1

a/ $\int_0^T |s(t) - \hat{s}(t)|^2 dt = \int_0^T |s(t) - \sum_{k=1}^{K} s_k f_k(t)|^2 dt$

$$= \underbrace{\int s^2(t)dt}_{E} + \int |\sum s_k f_k(t)|^2 dt - 2\int s(t) \sum s_k f_k(t) dt$$

$$= E + \sum_{k=1}^{K} s_k^2 - 2\sum s_k \underbrace{\int s(t) f_k(t)\, dt}_{\tilde{s}_k}$$

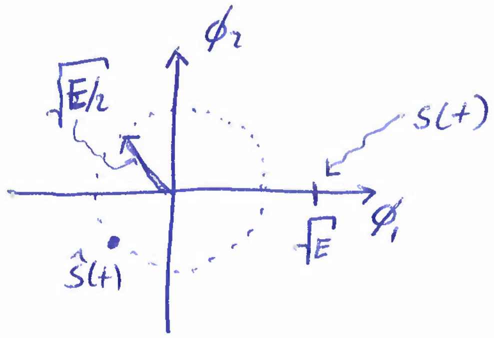
$$= \underbrace{E + \sum_{k=1}^{K} s_k^2 - 2\sum_{k=1}^{K} s_k \tilde{s}_k}$$

$$\frac{\partial}{\partial s_k}[\,\cdot\,] = 2s_k - 2\tilde{s}_k = 0 \qquad \boxed{s_k = \tilde{s}_k = \int_0^T s(t) f_k(t) dt}$$

b/ • Let $s(t) = \sqrt{E}\,\phi_1(t)$ for some $\phi_1(t)$

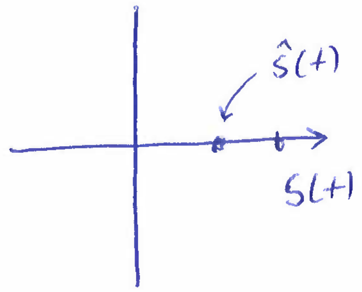
with $\int \phi_1^2(t)\, dt = 1$

• Let $\hat{s}(t) = s_1 \phi_1(t) + s_2 \phi_2(t)$, $s_1^2 + s_2^2 = E/2$

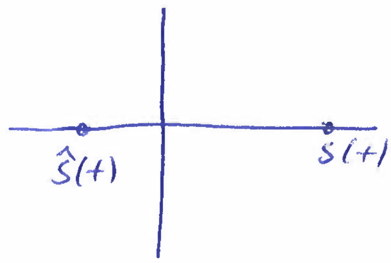


$\hat{s}(t)$ is on circle

$|s(t) - \hat{s}(t)|^2$ minimized



$|s(t) - \hat{s}(t)|^2$ maximized



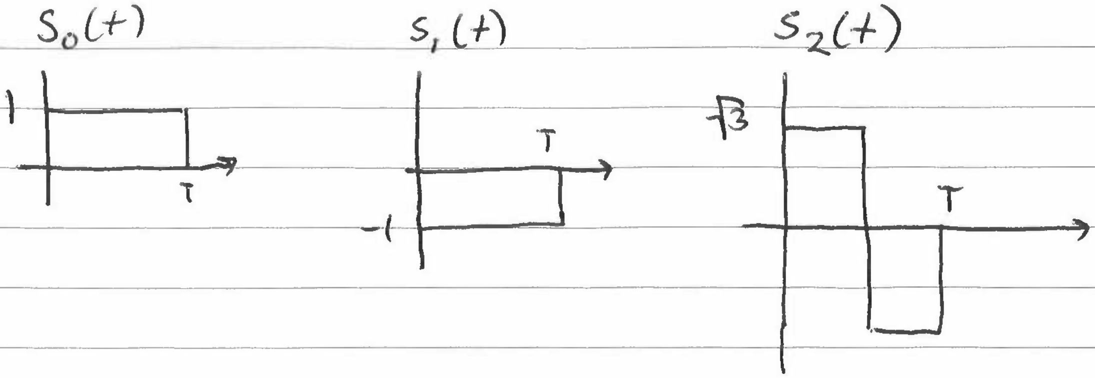
$$|s(t) - \hat{s}(t)|^2 = \dots = \left(\sqrt{E} - \sqrt{E/2}\right)^2$$

$$\approx 0.1E$$

$$\dots \left(\sqrt{E} + \sqrt{\tfrac{E}{2}}\right)^2 \approx 3E$$

c/ $$\int_0^T |s(t) - \sum s_k f_k(t)|^2 dt = \boxed{\int s^2(t)\,dt + \sum_{k=1}^{k} s_k^2}$$

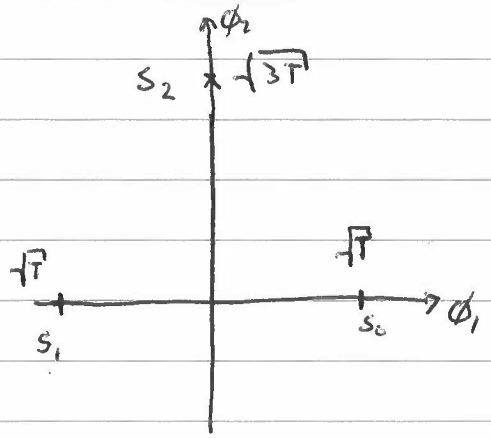
$$- \underbrace{2\int s(t) \sum s_k f_k(t)}_{0}$$



$$E_0 = T = E_1 \qquad E_2 = 3T$$

since $\int s_0(t)\, s_2(t)\, dt = 0 \implies$

$$\phi_1(t) = \frac{s_0(t)}{\sqrt{T}} \qquad \phi_2(t) = \frac{s_2(t)}{\sqrt{3T}}$$



Let $D_{i,j}^2 = \int |s_i(t) - s_j(t)|^2 \, dt$

$$\tilde{D}_{i,j}^2 = \int | F_i(t) - f_j(t)|^2 \, dt$$

$$E_i = \int s_i^2(t) \, dt \qquad \tilde{E}_i = \int F_i^2(t) \, dt$$

It follows that $\tilde{D}_{i,j}^2 = D_{i,j}^2$ where fore

error performance is <u>not</u> affected.

$$\tilde{E}_i = \int |s_i(t) - c(t)|^2 \, dt = \int s_i^2(t) \, dt + \int c^2(t) \, dt - 2 \int s_i(t) c(t) \, dt$$

$$= E_i + \int c^2(t) \, dt - 2 \int s_i(t) c(t) \, dt.$$

transmit Energy <u>is</u> affected.

Reasonable goal: find $c(t)$ that results in least average transmit energy, i.e.,

$\min\limits_{c(t)} \sum_{i=1}^{3} P_i \tilde{E}_i$ where P_i is the probability of message i

Let $c(t) = c_1 \phi_1(t) + c_2 \phi_2(t) + c_3 \phi_3(t)$

$c(t)$ possibly not in same space

$\tilde{E}_1 = T + c_1^2 + c_2^2 + c_3^2 - 2\sqrt{T} c_1$

$\tilde{E}_2 = T + c_1^2 + c_2^2 + c_3^2 + 2\sqrt{T} c_1$

$\tilde{E}_3 = 3T + c_1^2 + c_2^2 + c_3^2 - 2\sqrt{3}\sqrt{T} c_2$

Av. energy $= \frac{1}{3}(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3) = \frac{5T}{3} + c_1^2 + c_2^2 + c_3^2 - 2\sqrt{\frac{T}{3}} c_2$

clearly, mimimal for $c_1 = c_3 = 0$

$\frac{\partial}{\partial c_2} = 2c_2 - 2\sqrt{\frac{T}{3}} = 0 \rightarrow c_2 = \sqrt{\frac{T}{3}}$

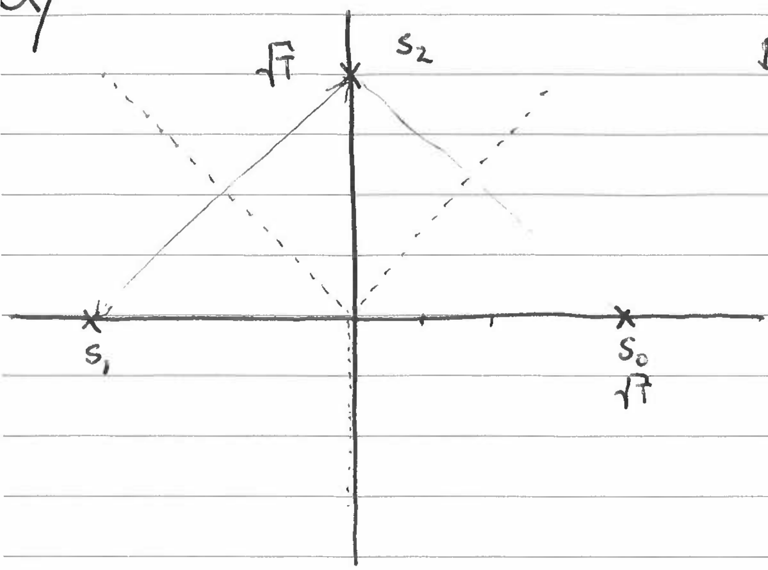
plug back $\rightarrow$ average power $= \frac{5T}{3} + \frac{T}{3} - 2\frac{T}{3} = \underline{\underline{\frac{4T}{3}}}$

average power for $c(t) = 0$: $\frac{5T}{3}$

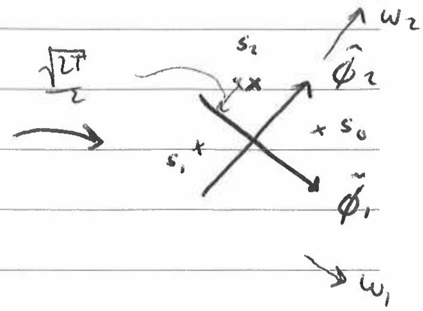
gain (saving) $\frac{4}{5}$

<u>Yes</u> since we used $P_i = \frac{1}{3} \; \forall i$

a/



$D_{2,1}^2 = D_{2,0}^2 = 2T$



w_1, w_2 noise in new basis

$P(\text{error}|s_2) = 1 - P(\text{correct}|s_2) = 1 - P\left(w_2 > -\sqrt{\frac{T}{2}} \text{ and } w_1 < \sqrt{\frac{T}{2}}\right)$

$= 1 - P\left(w_2 > -\sqrt{\frac{T}{2}}\right) P\left(w_1 < \sqrt{\frac{T}{2}}\right)$

$= 1 - \left(1 - P\left(w_2 > \sqrt{\frac{T}{2}}\right)\right)\left(1 - P\left(w_1 > \sqrt{\frac{T}{2}}\right)\right)$

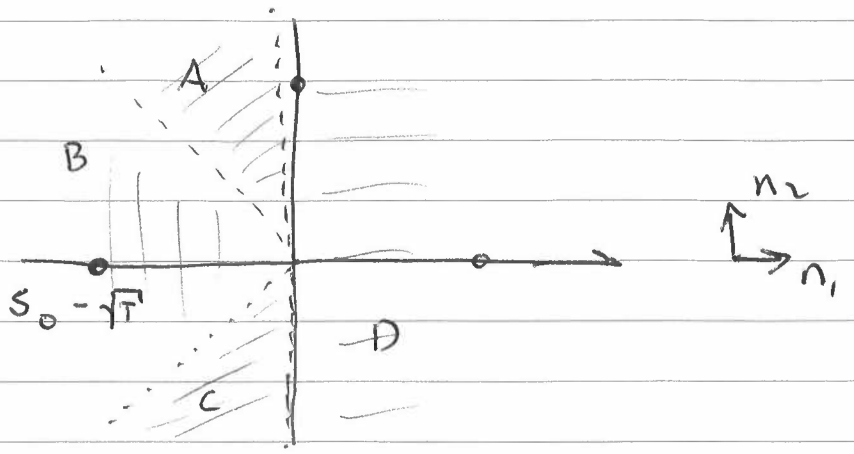
$P\left(w_2 > \sqrt{\frac{T}{2}}\right) = P\left(w_1 > \sqrt{\frac{T}{2}}\right) = P\left(x > \sqrt{\frac{T}{2}} \Big/ \sqrt{\frac{N_0}{2}}\right)$

$x \sim N(0,1)$

$= Q\left(\sqrt{\frac{T}{N_0}}\right)$

$\Rightarrow P(\text{error}|s_2) = 1 - (1 - Q(\cdot))(1 - Q(\cdot))$

$= 1 - [1 - 2Q(\cdot) + Q^2(\cdot)] = \boxed{2Q\left(\sqrt{\frac{T}{N_0}}\right) - Q^2\left(\sqrt{\frac{T}{N_0}}\right)}$



Consider the four regions A – D

Decision region for s_0 is $B \cup C$

union

received signal

$\Pr(r \in B) = P(\text{correct} \mid s_2)$

symmetry

$\Pr(r \in A \cup C) = \Pr(r \in A \cup B \cup C) - \Pr(r \in B)$

$= \Pr(n_1 < \sqrt{T}) - P(\text{correct} \mid s_2)$

$P(r \in C) = \frac{1}{2} P(r \in A \cup C)$

symmetry

$\Pr(r \in B \cup C) = \Pr(r \in B) + \Pr(r \in C)$

$$= \underbrace{\frac{1}{2} Pr(\text{correct} | s_2) + \frac{1}{2} Pr(n_1 < \sqrt{T})}_{P(\text{correct} | s_0)}$$

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y/ $C = R_s \log_2\left(1 + \frac{P}{N_0 R_s}\right)$

$R_s = R_b/3$ (3 bit per symbol due to the 3/4 code)

$= \frac{10^4}{3}$

$\rightarrow$

$$C = \frac{10^4}{3} \log_2\left(1 + \frac{10^4}{10^4/3}\right) = \frac{10^4}{3} \log_2(4) = \frac{2}{3} 10^4 < R_b$$

<u>breaks capacity</u>

can be done via Water filling, but also using "$W\log_2(1+\frac{P}{N_0W})$"

Let P be power in $[-1000, 1000]$

$\rightarrow$ $4-P$ power in $\pm[1000, 2000]$

Rx power $\boxed{\begin{matrix} P|H(f)|^2 = P \cdot \frac{1000 N_0}{2} \\ (4-P)|H(f)|^2 \quad (4-P)\frac{500 N_0}{2} \end{matrix}}$

$$C = 1000\log_2\left(1+\frac{P\cdot 1000\frac{N_0}{2}}{1000N_0}\right) + 1000\log_2\left(1+\frac{(4-P)\frac{500N_0}{2}}{1000N_0}\right)$$

$$= 1000\left[\log_2\left(1+\frac{P}{2}\right)+\log_2\left(1+\frac{4-P}{4}\right)\right]$$

$$= 1000\log_2\Big(\underbrace{\left(1+\tfrac{P}{2}\right)\left(2-\tfrac{P}{4}\right)}_{f(P)}\Big)$$

$$\frac{\partial f(P)}{\partial P} = 0 \rightarrow P=3$$

$$C|_{P=3} = 1000\left(\log_2\left(\tfrac{5}{2}\right)+\log_2\left(\tfrac{5}{4}\right)\right) < 2002$$

breaks

a/ $T_{obs} = \frac{1}{f_\Delta}$ $T_s = \frac{1}{f_\Delta} + 5\mu s$

$K = \frac{9 \cdot 10^6}{f_\Delta}$

16-QAM

$R_b = \frac{K \cdot 4}{T_s} = \frac{4 \cdot 9 \cdot 10^6}{f_\Delta \left[\frac{1}{f_\Delta} + 5 \cdot 10^{-6}\right]}$

$= \frac{36 \cdot 10^6}{1 + 5 f_\Delta \cdot 10^{-6}}$ (i) $\approx 3.47 \cdot 10^7$ bit/s

(ii) $\approx 3.35 \cdot 10^7$ bits

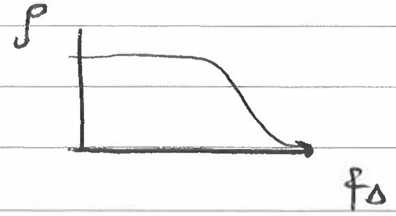
b/

$R_b = \frac{K \log_2(M)}{T_{obs} + T_{cp}}$ $W = f_\Delta K$

$\rho = \frac{R_b}{W} = \frac{\cancel{K} \log_2(M)}{\cancel{K} f_\Delta \left(\frac{1}{f_\Delta} + T_{cp}\right)} = \frac{\log_2(M)}{1 + f_\Delta T_{cp}}$

i) $T_{cp} \nearrow \rightarrow \rho \searrow$ (iii) No dep.

ii) (iv) No dep



c/ Yes. The channel must be constant over one OFDM symbol (T_s)

At high SNR, $BER \sim \dfrac{1}{2d_{min}^2 \frac{E_b}{N_0}} = \dfrac{1}{4\frac{E_b}{N_0}}$.

But this holds only for R_s where we have slow fading and frequency-non-selectivity.

$= 100 << R_s << f_{coh} = 100000$

$E_b R_b \rightarrow \dfrac{E_b}{N_0} = \dfrac{P}{R_b N_0} = \dfrac{10^7}{R_b} >> 1$ for all R_b

$2 = \frac{1}{4} R_b \frac{1}{10^7}$