Final exam in

Digital Communications, Advanced Course (ETTN01)

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on January 12, 2023, 8–13.

- During this final exam, you are allowed to use all tools (book, documents, old exams, notes, etc.)
- Each solution should be written on a separate sheet of paper.

  Please add Your name on each sheet.
- Show the line of reasoning clearly, and use the methods presented in the course.

  If You use results from the textbook, add a reference in Your solution.
- If any data is lacking, make reasonable presumptions.

Good Luck!

Problem 1:

We are given a signal $s(t)$ where $s(t) = 0,\ t > T$ and $t < 0$. We want to represent this signal with another one

$$\hat{s}(t) = \sum_{k=1}^{K} s_k f_k(t)$$

where $f_k(t)$ satisfies

$$\int_0^T f_k(t) f_\ell \mathrm{d}t = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell. \end{cases}$$

a) Determine the expressions for the coefficients $\{s_k\}$ that minimize the expression

$$\int_0^T |s(t) - \hat{s}(t)|^2 \mathrm{d}t.$$

b) Assume that we have chosen a set of values $\{s_k\}$ which results in

$$\int_0^T |\hat{s}(t)|^2 \mathrm{d}t = \frac{1}{2} \int_0^T |s(t)|^2 \mathrm{d}t.$$

What are the possible values of

$$\int_0^T |s(t) - \hat{s}(t)|^2 \mathrm{d}t.$$

c) Suppose

$$\int_0^T s(t) f_k \mathrm{d}t = 0, \forall k.$$

Simplify the expression

$$\int_0^T |s(t) - \hat{s}(t)|^2 \mathrm{d}t.$$

(5+3+2 points)

Problem 2:
Consider a signal set comprising three signals,

$$s_0(t) = 1,\ 0 \le t \le T \qquad s_1(t) = -1,\ 0 \le t \le T \qquad s_2(t) = \begin{cases} \sqrt{3}, & 0 \le t \le 0.5 \\ -\sqrt{3}, & 0.5 \le t \le 1. \end{cases}$$

a) Draw a signal space diagram.

b) Assume that the three signals are equiprobable. Define the following three signals

$$f_m(t) = s_m(t) - c(t),\ 0 \le k \le 2.$$

Briefly discuss the impact of $c(t)$ on (i) error probability, and (ii) transmit energy. Find the optimal function $c(t)$ and discuss in what sense the found $c(t)$ is optimal. Also, quantify the gain compared to the choice $c(t) = 0$

c) Would the calculations in b) change if the three signals are not equiprobable?

(2+4+1 points)

Problem 3:
The signal set in this problem is very similar to that of Problem 2, but $s_2(t)$ is different. Consider a signal set comprising three signals,

$$s_0(t) = 1,\ 0 \le t \le T \qquad s_1(t) = -1,\ 0 \le t \le T \qquad s_2(t) = \begin{cases} 1, & 0 \le t \le 0.5 \\ -1, & 0.5 \le t \le 1. \end{cases}$$

a) Calculate the error probability given that $s_2(t)$ has been transmitted.

b) Very hard! Calculate the error probability given that $s_0(t)$ has been transmitted.

Hint: As decision regions are not rectangularly shaped, you need to separate the decision regions into disjoint parts which you can handle separately. One part is a bit tricky.

(2+1 points)

Problem 4: For each of the following problems, determine if capacity is broken or not (i.e., determine whether the data rate exceeds the channel capacity). Although capacity formulas we have studied are derived under assumption of Gaussian signaling, you may use them also for other constellations, e.g., PAM and QAM.

a) 16-QAM with error correcting code rate $R = 3/4$, $P/N_0 = 10^4$, and bit rate $R_b = 10^4$ bit/s. You may assume that occupied bandwidth is $W = R_s$ where R_s is the symbol rate. The assumed channel in this problem is $r(t) = s(t) + n(t)$ where $s(t)$ is the transmitted signal and $n(t)$ is white Gaussian noise with spectral density $N_0/2$.

b) Transmission over a channel satisfying

$$\frac{2|H(f)|^2}{N_0} = \begin{cases} 1000, & 0 \le f \le 1000 \\ 500, & 1000 \le f \le 2000 \\ 0, & f > 2000. \end{cases}$$

with $P = 4$ [W] and a bitrate $R_b = 2000$ bit/s.

(5+5 points)

Problem 5: a) Consider an OFDM signal with carrier frequency 2 GHz, bandwidth 9 MHz and with cyclic prefix 5 μs. Furthermore, uncoded 16-QAM is used at each subcarrier.

Calculate the bit rate transmitted by this OFDM signal if the sub-carrier frequency separation is:

i) 7.5 kHz. ii) 15 kHz.

b) Comment (no essays) how the following parameters impact spectral efficiency of OFDM

i) Duration of cyclic prefix. ii) subcarrier spacing f_Δ. iii) Number of subcarriers, K. iv) carrier frequency, f_c.

c) For OFDM to work as intended, is $T_{obs} \ll t_{coh}$ a necessary condition. Please motivate. Note: I do not ask you whether it is sufficent to have $T_{obs} < t_{coh}$, I ask if there is any relation between T_{obs} and t_{coh}.

(3+4+3 points)

Problem 6:

Suppose a Rayleigh fading channel where $t_{coh} = 10$ ms and $f_{coh} = 100$ kHz. A QPSK system is applied with a bit rate that can be adjusted. Suppose that $P/N_0 = 10^7$. Make a plot with bit rate on the x-axis and error probability on the y-axis, and show where such a system operates. Fill in a region as large as your knowledge allows you to do.

(10 points)
