Lecture 4: Capacity

Project info

1. Each project group consists of two students.
2. Each project group should as soon as possible, send an email to fredrik.rusek@eit.lth.se and containing Name and email address to each project member. **NOTE: email should have subject: ETTN01PROJECT**
3. The project group should contact Fredrik Rusek to decide about project and articles!
4. Each group should write a project report.
5. The structure of the project report should follow journal articles published by IEEE. However, two columns are not needed.
6. The project report should be written in English *with your own words, tables and figures*, and contain 4-5 pages.
7. The report should be clearly written, and written to the other students in this course!
8. Observe copyright rules: ”copy & paste” is in general strictly forbidden!

Lecture 4: Capacity

Project info

9. The project report should be sent in .pdf format to Fredrik before Thursday 12 December, 17.00
10. Oral presentations in the week starting with Monday December 16
11. Each group should have relevant comments and questions on the project report and on the oral presentation of another group. NOTE! The project presentation should be clear and aimed to the other students in this course! After the oral presentation the project report and the presentation will be discussed (5 min).
12. Final report should be sent to Fredrik at latest January 10, 2020.

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

Power efficiency

We know from before (e.g., union bound) that $P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

Now, divide both sides with the bandwidth W $\frac{R_b}{W} \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

We have seen this before, it is defined as…

Now, divide both sides with the bandwidth W $\frac{R_b}{W} \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

We have seen this before, it is defined as bandwidth efficiency

Now, divide both sides with the bandwidth W $\rho = \frac{R_b}{W} \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_s \le cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \ge \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \ge \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \le \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

We have seen this before, it is defined as bandwidth efficiency

Now, divide both sides with the bandwidth W $\rho \le \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$ Power efficiency

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_s \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

We have seen this before, it is defined as bandwidth efficiency

Now, divide both sides with the bandwidth W $\rho \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$ Performance req

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

Bandwidth and power efficiencies are linked

Now, divide both sides with the bandwidth W $\rho \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

Unit ?

Now, divide both sides with the bandwidth W $\rho \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_\mathrm{s} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

Power

Now, divide both sides with the bandwidth W $\rho \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$

W

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_s \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

Bandwidth

Now, divide both sides with the bandwidth W $\rho \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$

$\frac{W}{\text{Hz}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_s \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

Spectral density

Now, divide both sides with the bandwidth W $\rho \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$

$\frac{\text{W}}{\text{? Hz}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_s \leq cQ\left(\sqrt{d^2_{\min}\frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d^2_{\min}}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d^2_{\min}}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0}\frac{d^2_{\min}}{\mathcal{X}}$

Spectral density

Now, divide both sides with the bandwidth W $\rho \leq \frac{\mathcal{P}}{N_0 W}\frac{d^2_{\min}}{\mathcal{X}}$

$\frac{\text{W}}{\text{W/Hz Hz}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_\mathrm{s} \leq cQ\left(\sqrt{d^2_\mathrm{min}\frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d^2_\mathrm{min}}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d^2_\mathrm{min}}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0}\frac{d^2_\mathrm{min}}{\mathcal{X}}$

Has no unit (dimensionless)

Now, divide both sides with the bandwidth W $\rho \leq \frac{\mathcal{P}}{N_0 W}\frac{d^2_\mathrm{min}}{\mathcal{X}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_\mathrm{s} \leq cQ\left(\sqrt{d^2_{\min}\frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d^2_{\min}}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d^2_{\min}}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0}\frac{d^2_{\min}}{\mathcal{X}}$

Dito

Now, divide both sides with the bandwidth W $\rho \leq \frac{\mathcal{P}}{N_0 W}\frac{d^2_{\min}}{\mathcal{X}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_\mathrm{s} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

Dito

Now, divide both sides with the bandwidth W $\rho \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$

Power efficiency

We know from before (e.g., union bound) that $P_{\mathrm{s}} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

Received signal-to-noise-power-ratio

Now, divide both sides with the bandwidth W $\rho \leq \frac{\mathcal{P}}{N_0 W} \frac{d_{\min}^2}{\mathcal{X}}$

Lecture 4: Capacity

Power efficiency

We know from before (e.g., union bound) that $P_\mathrm{s} \leq cQ\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}}\right)$

To meet a specific error probability target, this implies $\frac{E_b}{N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$

We also know that the transmit power satisfies $\mathcal{P} = E_b R_b$

Thus, $\frac{E_b}{N_0} = \frac{\mathcal{P}}{R_b N_0} \geq \frac{\mathcal{X}}{d_{\min}^2}$ or, equivalently, $R_b \leq \frac{\mathcal{P}}{N_0} \frac{d_{\min}^2}{\mathcal{X}}$

Definition

Now, divide both sides with the bandwidth W $\boxed{\rho \leq \mathcal{SNR}_r \frac{d_{\min}^2}{\mathcal{X}}}$

Lecture 4: Capacity

Power efficiency

"BW efficiency" = "Signal-to-noise-power-ratio" x "Power efficiency"

$$\boxed{\rho \leq \mathcal{SNR}_r \frac{d_{\min}^2}{\mathcal{X}}}$$

Lecture 4: Capacity

Shannon Capacity

Before going on, we go through what the term capacity means

Given a scalar channel of form $y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$

We know that the capacity is $C = \log_2 \left(1 + \frac{A}{N_0}\right)$

But what does this mean?

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Build a codebook of all information sequences possible to send of length K

$$\underbrace{\begin{matrix} 000000 \; \ldots\ldots \; 00 \\ 000000 \; \ldots\ldots \; 01 \\ 000000 \; \ldots\ldots \; 10 \\ \vdots \\ 1111111 \; \ldots\ldots \; 10 \\ 1111111 \; \ldots\ldots \; 11 \end{matrix}}_{K}$$

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Build a codebook of all information sequences possible to send of length K

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Sending K bits of information means:

pick one of the rows, and tell the receiver which row it is

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \ n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Build a codebook of codewords to send for each information word, length N

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00

$\boxed{000000 \text{ ..... } 01}$

000000 10

If this is my data

1111111 10

1111111 11

$\underbrace{\hphantom{1111111 \text{ ..... } 11}}_{K}$

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$

$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

$\underbrace{\hphantom{x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4} \text{ ..... } x_{2^K(N-1)}x_{2^KN}}}_{N}$

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

If this is my data

1111111 10
1111111 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

I send this one

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Shannon Capacity

As x over this channel used N times

$$y = \sqrt{A}x + n, \quad n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

If this is my data

1111111 10
1111111 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14} \ldots x_{1(N-1)}x_{1N}$

$x_{21}x_{22}x_{23}x_{24} \ldots x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4} \ldots x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Clearly, bit rate is K/N bits/channel use

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \cdots\cdots y_{(N-1)} y_N$

$$y = \sqrt{A}x + n, \; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

$\underbrace{\begin{matrix} 000000 \;\cdots\cdots\; 00 \\ 000000 \;\cdots\cdots\; 01 \\ 000000 \;\cdots\cdots\; 10 \\ \vdots \\ 1111111 \;\cdots\cdots\; 10 \\ 1111111 \;\cdots\cdots\; 11 \end{matrix}}_{K}$

Codebook

$x_{11}x_{12}x_{13}x_{14} \cdots\cdots x_{1(N-1)}x_{1N}$

$x_{21}x_{22}x_{23}x_{24} \cdots\cdots x_{2(N-1)}x_{2N}$

$\vdots$

$\underbrace{x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4} \cdots\cdots x_{2^K(N-1)}x_{2^KN}}_{N}$

Lecture 4: Capacity

$$y = \sqrt{A}x + n,\ n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1y_2y_3y_4$ ..... $y_{(N-1)}y_N$

Compare with this one

$$\boxed{d_1 = \sum_{n=1}^{N} |y_n - x_{1n}|^2}$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \;\cdots\cdots\; y_{(N-1)} y_N$

Compare with this one

$$\boxed{d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2}$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \;\cdots\cdots\; x_{1(N-1)} x_{1N}$
$x_{21} x_{22} x_{23} x_{24} \;\cdots\cdots\; x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \;\cdots\cdots\; x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \;\cdots\; y_{(N-1)} y_N$

Compare with this one

$$\boxed{d_{2^K} = \sum_{n=1}^{N} |y_n - x_{2^K n}|^2}$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \;\cdots\; x_{1(N-1)} x_{1N}$

$x_{21} x_{22} x_{23} x_{24} \;\cdots\; x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \;\cdots\; x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \; ..... \; y_{(N-1)} y_N$

Take smallest

$$\boxed{d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2}$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \; ..... \; x_{1(N-1)} x_{1N}$

$\boxed{x_{21} x_{22} x_{23} x_{24} \; ..... \; x_{2(N-1)} x_{2N}}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \; ..... \; x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \;.....\; y_{(N-1)} y_N$

Take smallest

$$\boxed{d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2}$$

Information book

000000 00
$\boxed{000000 \;.....\; 01}$
000000 10

So data is this one

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \;.....\; x_{1(N-1)} x_{1N}$
$\boxed{x_{21} x_{22} x_{23} x_{24} \;.....\; x_{2(N-1)} x_{2N}}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \;.....\; x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

This is ML decoding and is optimal

Capacity means the following

Receiver observes

$y_1 y_2 y_3 y_4 \cdots\cdots y_{(N-1)} y_N$

Take smallest

$$\boxed{d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2}$$

Information book

000000 00

$\boxed{000000 \; \ldots\ldots \; 01}$

000000 10

So data is this one

1111111 10

1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \cdots\cdots x_{1(N-1)} x_{1N}$

$\boxed{x_{21} x_{22} x_{23} x_{24} \cdots\cdots x_{2(N-1)} x_{2N}}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \cdots\cdots x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \cdots y_{(N-1)} y_N$

Take smallest

$$\boxed{d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2}$$

This is ML decoding and is optimal

Capacity means the following

1. If K/N ≤ C, and K->∞ then
   Prob(Correct detection)=1

Information book

000000 00
$\boxed{\text{000000 ..... 01}}$
000000 10

So data is this one

1111111 10
1111111 11

$\underbrace{\hphantom{1111111 \cdots 11}}_{K}$

Codebook

$x_{11} x_{12} x_{13} x_{14} \cdots x_{1(N-1)} x_{1N}$
$\boxed{x_{21} x_{22} x_{23} x_{24} \cdots x_{2(N-1)} x_{2N}}$

$\underbrace{x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \cdots x_{2^K (N-1)} x_{2^K N}}_{N}$

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \ldots\ldots y_{(N-1)} y_N$

Take smallest

$$\boxed{d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2}$$

This is ML decoding and is optimal

Capacity means the following

1. If $K/N \leq C$, and $K \to \infty$ then
   Prob(Correct detection)=1
2. If $K/N > C$, then
   Prob(Incorrect detection)=1

Information book

000000 00
000000 01
000000 10

So data is this one

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \ldots\ldots x_{1(N-1)} x_{1N}$

$x_{21} x_{22} x_{23} x_{24} \ldots\ldots x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \ldots\ldots x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \; ..... \; y_{(N-1)} y_N$

Take smallest

$$\boxed{d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2}$$

Information book

000000 00
$\boxed{000000 \; ..... \; 01}$
000000 10

So data is this one

1111111 10
1111111 11

K

To reach C, code-symbols must be
Random complex Gaussian variables
That is, generate codebook randomly

If it is generated with, say, 16QAM
C cannot be reached

Codebook

$x_{11} x_{12} x_{13} x_{14} \; ..... \; x_{1(N-1)} x_{1N}$
$\boxed{x_{21} x_{22} x_{23} x_{24} \; ..... \; x_{2(N-1)} x_{2N}}$

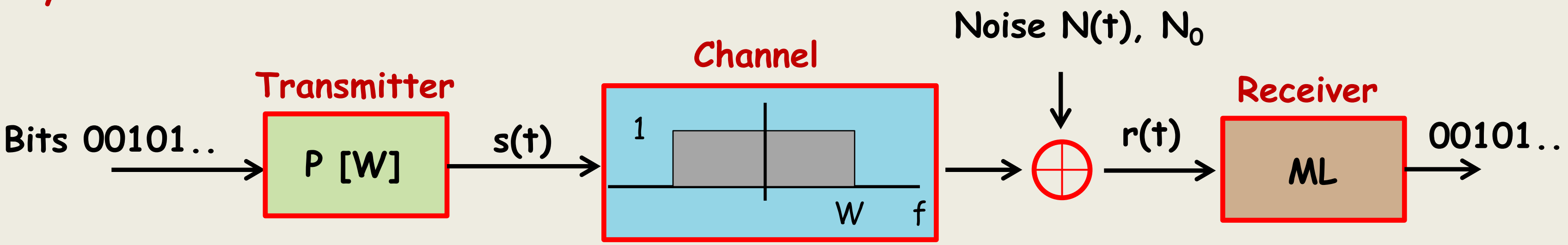
$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \; ..... \; x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

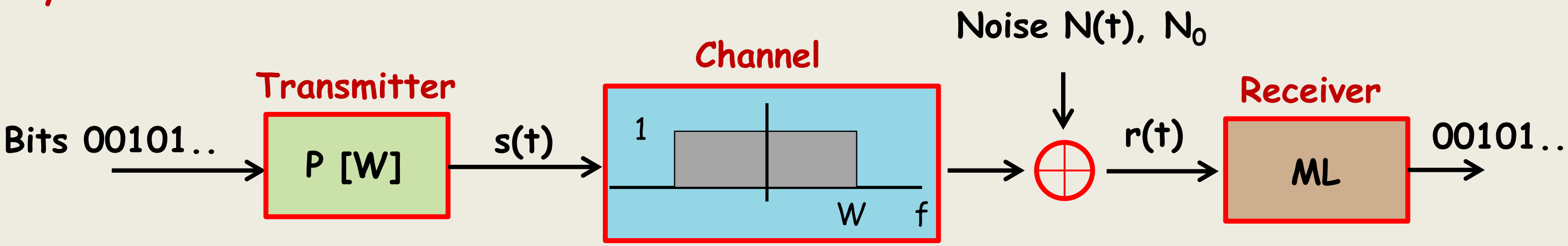
System model:



Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



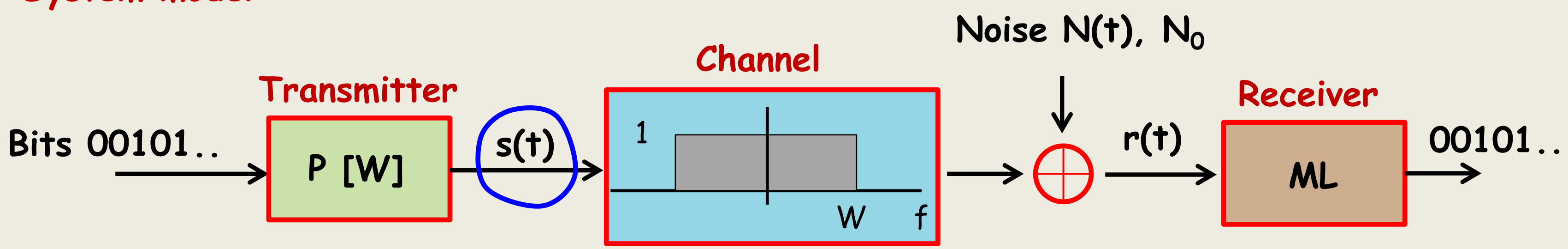
Interpretation of capacity:

Given a transmission of length T (seconds)

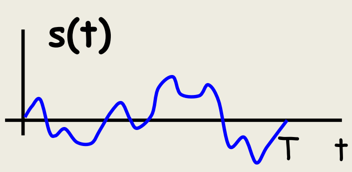
Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



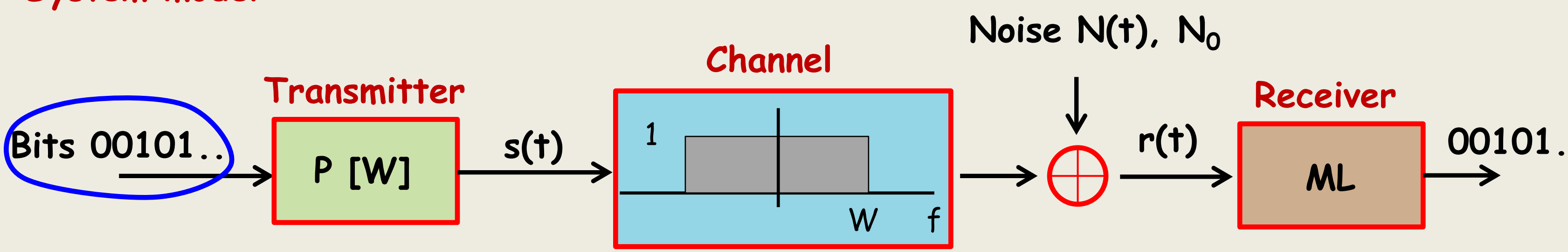
Interpretation of capacity:
Given a transmission of length T (seconds)



Lecture 4: Capacity

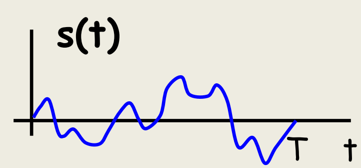
Extension to continuous channel (Shannon '48)

System model:



Interpretation of capacity:
Given a transmission of length T (seconds)

And a number of bits K

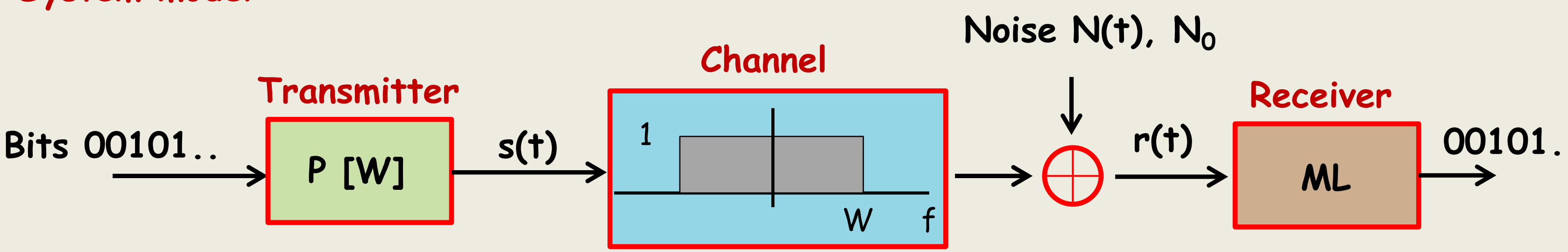


Bits $\underbrace{0010111010110100...010011}_{K}$

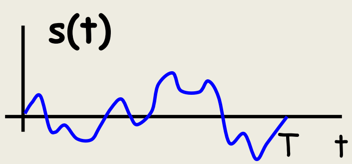
Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Interpretation of capacity:
Given a transmission of length T (seconds)



And a number of bits K

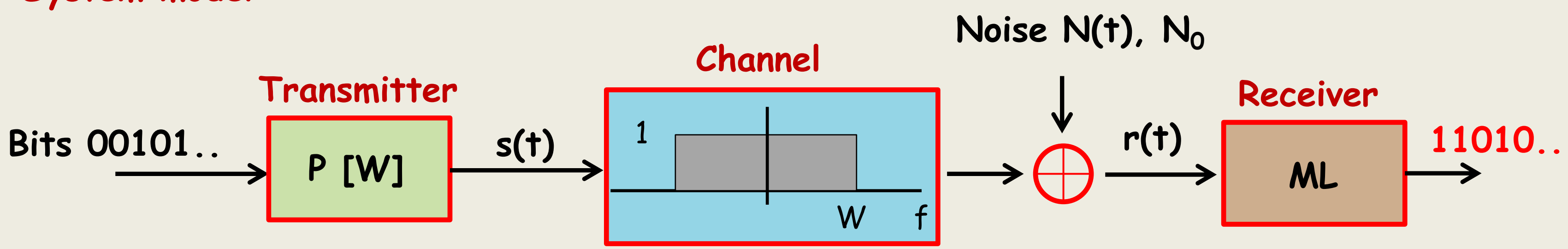
Bits $\underbrace{0010111010110100\ldots010011}_{K}$

The bitrate is: K/T [bit/sec]

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:

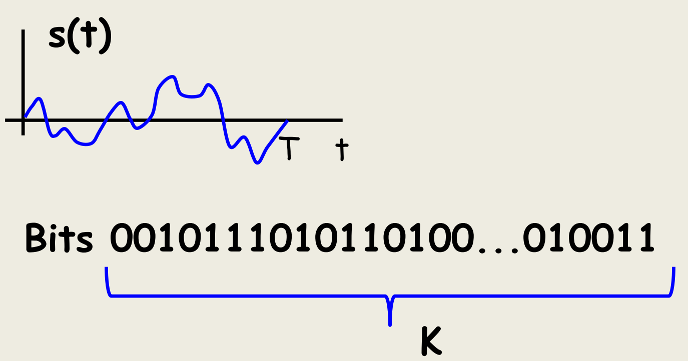


Interpretation of capacity:
Given a transmission of length T (seconds)

And a number of bits K

The bitrate is: K/T [bit/sec]

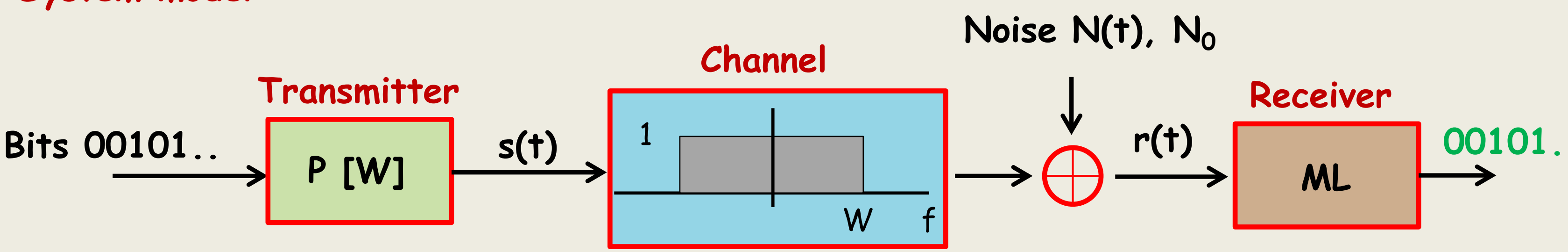
If K/T is too high, then many errors



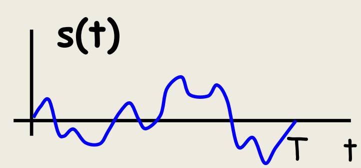
Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Interpretation of capacity:
Given a transmission of length T (seconds)



And a number of bits K

$$\text{Bits } \underbrace{0010111010110100\ldots010011}_{K}$$

The bitrate is: K/T [bit/sec]

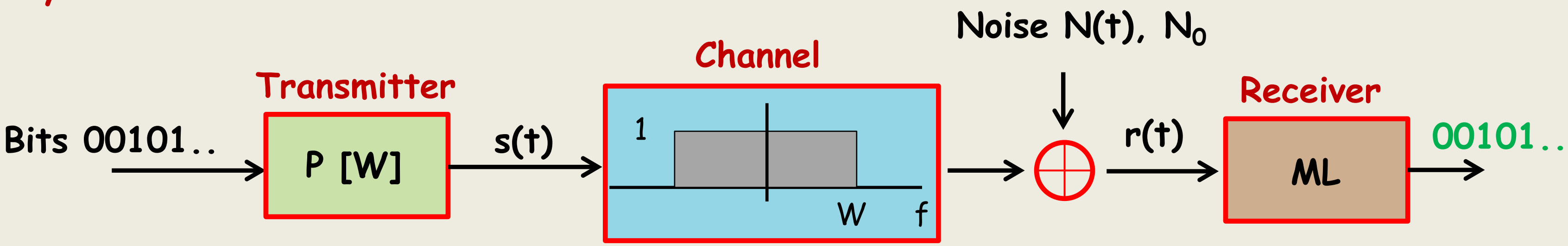
If K/T is too high, then many errors
Shannon proved: Possible to have NO ERRORS if,

1) $T \to \infty$ 2) $\lim_{T\to\infty} \frac{K}{T} = C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

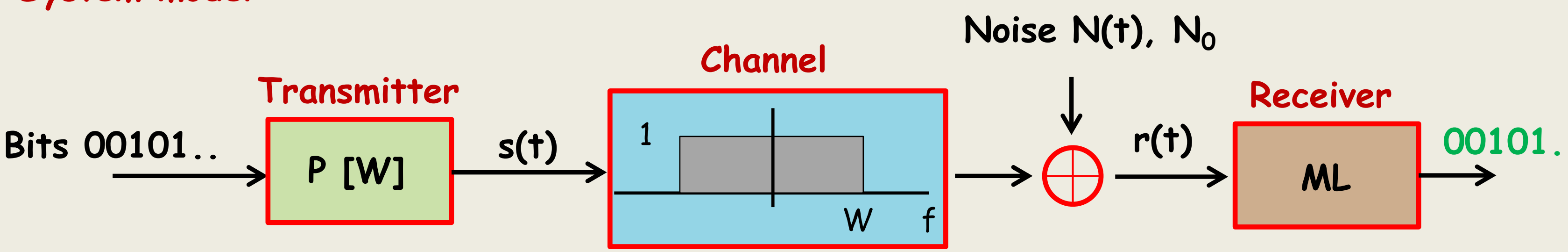
Facts:

1. C is not power, nor bandwidth efficiency
   (C is not dimensionless)

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

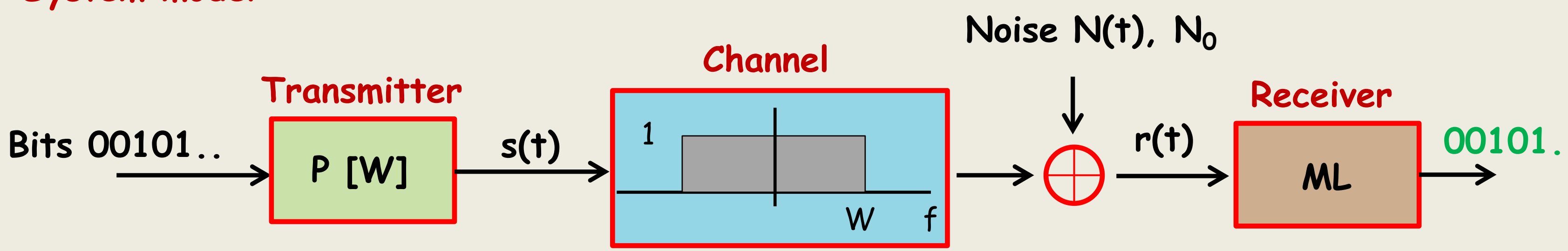
Facts:

1. C is not power, nor bandwidth efficiency
   (C is not dimensionless)
2. Not easy to reach C
   (i.e., to find a set of s(t) signals)

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

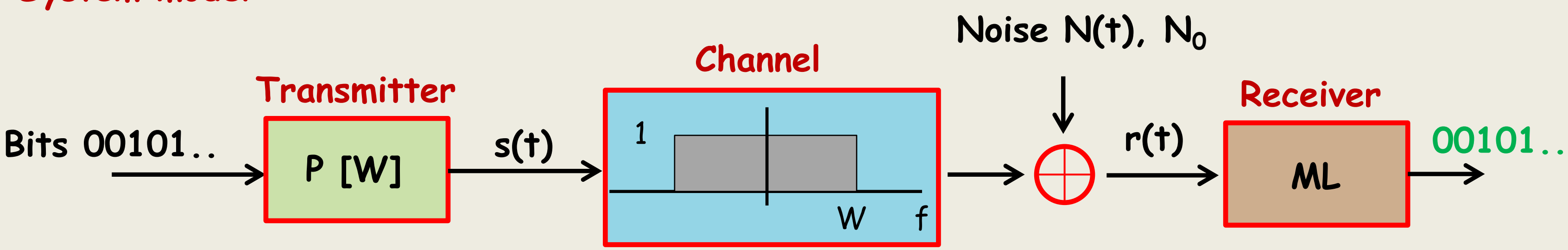
Facts:

1. C is not power, nor bandwidth efficiency
   (C is not dimensionless)
2. Not easy to reach C
   (i.e., to find a set of s(t) signals)
3. There is no parameter called $d^2_{\min}$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

Facts:

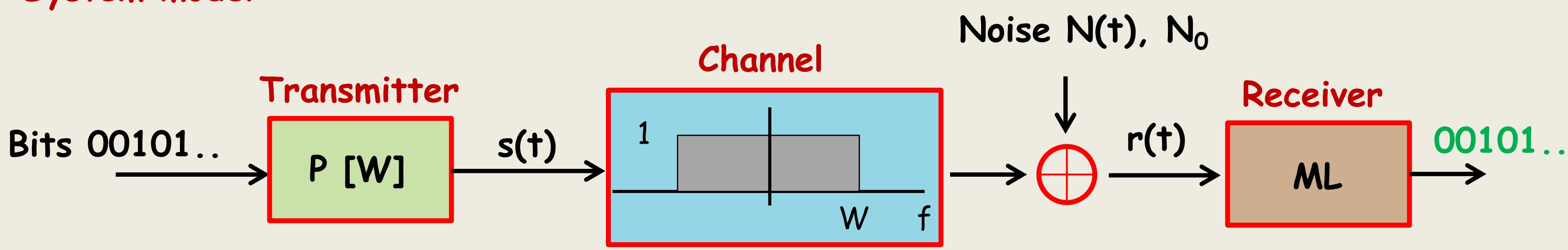
1. C is not power, nor bandwidth efficiency
   (C is not dimensionless)
2. Not easy to reach C
   (i.e., to find a set of s(t) signals)
3. There is no parameter called $d^2_{\min}$

4. When W grows:

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

Grows linearly

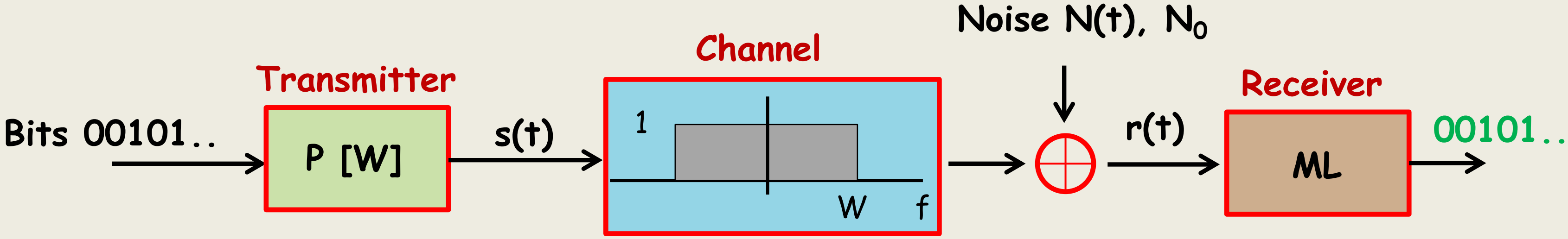
Facts:

1. C is not power, nor bandwidth efficiency
   (C is not dimensionless)
2. Not easy to reach C
   (i.e., to find a set of s(t) signals)
3. There is no parameter called $d^2_{\min}$
4. When W grows:

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

Grows linearly

decreases

Facts:

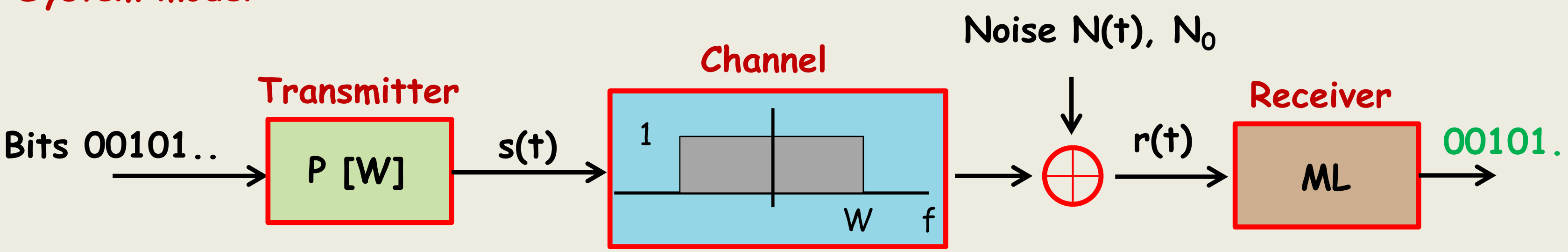
1. C is not power, nor bandwidth efficiency
   (C is not dimensionless)
2. Not easy to reach C
   (i.e., to find a set of s(t) signals)
3. There is no parameter called $d^2_{\min}$

4. When W grows:

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Decreases logarihtmically

$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

Grows linearly

Facts:

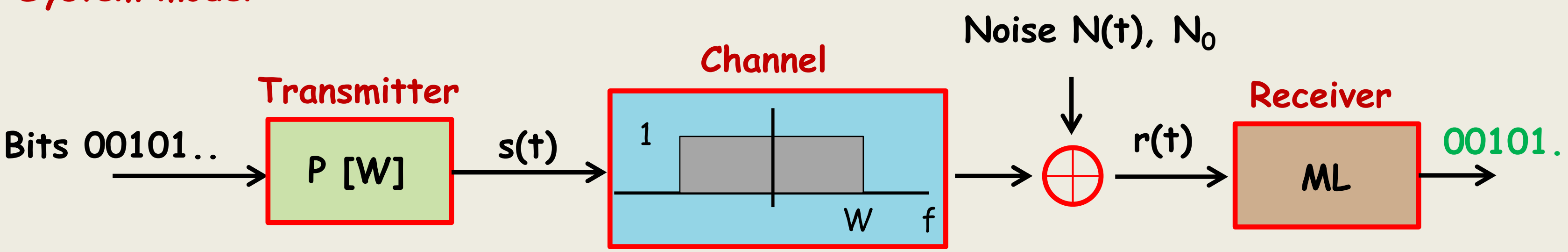
1. C is not power, nor bandwidth efficiency
   (C is not dimensionless)
2. Not easy to reach C
   (i.e., to find a set of s(t) signals)
3. There is no parameter called $d^2_{\min}$

4. When W grows:

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

Grows

Facts:

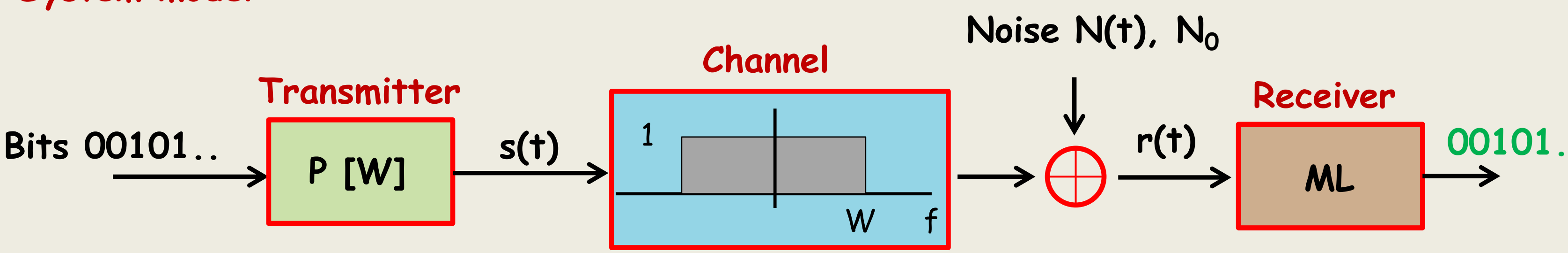
1. C is not power, nor bandwidth efficiency
   (C is not dimensionless)
2. Not easy to reach C
   (i.e., to find a set of s(t) signals)
3. There is no parameter called $d^2_{\min}$

4. When W grows: C grows

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

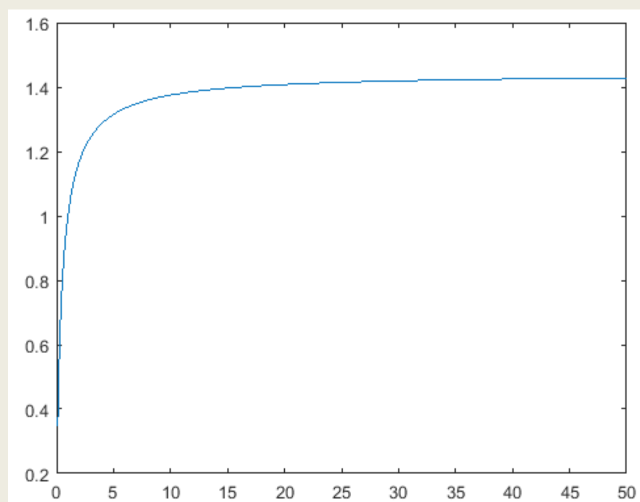
System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

Example, $P/N_0 = 1$

C



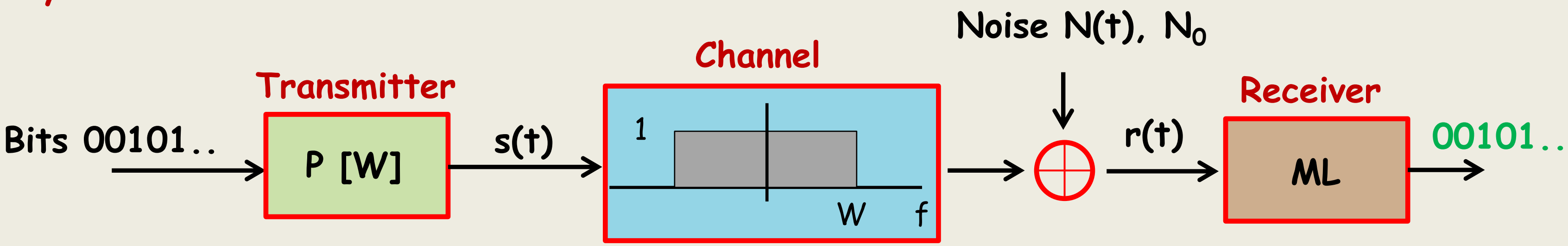
W

But it grows to a limit

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

What is the limit?

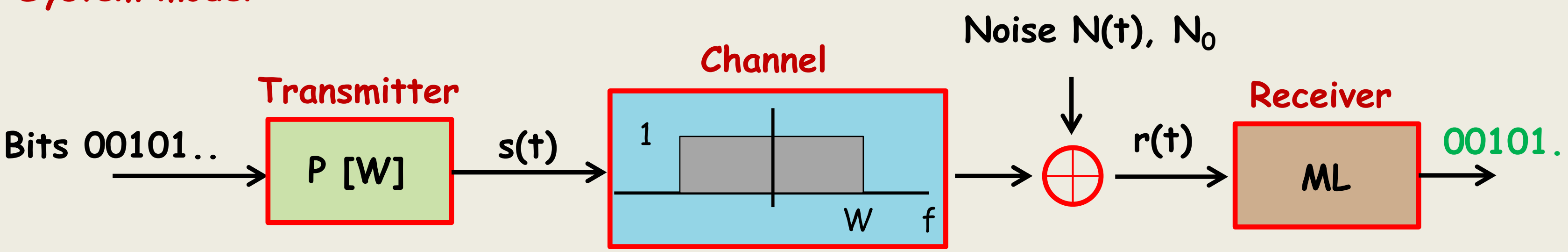
Standard limit

$$\lim_{x \to \infty} x \ln \left(1 + \frac{A}{x}\right) = A$$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

What is the limit?

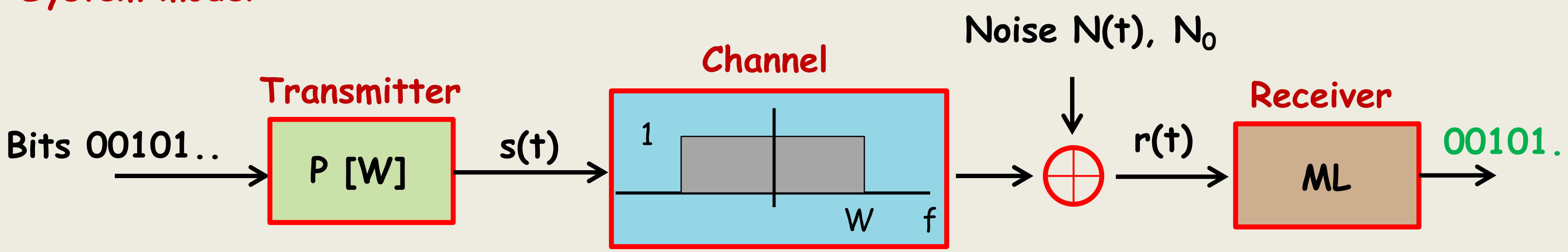
Standard limit

Identify x with W

$$\lim_{x\to\infty} x \ln\left(1 + \frac{A}{x}\right) = A$$

Extension to continuous channel (Shannon '48)

System model:



$$C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

What is the limit?

Standard limit

Identify x with W

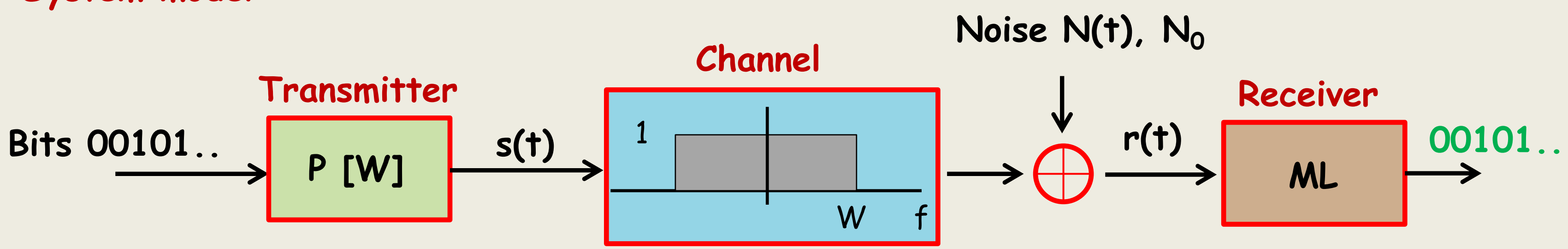
Identify A with P/N_0

$$\lim_{x \to \infty} x \ln \left(1 + \frac{A}{x}\right) = A$$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

What is the limit?

Standard limit

Identify x with W

Identify A with P/N_0

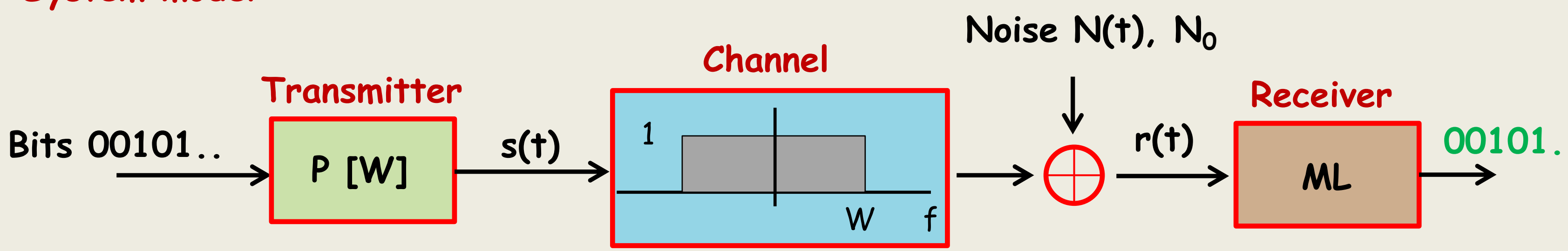
Express $\log_2(x)$ as $\ln(x)/\ln(2)$

$$\lim_{x \to \infty} x \ln\left(1 + \frac{A}{x}\right) = A$$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

What is the limit?

Standard limit

$$\lim_{x \to \infty} x \ln\left(1 + \frac{A}{x}\right) = A$$

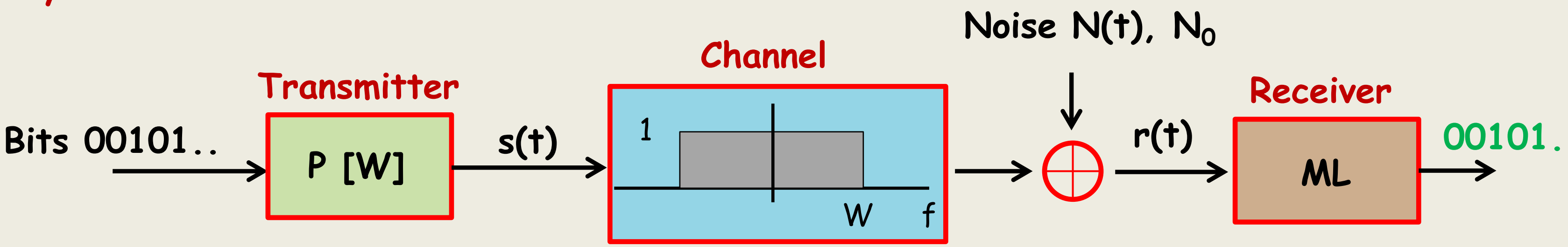
Carry out limit

$$C_{\max} = \lim_{W \to \infty} \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right) =$$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

What is the limit?

Standard limit

$$\lim_{x\to\infty} x \ln\left(1 + \frac{A}{x}\right) = A$$

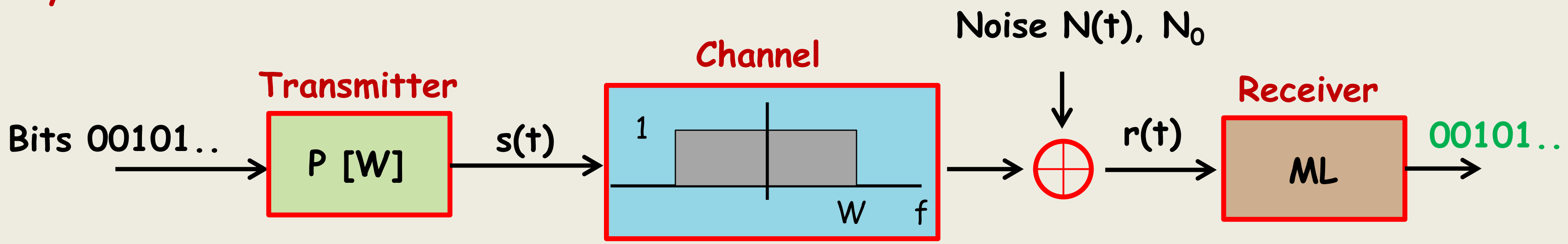
Carry out limit

$$C_{\max} = \lim_{W\to\infty} \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right) = \frac{\mathcal{P}}{N_0 \ln(2)}$$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

What is the limit?

Standard limit

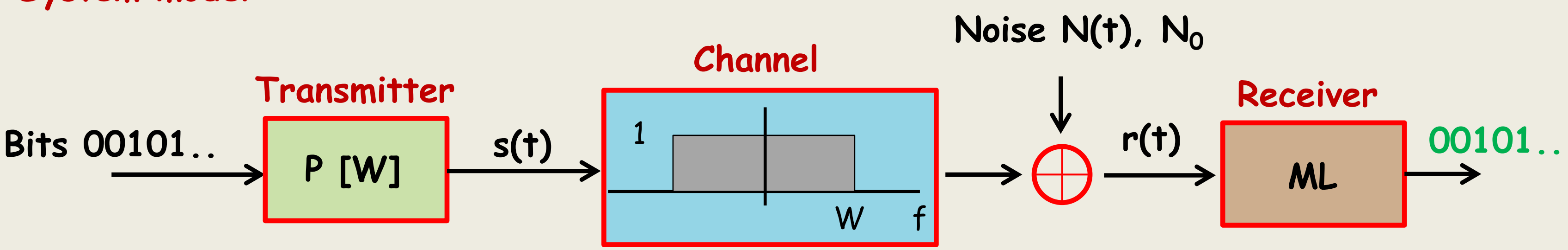
$$\lim_{x \to \infty} x \ln\left(1 + \frac{A}{x}\right) = A$$

Carry out limit

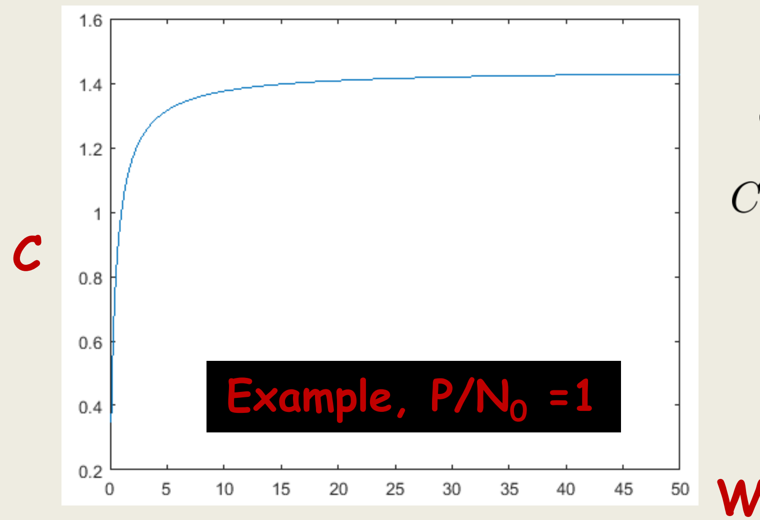
$$C_{\max} = \lim_{W \to \infty} \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right) = \frac{\mathcal{P}}{N_0 \ln(2)}$$

Extension to continuous channel (Shannon '48)

System model:



$$C = \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$



But it grows to a limit

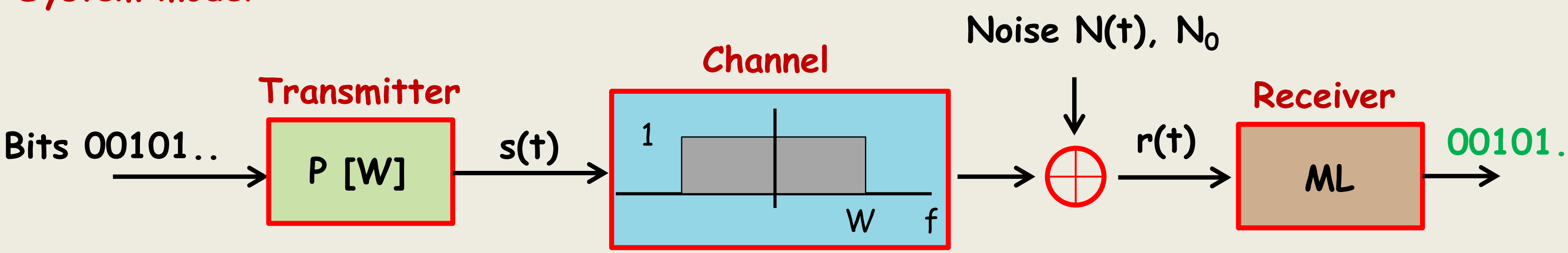
$$C_{\max} = \frac{\mathcal{P}}{N_0 \ln(2)}$$

Standard limit

$$\lim_{x\to\infty} x \ln\left(1 + \frac{A}{x}\right) = A$$

Extension to continuous channel (Shannon '48)

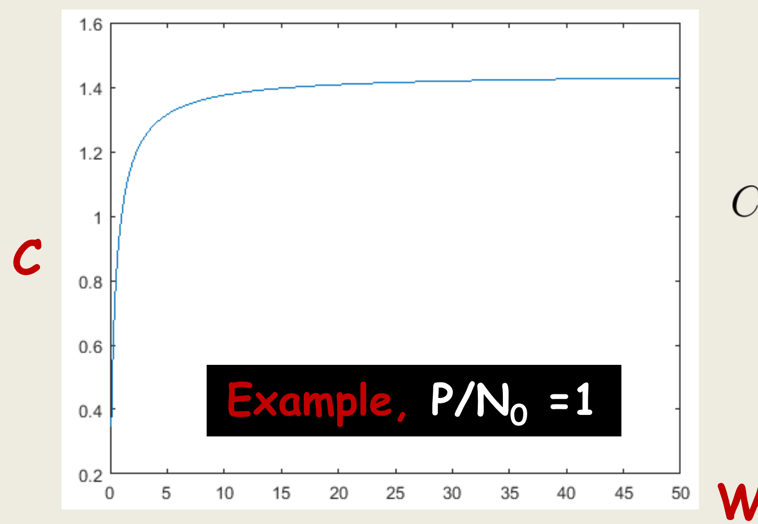
System model:



$$C = \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

<u>Standard limit</u>

$$\lim_{x \to \infty} x \ln\left(1 + \frac{A}{x}\right) = A$$



But it grows to a limit

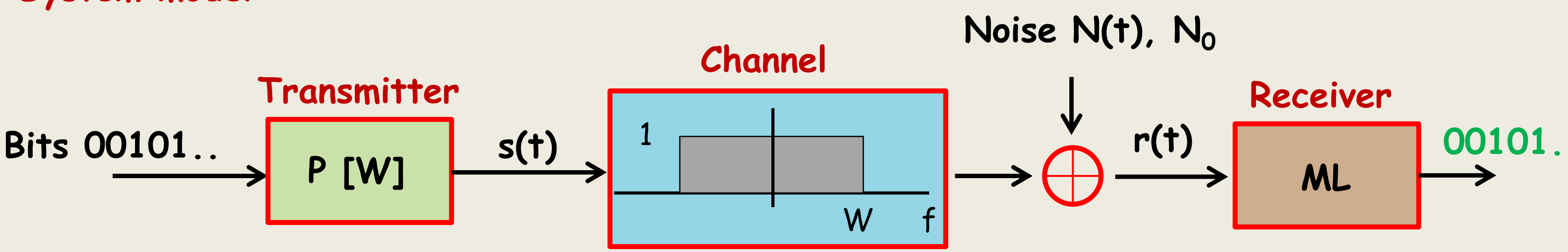
$$C_{\max} = \frac{\mathcal{P}}{N_0 \ln(2)}$$

$$= \frac{1}{\ln(2)}$$

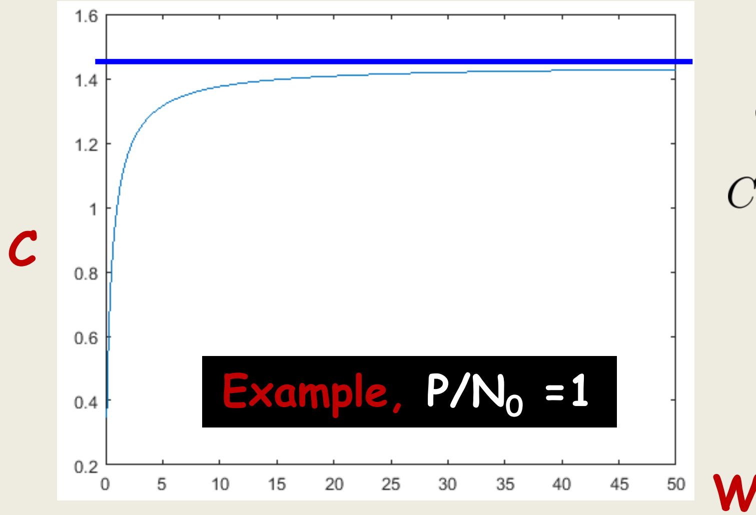
Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



$$C = \frac{W}{\ln(2)} \ln\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$



But it grows to a limit

$$C_{\max} = \frac{\mathcal{P}}{N_0 \ln(2)}$$

$$= \frac{1}{\ln(2)}$$

$$= 1.4427$$

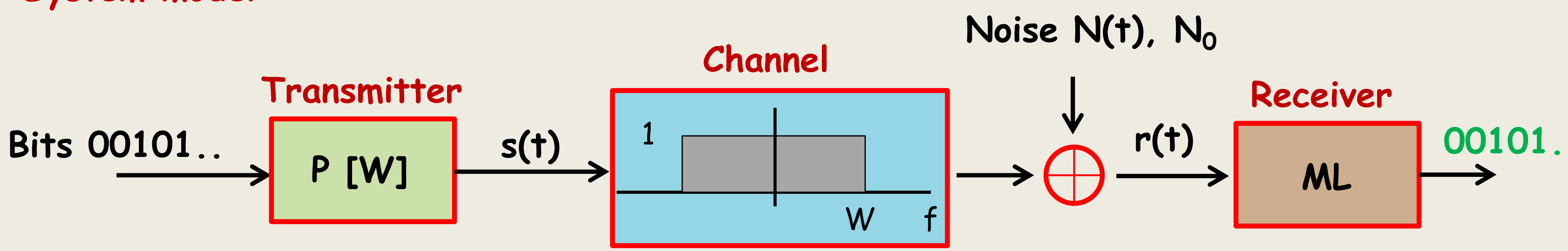
__Standard limit__

$$\lim_{x\to\infty} x \ln\left(1 + \frac{A}{x}\right) = A$$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



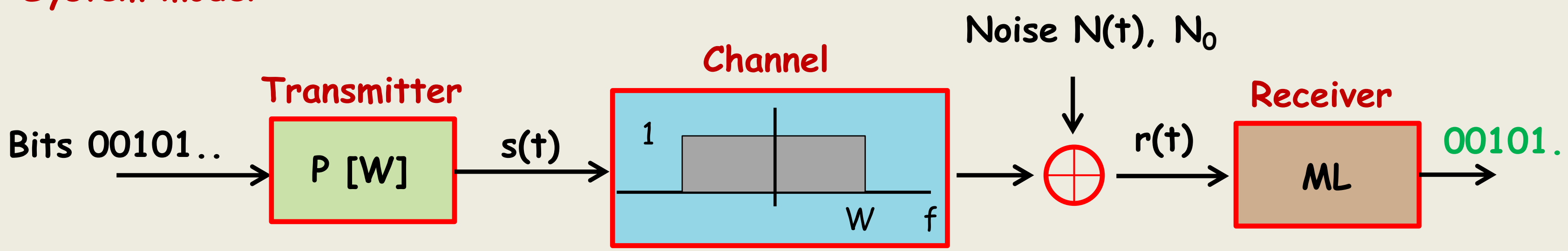
Summary

1. We stated that the capacity of the above is $C = W \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$ bits/second
2. We proved that for infinite bandwidth, the capacity is $C_{\max} = \frac{\mathcal{P}}{N_0 \ln(2)}$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Summary

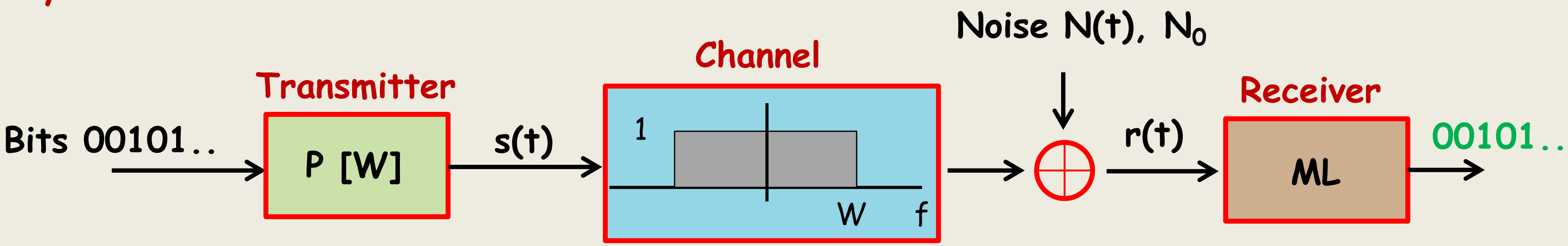
1. We stated that the capacity of the above is $C = W \log_2 \left(1 + \frac{\mathcal{P}}{N_0 W}\right)$ bits/second

2. We proved that for infinite bandwidth, the capacity is $C_{\max} = \frac{\mathcal{P}}{N_0 \ln(2)}$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



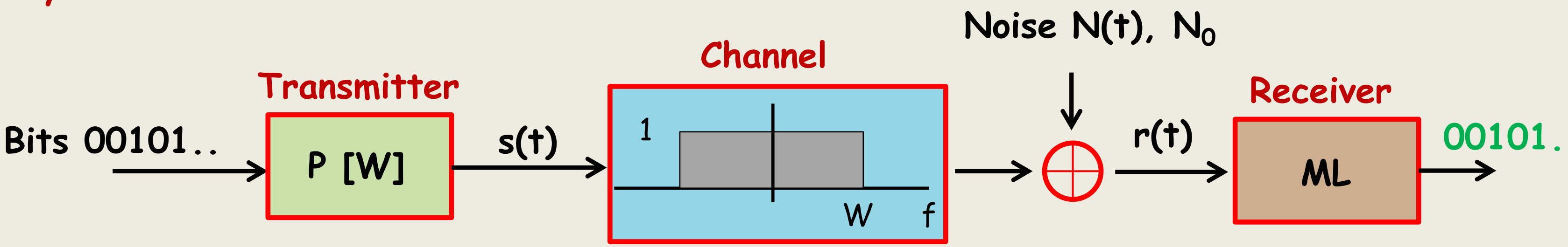
Bandwidth efficiency

By definition, $\frac{C}{W} = \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Bandwidth efficiency

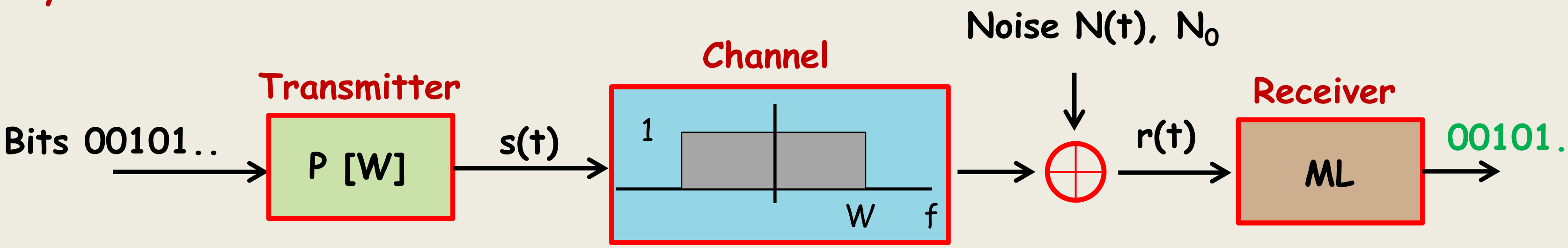
By definition, $\frac{C}{W} = \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$

Effect of increasing/decreasing W ?

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Bandwidth efficiency

By definition, $$\frac{C}{W} = \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

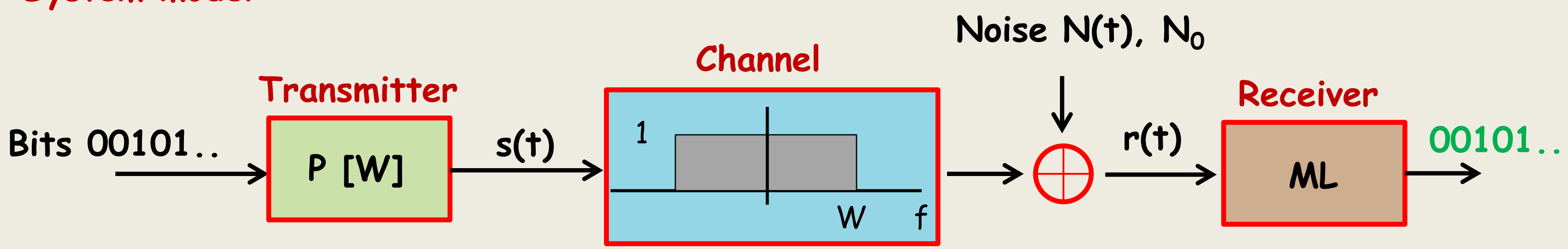
Effect of increasing/decreasing W ?

For large W, BW efficiency = 0
For small W, BW efficiency = ∞

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Bandwidth efficiency

By definition, $\frac{C}{W} = \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$

Effect of increasing/decreasing W ?

For large W, BW efficiency = 0
For small W, BW efficiency = ∞

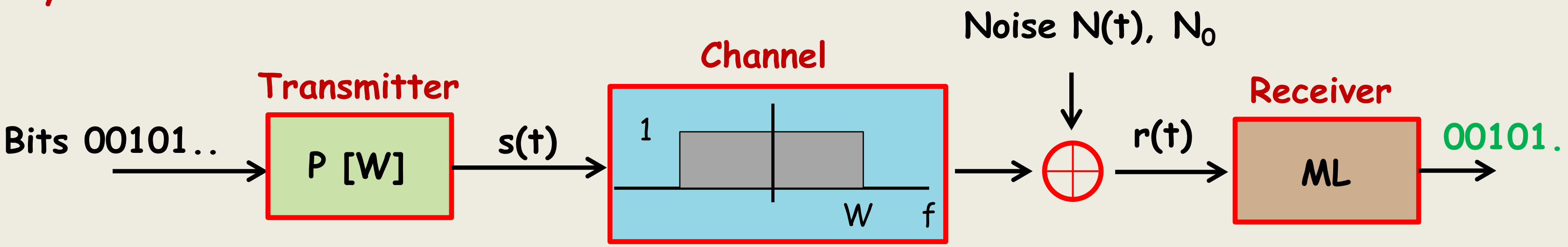
Bandwidth vs. Power efficiency

However $\mathcal{P} = C E_b$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Bandwidth efficiency

By definition, $\frac{C}{W} = \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$

Effect of increasing/decreasing W ?

For large W, BW efficiency = 0
For small W, BW efficiency = ∞

Bandwidth vs. Power efficiency

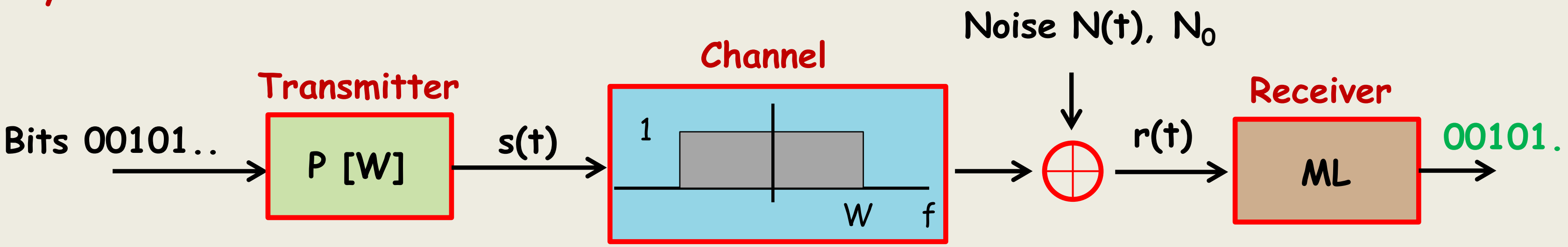
However $\mathcal{P} = C E_b$

So, $\frac{C}{W} = \log_2\left(1 + \frac{C}{W}\frac{E_b}{N_0}\right)$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Bandwidth efficiency

By definition, $$\frac{C}{W} = \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

Effect of increasing/decreasing W ?

For large W, BW efficiency = 0
For small W, BW efficiency = ∞

Bandwidth vs. Power efficiency

However $\mathcal{P} = C E_b$

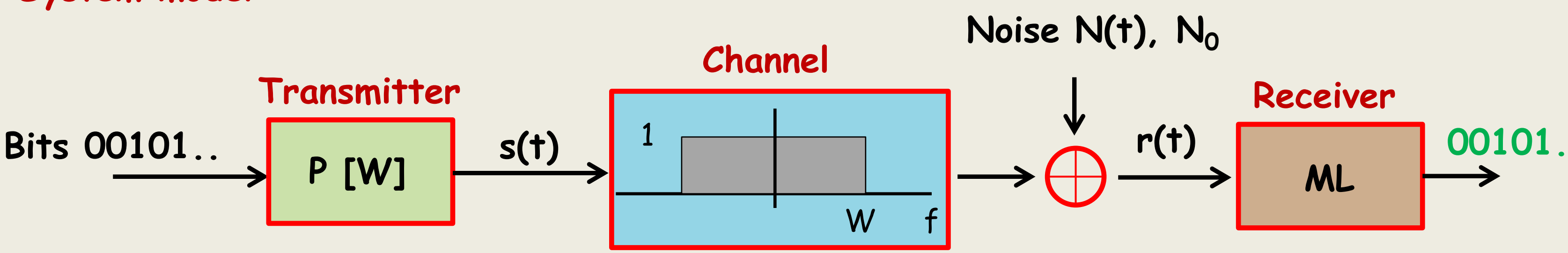
So, $$\frac{C}{W} = \log_2\left(1 + \frac{C}{W}\frac{E_b}{N_0}\right)$$

Or, equivalently $$\boxed{\frac{E_b}{N_0} = \frac{2^{\frac{C}{W}} - 1}{\frac{C}{W}}}$$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Bandwidth vs. Power efficiency

What happens if C/W grows?

However $\mathcal{P} = C E_b$

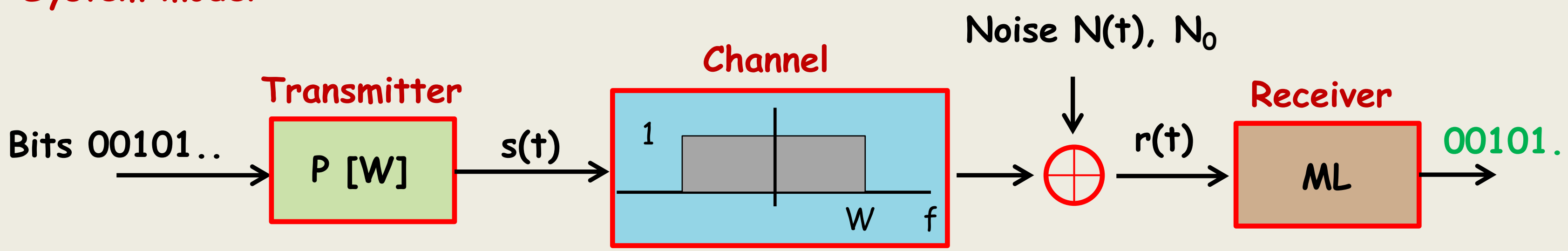
So, $\frac{C}{W} = \log_2\left(1 + \frac{C}{W}\frac{E_b}{N_0}\right)$

Or, equivalently $\boxed{\frac{E_b}{N_0} = \frac{2^{\frac{C}{W}} - 1}{\frac{C}{W}}}$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Bandwidth vs. Power efficiency

What happens if C/W grows?

E_b/N_0 grows as well

In fact, we have (check at home) that to have 0 error probability

However $\mathcal{P} = CE_b$

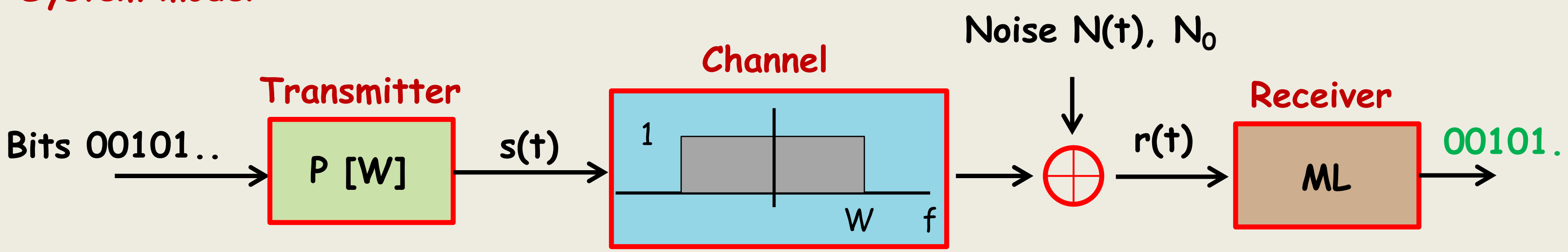
So, $\frac{C}{W} = \log_2\left(1 + \frac{C}{W}\frac{E_b}{N_0}\right)$

Or, equivalently $\boxed{\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}} - 1}{\frac{C}{W}}}$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Bandwidth vs. Power efficiency

What happens if C/W grows?

E_b/N_0 grows as well

In fact, we have (check at home) that to have 0 error probability

But, since E_b/N_0 grows with C/W, there must be a minimum E_b/N_0 achieved at vanishing C/W

Standard limit: $\lim_{x\to 0} \frac{2^x - 1}{x} = \ln(2)$

However $\mathcal{P} = C E_b$

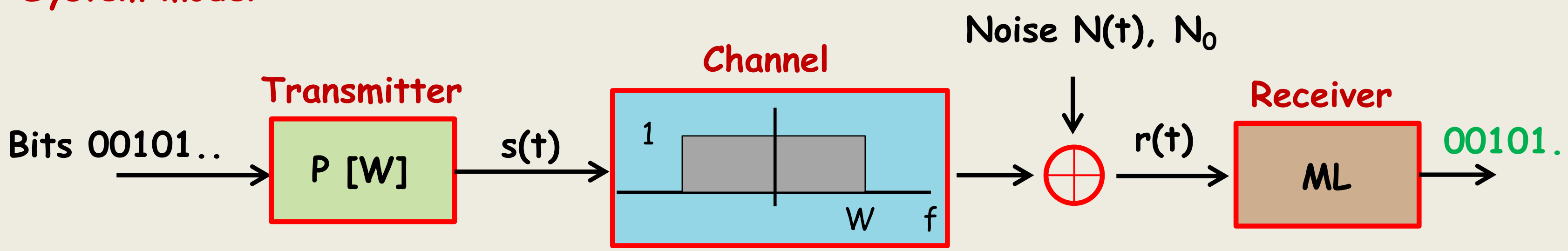
So, $\frac{C}{W} = \log_2\left(1 + \frac{C}{W}\frac{E_b}{N_0}\right)$

Or, equivalently $\boxed{\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}} - 1}{\frac{C}{W}}}$

Lecture 4: Capacity

Extension to continuous channel (Shannon '48)

System model:



Bandwidth vs. Power efficiency

However $\mathcal{P} = C E_b$

So, $\frac{C}{W} = \log_2\left(1 + \frac{C}{W}\frac{E_b}{N_0}\right)$

Thus $\boxed{\lim_{C/W \to 0} \frac{E_b}{N_0} \geq \ln(2)}$ $(= -1.6\text{dB})$

What happens if C/W grows?

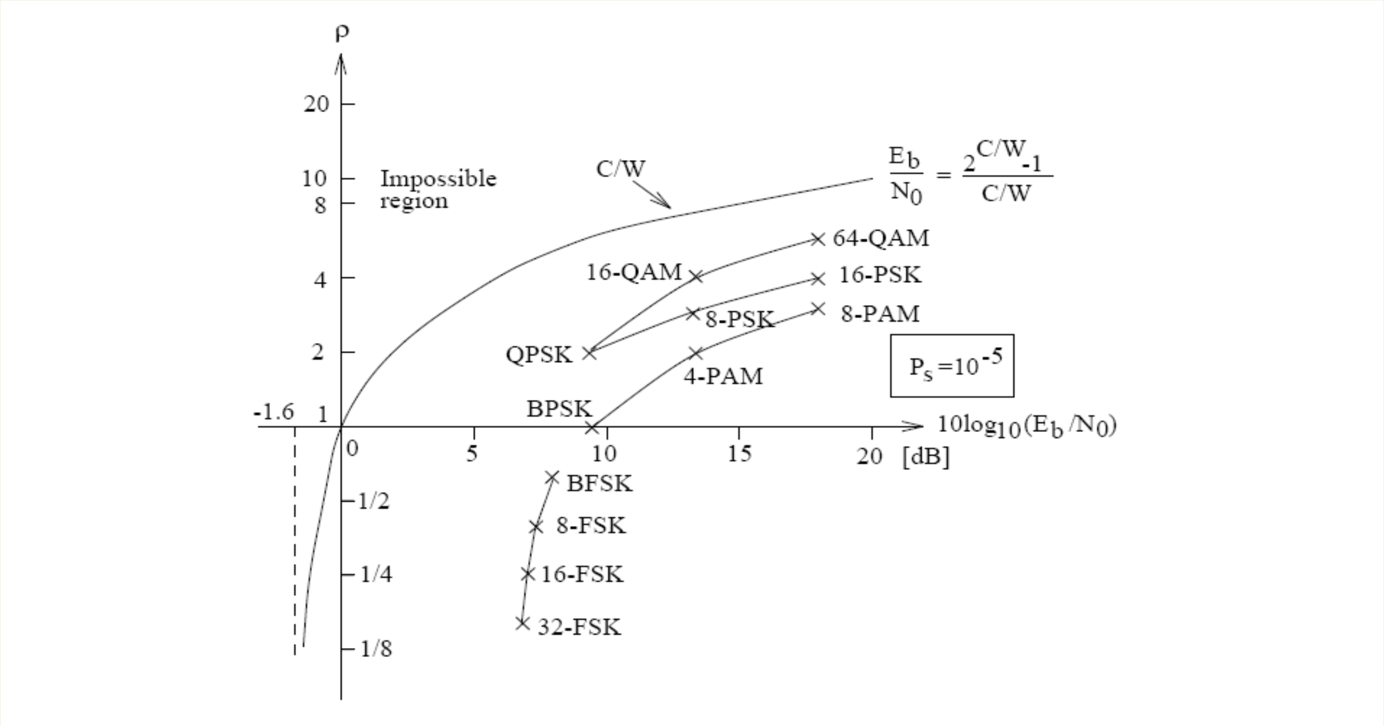
E_b/N_0 grows as well

In fact, we have (check at home) that to have 0 error probability

But, since E_b/N_0 grows with C/W, there must be a minimum E_b/N_0 achieved at vanishing C/W

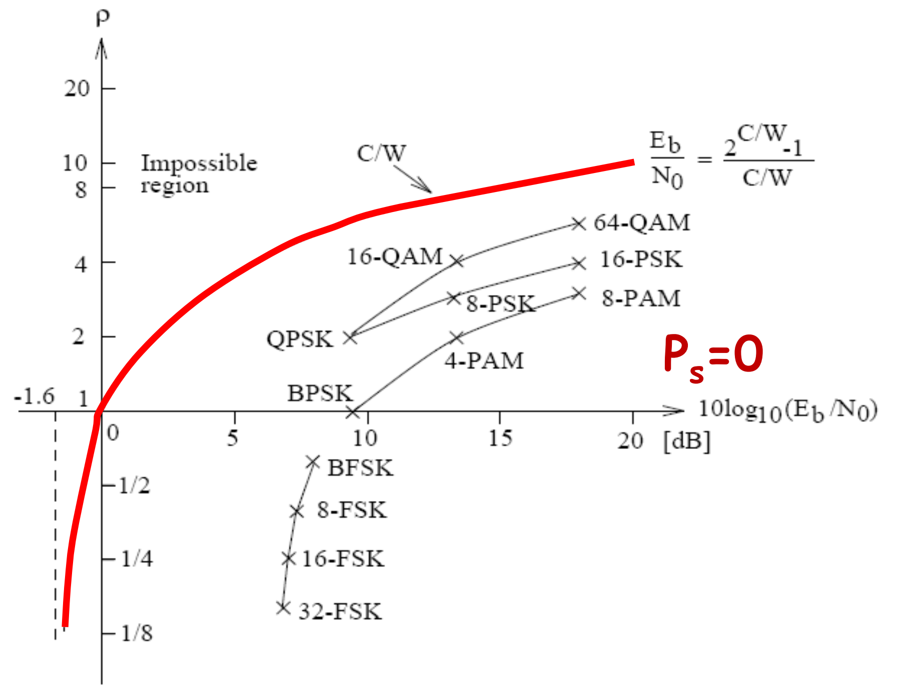
Standard limit: $\lim_{x \to 0} \frac{2^x - 1}{x} = \ln(2)$

Lecture 4: Capacity



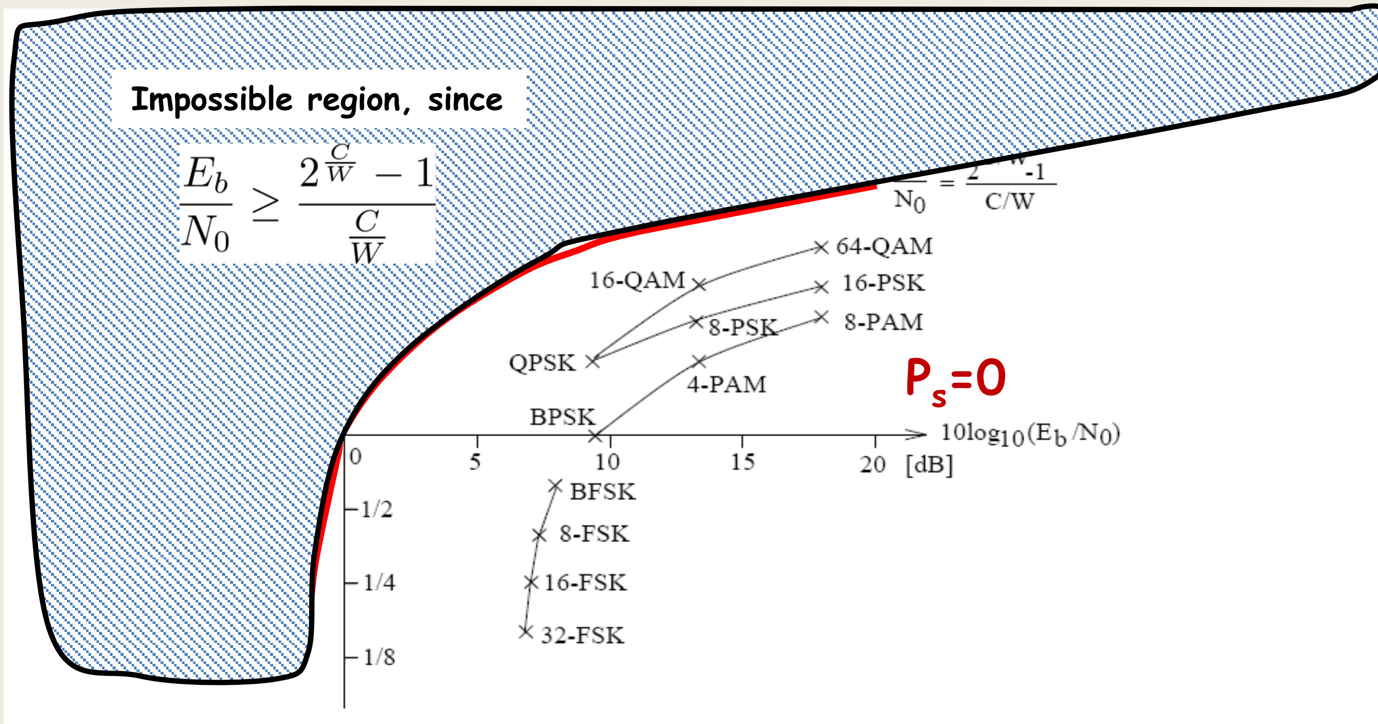
$$\boxed{\lim_{C/W\to 0}\frac{E_b}{N_0}\geq \ln(2)}\quad (= -1.6\text{dB})\qquad \boxed{\frac{E_b}{N_0}\geq \frac{2^{\frac{C}{W}}-1}{\frac{C}{W}}}$$

Lecture 4: Capacity



$$\boxed{\lim_{C/W\to 0} \frac{E_b}{N_0} \geq \ln(2)} \; (= -1.6\text{dB}) \qquad \boxed{\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}}-1}{\frac{C}{W}}}$$

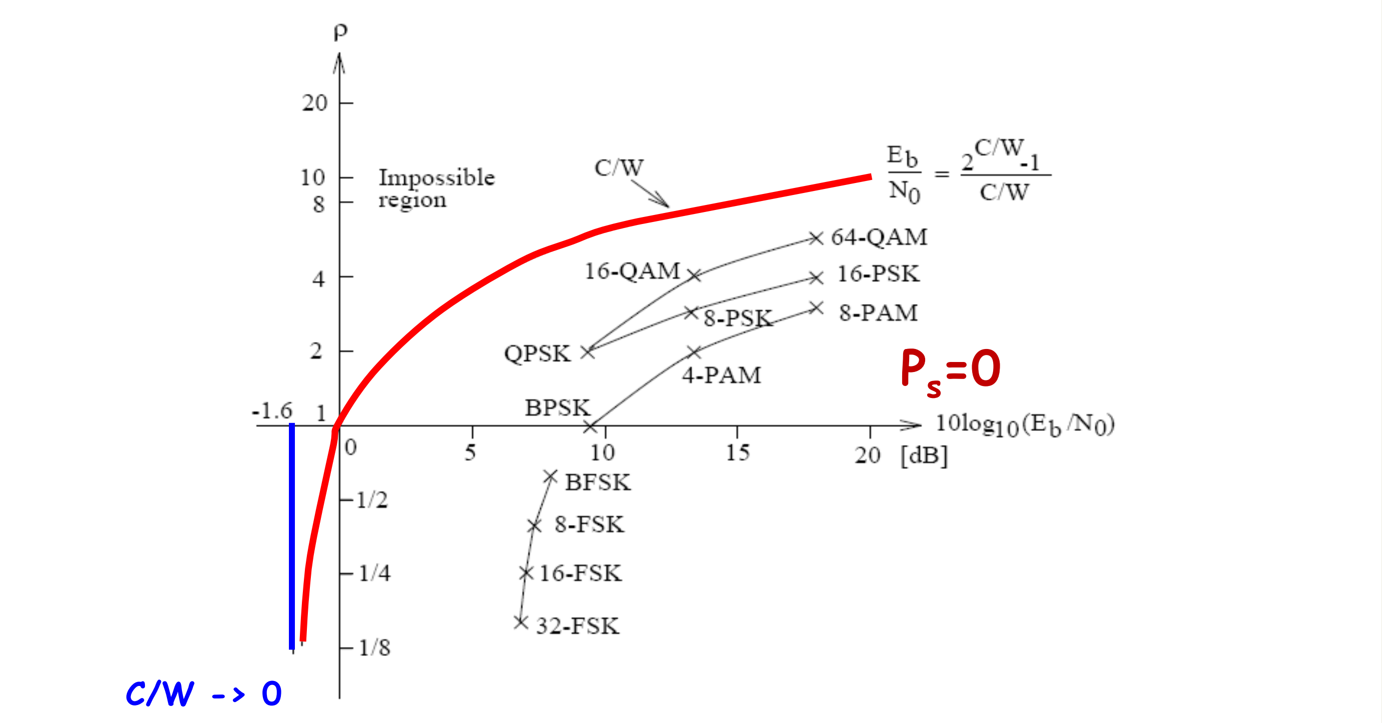
Lecture 4: Capacity



$$\boxed{\lim_{C/W \to 0} \frac{E_b}{N_0} \geq \ln(2)} \quad (= -1.6\text{dB}) \qquad \boxed{\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}} - 1}{\frac{C}{W}}}$$

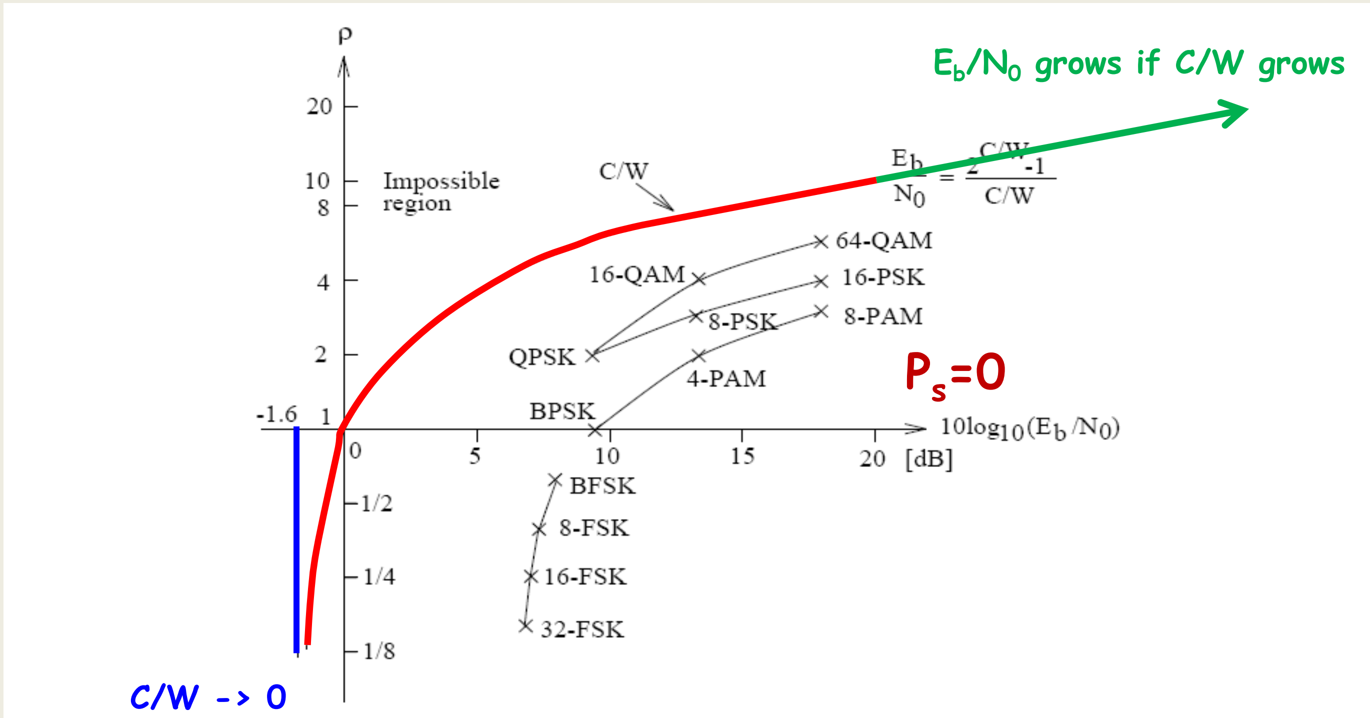
NOTE: Plot does not tell what the capacity is in bit/sec

Lecture 4: Capacity



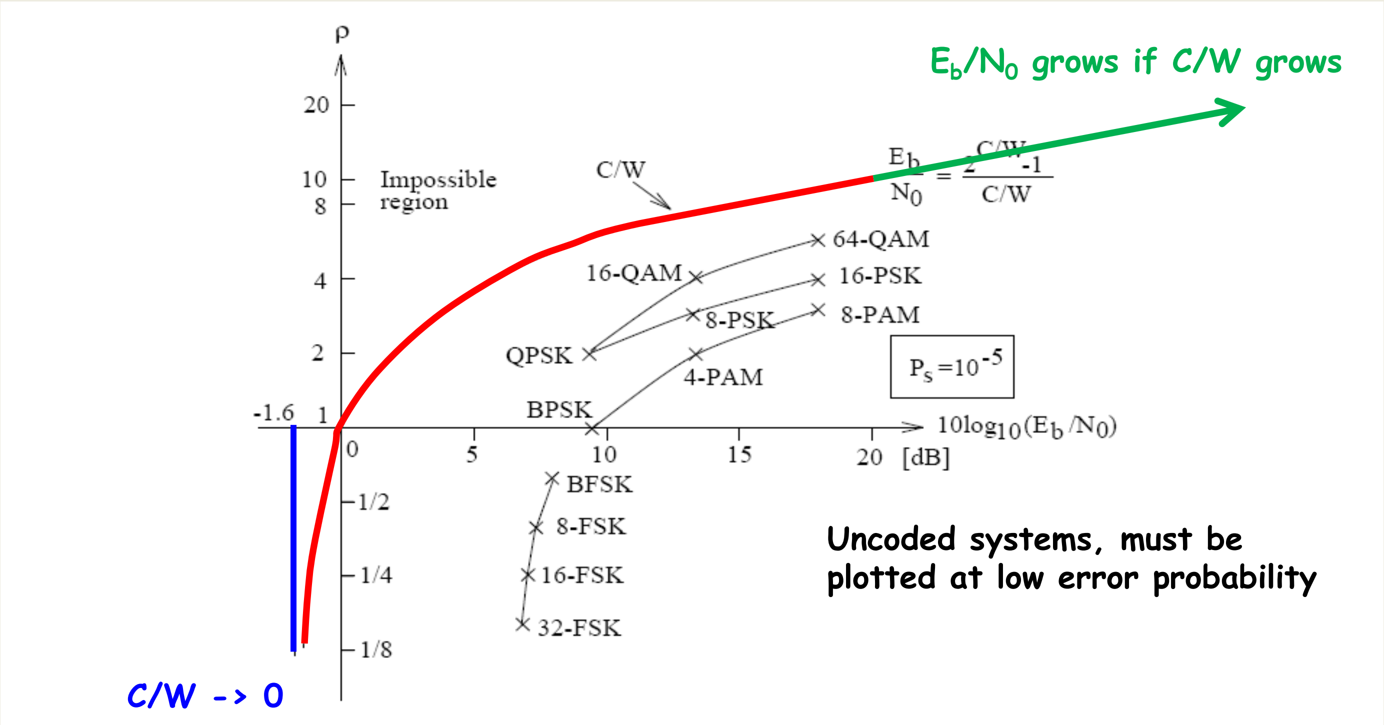
$$\boxed{\lim_{C/W \to 0} \frac{E_b}{N_0} \geq \ln(2)} \quad (= -1.6\text{dB}) \quad \boxed{\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}}-1}{\frac{C}{W}}}$$

Lecture 4: Capacity



$$\boxed{\lim_{C/W\to 0}\frac{E_b}{N_0} \geq \ln(2)}\ (= -1.6\text{dB})\qquad \boxed{\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}}-1}{\frac{C}{W}}}$$

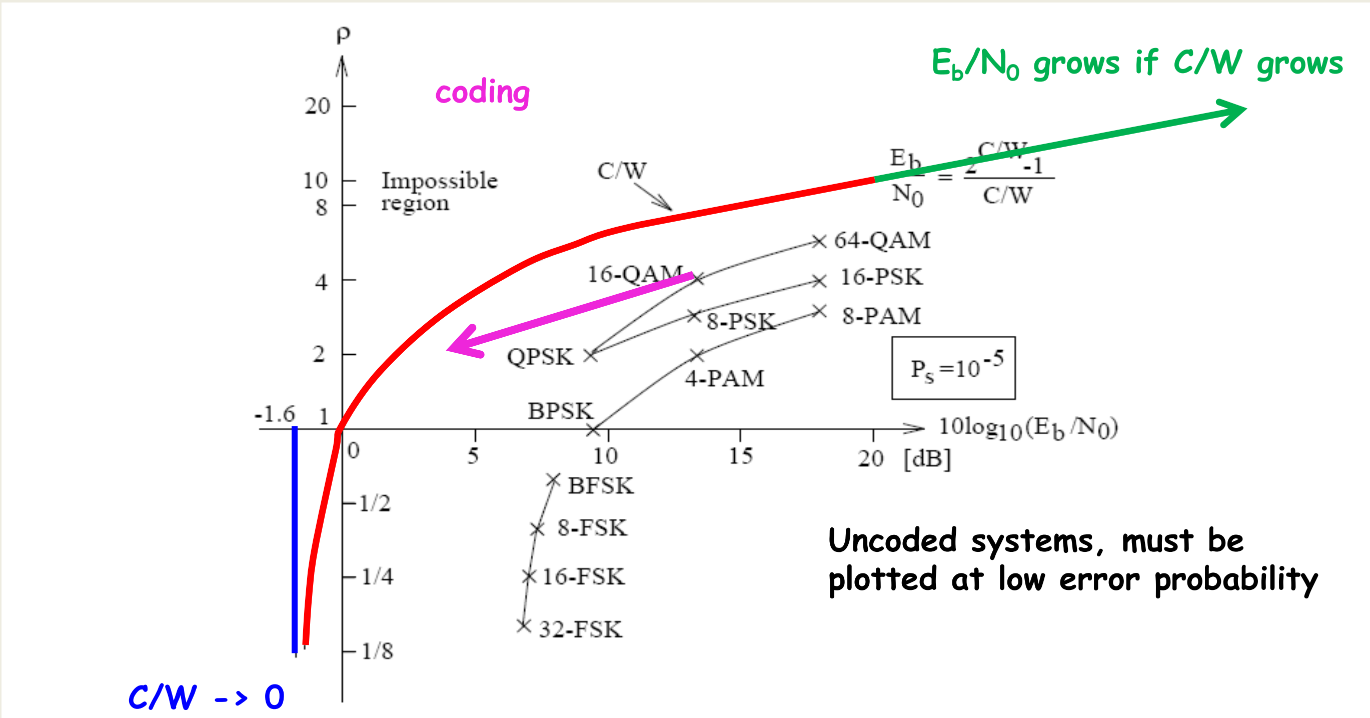
Lecture 4: Capacity



$$\boxed{\lim_{C/W \to 0} \frac{E_b}{N_0} \geq \ln(2)} \; (= -1.6\text{dB})$$

$$\boxed{\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}} - 1}{\frac{C}{W}}}$$

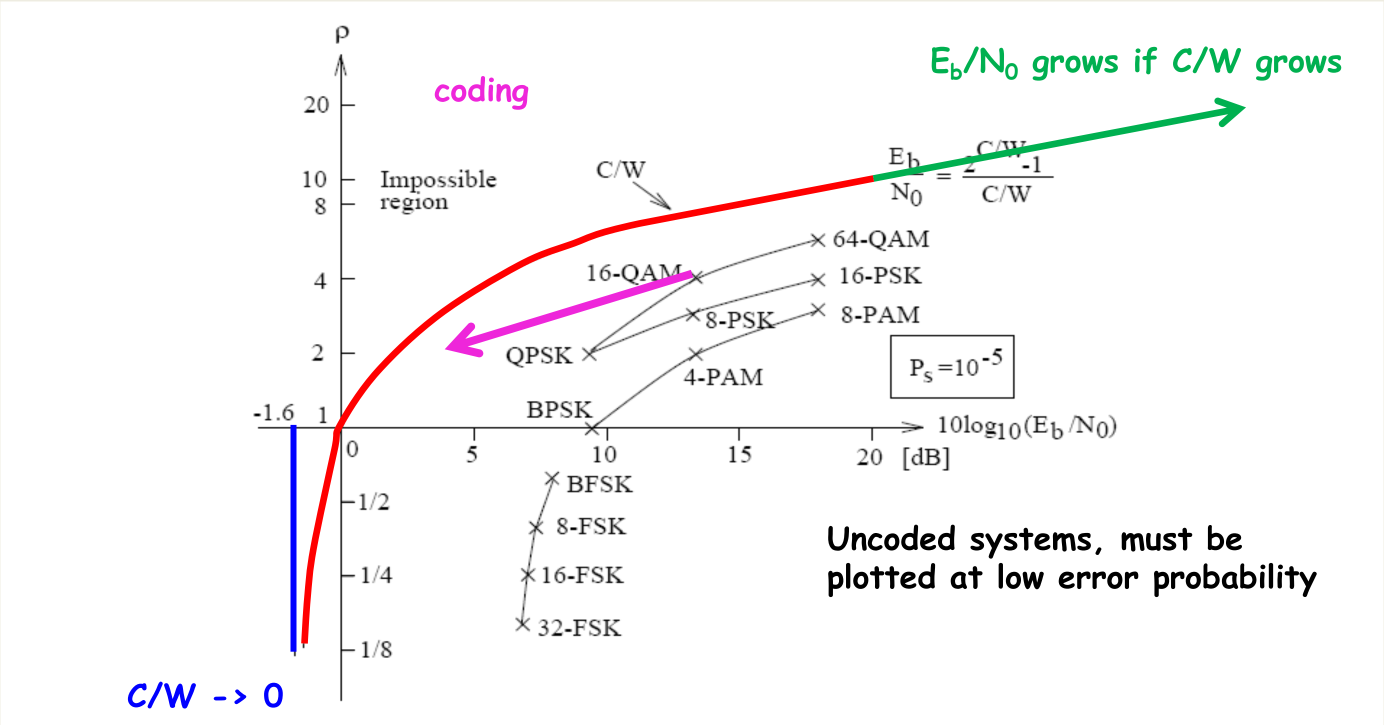
Lecture 4: Capacity



$$\lim_{C/W \to 0} \frac{E_b}{N_0} \geq \ln(2)$$ $(= -1.6\text{dB})$

$$\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}} - 1}{\frac{C}{W}}$$

Lecture 4: Capacity



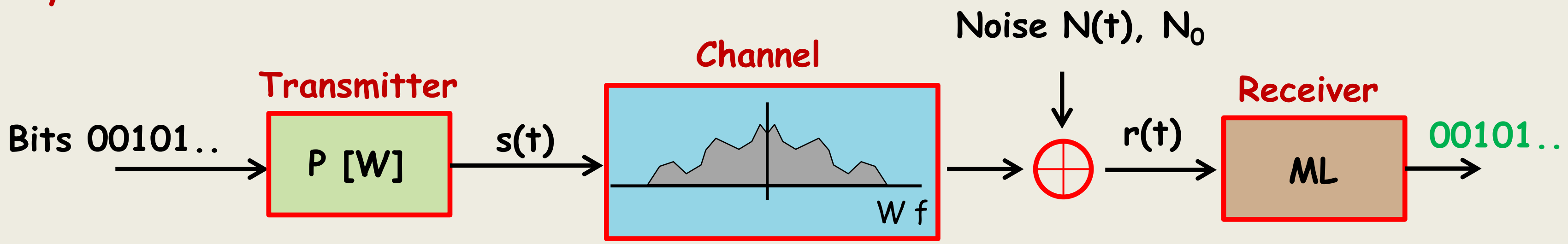
$$\boxed{\lim_{C/W \to 0} \frac{E_b}{N_0} \geq \ln(2)} \; (= -1.6\text{dB}) \qquad \boxed{\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}} - 1}{\frac{C}{W}}}$$

NOTE: Plot does not tell what the capacity is in bit/sec

Lecture 4: Capacity

Extension to frequency dependent channel

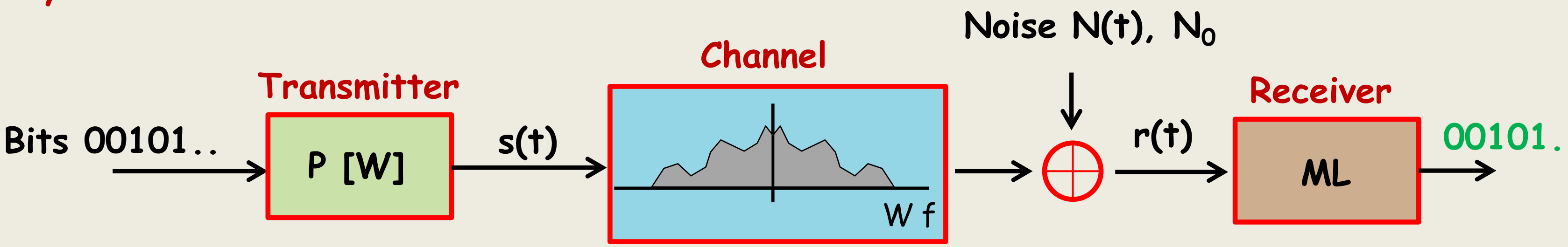
System model:



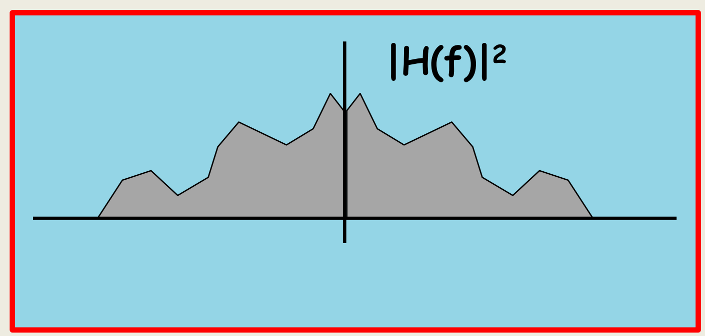
Lecture 4: Capacity

Extension to frequency dependent channel

System model:



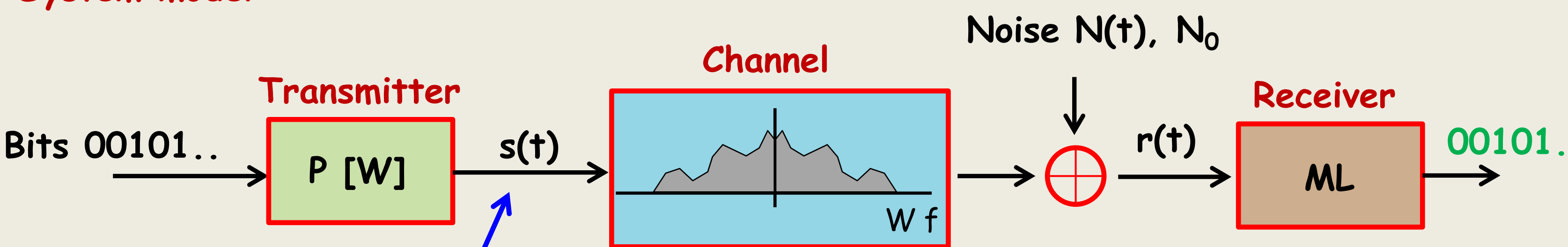
Frequency response of channel



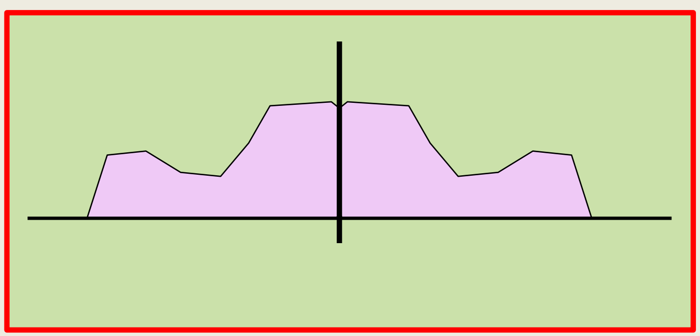
Lecture 4: Capacity

Extension to frequency dependent channel

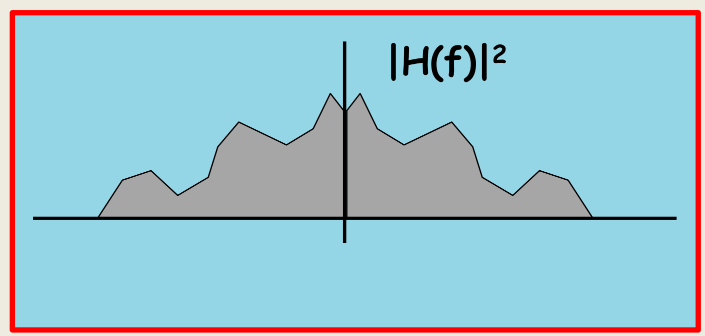
System model:



Signal also have frequency representation



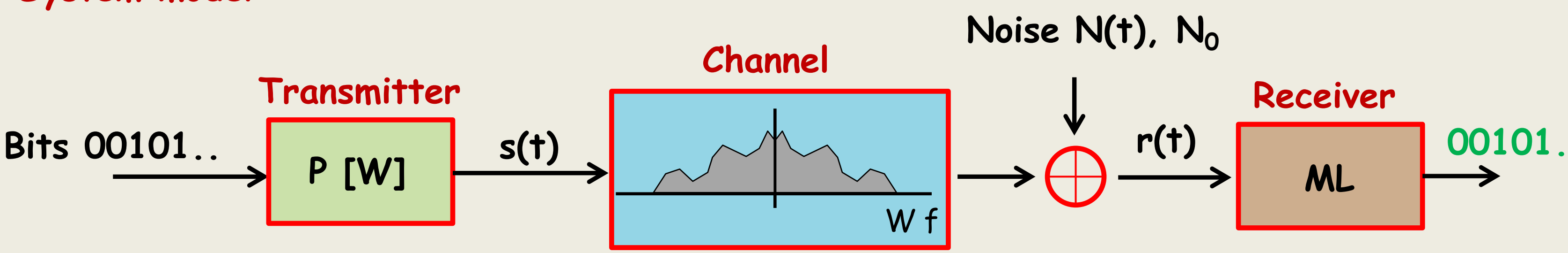
Frequency response of channel



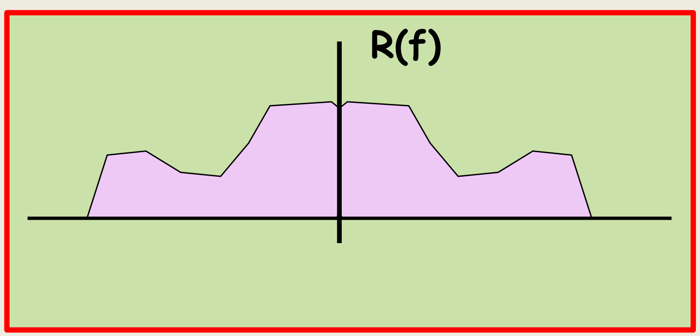
Lecture 4: Capacity

Extension to frequency dependent channel

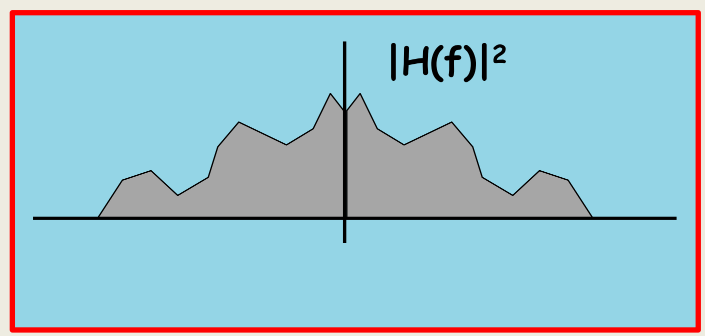
System model:



Power spectral density is what matters



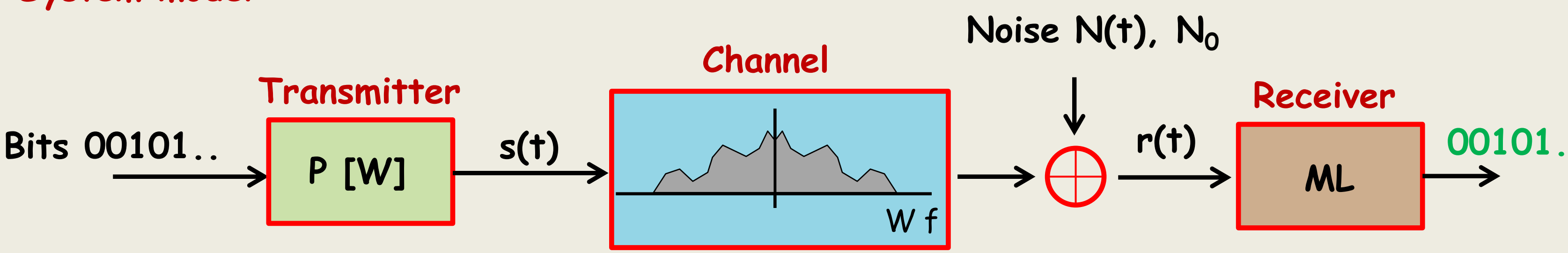
Frequency response of channel



Lecture 4: Capacity

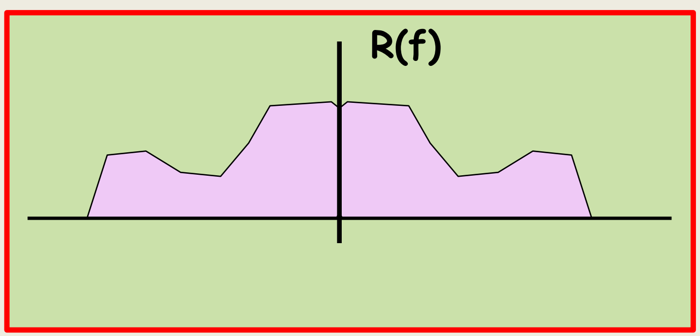
Extension to frequency dependent channel

System model:

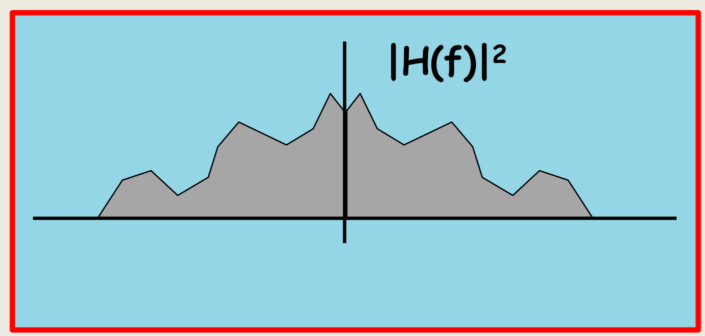


Constraint on R(f) ?

Power spectral density is what matters



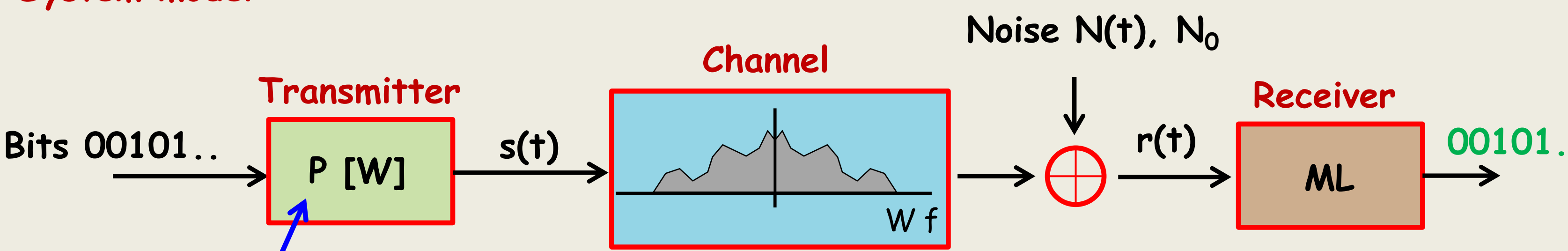
Frequency response of channel



Lecture 4: Capacity

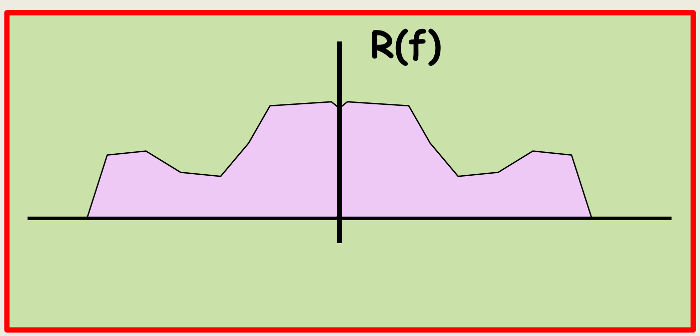
Extension to frequency dependent channel

System model:

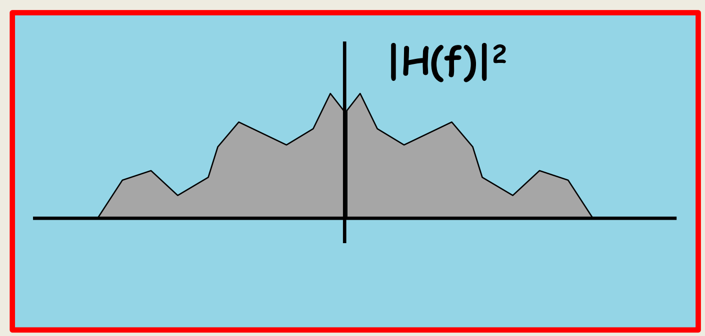


Constraint on R(f) ? $\int_{-\infty}^{\infty} R(f)\mathrm{d}f = P$

Power spectral density is what matters



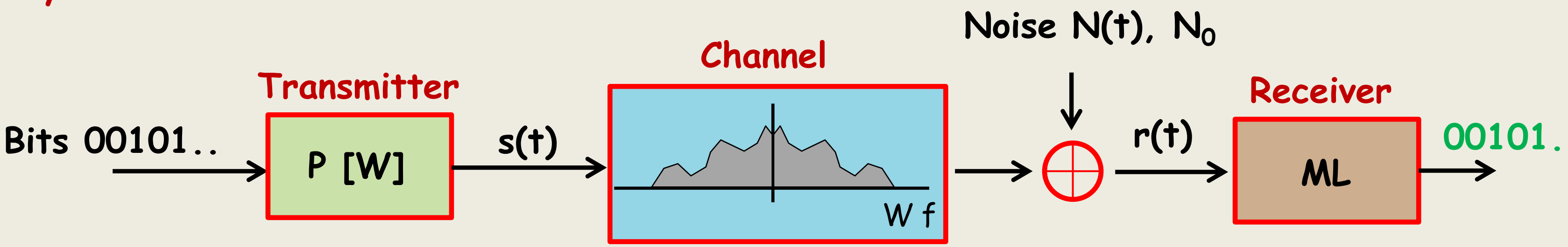
Frequency response of channel



Lecture 4: Capacity

Extension to frequency dependent channel

System model:

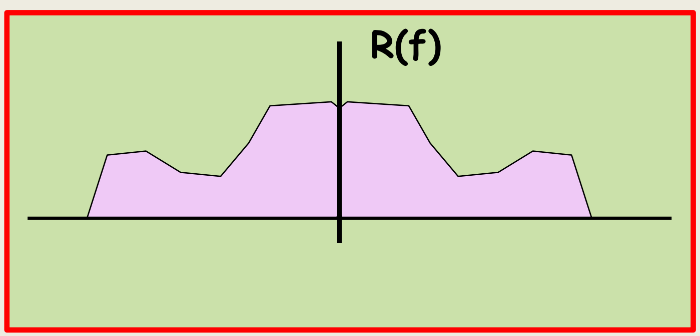


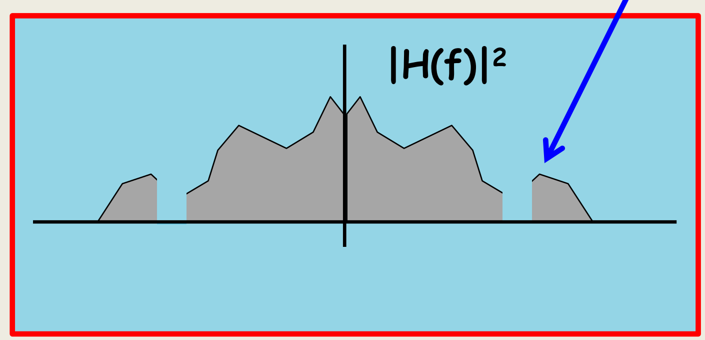
Constraint on R(f) ? $\int_{-\infty}^{\infty} R(f)\mathrm{d}f = P$

If deep fade here

Power spectral density is what matters

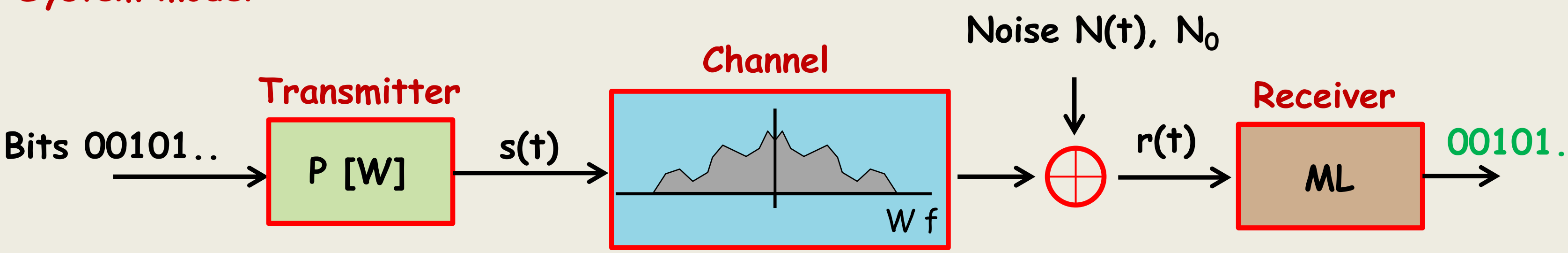
Frequency response of channel





Extension to frequency dependent channel

System model:

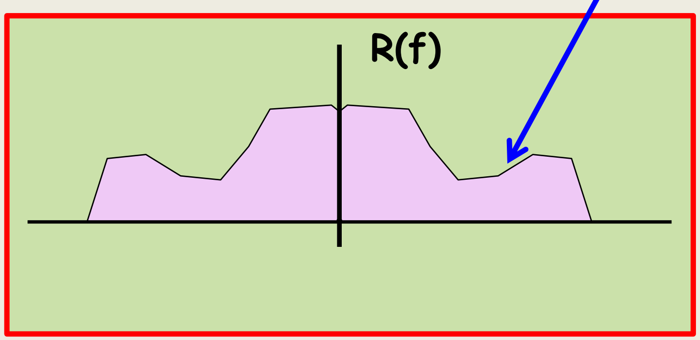


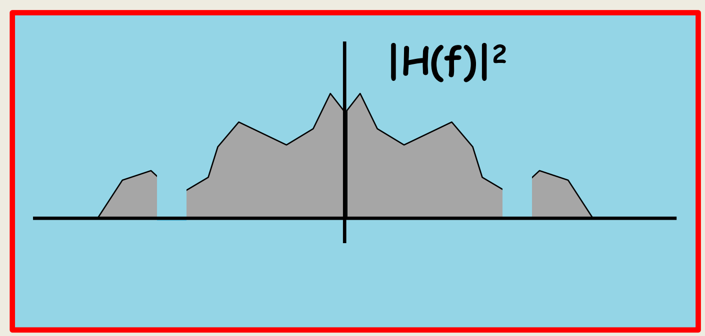
Stupidity to put power here

Constraint on R(f) ? $\int_{-\infty}^{\infty} R(f)\mathrm{d}f = P$

Power spectral density is what matters

Frequency response of channel

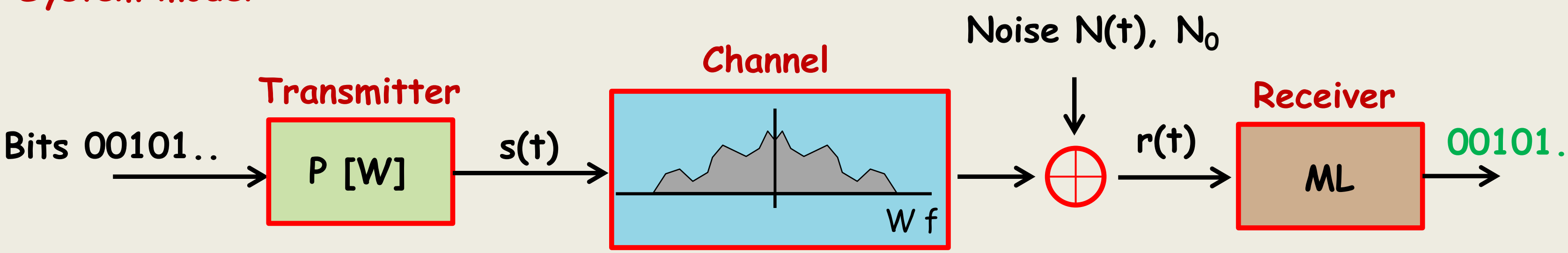




Lecture 4: Capacity

Extension to frequency dependent channel

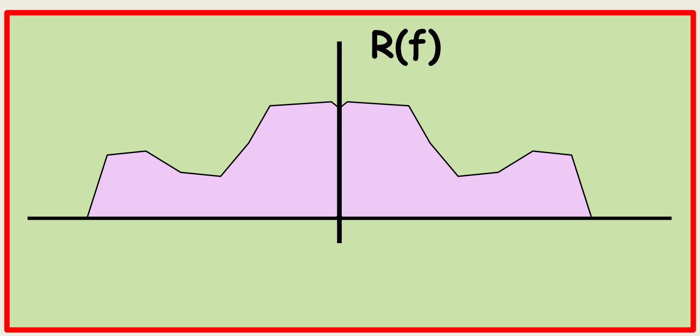
System model:



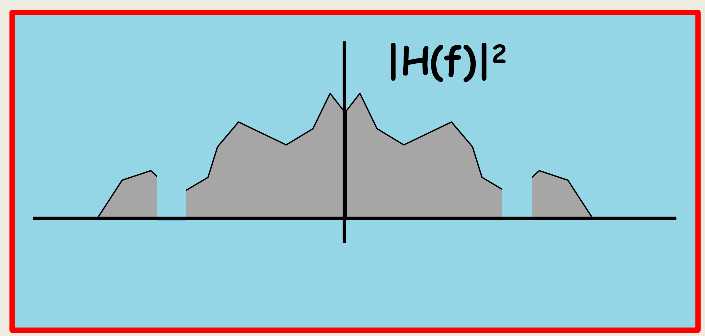
Conclusion: We should optimize the left plot, for the given right plot

Constraint on left plot is $\int_{-\infty}^{\infty} R(f)\mathrm{d}f = P$

Power spectral density is what matters

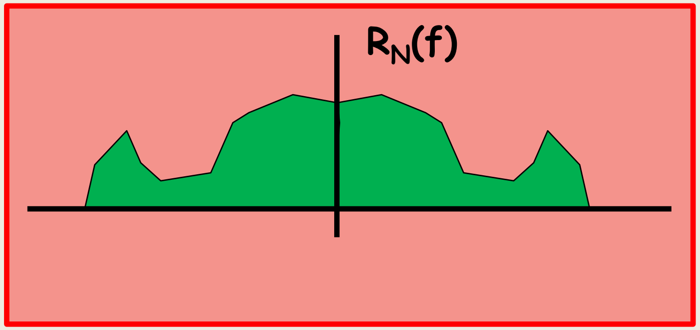


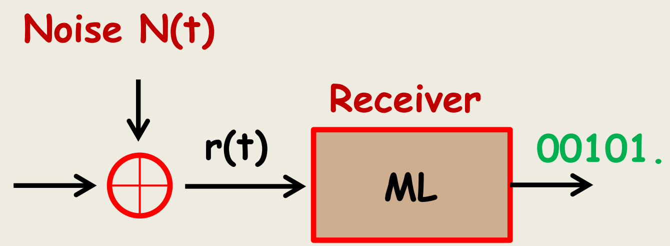
Frequency response of channel



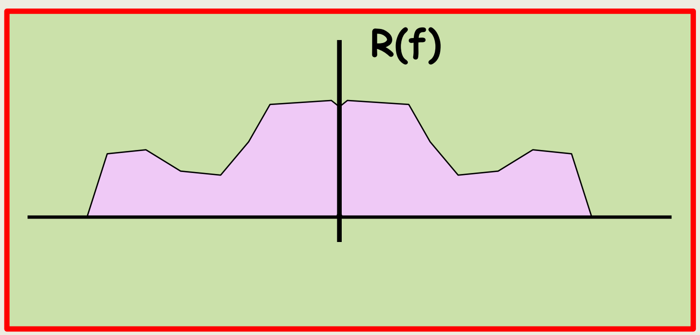
Lecture 4: Capacity

Frequency response of Noise

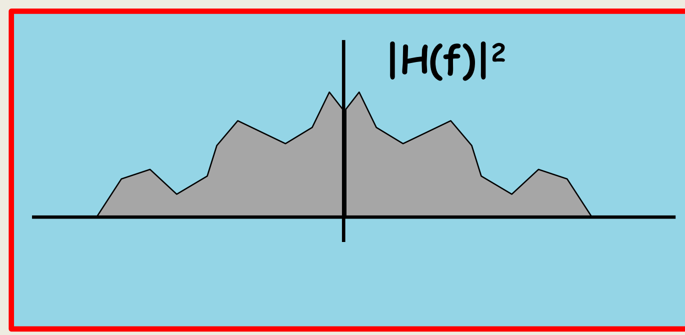




Power spectral density is what matters

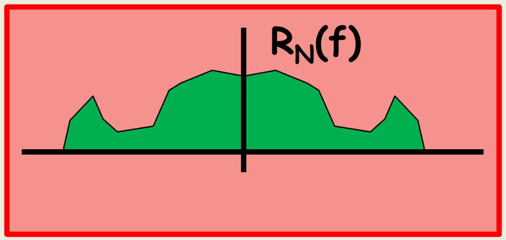


Frequency response of channel

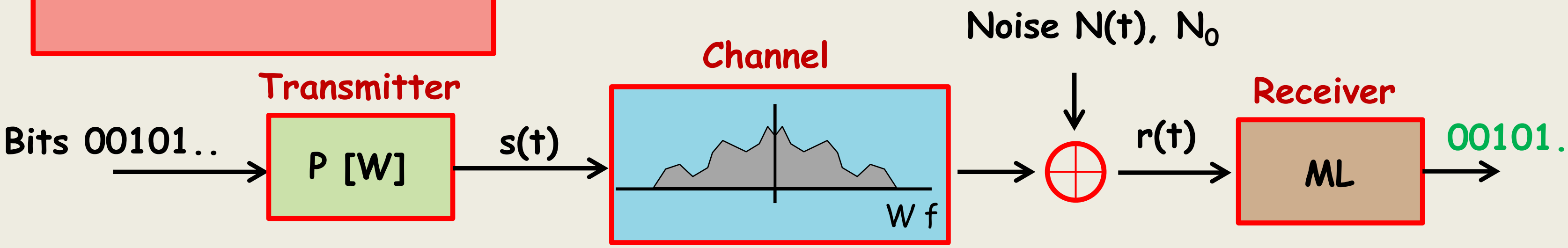


Lecture 4: Capacity

Frequency response of Noise



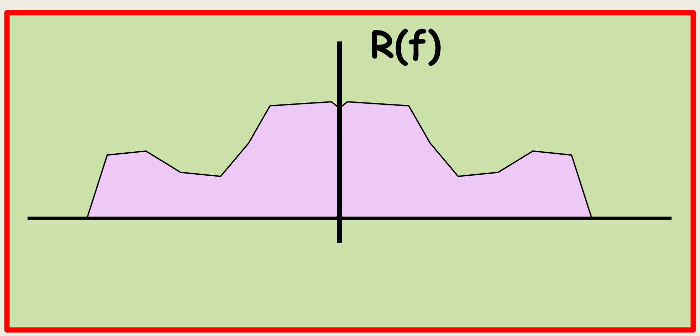
$$C = \max_{R(f): \int R(f)\mathrm{d}f = P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$



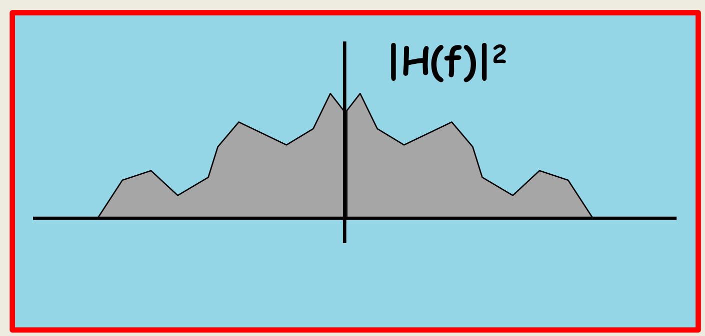
Conclusion: We should optimize the left plot, for the given right plot

Constraint on left plot is $\int_{-\infty}^{\infty} R(f)\mathrm{d}f = P$

Power spectral density is what matters

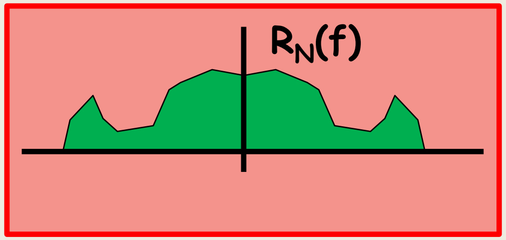


Frequency response of channel



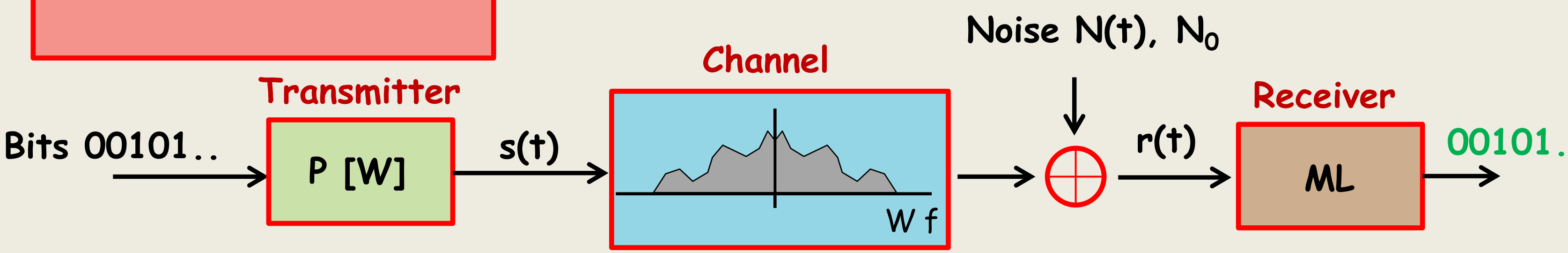
Lecture 4: Capacity

Frequency response of Noise



We need to find this formula

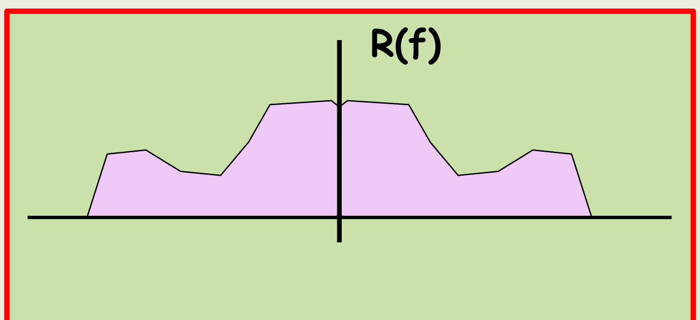
$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$



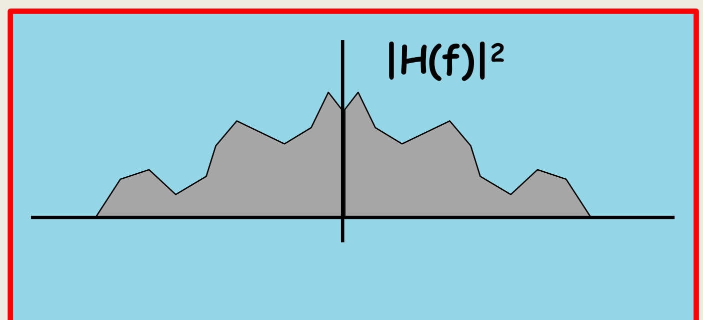
Conclusion: We should optimize the left plot, for the given right plot

Constraint on left plot is $\int_{-\infty}^{\infty} R(f)\mathrm{d}f = P$

Power spectral density is what matters

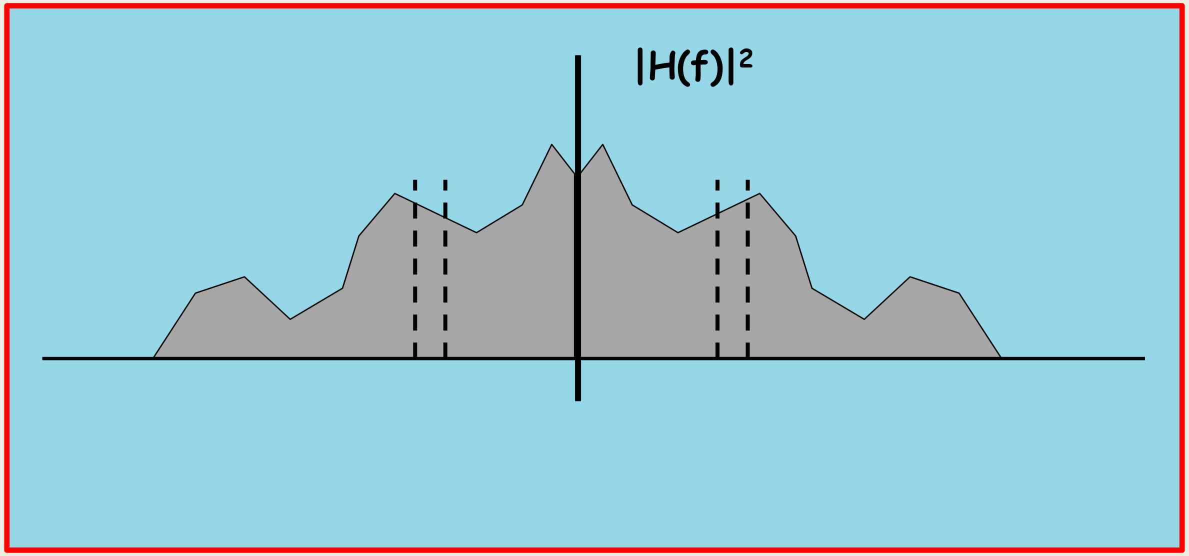


Frequency response of channel



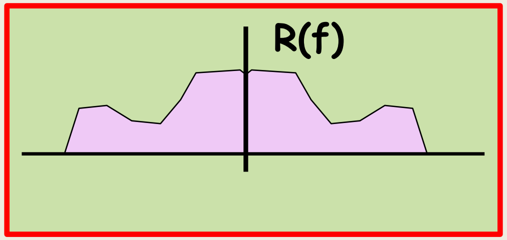
Lecture 4: Capacity

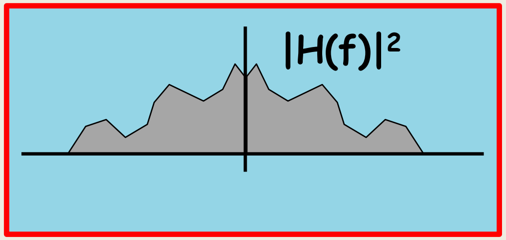
$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \text{Capacity}(|H(f)|^2, R_N(f), R(f))$$

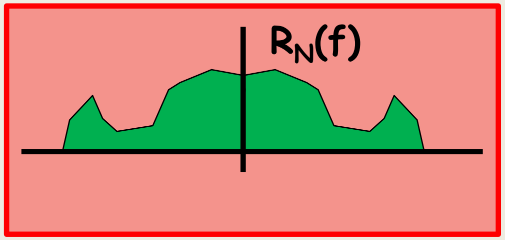


In this small piece
We can use

$$C = W \log_2\left(1 + \frac{\mathcal{P}}{N_0 W}\right)$$

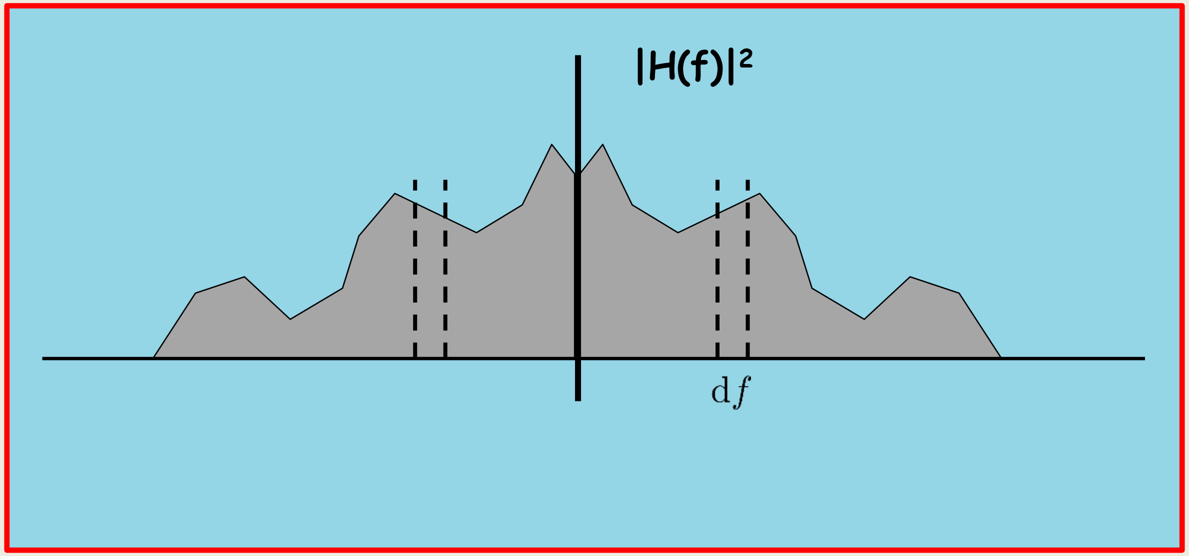






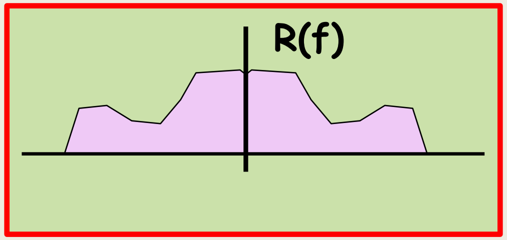
Lecture 4: Capacity

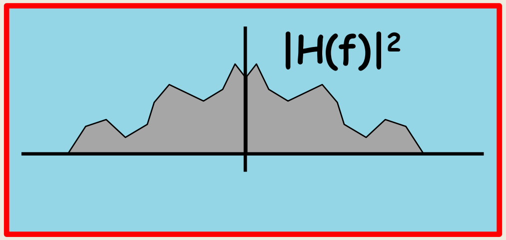
$$C = \max_{R(f): \int R(f)\mathrm{d}f = P} \text{Capacity}(|H(f)|^2, R_N(f), R(f))$$

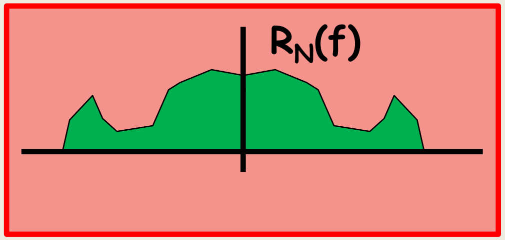


In this small piece
We can use

$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{N_0 \mathrm{d}f}\right)$$

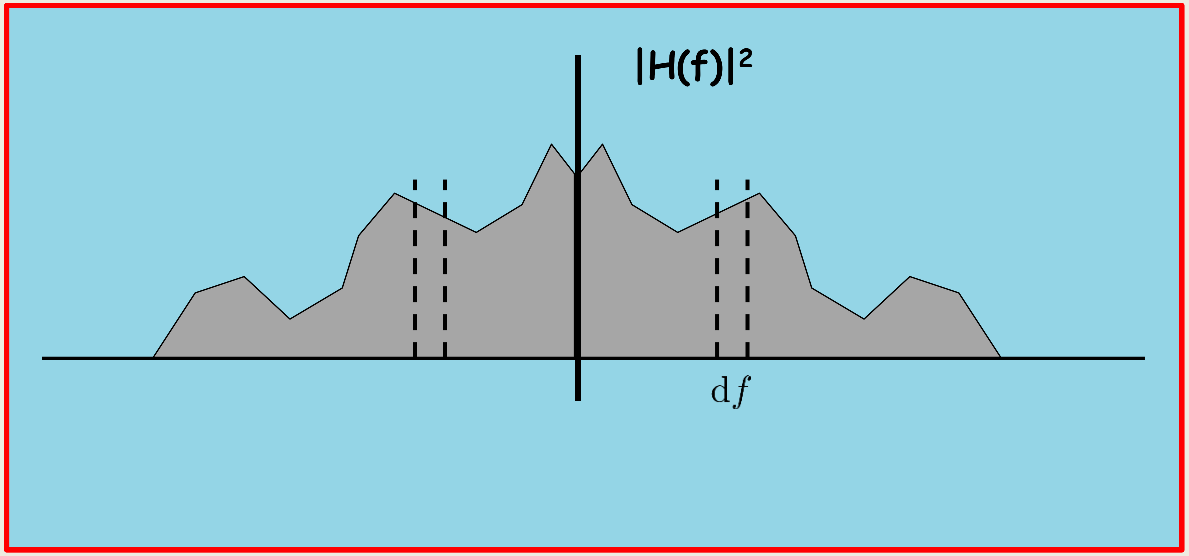






Lecture 4: Capacity

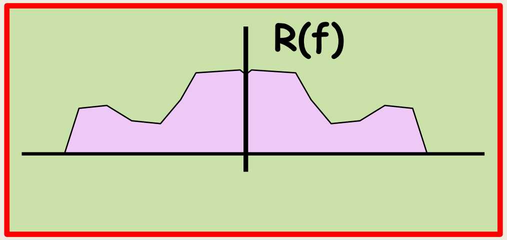
$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \text{Capacity}(|H(f)|^2, R_N(f), R(f))$$

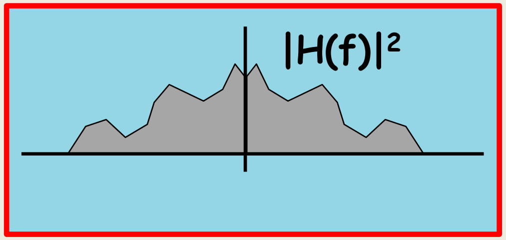


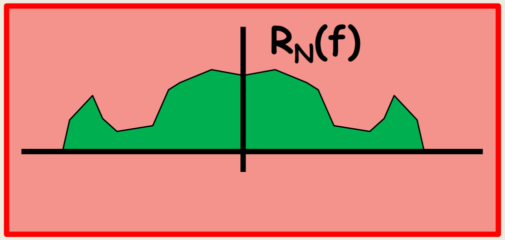
In this small piece
We can use

$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{N_0 \mathrm{d}f}\right)$$

For flat noise,
$R_N(f) = N_0/2$

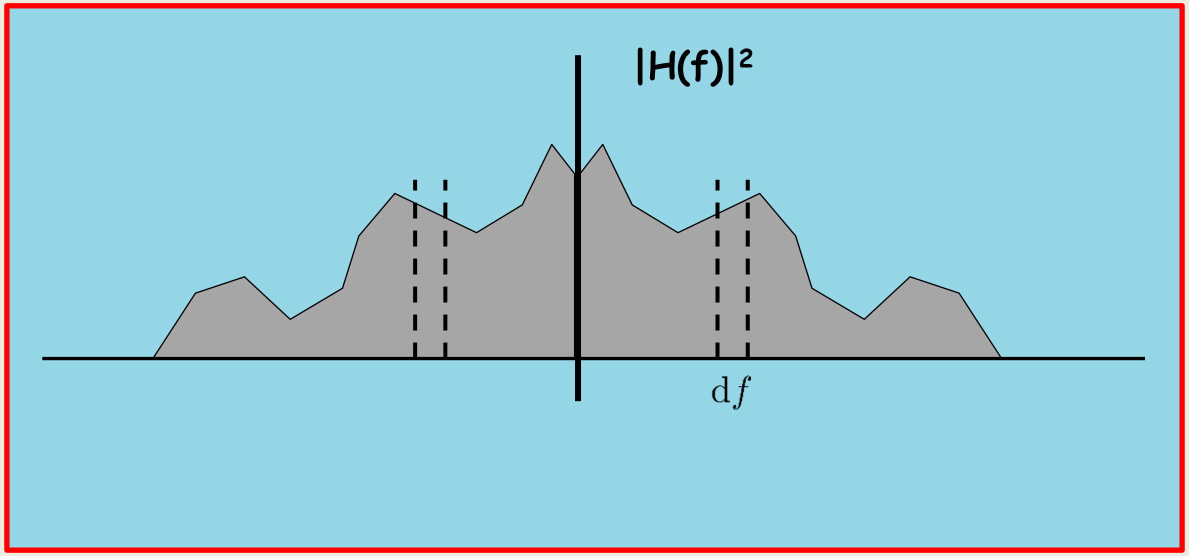






Lecture 4: Capacity

$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

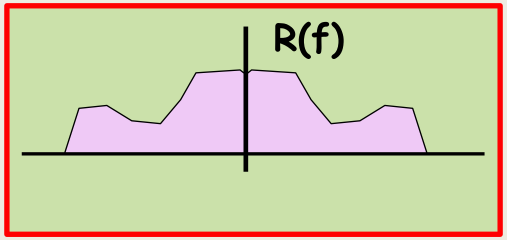


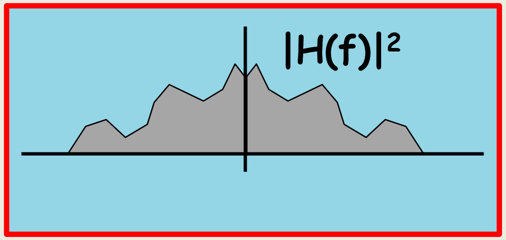
In this small piece
We can use

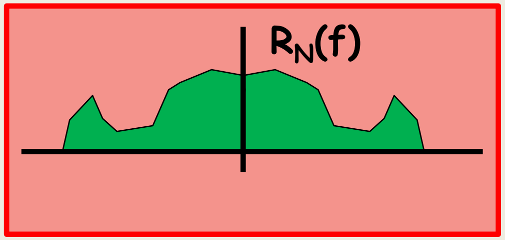
$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{N_0 \mathrm{d}f}\right)$$

$=2R_N(f)$

For flat noise,
$R_N(f) = N_0/2$

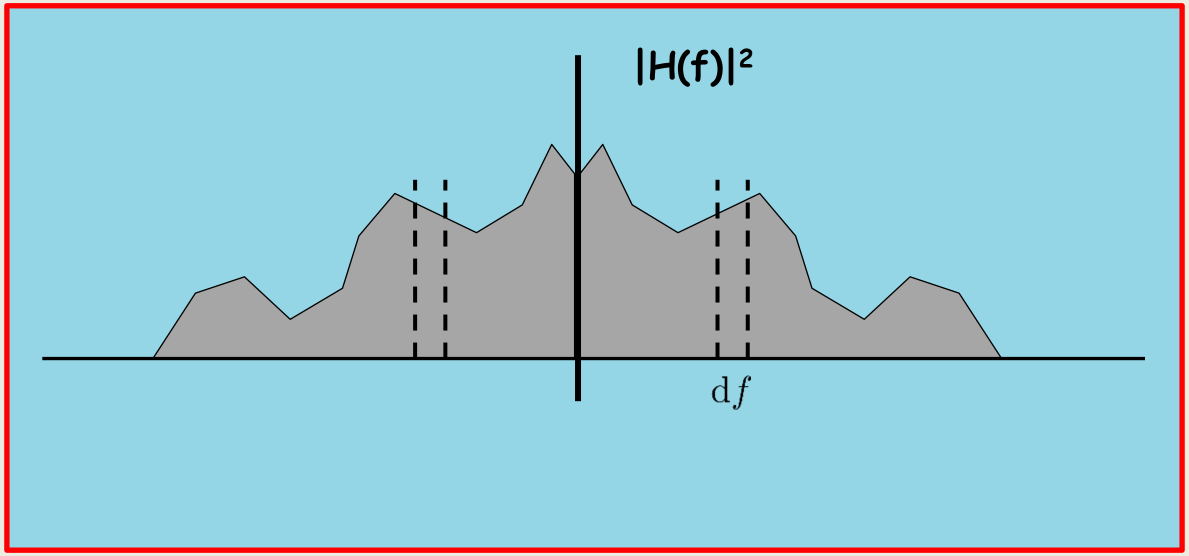






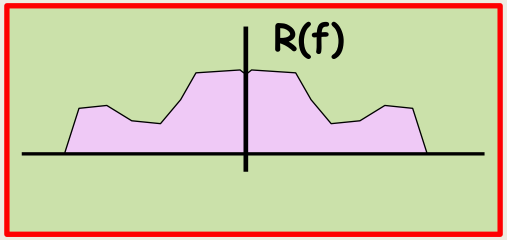
Lecture 4: Capacity

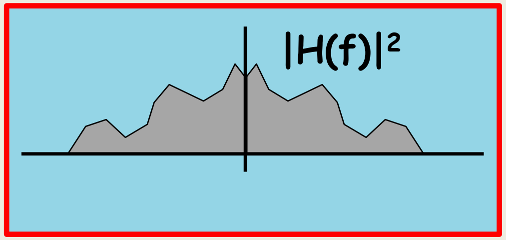
$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

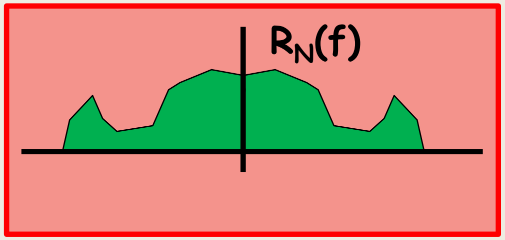


In this small piece
We can use

$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$

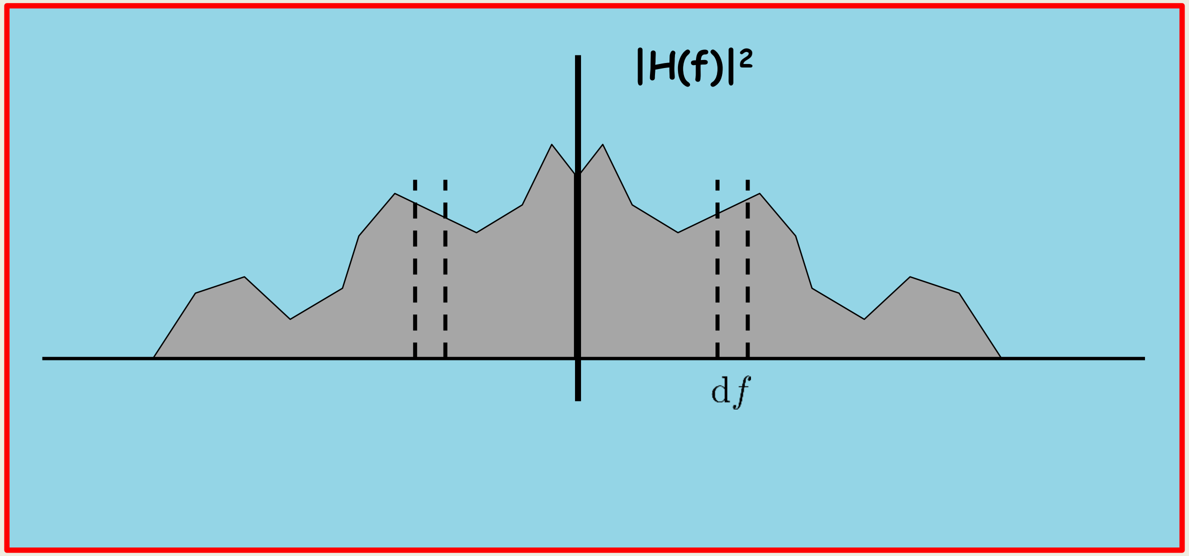






Lecture 4: Capacity

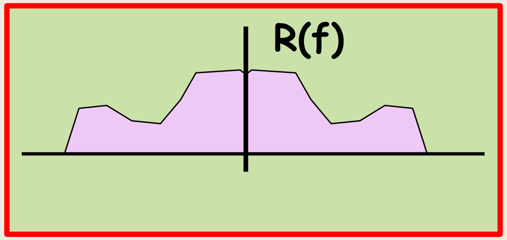
$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

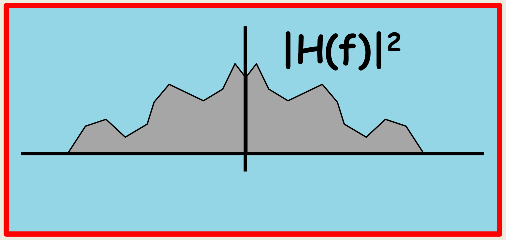


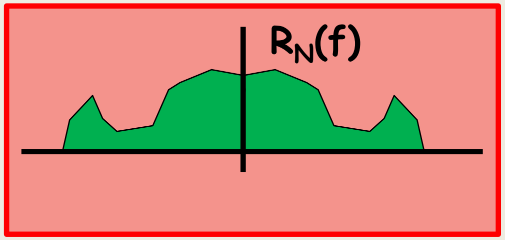
In this small piece
We can use

$$C = \mathrm{d}f \log_2\left(1+\frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$

How much power do
we have?

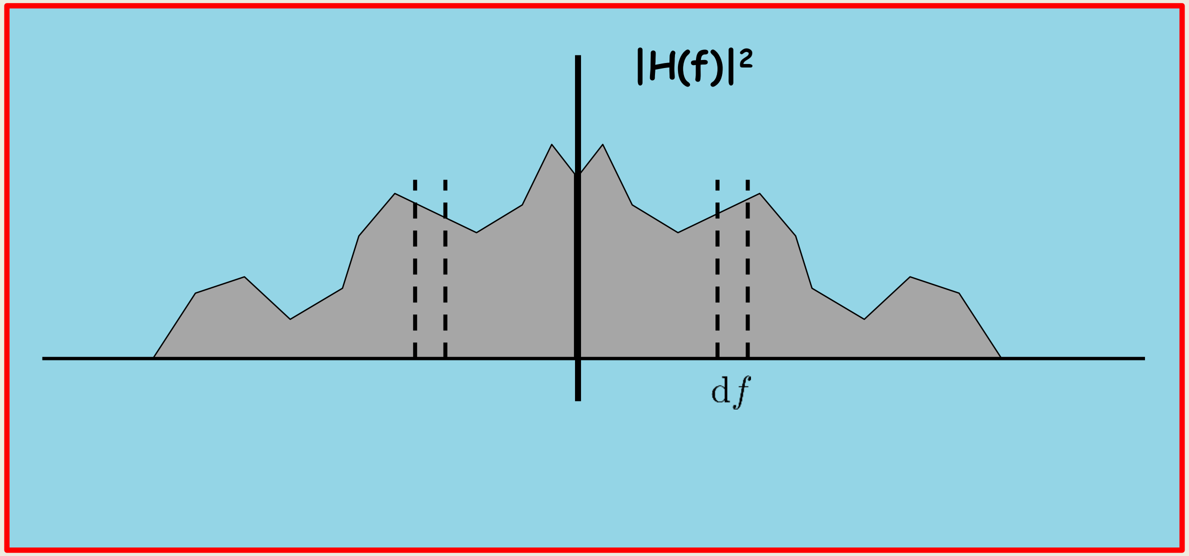






Lecture 4: Capacity

$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

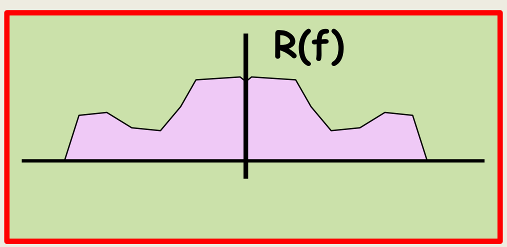


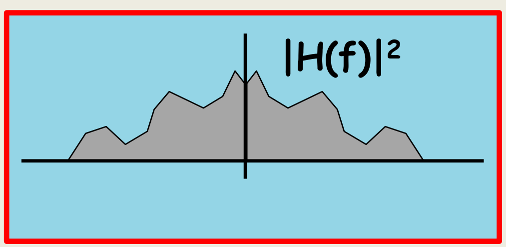
In this small piece
We can use

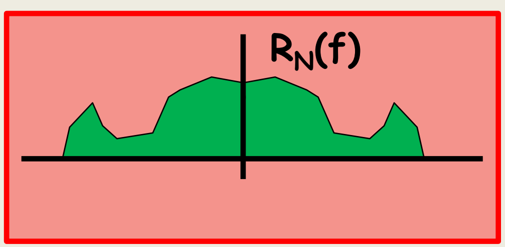
$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$

How much power do
we have?

$$2\mathrm{d}f\, R(f)|H(f)|^2$$

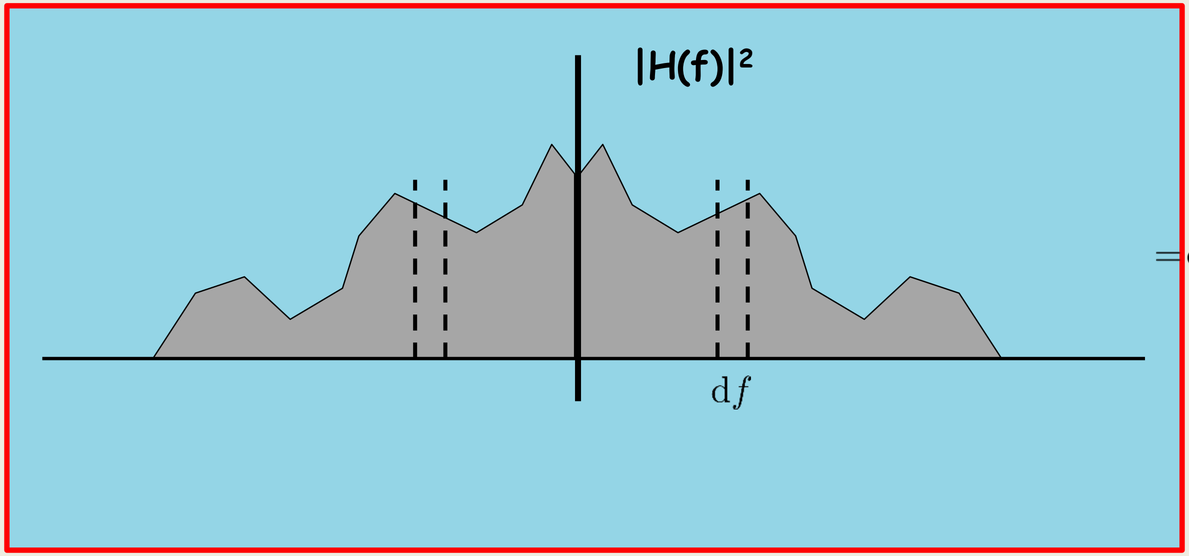






Lecture 4: Capacity

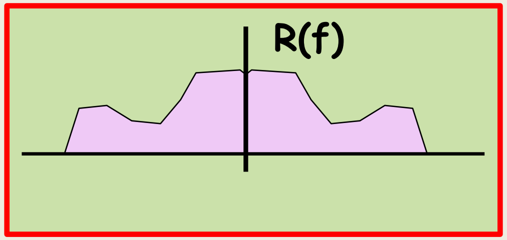
$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

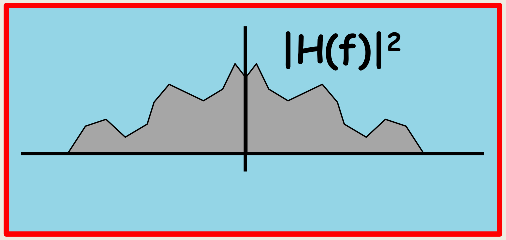


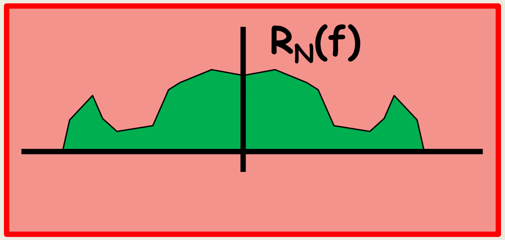
In this small piece
We can use

$$C = \mathrm{d}f \log_2\left(1+\frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$

$$= \mathrm{d}f \log_2\left(1+\frac{2R(f)|H(f)|^2\mathrm{d}f}{2R_N(f)\mathrm{d}f}\right)$$

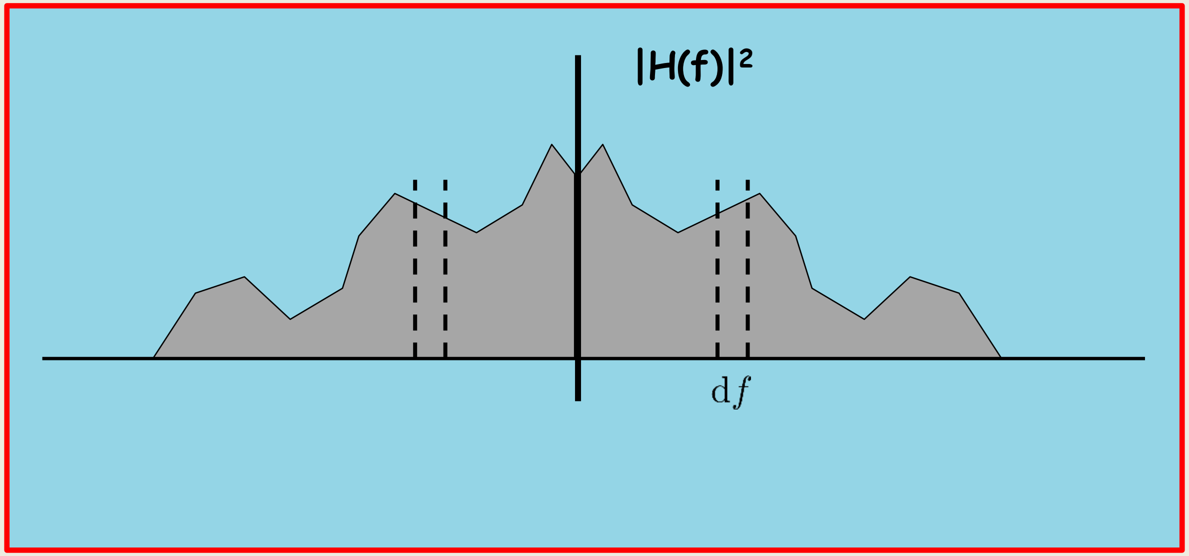






Lecture 4: Capacity

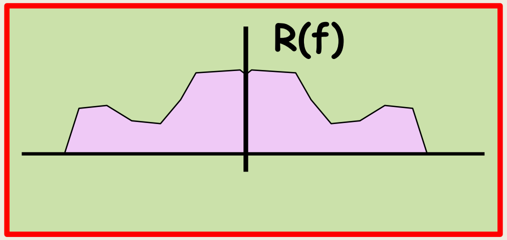
$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

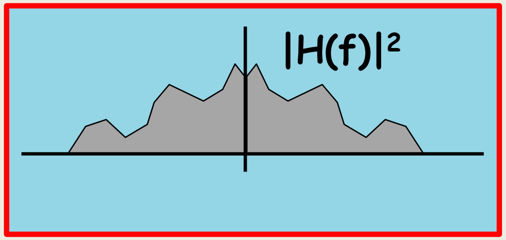


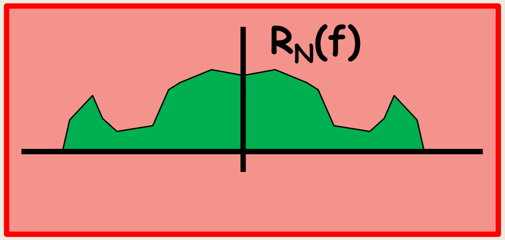
In this small piece
We can use

$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$

$$= \mathrm{d}f \log_2\left(1 + \frac{R(f)|H(f)|^2}{R_N(f)}\right)$$







Lecture 4: Capacity

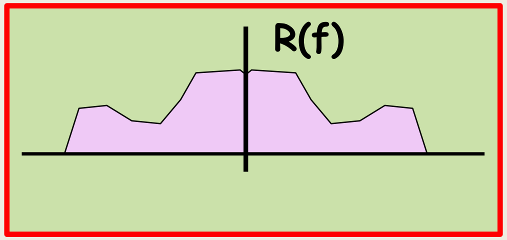
$$C = \max_{R(f): \int R(f)\mathrm{d}f = P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

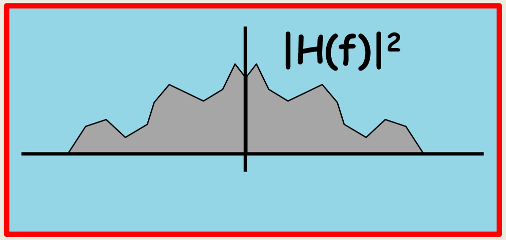
In this small piece
We can use

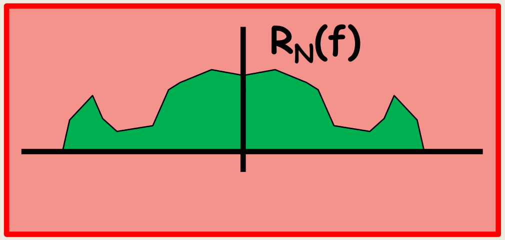
$$C = \mathrm{d}f \log_2\left(1 + \frac{\mathcal{P}}{2R_N(f)\mathrm{d}f}\right)$$
$$= \mathrm{d}f \log_2\left(1 + \frac{R(f)|H(f)|^2}{R_N(f)}\right)$$

Sum up

$$\mathrm{Capacity}(|H(f)|^2, R_N(f), R(f)) = \int_0^\infty \log_2\left(1 + \frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$





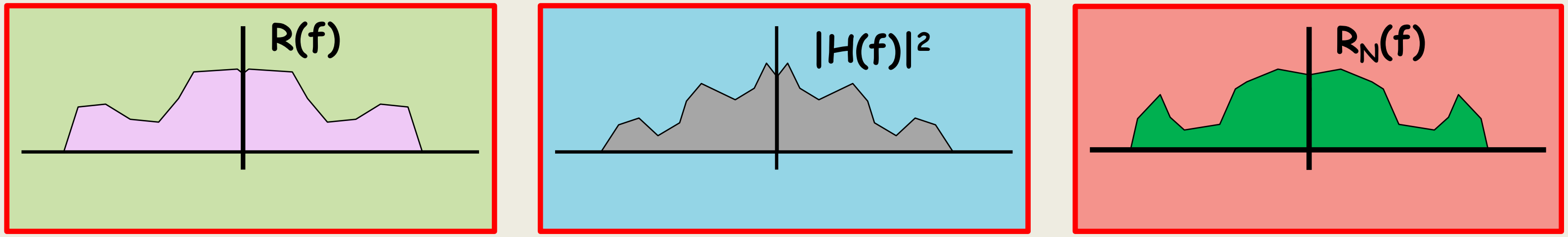


Lecture 4: Capacity

$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \mathrm{Capacity}(|H(f)|^2, R_N(f), R(f))$$

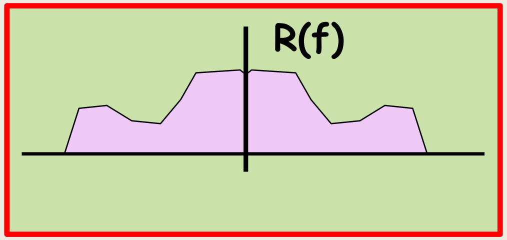
Sum up

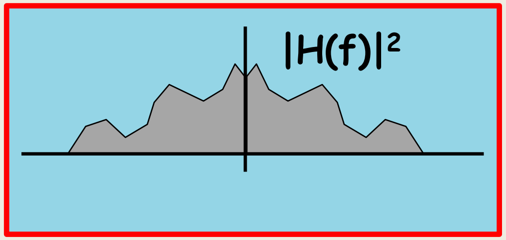
$$\mathrm{Capacity}(|H(f)|^2, R_N(f), R(f)) = \int_0^\infty \log_2\left(1+\frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$
$$= \frac{1}{2}\int_{-\infty}^\infty \log_2\left(1+\frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$

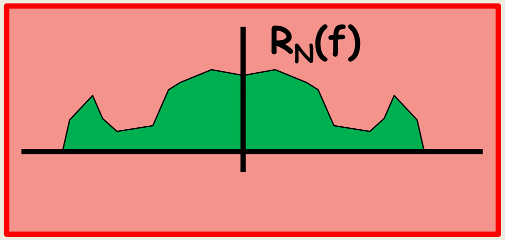


Lecture 4: Capacity

$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1+\frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$



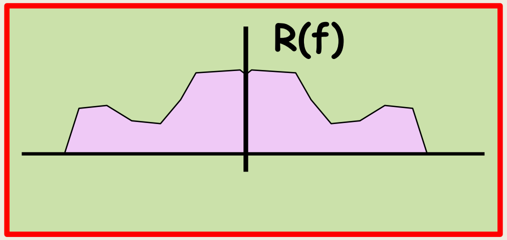


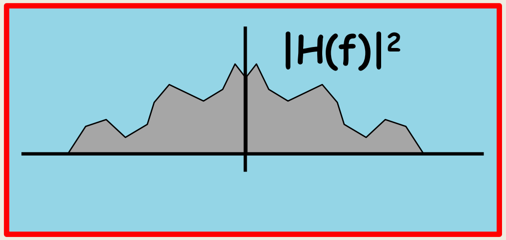


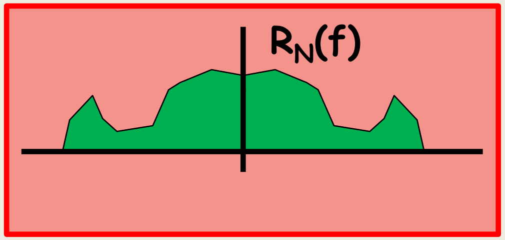
Lecture 4: Capacity

How to solve the below problem? WATERFILLING

$$C = \max_{R(f): \int R(f)\mathrm{d}f = P} \frac{1}{2} \int_{-\infty}^{\infty} \log_2 \left(1 + \frac{R(f)|H(f)|^2}{R_N(f)}\right) \mathrm{d}f$$



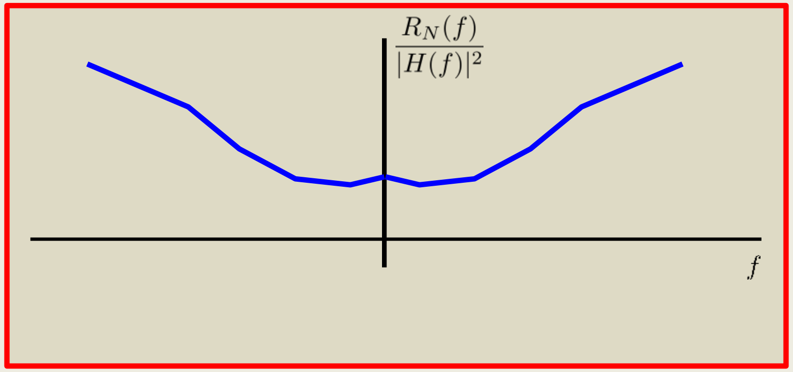




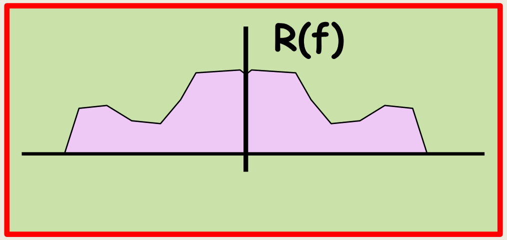
Lecture 4: Capacity

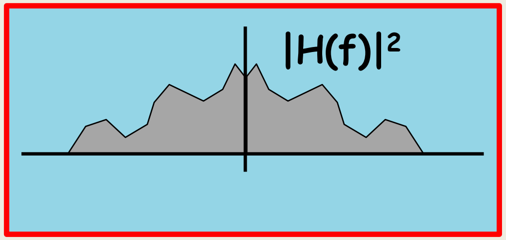
How to solve the below problem? WATERFILLING

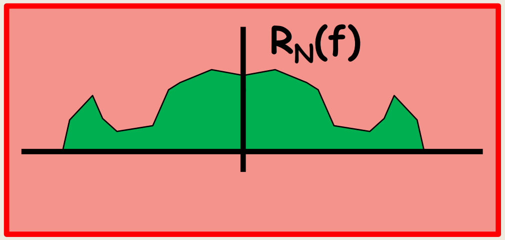
Step 1. Find and plot $\dfrac{R_N(f)}{|H(f)|^2}$



$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1 + \frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$





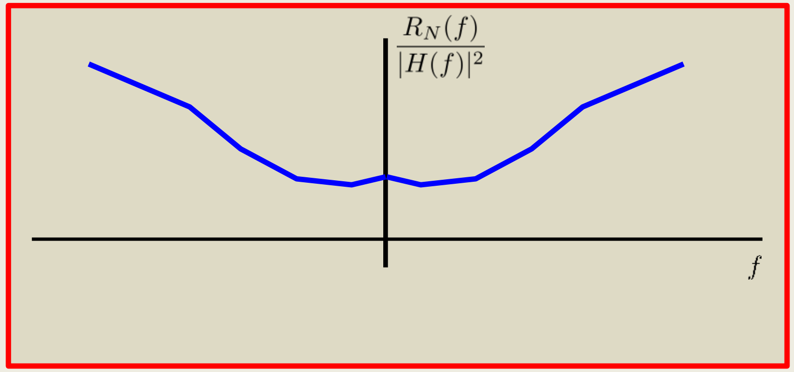


Lecture 4: Capacity

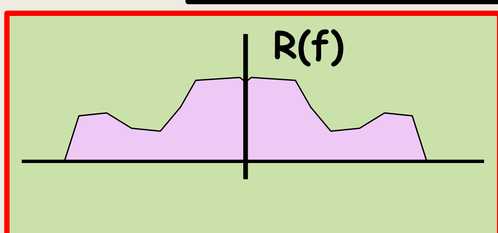
How to solve the below problem? **WATERFILLING**

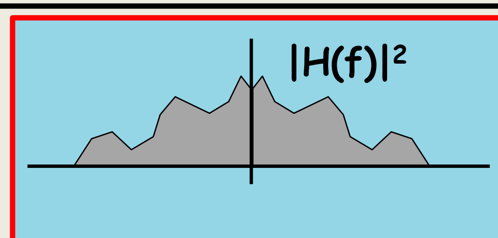
Step 1. Find and plot $\dfrac{R_N(f)}{|H(f)|^2}$

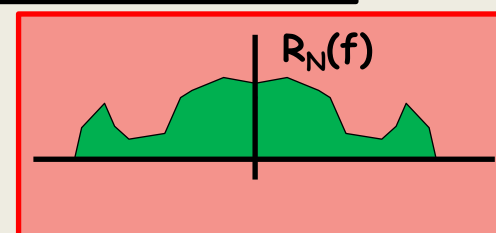
Step 2. Fill a bucket with P units of water



$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1 + \frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$







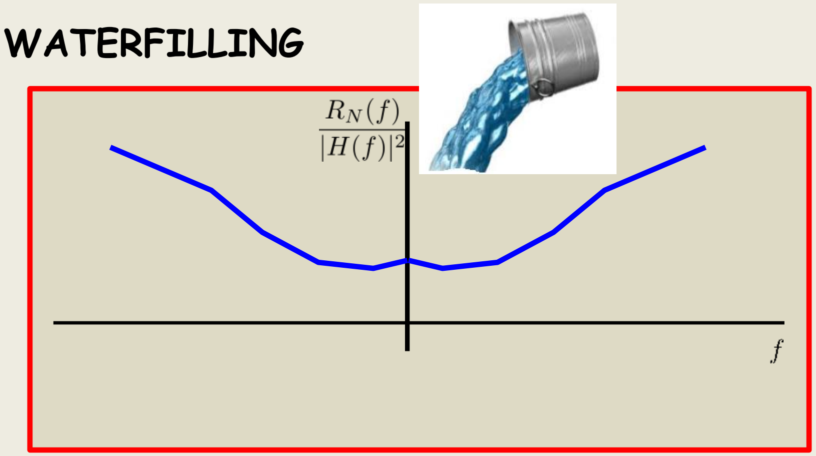
Lecture 4: Capacity

How to solve the below problem? WATERFILLING

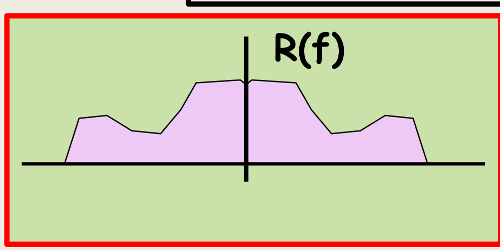
Step 1. Find and plot $\frac{R_N(f)}{|H(f)|^2}$

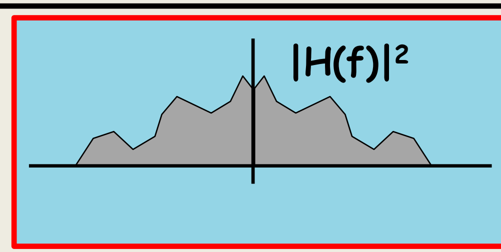
Step 2. Fill a bucket with P units of water

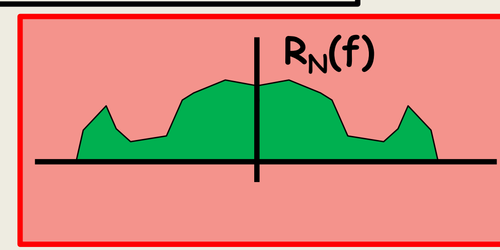
Step 3. Pour it in the shape



$$C = \max_{R(f):\int R(f)\mathrm{d}f=P} \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1+\frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$







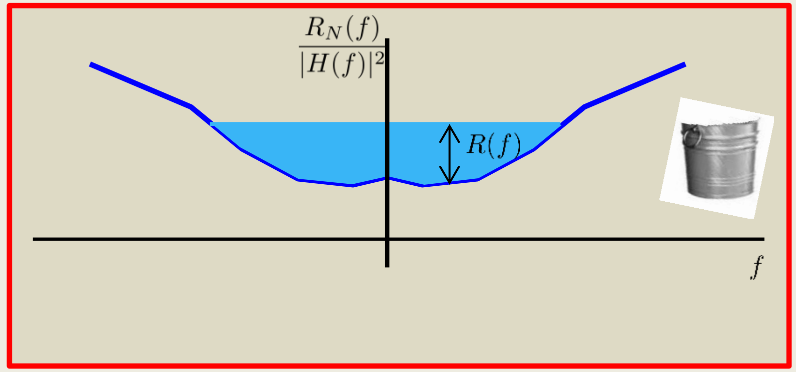
Lecture 4: Capacity

How to solve the below problem? WATERFILLING

Step 1. Find and plot $\dfrac{R_N(f)}{|H(f)|^2}$

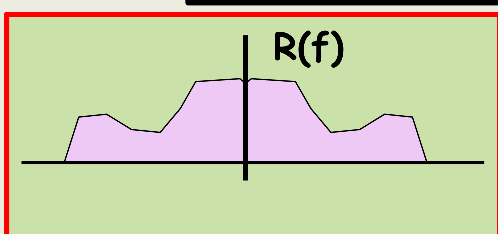
Step 2. Fill a bucket with P units of water

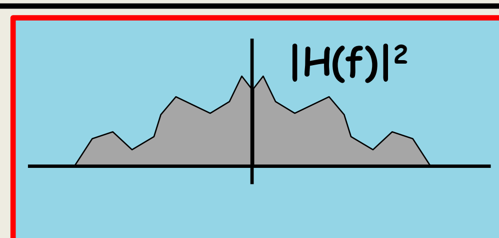
Step 3. Pour it in the shape

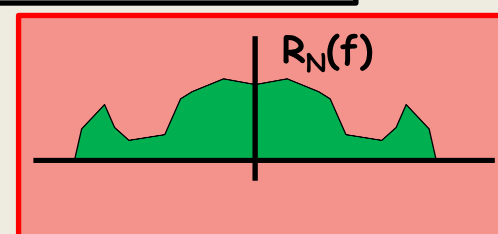


Step 4. R(f) is the water-level

$$C = \max_{R(f):\int R(f)\mathrm{d}f = P} \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1 + \frac{R(f)|H(f)|^2}{R_N(f)}\right)\mathrm{d}f$$



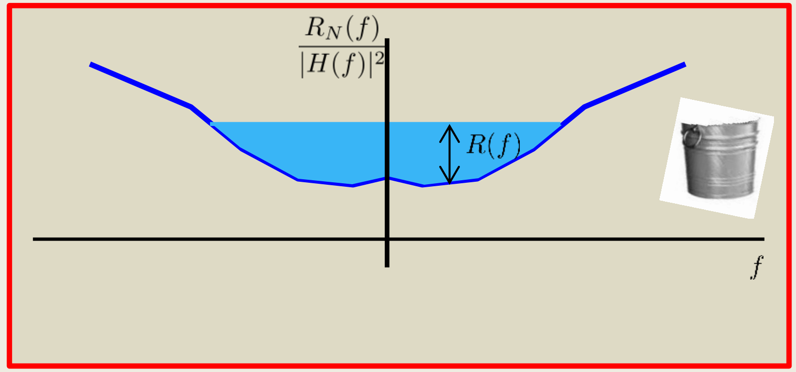




Lecture 4: Capacity

How to solve the below problem? WATERFILLING

Step 1. Find and plot $\dfrac{R_N(f)}{|H(f)|^2}$

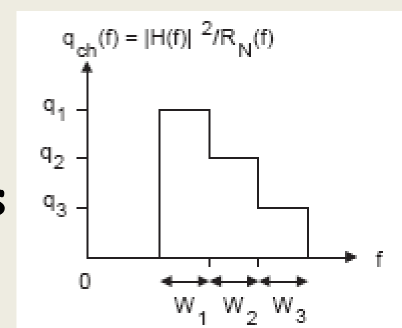


Step 2. Fill a bucket with P units of water

Step 3. Pour it in the shape

Step 4.
R(f) is the water-level

On Exam, |H(f)|^2 would be "nice", such as



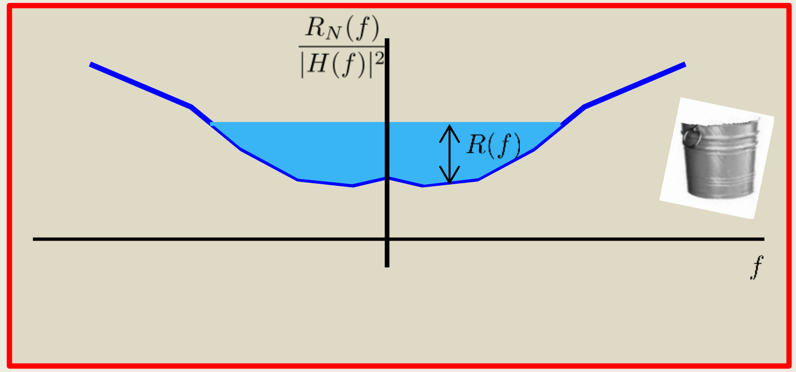
Lecture 4: Capacity

How to solve the below problem? WATERFILLING

Step 1. Find and plot $\frac{R_N(f)}{|H(f)|^2}$

Step 2. Fill a bucket with P units of water

Step 3. Pour it in the shape



Step 4. R(f) is the water-level

Observations:

1. Good channels get more power than bad
2. At very high SNRs, all channels get, roughly, the same power