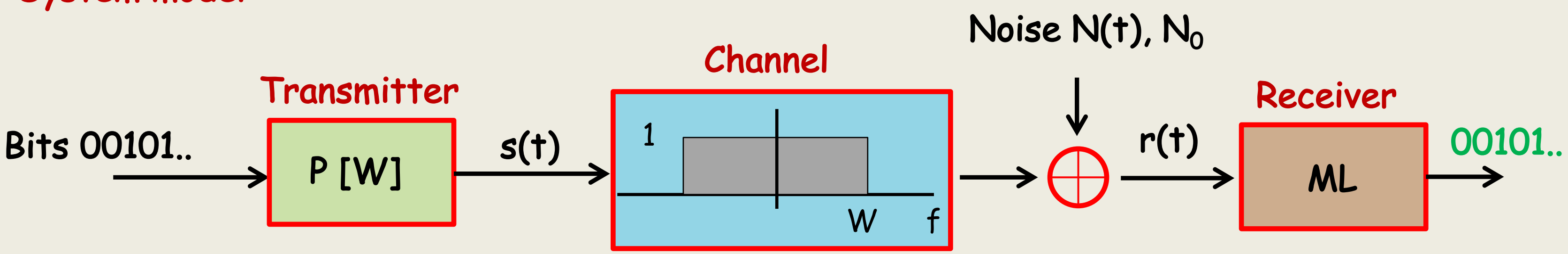


But it grows to a limit

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

What is the limit?

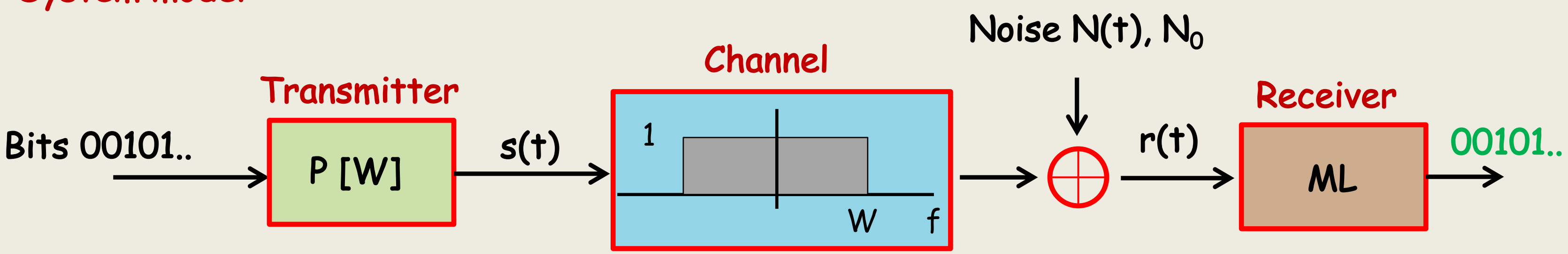
Standard limit

$$\lim_{x \to \infty} x \ln \left(1 + \frac{A}{x}\right) = A$$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



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Identify x with W

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Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

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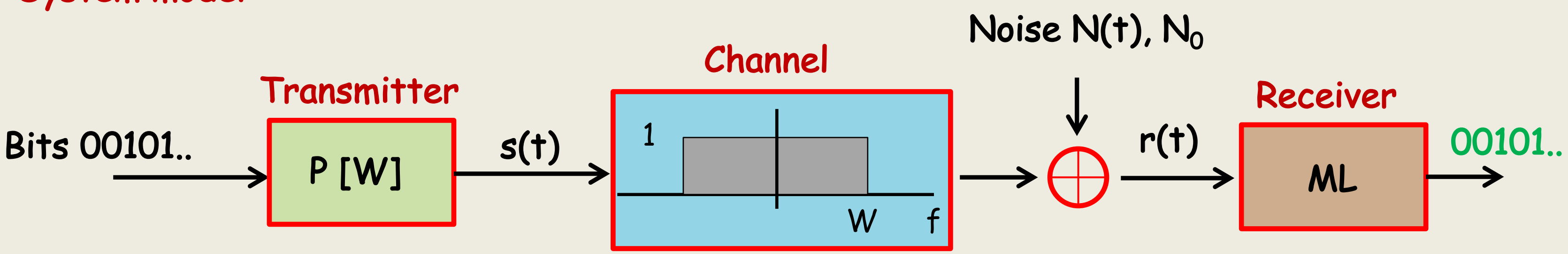

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Identify A with P/N_0

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Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

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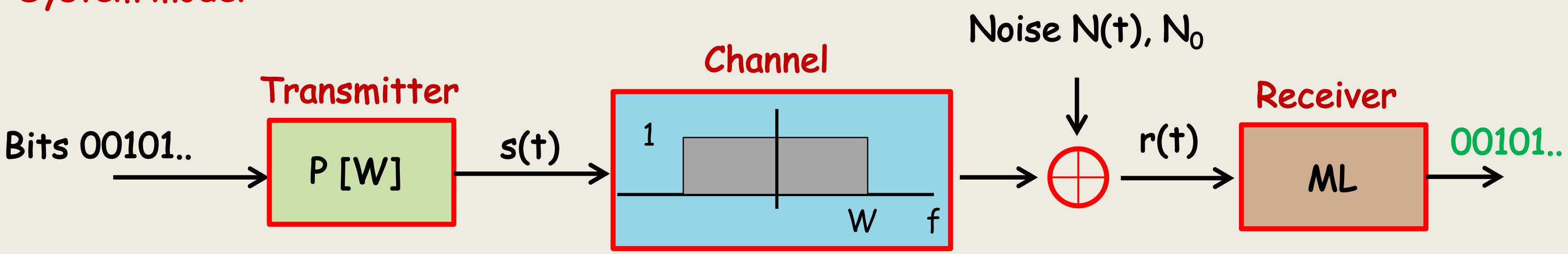

$$C = \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

What is the limit?

Standard limit

Identify x with W

Identify A with P/N_0

Express $\log_2(x)$ as $\ln(x)/\ln(2)$

$$\lim_{x\to\infty} x \ln\left(1 + \frac{A}{x}\right) = A$$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

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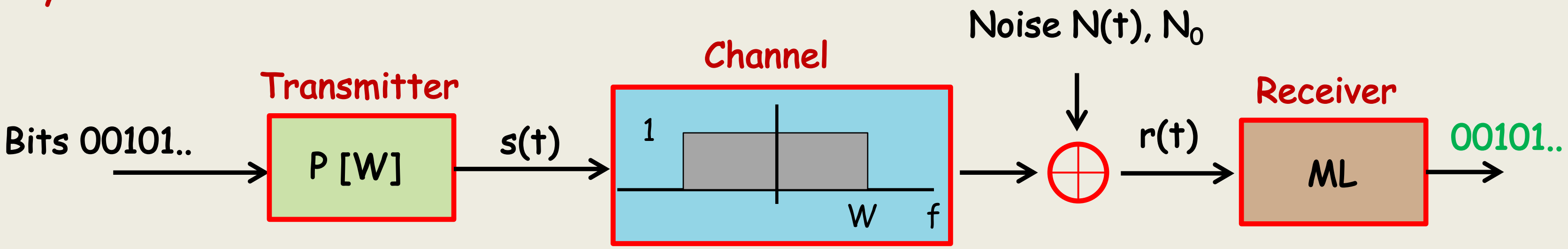

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Standard limit

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Carry out limit

$$C_{\max} = \lim_{W\to\infty} \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right) = \frac{\mathcal{P}}{N_0 \ln(2)}$$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

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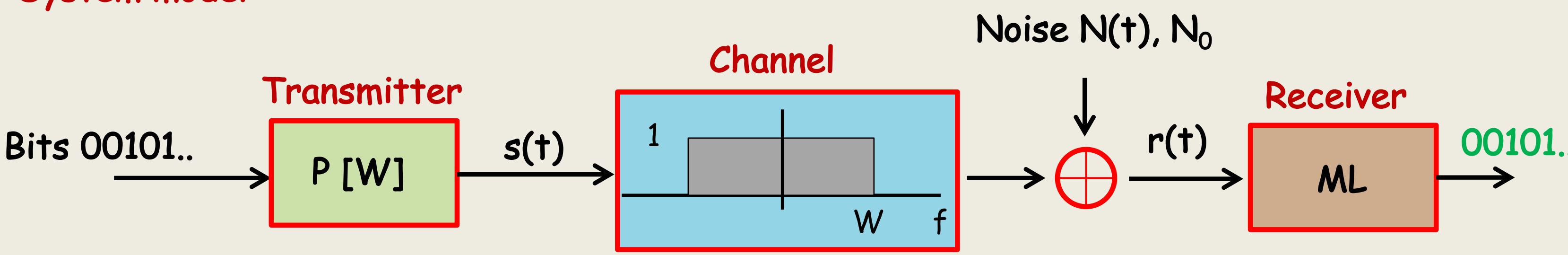

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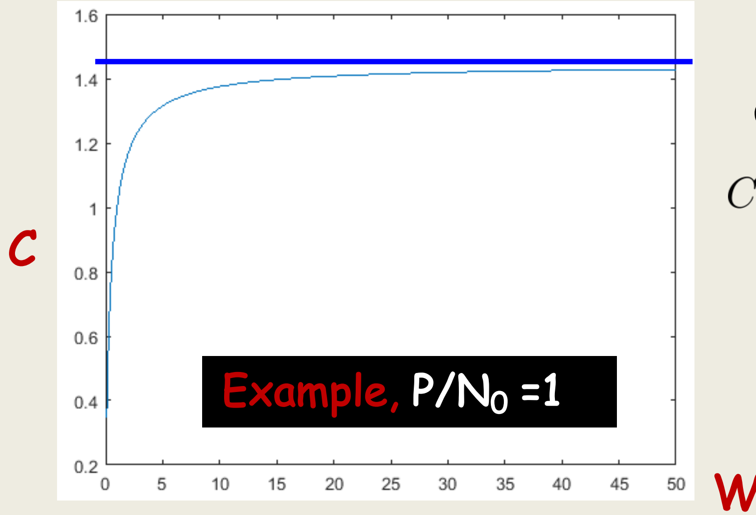

But it grows to a limit

$$C_{\max} = \frac{\mathcal{P}}{N_0 \ln(2)}$$

$$= \frac{1}{\ln(2)}$$

$$= 1.4427$$

Standard limit

$$\lim_{x \to \infty} x \ln\left(1 + \frac{A}{x}\right) = A$$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

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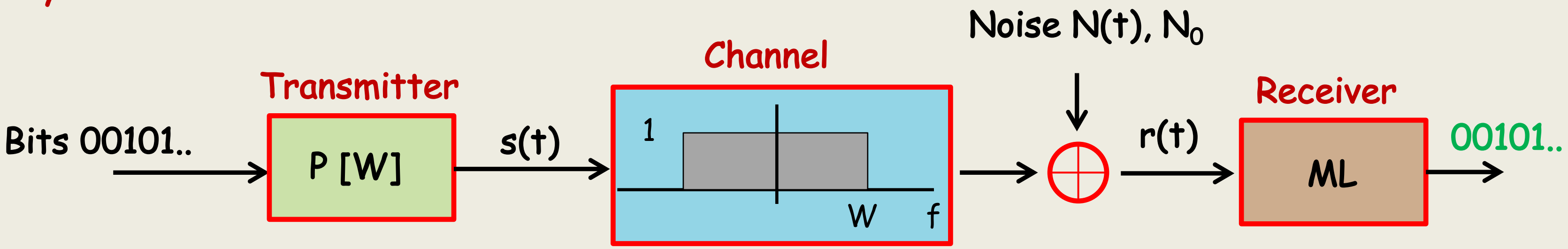


Summary

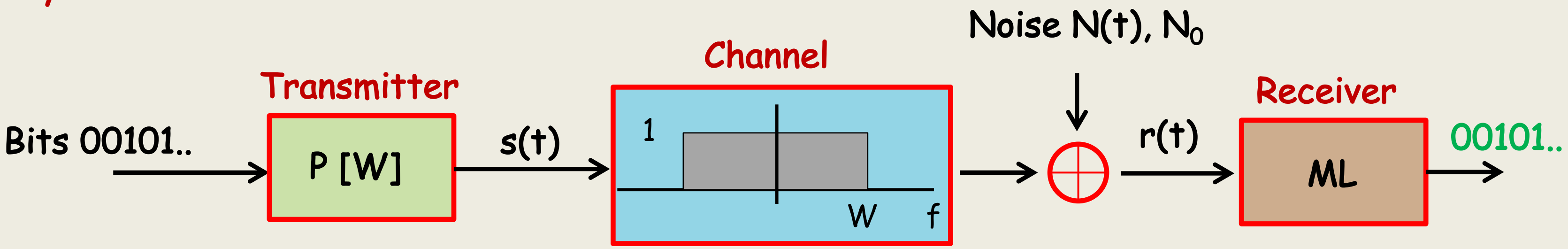
1. We stated that the capacity of the above is $C = W \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$ bits/second

2. We proved that for infinite bandwidth, the capacity is $C_{\max} = \frac{\mathcal{P}}{N_0 \ln(2)}$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Bandwidth efficiency

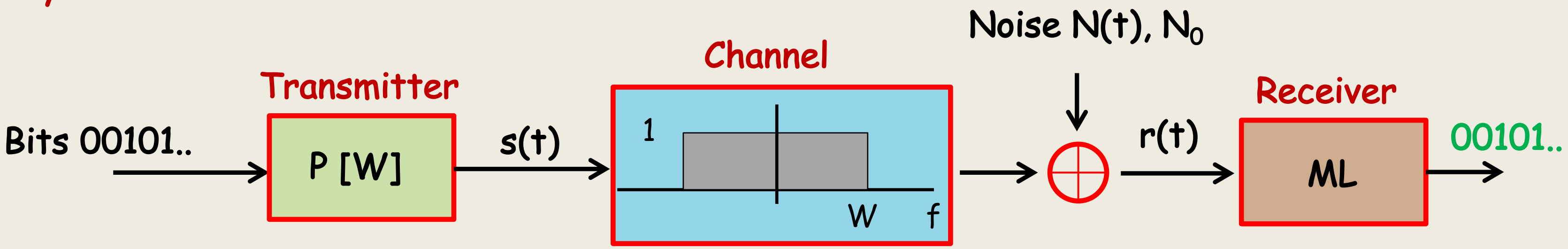
By definition, $\frac{C}{W} = \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$

Effect of increasing/decreasing W ?

For large W, BW efficiency = 0
For small W, BW efficiency = ∞

Extension to continuous channel (Shannon '48)

System model:



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Bandwidth vs. Power efficiency

However $\mathcal{P} = C E_b$

So, $\frac{C}{W} = \log_2\left(1 + \frac{C}{W}\frac{E_b}{N_0}\right)$

Or, equivalently $\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}} - 1}{\frac{C}{W}}$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

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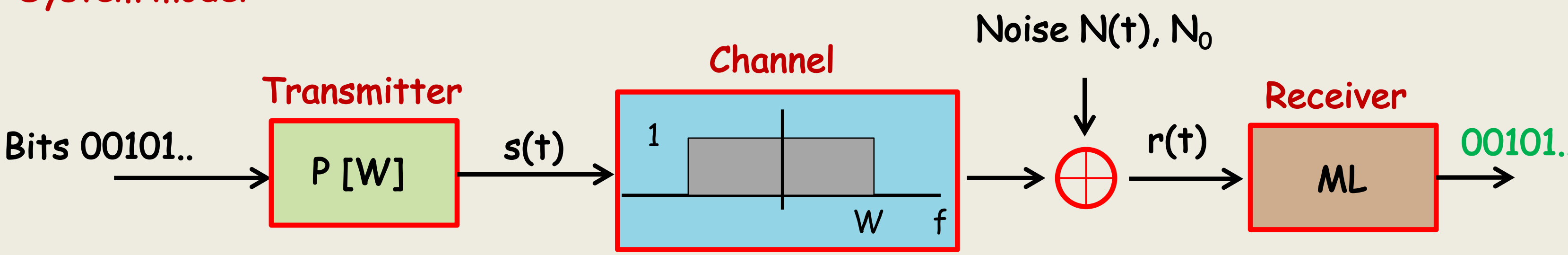

What happens if C/W grows?

E_b/N_0 grows as well

Bandwidth vs. Power efficiency

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Lecture 4: Capacity

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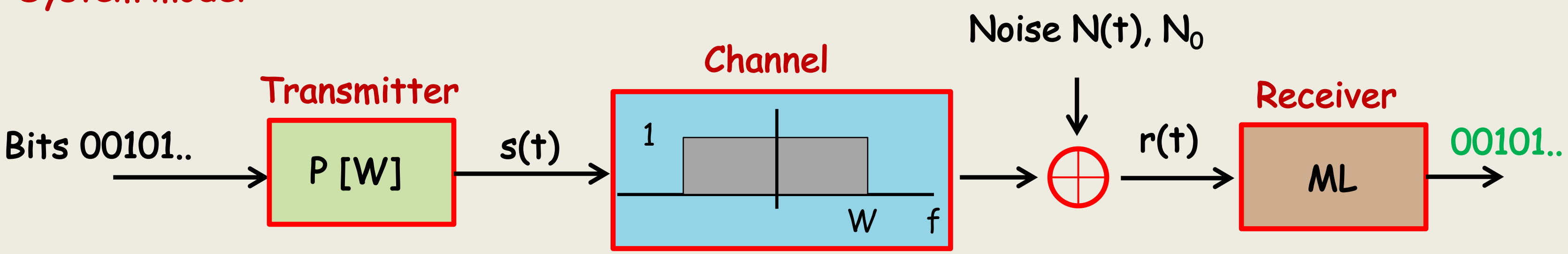

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Lecture 4: Capacity

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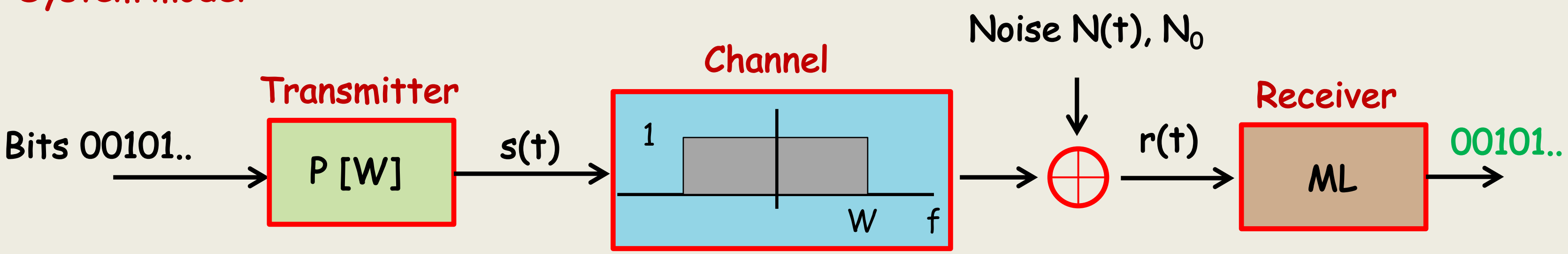

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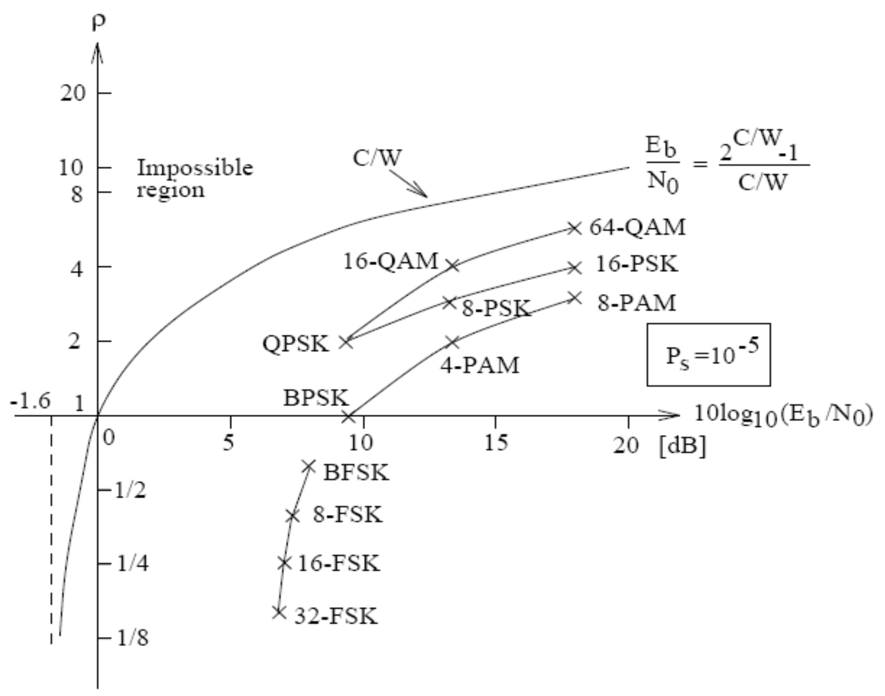
However $\mathcal{P} = C E_b$

So, $\frac{C}{W} = \log_2 \left(1 + \frac{C}{W} \frac{E_b}{N_0} \right)$

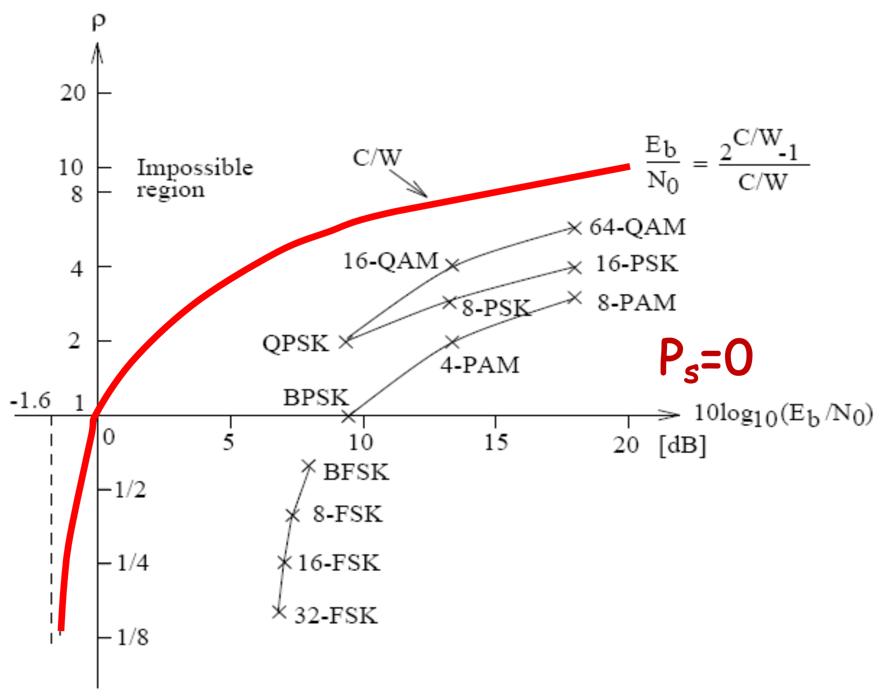
Thus $\lim_{C/W \to 0} \frac{E_b}{N_0} \geq \ln(2)$ $(= -1.6\text{dB})$

Lecture 4: Capacity



$$\lim_{C/W \to 0} \frac{E_b}{N_0} \geq \ln(2) \quad (= -1.6\text{dB}) \qquad \frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}} - 1}{\frac{C}{W}}$$

Lecture 4: Capacity



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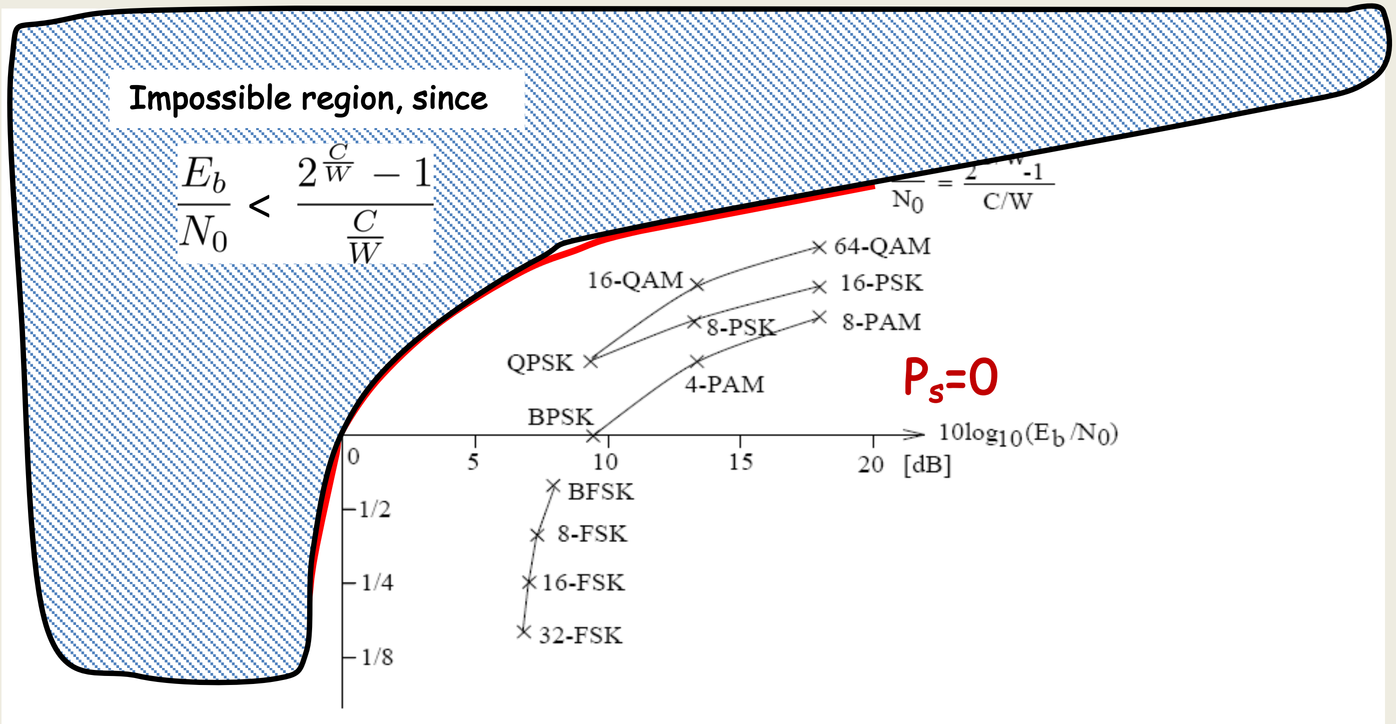

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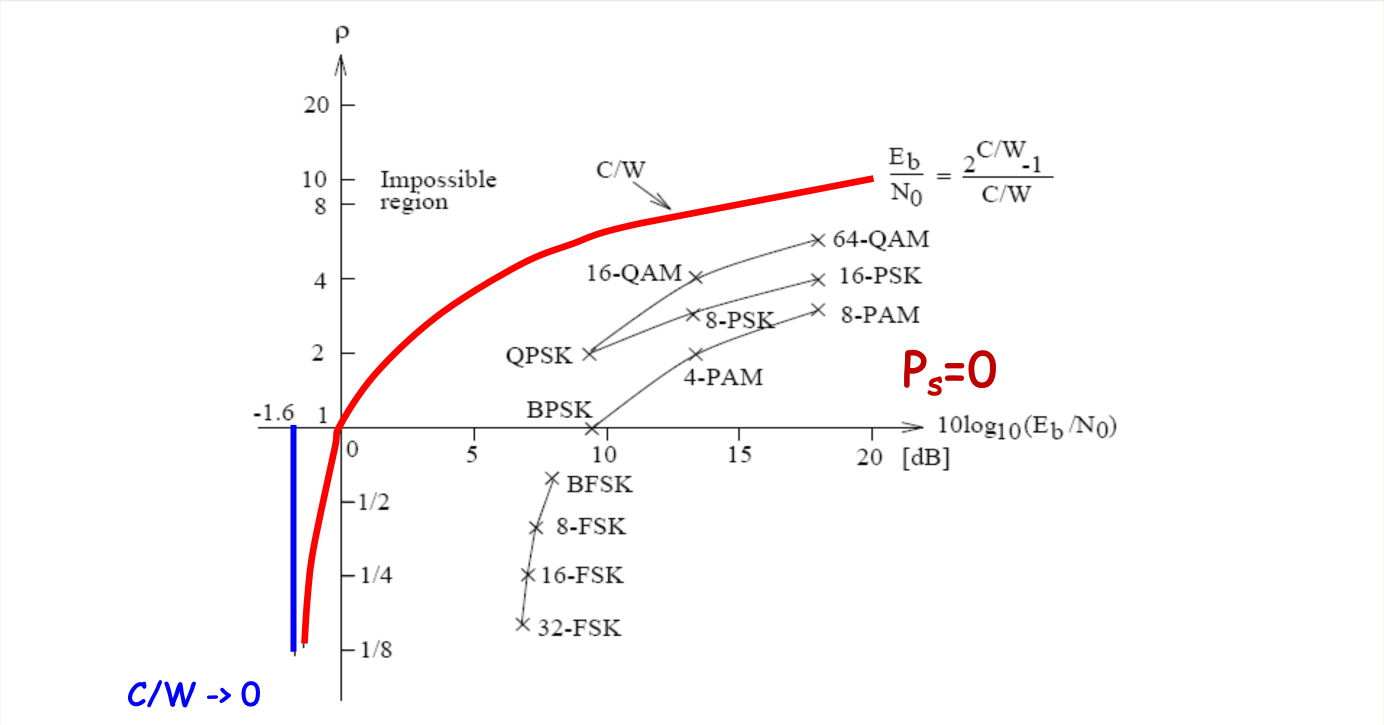
$$\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}} - 1}{\frac{C}{W}}$$

NOTE: Plot does not tell what the capacity is in bit/sec

Lecture 4: Capacity



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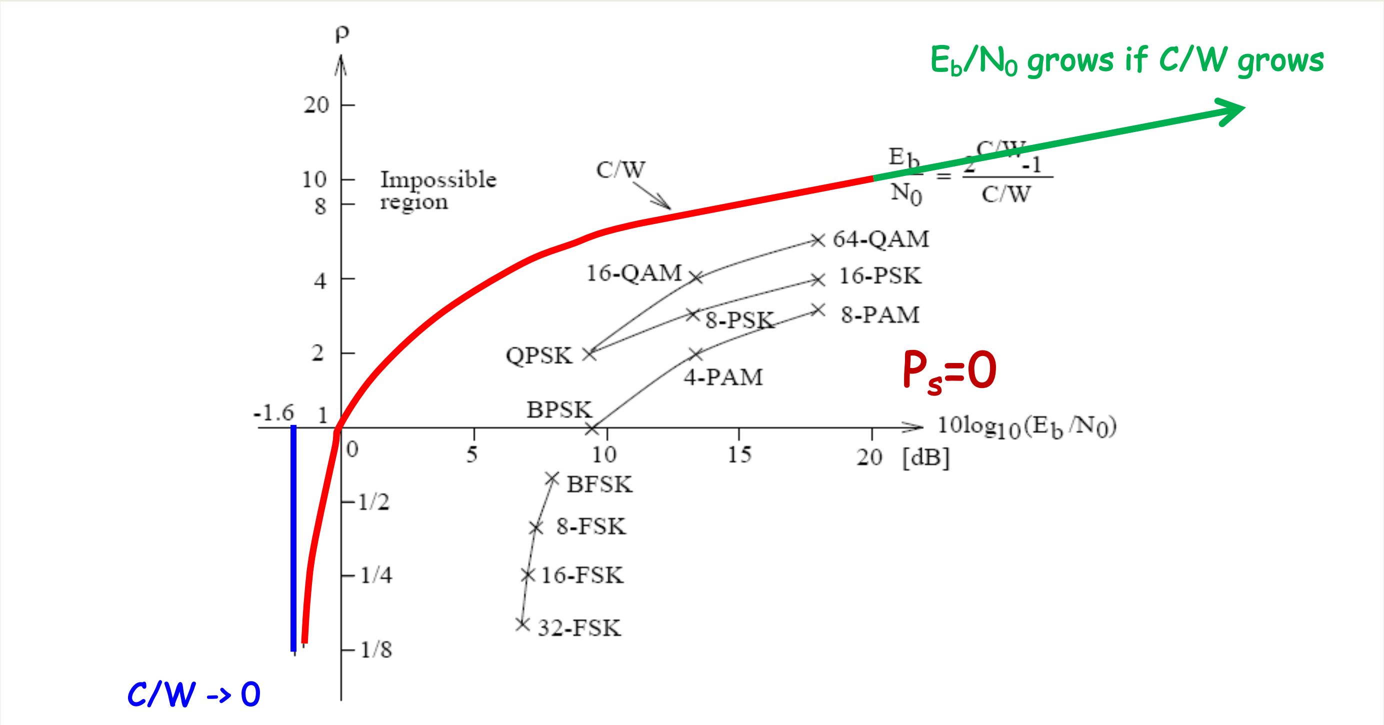

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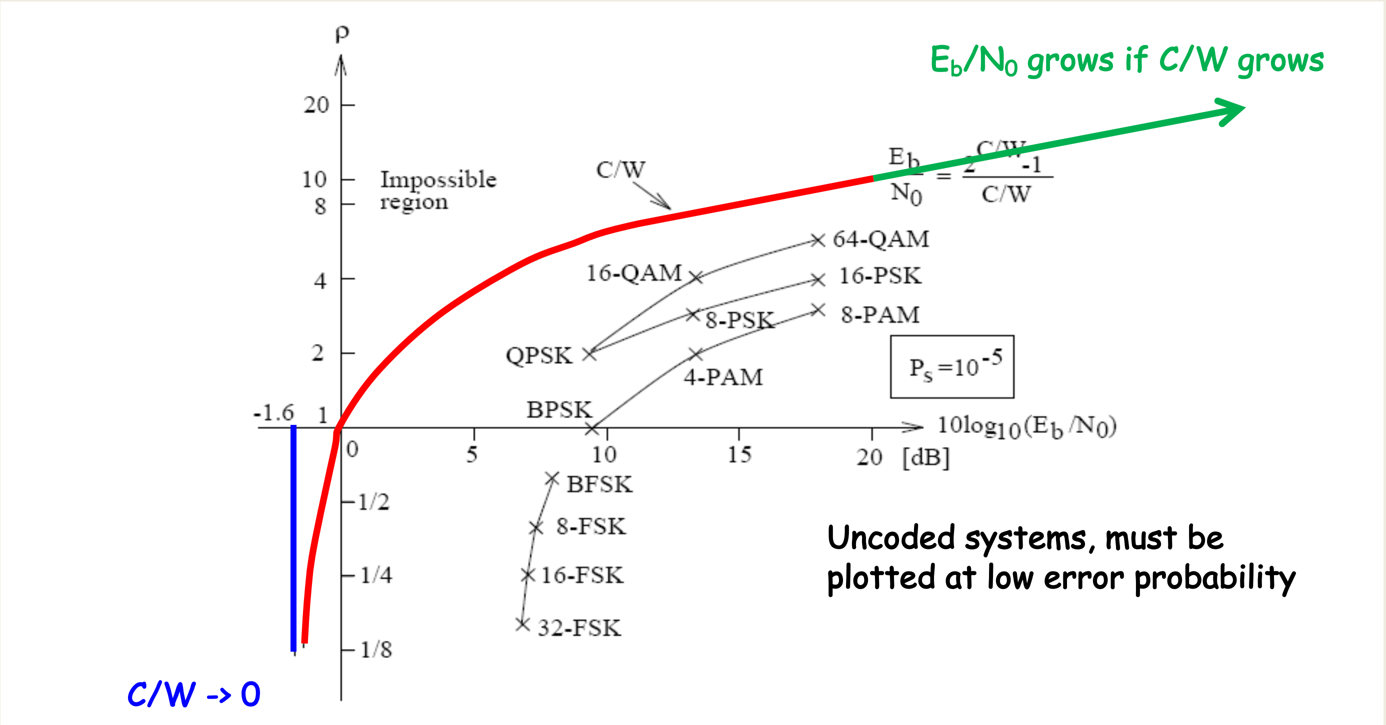

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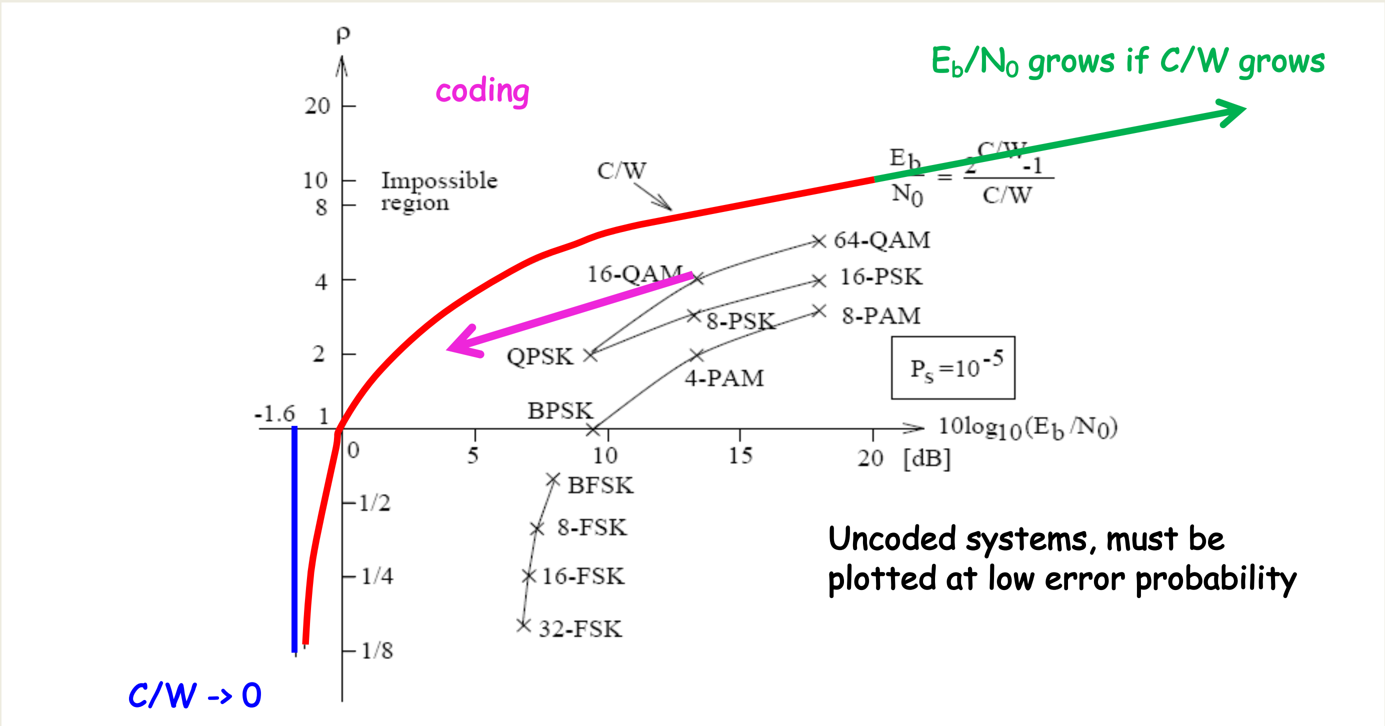

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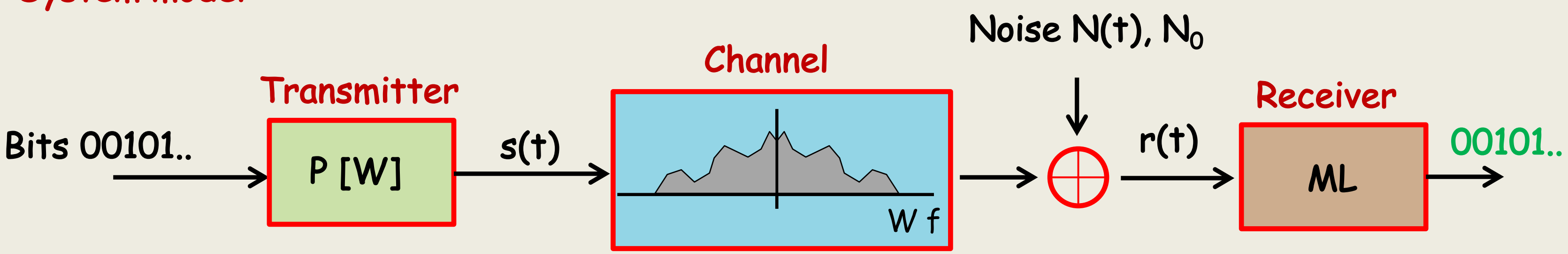
Lecture 4: Capacity

Extension to frequency dependent channel

System model:



Lecture 4: Capacity

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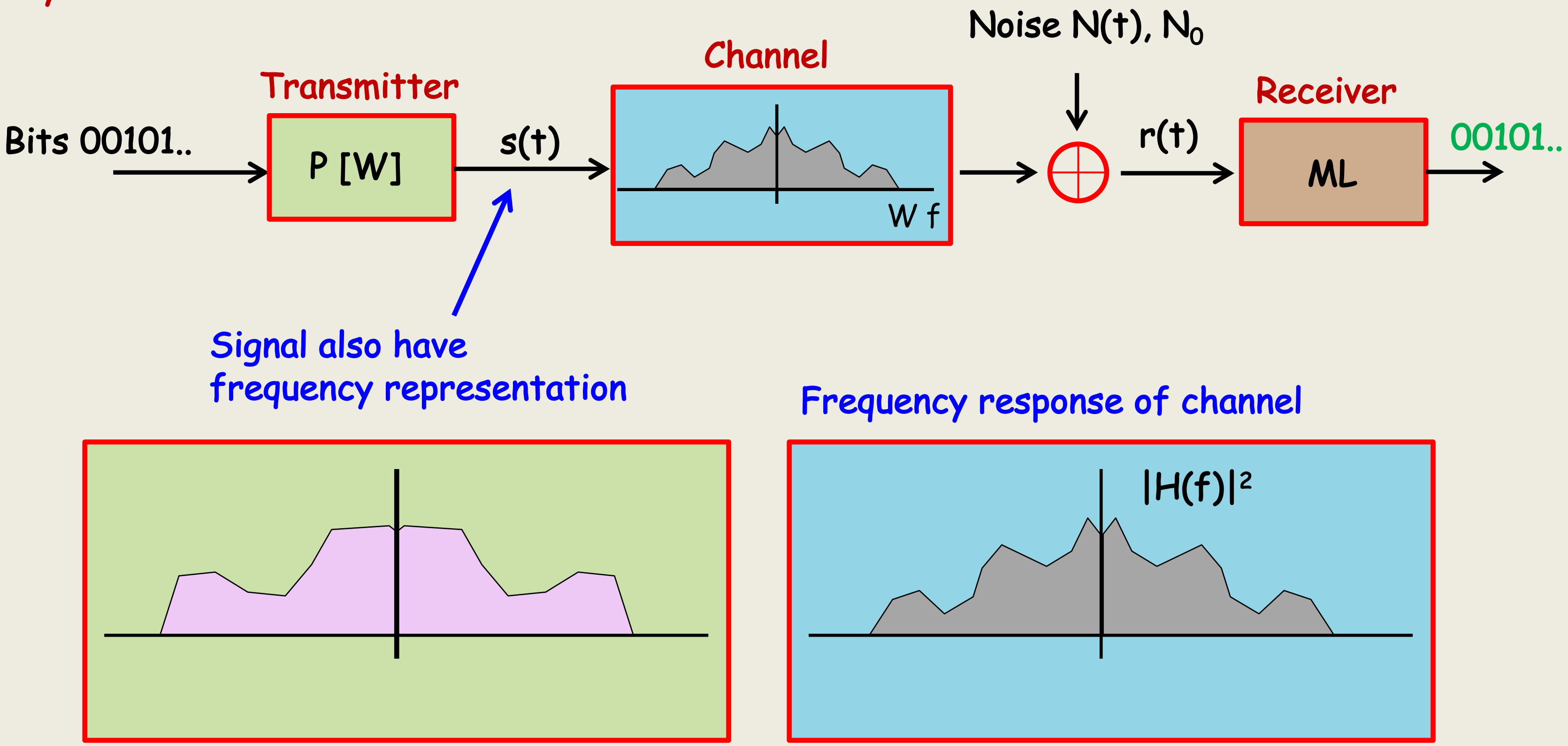

Lecture 4: Capacity

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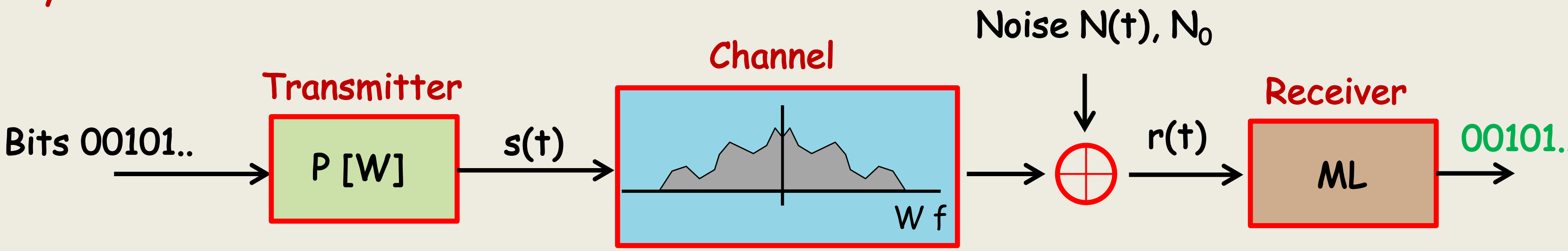
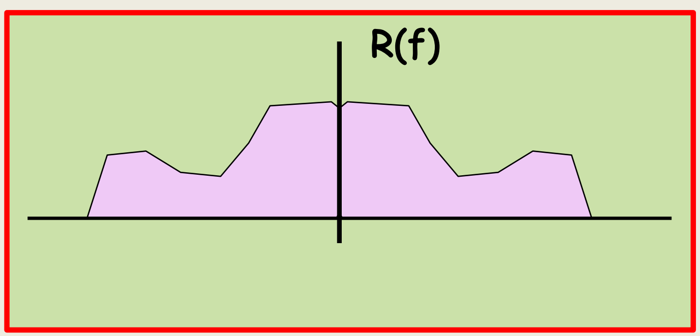
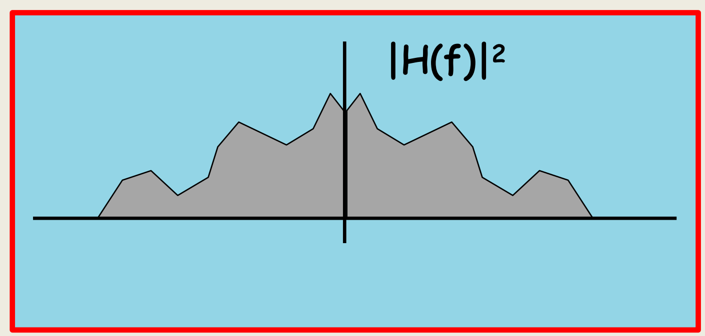


Constraint on R(f) ?

Power spectral density is what matters



Frequency response of channel



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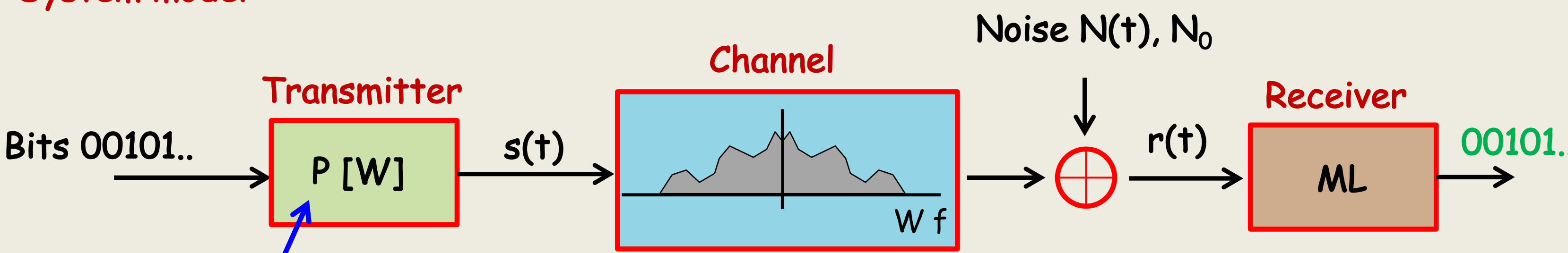
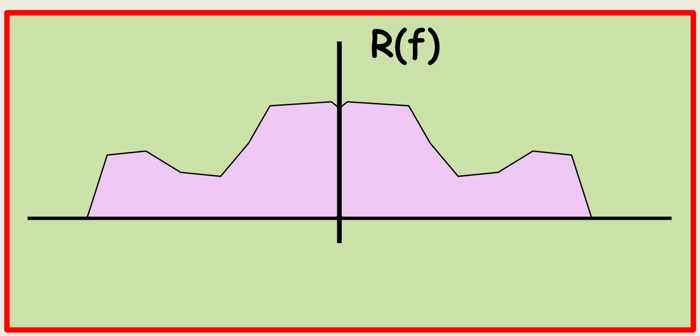


Constraint on R(f) ? $\int_{-\infty}^{\infty} R(f)\mathrm{d}f = P$

Power spectral density is what matters



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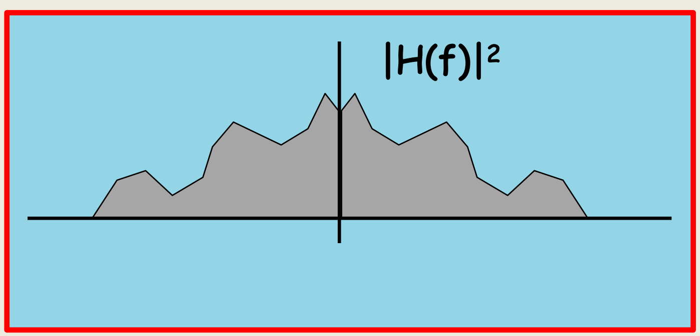

Lecture 4: Capacity

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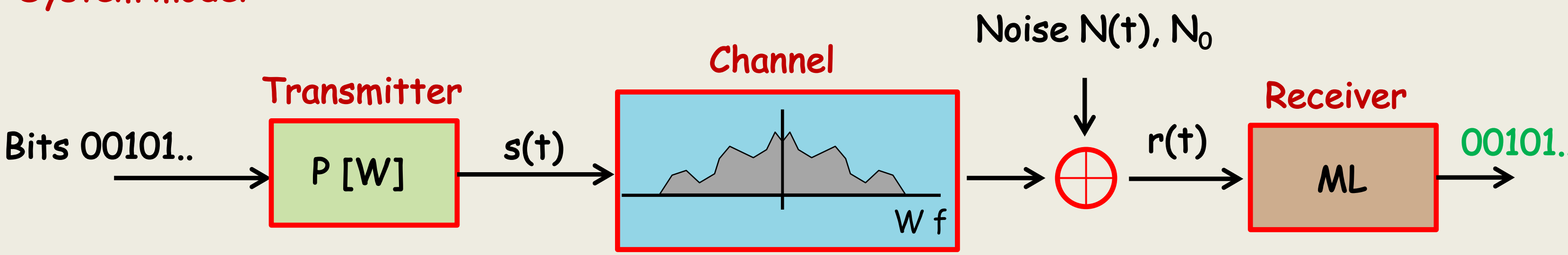
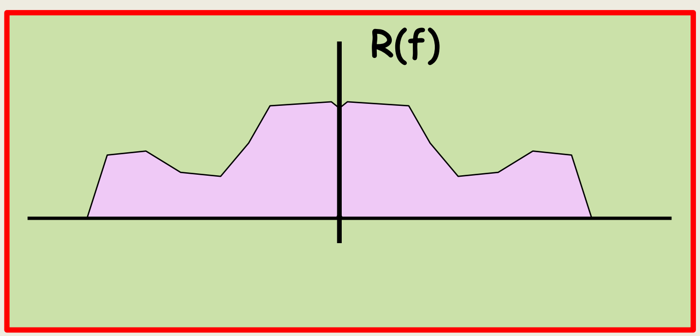


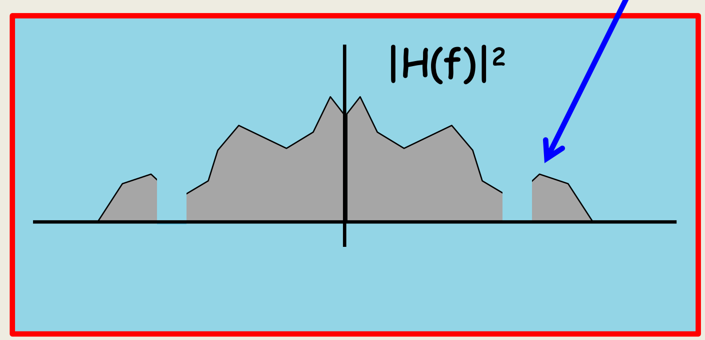
Constraint on R(f) ? $\int_{-\infty}^{\infty} R(f)\mathrm{d}f = P$

If deep fade here

Power spectral density is what matters



Frequency response of channel



Lecture 4: Capacity

Extension to frequency dependent channel

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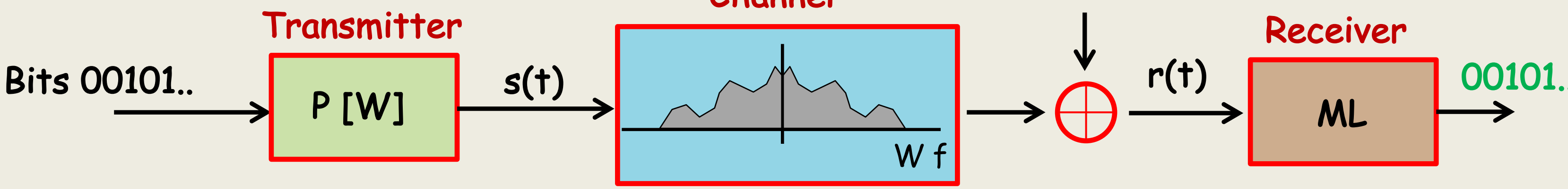
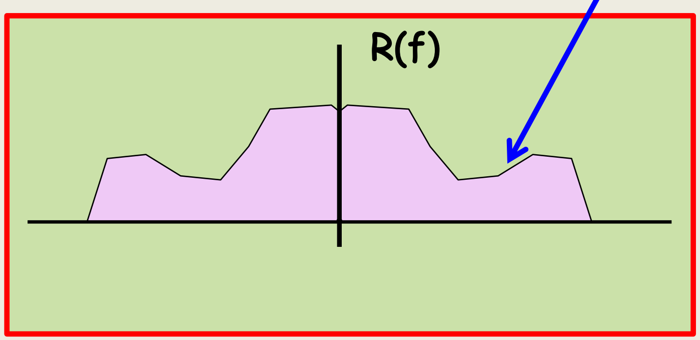


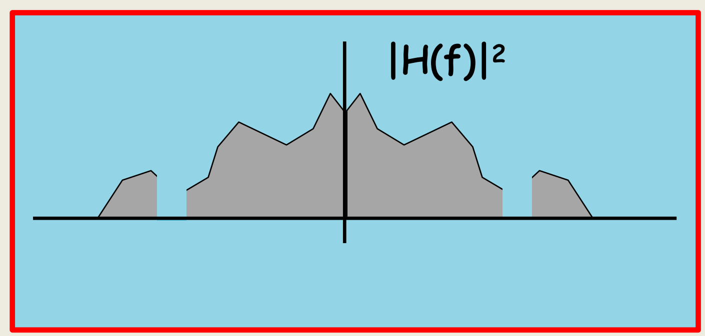
Stupidity to put power here

Constraint on R(f) ? $\int_{-\infty}^{\infty} R(f)\mathrm{d}f = P$

Power spectral density is what matters



Frequency response of channel



Lecture 4: Capacity

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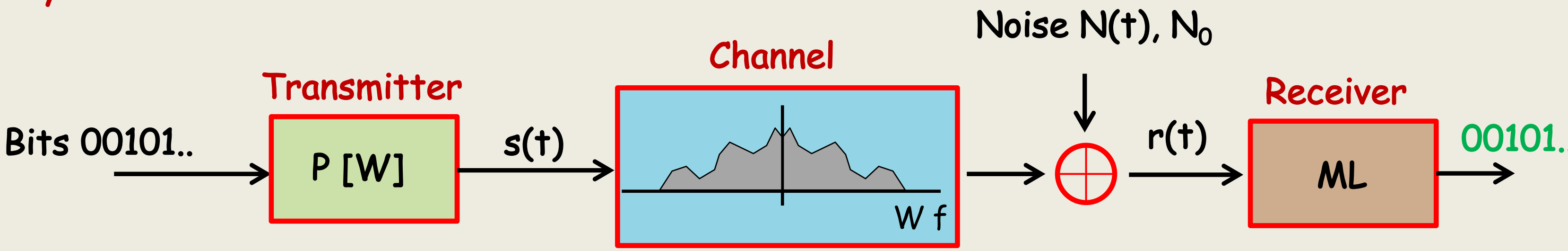
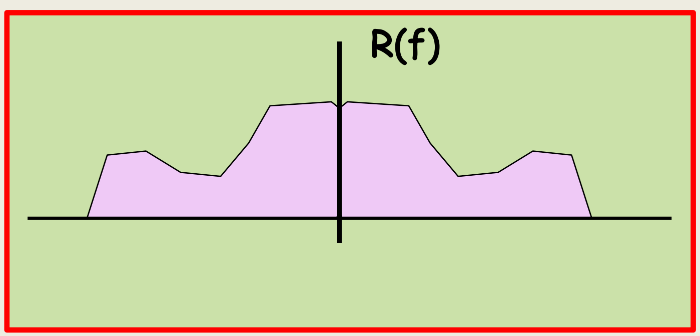


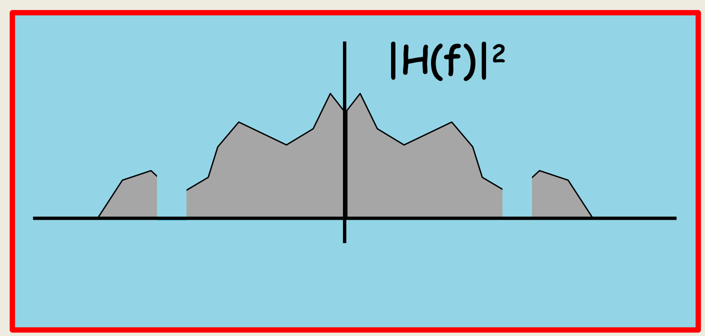
Conclusion: We should optimize the left plot, for the given right plot

Constraint on left plot is $\int_{-\infty}^{\infty} R(f)\mathrm{d}f = P$

Power spectral density is what matters

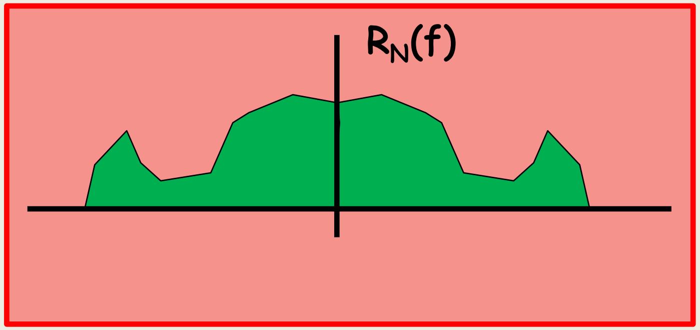


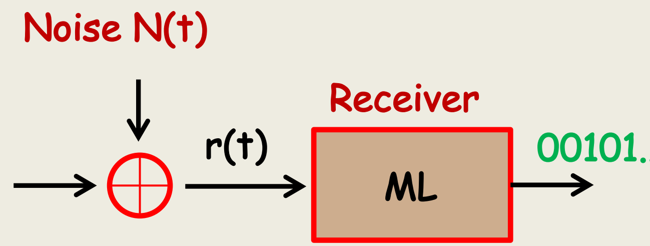
Frequency response of channel



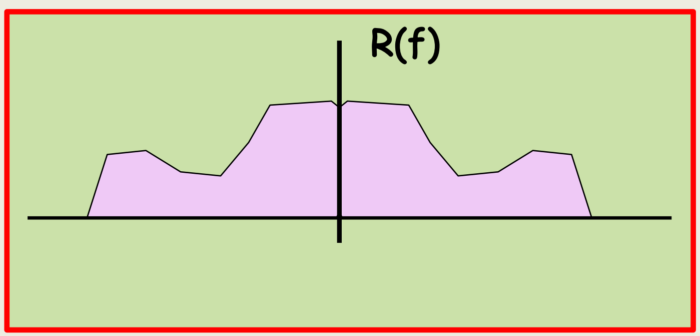
Lecture 4: Capacity

Frequency response of Noise

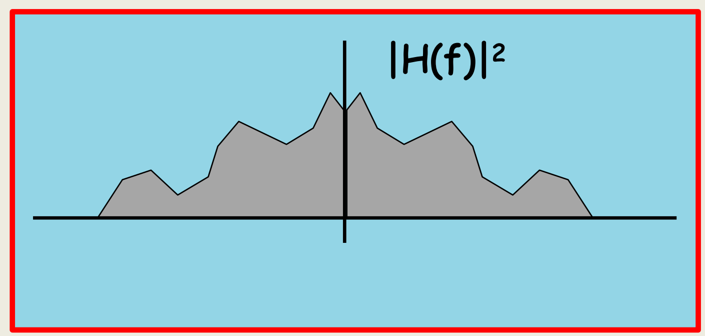




Power spectral density is what matters

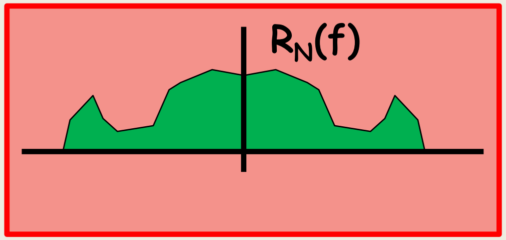


Frequency response of channel

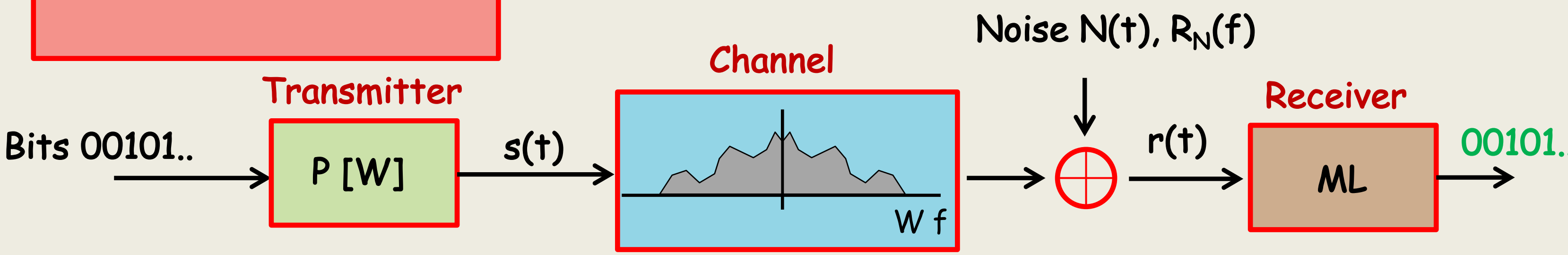


Lecture 4: Capacity

Frequency response of Noise



$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \text{Capacity}(|H(f)|^2, R_N(f), R(f))$$



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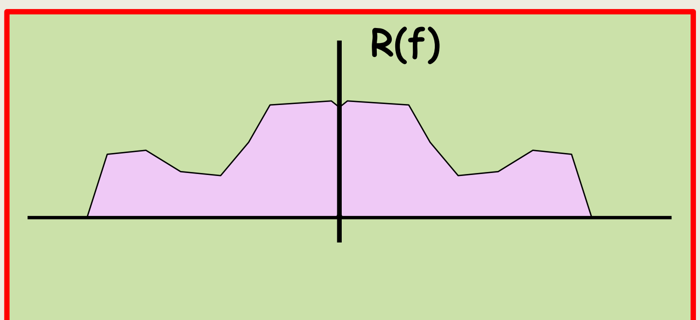
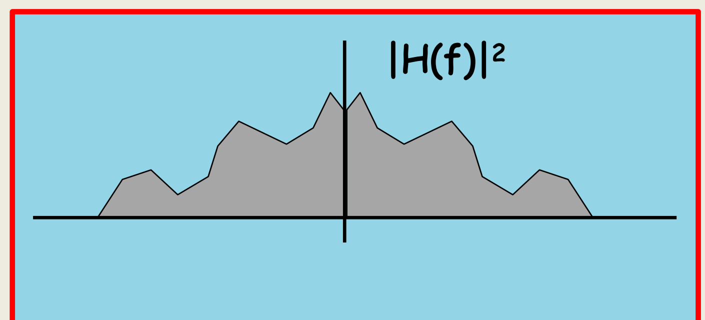
Power spectral density is what matters

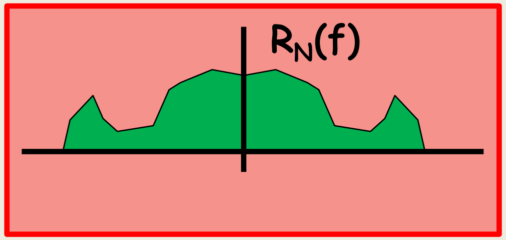


Frequency response of channel



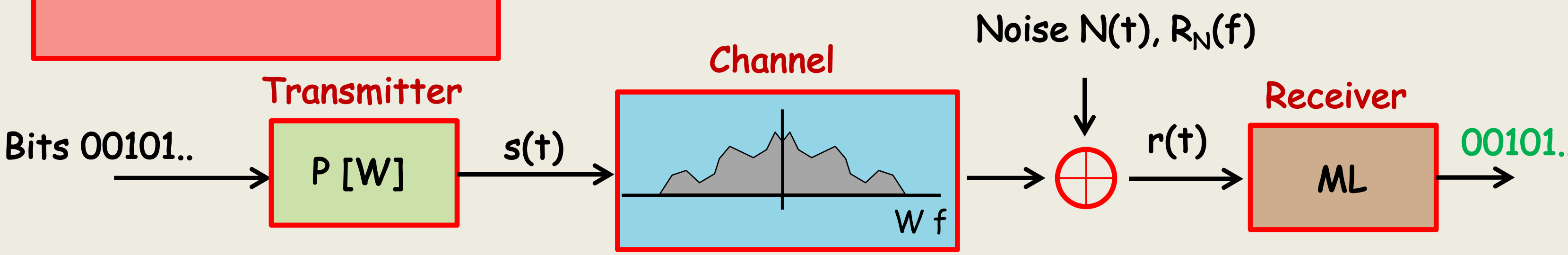
Lecture 4: Capacity

Frequency response of Noise



We need to find this formula

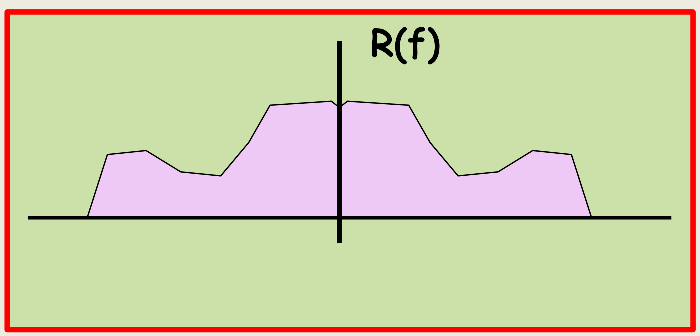
$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \text{Capacity}(|H(f)|^2, R_N(f), R(f))$$



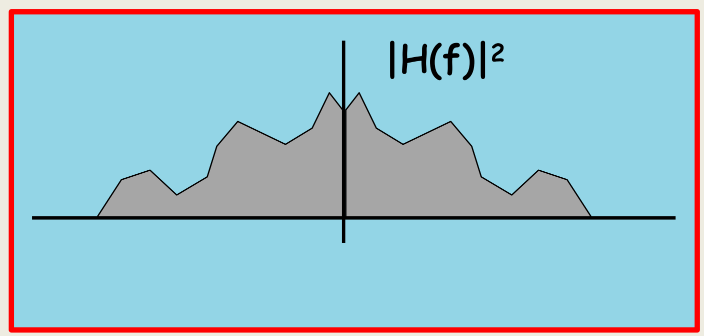
Conclusion: We should optimize the left plot, for the given right plot

Constraint on left plot is $\int_{-\infty}^{\infty} R(f)\mathrm{d}f = P$

Power spectral density is what matters

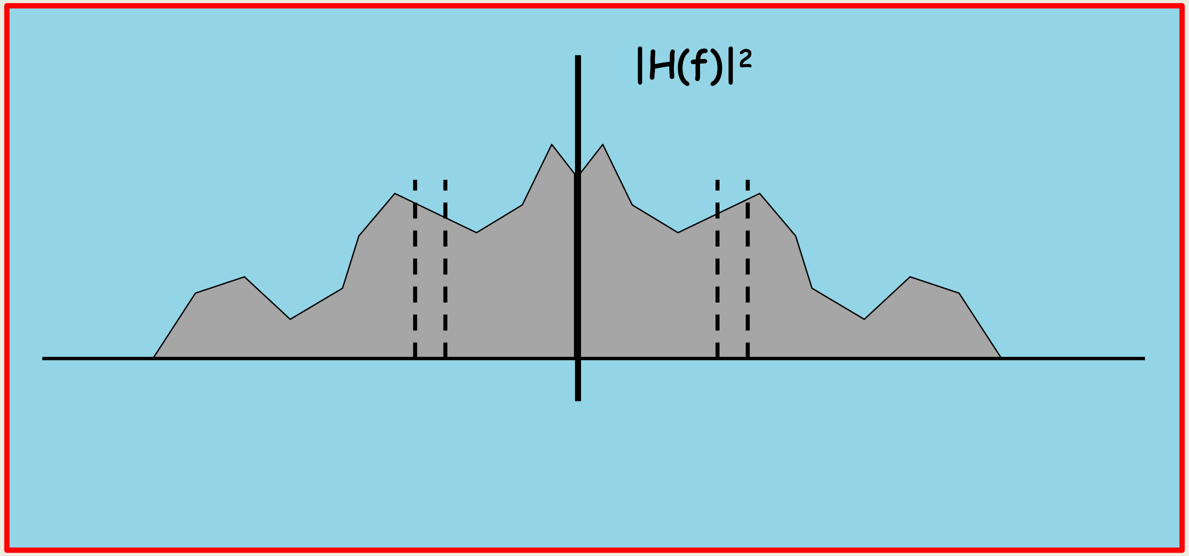


Frequency response of channel



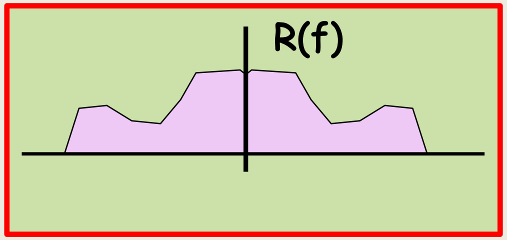
Lecture 4: Capacity

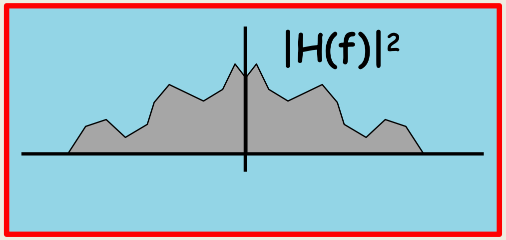
$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

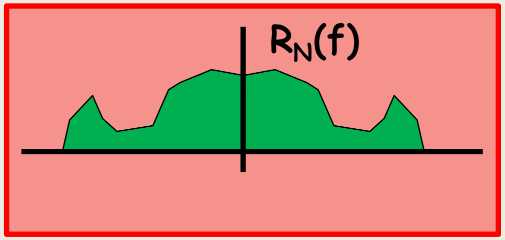


In this small piece
We can use

$$C = W\log_2\left(1+\frac{\mathcal{P}}{N_0 W}\right)$$

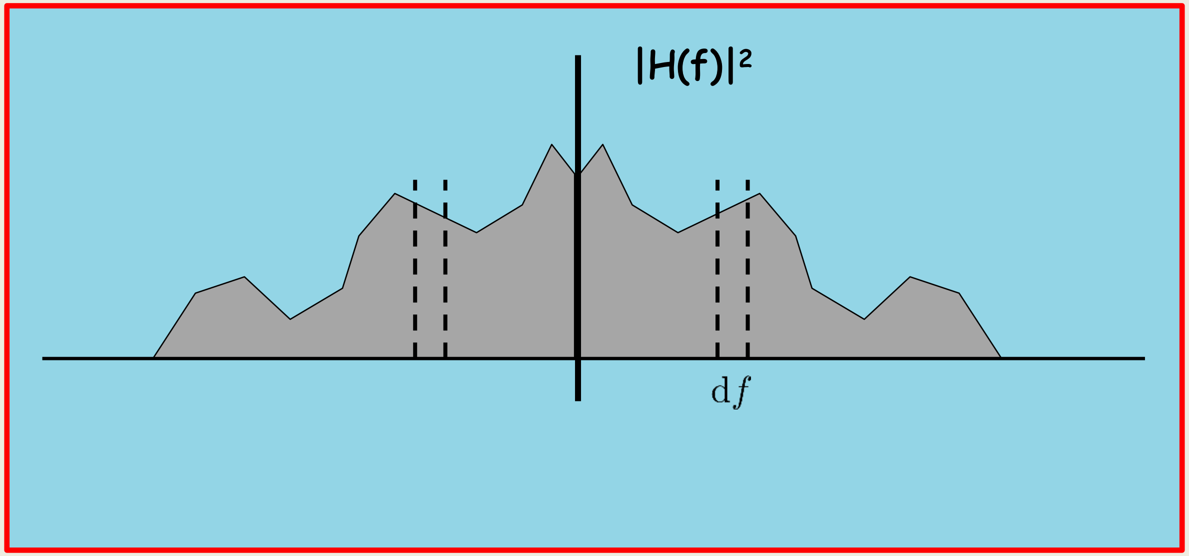






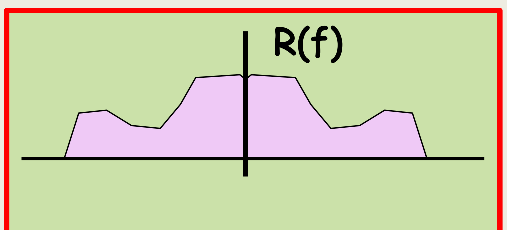
Lecture 4: Capacity

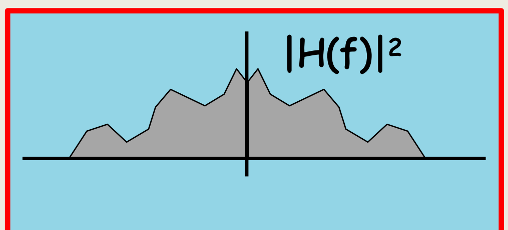
$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \text{Capacity}(|H(f)|^2, R_N(f), R(f))$$

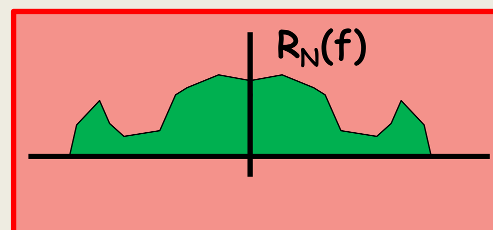


In this small piece
We can use

$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{N_0 \mathrm{d}f}\right)$$

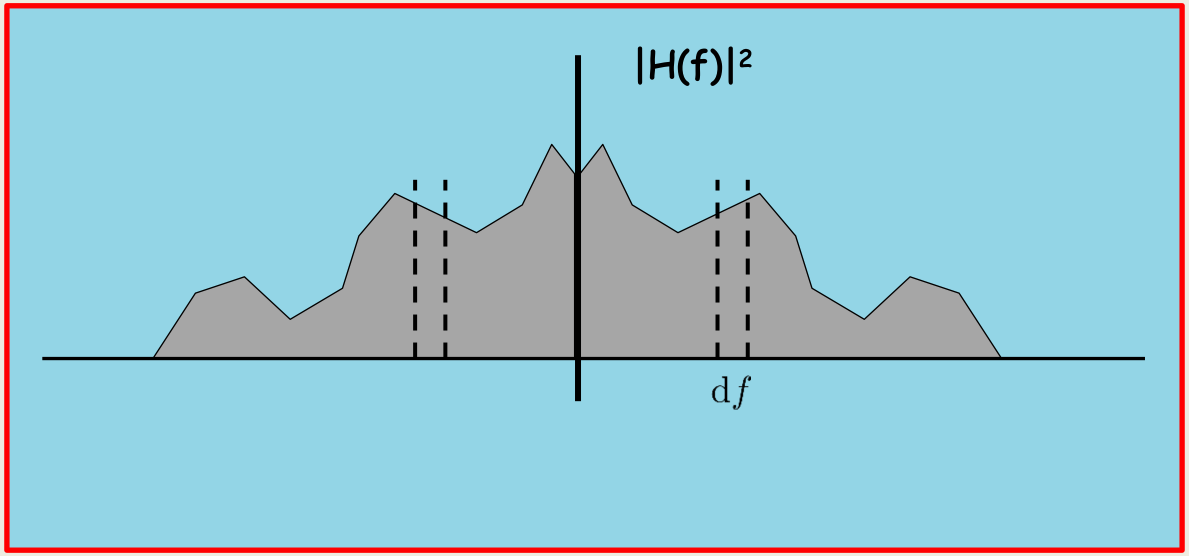






Lecture 4: Capacity

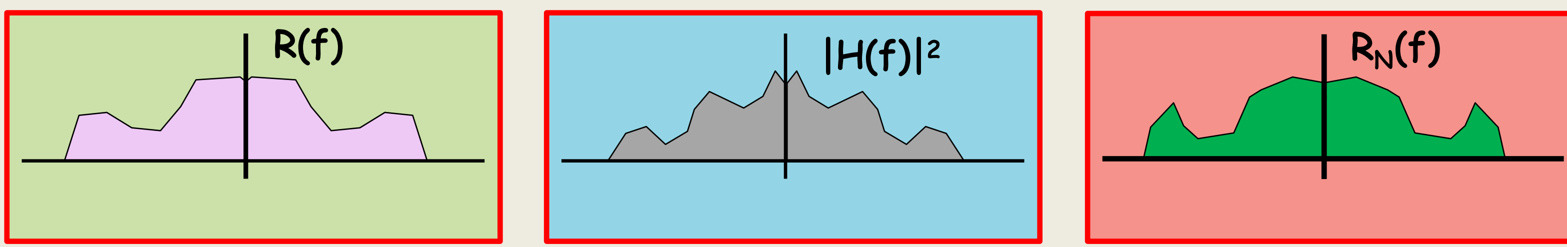
$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$



In this small piece
We can use

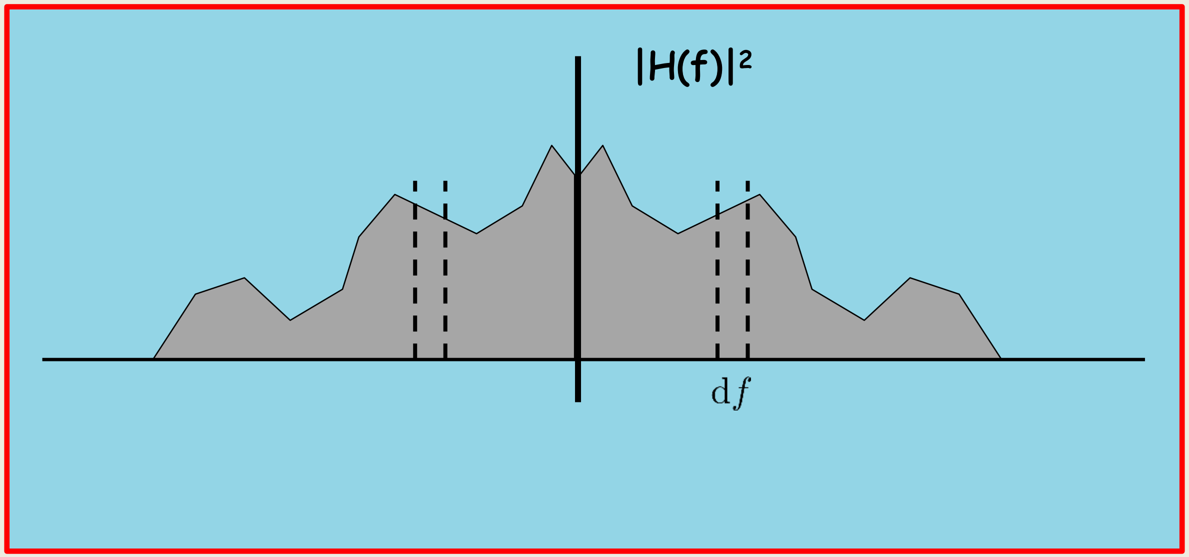
$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{N_0 \mathrm{d}f}\right)$$

For flat noise,
$R_N(f) = N_0/2$



Lecture 4: Capacity

$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

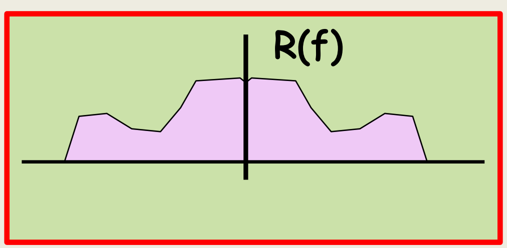


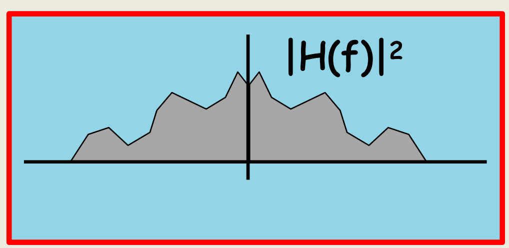
In this small piece
We can use

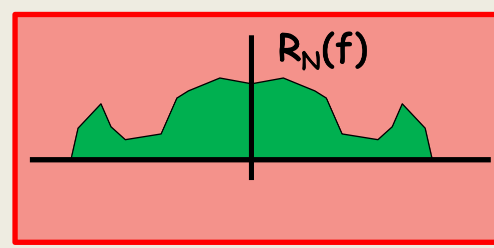
$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{N_0 \mathrm{d}f}\right)$$

$=2R_N(f)$

For flat noise,
$R_N(f) = N_0/2$

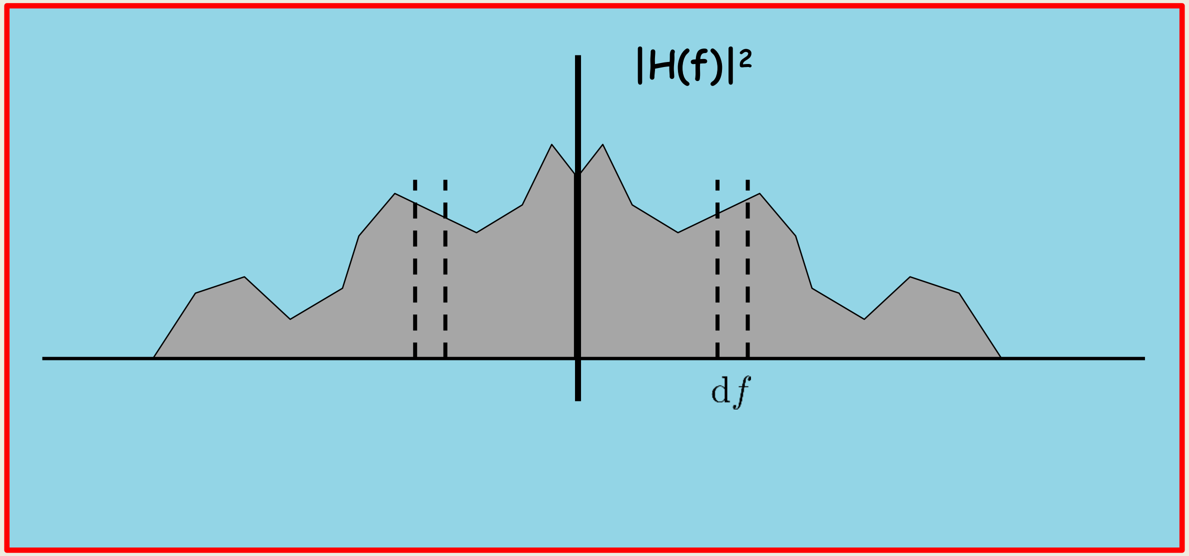






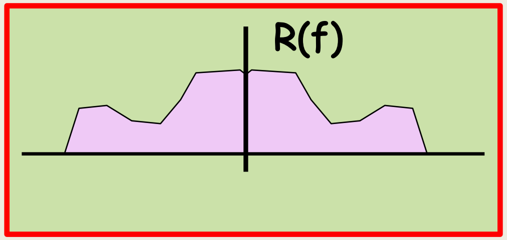
Lecture 4: Capacity

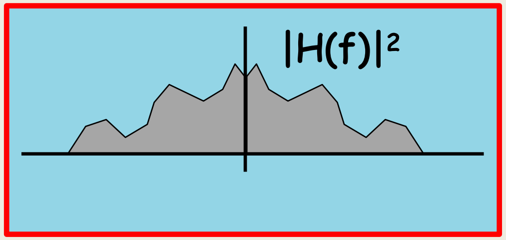
$$C = \max_{R(f): \int R(f)\mathrm{d}f = P} \text{Capacity}(|H(f)|^2, R_N(f), R(f))$$

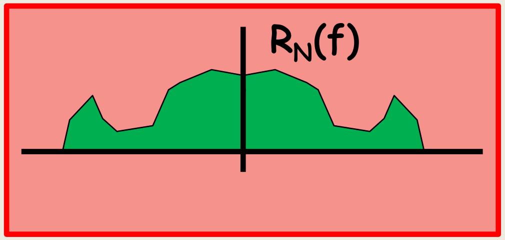


In this small piece
We can use

$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$

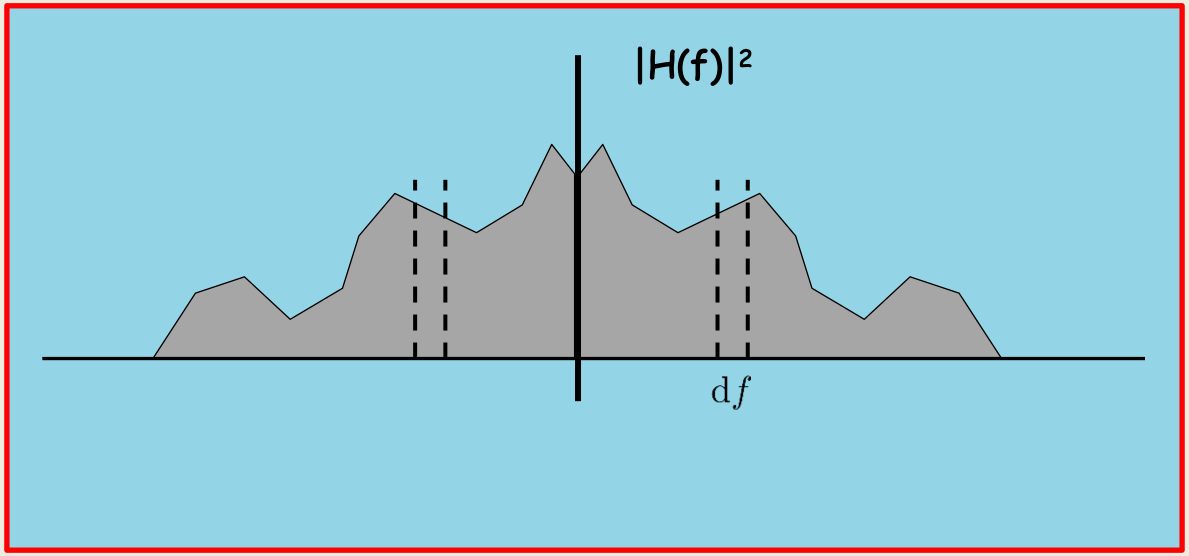






Lecture 4: Capacity

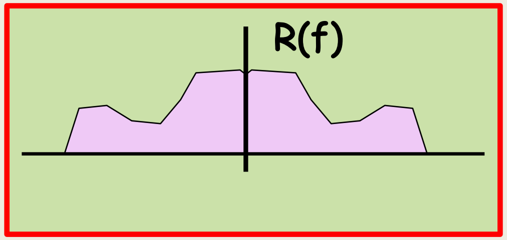
$$C = \max_{R(f): \int R(f)\mathrm{d}f = P} \text{Capacity}(|H(f)|^2, R_N(f), R(f))$$

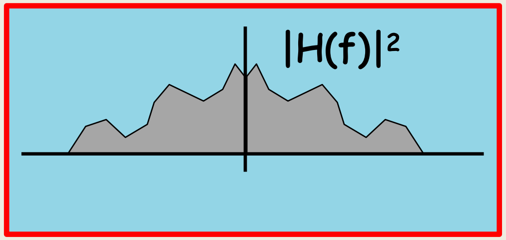


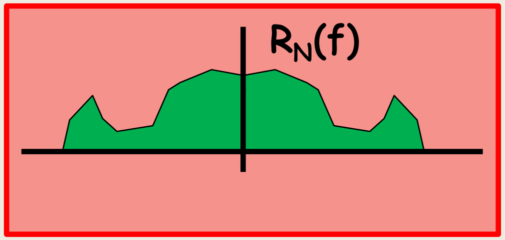
In this small piece
We can use

$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$

How much power do we have?

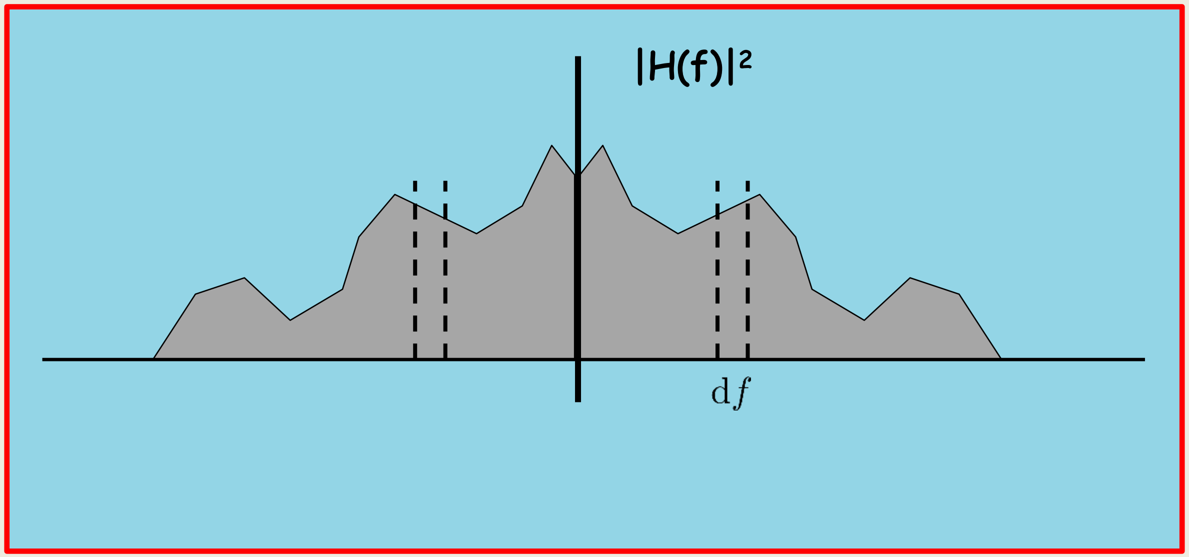






Lecture 4: Capacity

$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

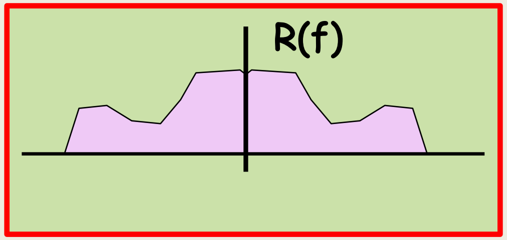


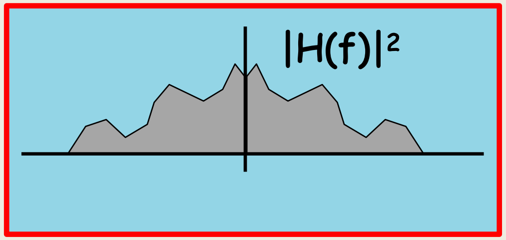
In this small piece
We can use

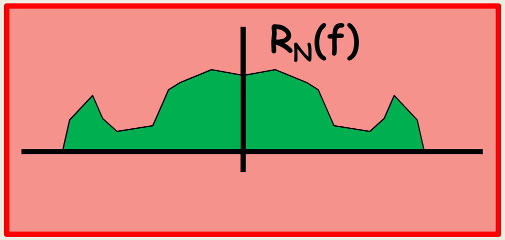
$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$

How much power do we have?

$$2\mathrm{d}f R(f)|H(f)|^2$$

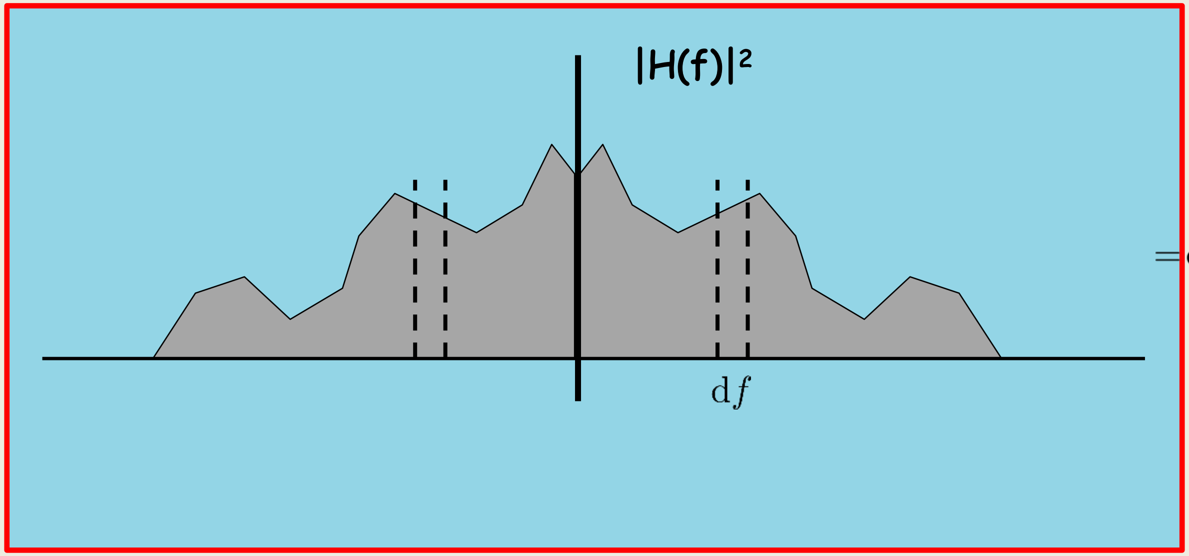






Lecture 4: Capacity

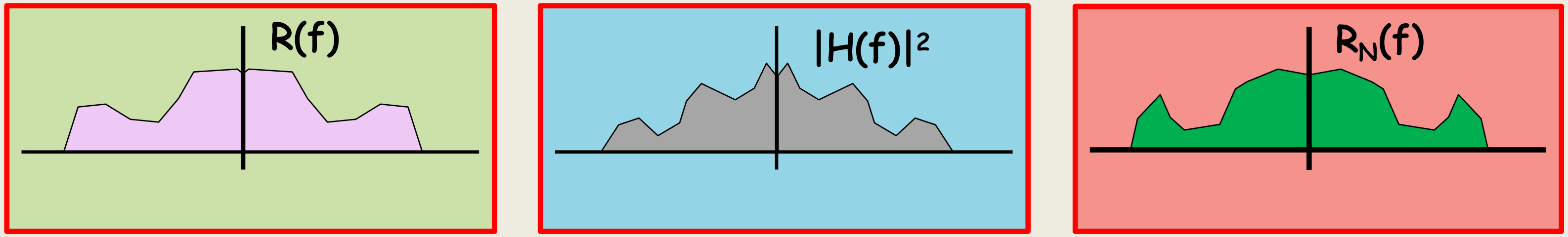
$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \text{Capacity}(|H(f)|^2, R_N(f), R(f))$$



In this small piece
We can use

$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$

$$= \mathrm{d}f \log_2\left(1 + \frac{2R(f)|H(f)|^2\mathrm{d}f}{2R_N(f)\mathrm{d}f}\right)$$



Lecture 4: Capacity

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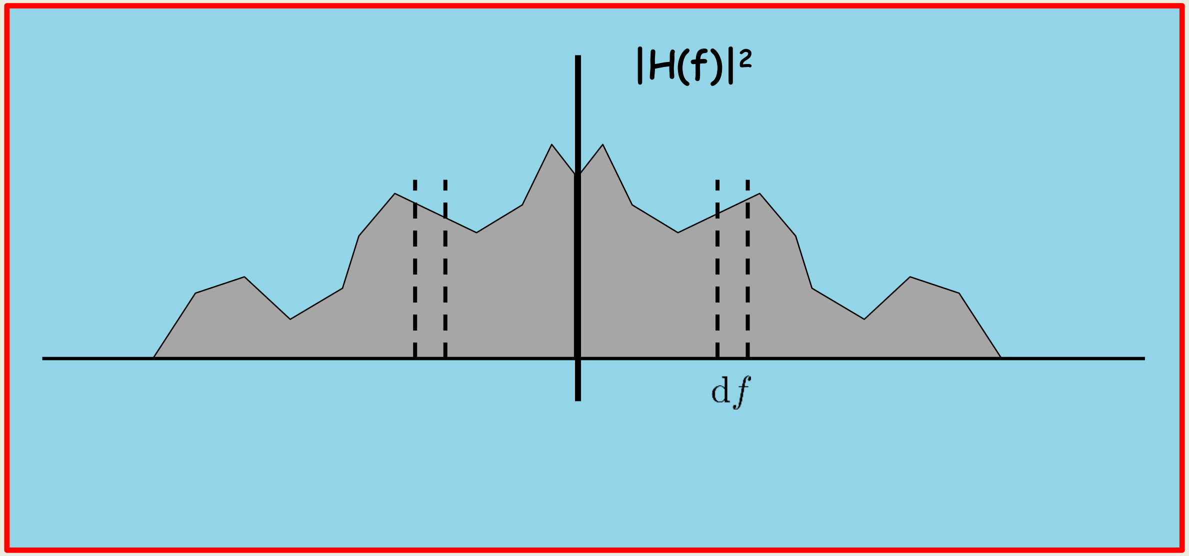
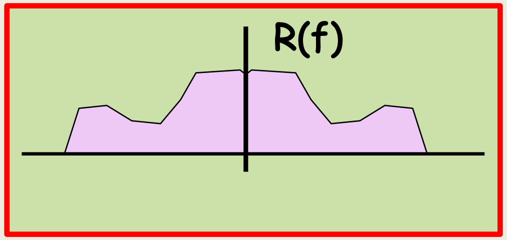
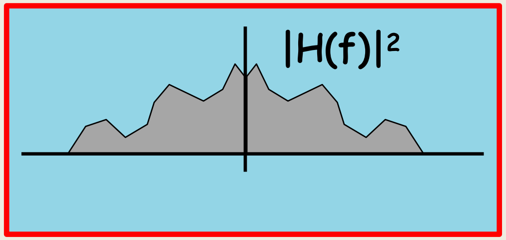


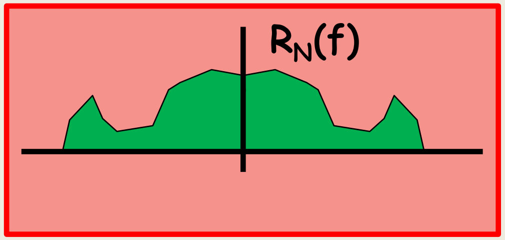
In this small piece
We can use

$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$

$$= \mathrm{d}f \log_2\left(1 + \frac{R(f)|H(f)|^2}{R_N(f)}\right)$$







Lecture 4: Capacity

$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

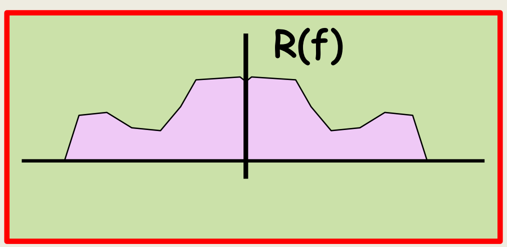
In this small piece
We can use

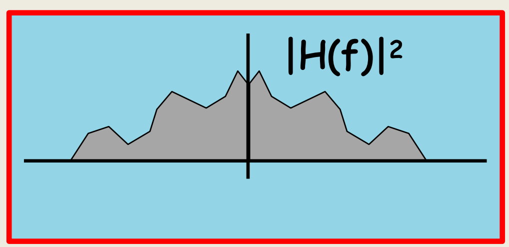
$$C = \mathrm{d}f\,\log_2\left(1+\frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$

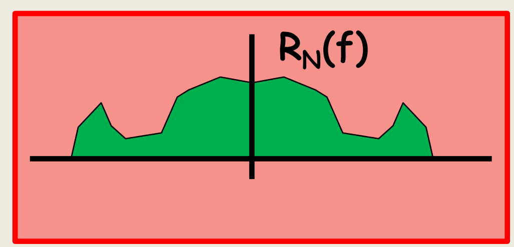
$$= \mathrm{d}f\,\log_2\left(1+\frac{R(f)|H(f)|^2}{R_N(f)}\right)$$

Sum up

$$\mathrm{Capacity}(|H(f)|^2, R_N(f), R(f)) = \int_0^\infty \log_2\left(1+\frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$







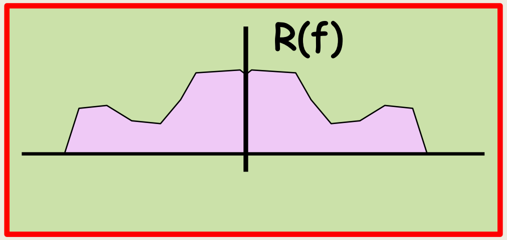
Lecture 4: Capacity

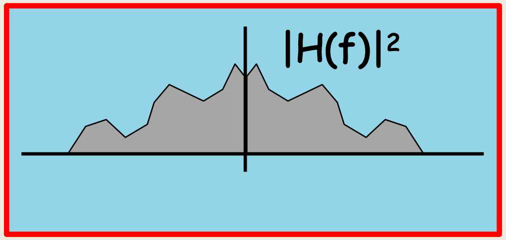
$$C = \max_{R(f): \int R(f)\mathrm{d}f = P} \text{Capacity}(|H(f)|^2, R_N(f), R(f))$$

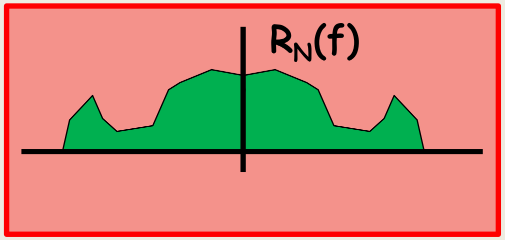
Sum up

$$\text{Capacity}(|H(f)|^2, R_N(f), R(f)) = \int_0^\infty \log_2\left(1 + \frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1 + \frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$

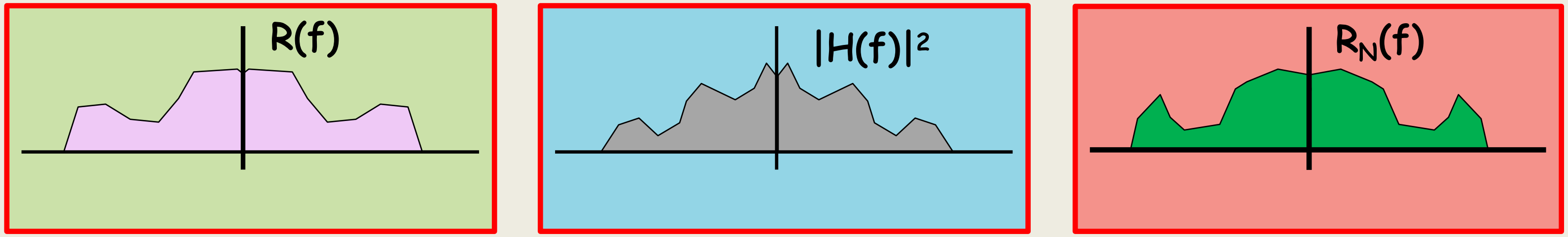






Lecture 4: Capacity

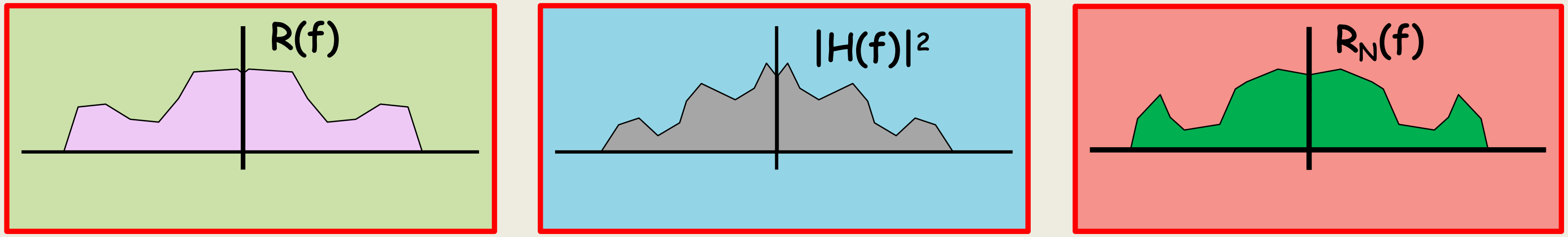
$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1+\frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$



Lecture 4: Capacity

How to solve the below problem? WATERFILLING

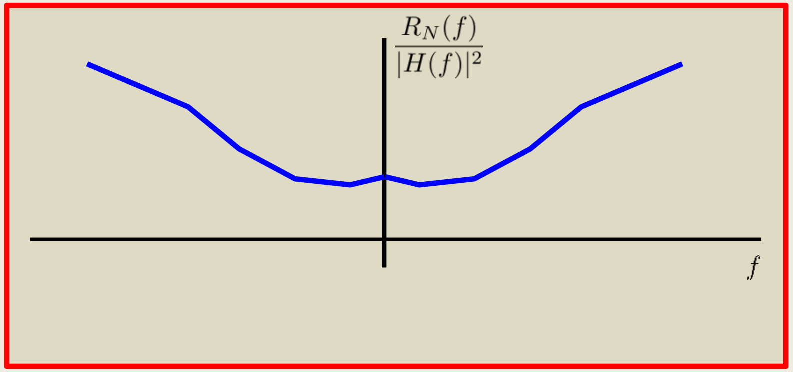
$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1 + \frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$



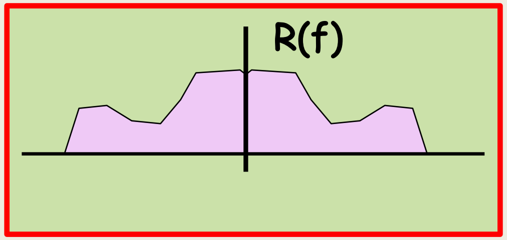
Lecture 4: Capacity

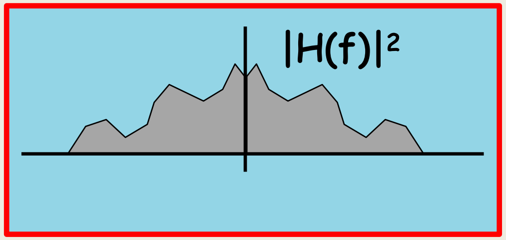
How to solve the below problem? WATERFILLING

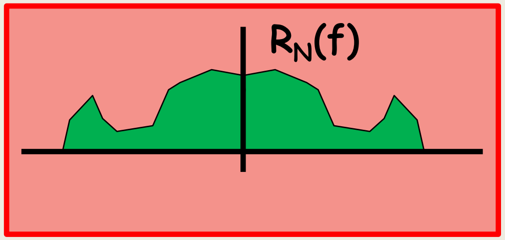
Step 1. Find and plot $\dfrac{R_N(f)}{|H(f)|^2}$



$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1+\frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$



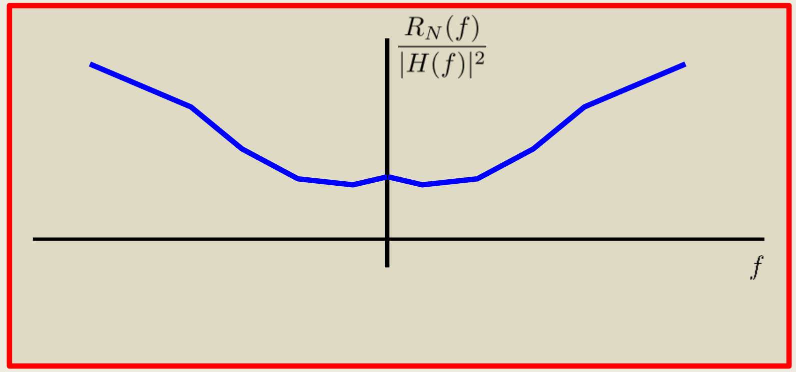




Lecture 4: Capacity

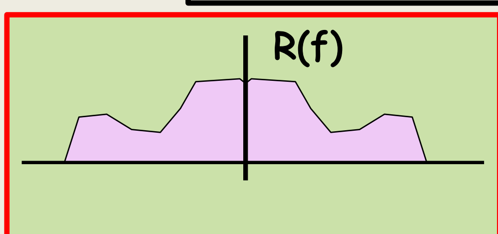
How to solve the below problem? WATERFILLING

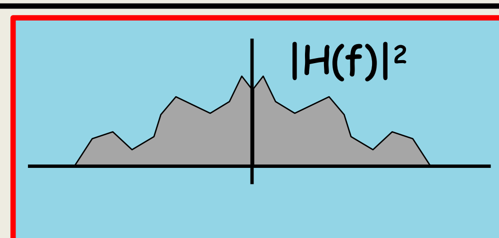
Step 1. Find and plot $\dfrac{R_N(f)}{|H(f)|^2}$

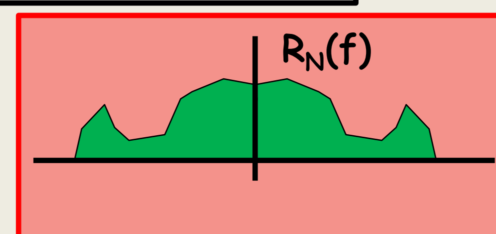


Step 2. Fill a bucket with P units of water

$$C = \max_{R(f): \int R(f) \mathrm{d}f = P} \frac{1}{2} \int_{-\infty}^{\infty} \log_2 \left(1 + \frac{R(f)|H(f)|^2}{R_N(f)} \right) \mathrm{d}f$$



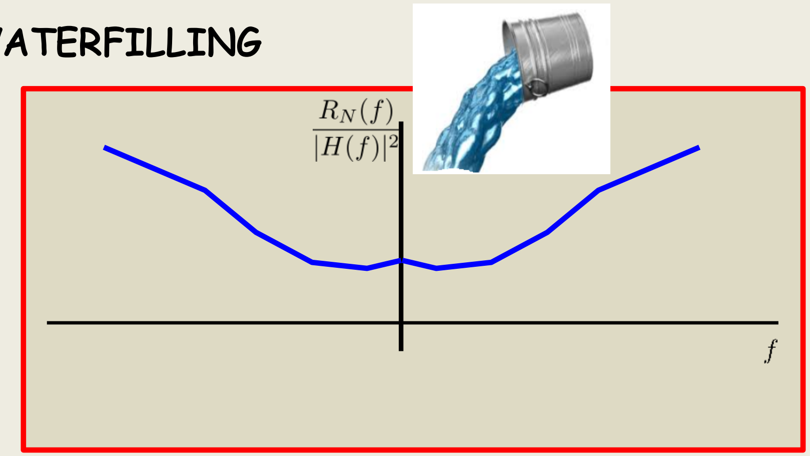




Lecture 4: Capacity

How to solve the below problem? WATERFILLING

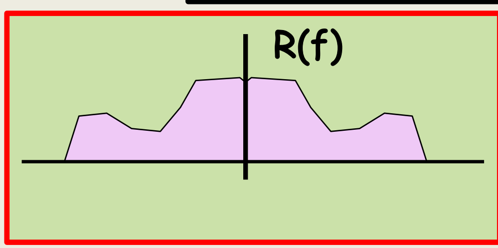
Step 1. Find and plot $\frac{R_N(f)}{|H(f)|^2}$

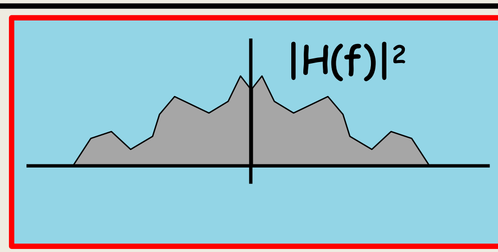


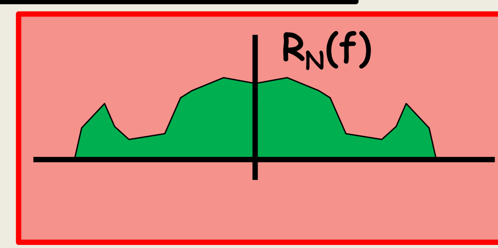
Step 2. Fill a bucket with P units of water

Step 3. Pour it in the shape

$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1+\frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$



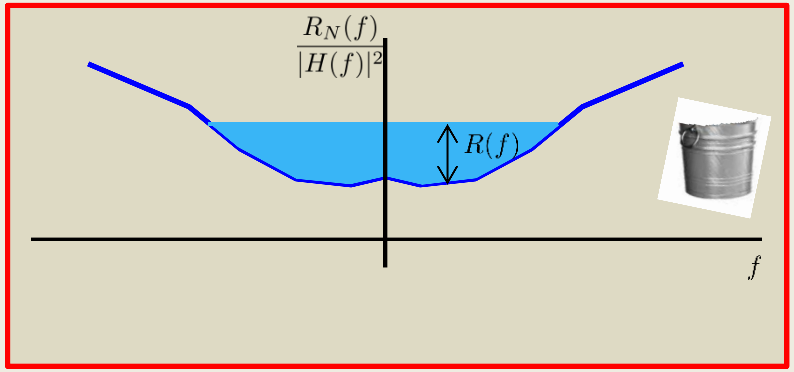




Lecture 4: Capacity

How to solve the below problem? WATERFILLING

Step 1. Find and plot $\dfrac{R_N(f)}{|H(f)|^2}$

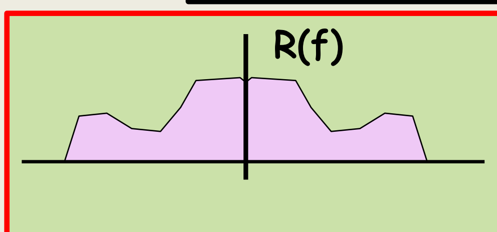


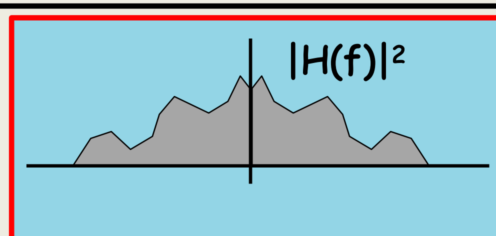
Step 2. Fill a bucket with P units of water

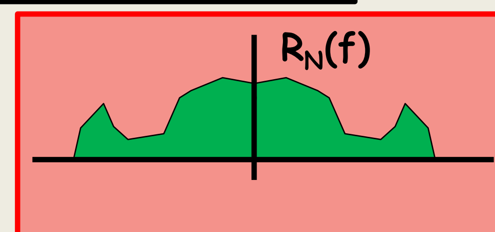
Step 3. Pour it in the shape

Step 4. R(f) is the water-level

$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1+\frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$







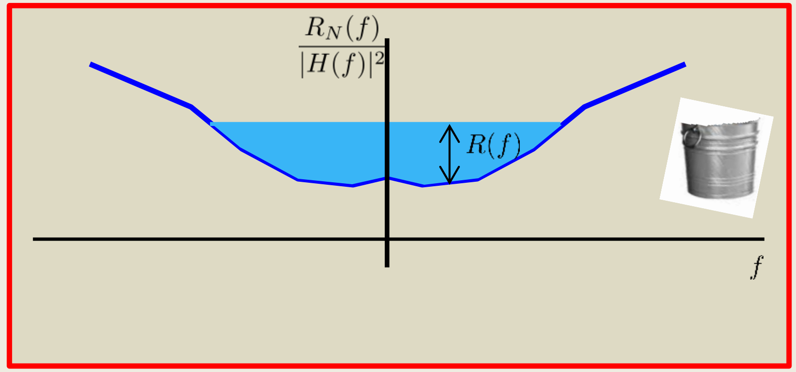
Lecture 4: Capacity

How to solve the below problem? WATERFILLING

Step 1. Find and plot $\dfrac{R_N(f)}{|H(f)|^2}$

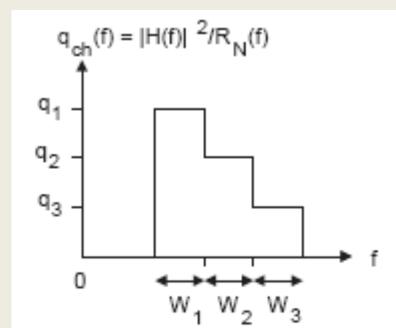
Step 2. Fill a bucket with P units of water

Step 3. Pour it in the shape



Step 4.
R(f) is the water-level

On Exam, |H(f)|^2 would be "nice", such as



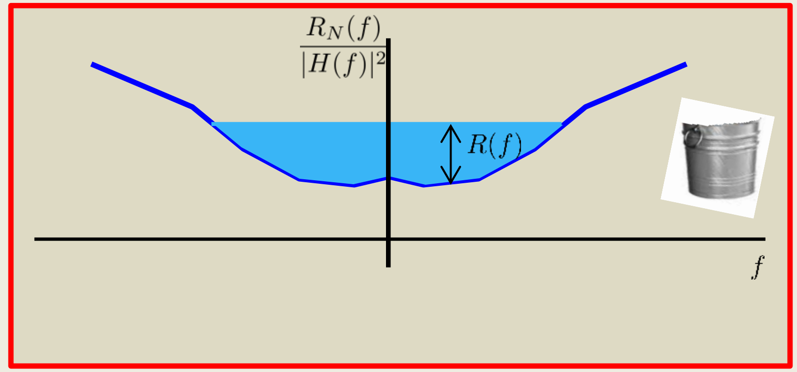
Lecture 4: Capacity

How to solve the below problem? WATERFILLING

Step 1. Find and plot $\frac{R_N(f)}{|H(f)|^2}$

Step 2. Fill a bucket with P units of water

Step 3. Pour it in the shape



Step 4. R(f) is the water-level

Observations:

1. Good channels get more power than bad
2. At very high SNRs, all channels get, roughly, the same power