Exam in Advanced Digital Communications, ETTN01, Lund University

January 9, 2019

- Allowed tools: Book, OFDM compendium, pocket calculator.
- Write clearly. If I have problems reading what you wrote I will mark it as 0 points.
- Motivate your answers.

Problem 1. Some manipulations with a signal set [11p]

Assume the four equally likely signals

$$s_0(t) = -s_2(t) = \alpha t/T, \qquad s_1(t) = -s_3(t) = \alpha(1 - t/T).$$

Assume an AWGN channel and an ML receiver, and that all signals are equal to zero outside the interval $0 \leq t \leq T$. The following sub-problems do not build upon each other, e.g., in problem e), the text in d) does not longer apply.

a. Draw the signal space in a figure.

b. Determine the symbol error probability exactly.

c. Compute the minimum distance

d. Now add signals $s_4(t) = -s_5(t) = \alpha(2 + t/T)$. Draw the signal space in a figure.

e. Now assume that the four signals are not equally likely, but occur with probabilities 0.5, 0.3, 0.1, and 0.1, respectively. Draw the signal space in a figure.

f. Now replace signals $s_1(t)$ and $s_3(t)$ with two new signals. To maximize the normalized Euclidean distance, which two signals should be added ? Provide time-domain formulas.

g. In this problem you do not have to solve any integrals. Instead, you may label any integral that you need to solve by some variable name (I suggest I_1 for the first, I_2 for the second etc) and then use these variables in subsequent derivations. Now add a fifth signal $s_4(t)$ of the form $s_4(t) = \beta_1 + \beta_2 t^2/T$ where $\beta_1 \neq 0$. Provide derivations that would clarify whether or not the signal $s_4(t)$ can be fully contained in the signal space of signals $\{s_k(t)\}_{k=0}^{3}$.

Problem 2. A collection of largely unconnected signal space problems [12p]

a. Assume that the two signals $x_1(t)$ and $x_2(t)$ are orthogonal, and also that $y_1(t)$ and $y_2(t)$ are orthogonal. Define $z_1(t) = x_1(t) + y_1(t)$ and $z_2(t) = x_2(t) + y_2(t)$. Is $z_1(t)$ orthogonal to $z_2(t)$?

b. Define the signal

$$u(t) = \sum_{n=1}^{N} u_n \psi_n(t)$$

where

$$\int_{-\infty}^{\infty} \psi_k(t)\psi_\ell(t)\mathrm{d}t = \begin{cases} A_k, & k = \ell \\ 0, & k \neq \ell. \end{cases}$$

Compute the energy of the signal $u(t)$. Simplify your answer as far as possible.

c. Define the signal

$$u(t) = \sum_{n=1}^{2} u_n \psi_n(t)$$

where

$$\int_{-\infty}^{\infty} \psi_1^2(t)\mathrm{d}t = \int_{-\infty}^{\infty} \psi_2^2(t)\mathrm{d}t = 1, \qquad \int_{-\infty}^{\infty} \psi_1(t)\psi_2(t)\mathrm{d}t = 1/2.$$

Compute the energy of the signal $u(t)$. Simplify your answer as far as possible.

d. Assume an arbitrary signal $y(t)$ that we would like to approximate by another signal $x(t) = \sum_{n=1}^{N} x_n \phi_n(t)$ where

$$\int_{-\infty}^{\infty} \phi_k(t)\phi_\ell(t)\mathrm{d}t = \begin{cases} 1, & k = \ell \\ 0, & k \neq \ell \end{cases}$$

for some cleverly chosen values of $x_n,\ 1 \leq n \leq N$. Solve the following approximation-problem:

$$\arg \min_{\{x_n\}_{n=1}^{N}} \int_{-\infty}^{\infty} (y(t) - x(t))^2 \mathrm{d}t.$$

e. For the problem in d), provide a one sentence interpretation of the value

$$\min_{\{x_n\}_{n=1}^{N}} \int_{-\infty}^{\infty} (y(t) - x(t))^2 \mathrm{d}t.$$

f. Engineer A is operating a transmitter, and transmits signals of the form $s(t) = a_{k,1}\phi_1(t) + a_{k,2}\phi_2(t)$ where $a_{k,n}$ are signal space representations of the messages, and the functions $\phi_1(t)$ and $\phi_2(t)$ are orthonormal. Engineer B is receiving the signal that Engineer A is transmitting, by means of an ML receiver, and the error probability is small. One day, Engineer A and B happen to meet and they discuss their communcation system. In particular, Engineer A asks Engineer B which basis functions that Engineer B is using. Clearly, there are three possible alternatives: (1) Engineer B uses the very same two basis functions. (2) Engineer B uses $\phi_1(t)$ and/or $\phi_2(t)$, but also additional basis functions. (3) Engineer B do not use any of $\phi_1(t)$ or $\phi_2(t)$, but some others that are orthogonal to both $\phi_1(t)$ and $\phi_2(t)$. Are any of the alternatives (1) - (3) impossible? Motivate your answer.

Problem 3. OFDM [11p]

a. Using notation from the compendium, let $z(t)$ denote the received OFDM signal in the time interval $T_{\mathrm{cp}} \leq t \leq T_{\mathrm{s}}$. Provide a set of orthonormal basis functions for said signal $z(t)$. Illustrate the received signal space using figures. Be as precise as you can be.

b. This problem relates to Equations (3.3) and (3.5) in the OFDM compendium, and we assume for the sake of simplicity that $K = 2$ and that BPSK is used, i.e., $a_k \in \{\pm 1\}$. By close inspection of Equation (3.5), it can be seen that (3.5) can in fact be written as

$$s(t) = a_0\psi_0(t) + a_1\psi_1(t).$$

Now, the "O" in OFDM means "orthogonal". However, it is easy to show, either by hand or by means of Matlab, that $\int_{-\infty}^{\infty} \psi_0(t)\psi_1(t)\mathrm{d}t \neq 0$. Consequently, $\psi_0(t)$ and $\psi_1(t)$ do not constitute an orthonormal set of basis functions. Explain why this happens, but no essays please. Assume you are explaining it to someone that firmly believed that said integral would evalaute to 0.

(If you did not bring the compendium: K is the number if subcarriers, a_k is the data symbol sent on subcarrier k, and $s(t)$ is the transmitted signal.)

Problem 4. Misc. [5p]

Assume an OFDM system, and use notation from the OFDM compendium in your derivations. The compendium is describing the transmitted and received signal during one symbol interval, but in an actual system there are of course many OFDM symbols in time adjacent to each other. Let $a_{k,m}$ denote the symbol transmitted at the kth subcarrier during symbol period m. The received signal is affected by the channel response $h(t)$, and in order to decode the data it is important to estimate the effect of the channel. In order to do so, all practical systems include known training symbols (a.k.a. pilot symbols). This means that some of the symbols satisfy $a_{k,m} = 1$. Let us assume a regular spacing of the training symbols according to $a_{k'X,m'Y} = 1$ for some integer values X and Y, and where $k' = 0, 1, 2, ...$, $m' = 0, 1, 2, ...$ Discuss what it is that determines X and Y, and give an approximative formula for the spectral efficiency loss due to training. You may assume that the cyclic prefix duration is negliable and that the SNR is very high.

Problem 5. Capacity [11 p]

a. Engineer A claims the following: "According to Shannon's formula, if Bandwidth grows towards infinity, the capacity is anyway finite." Engineer B thinks about this for a while and then replies: "That cannot be correct, because if I use 2PAM with a basic pulse shape, then if I make the pulse width T smaller and smaller, bandwidth will grow towards infinity. Howerer, my transmission rate is linearly increasing in $1/T$, so I am not bounded by any finite limit."

Provide a brief explanation to them that clarifies the situation.

b. A communication system is operating in a channel with transfer function according to

$$|H(f)|^2 = \begin{cases} A, & 1000\alpha \le f \le 2000\alpha \\ 2A, & 2000\alpha \le f \le 3000\alpha \\ 4A, & 3000\alpha \le f \le 4000\alpha. \end{cases}$$

The PSD of the transmitted signal is given by

$$R(f) = \begin{cases} 1, & 1000\alpha \le f \le 2000\alpha \\ 2, & 2000\alpha \le f \le 3000\alpha \\ 4, & 3000\alpha \le f \le 4000\alpha. \end{cases}$$

Compute capacity of the system.

c. This is a direct continuation of b). Using the low and high SNR approximations,

$$\log_2(1+x) \approx \log_2(x), \quad x \gg 1 \qquad\qquad \log_2(1+x) \approx x/\ln(2), \quad x \approx 0,$$

analyze how much each one of the three bands of the channel (one band is 1000α Hz) is contributing to the overall capacity. As bandwidth is expensive, do we really need all bands in all cases?

d. This is a direct continuation of b). Is the PSD found by the waterfilling method? That is, is it the optimal PSD for the channel?