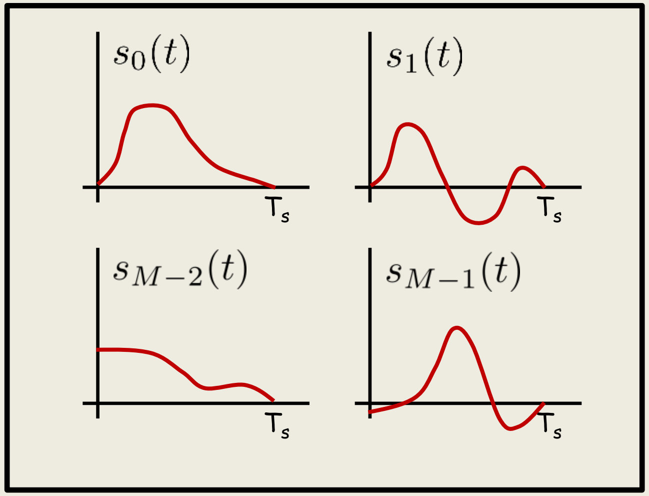
Lecture 3: Error Probabilities

Recap

System model:

1. A known signal set $\{s_\ell(t)\}_{\ell=0}^{M-1}$
2. White Gaussian noise $r(t) = s_a(t) + N(t)$



Lecture 3: Error Probabilities

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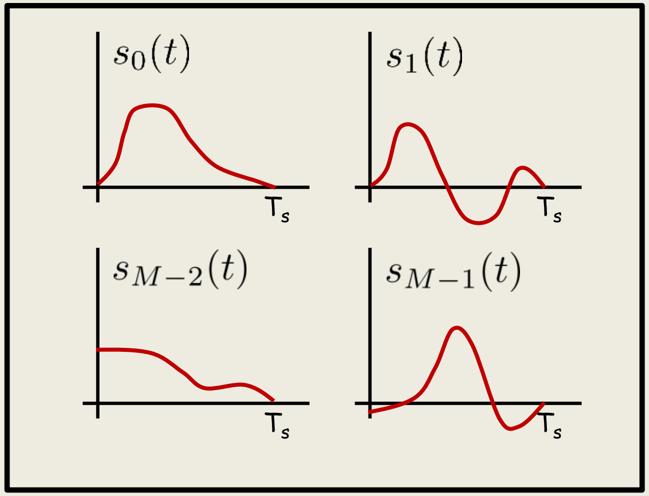
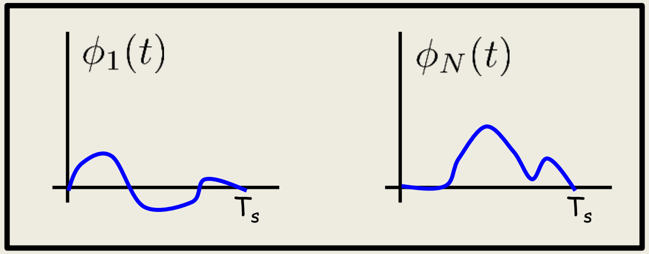
Signal space expansion

1. Find orthonormal basis functions giving smallest N such that

$$s_\ell(t) = \sum_n^N s_{\ell,n}\phi_n(t)$$





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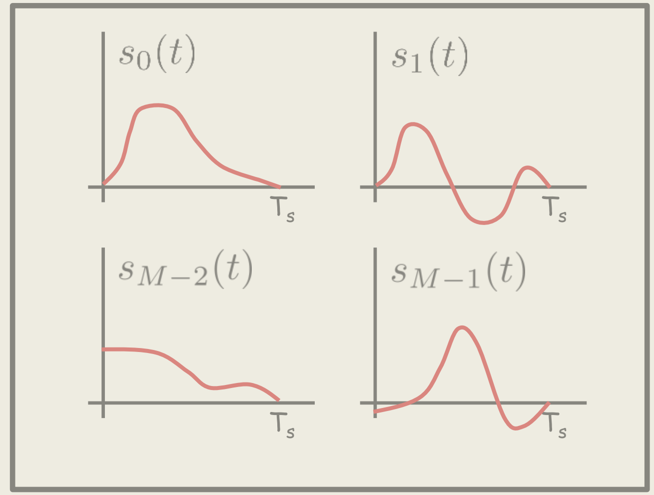

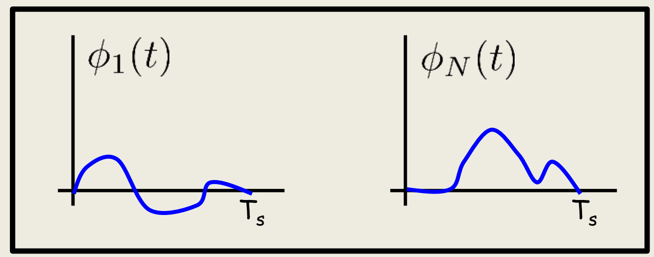

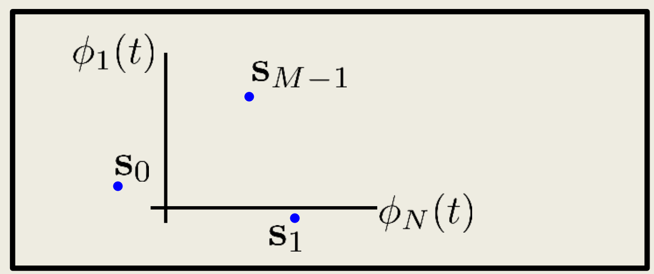

Lecture 3: Error Probabilities

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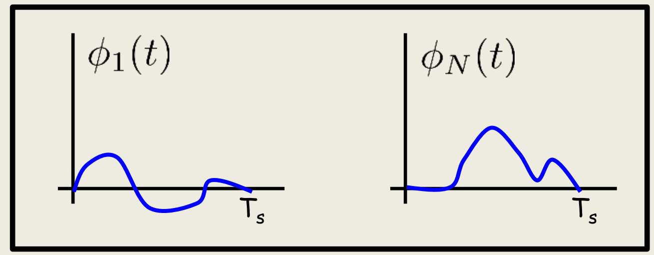
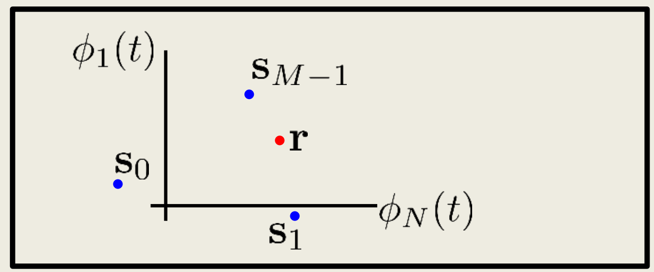
3. Map received signal to signal space

$$r_n = \int r(t)\phi_n(t)\mathrm{d}t$$





Lecture 3: Error Probabilities

Recap

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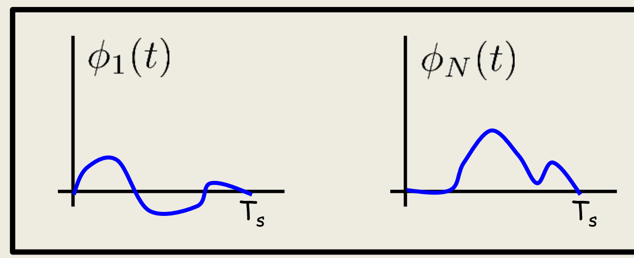
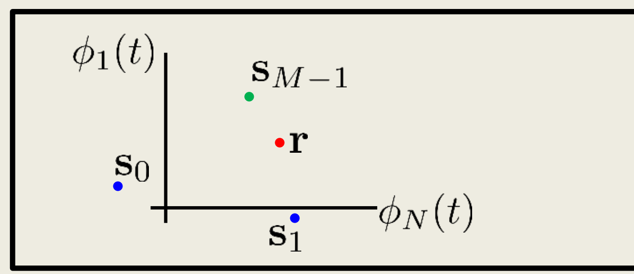
Execute ML/MAP receiver

1. ML $\hat{m} = \arg\min_\ell \|\mathbf{r} - \mathbf{s}_\ell\|^2$
2. MAP

$$\hat{m} = \arg\max_\ell P(\ell)\exp\left(-\frac{\|\mathbf{r} - \mathbf{s}_\ell\|^2}{N_0}\right)$$





Lecture 3: Error Probabilities

We now ask for the error probability, i.e.,

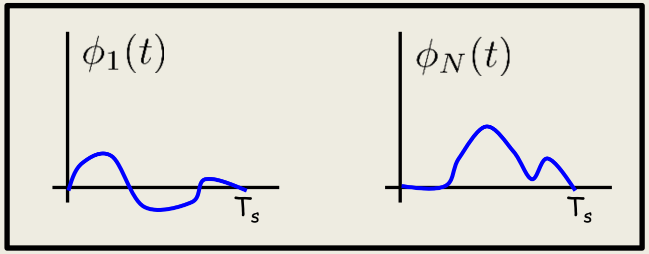
$$P_\mathrm{s} = \Pr(\hat{m} \neq m)$$

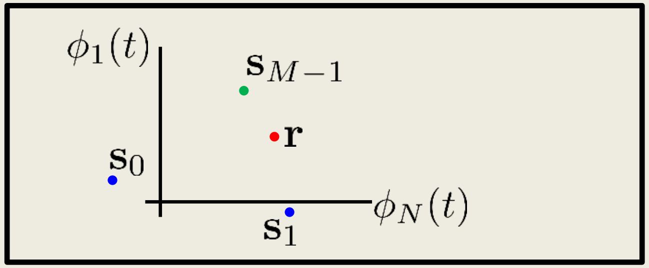
Execute ML/MAP receiver

1. **ML** $\hat{m} = \arg\min_{\ell} \|\mathbf{r} - \mathbf{s}_\ell\|^2$

2. **MAP**

$$\hat{m} = \arg\max_{\ell} P(\ell) \exp\left(-\frac{\|\mathbf{r} - \mathbf{s}_\ell\|^2}{N_0}\right)$$





Lecture 3: Error Probabilities

Case I: M-PAM

Lecture 3: Error Probabilities

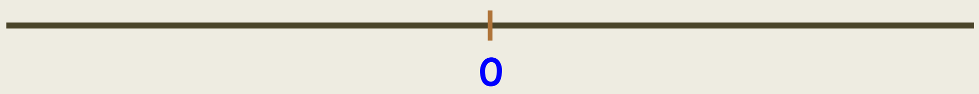
Case I: M-PAM

What is dimensionality, N?

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

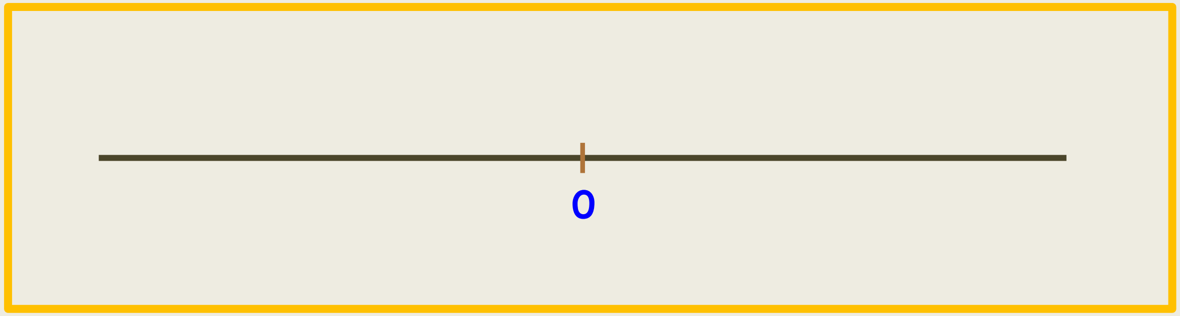


Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like?

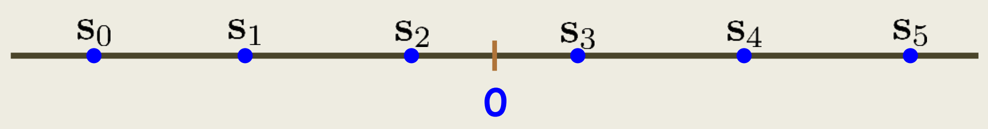


Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Lecture 3: Error Probabilities

Case I: M-PAM

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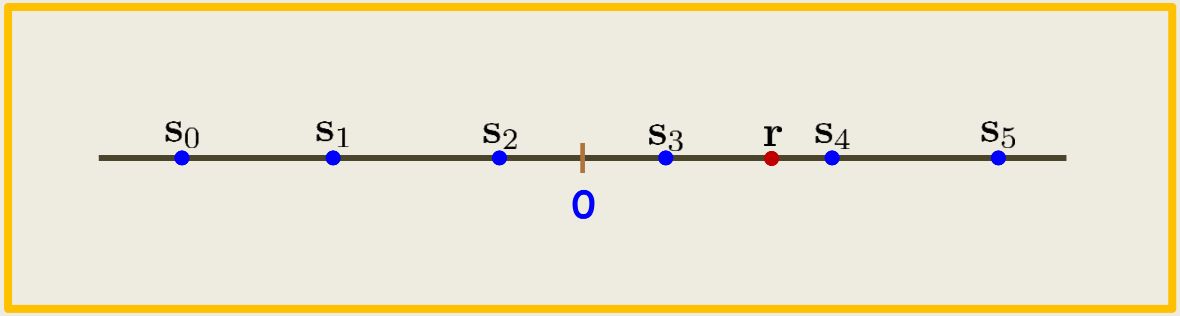


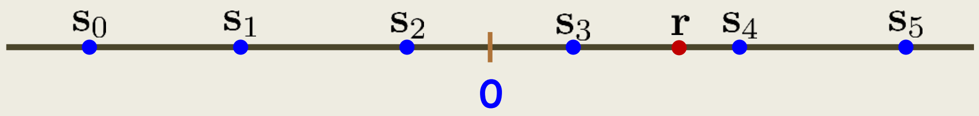
Example

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Example

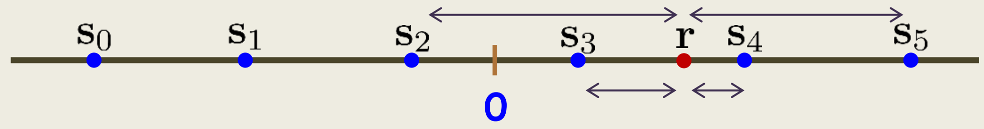
$$\hat{m} = \arg\min_{\ell} \|\mathbf{r} - \mathbf{s}_\ell\|^2$$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Example

$$\hat{m} = \arg\min_{\ell} \|\mathbf{r} - \mathbf{s}_\ell\|^2$$

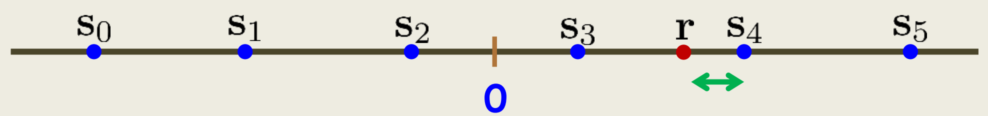
$=$ *" which arrow above is the shortest"*

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Example

$$\hat{m} = \arg\min_{\ell} \|\mathbf{r} - \mathbf{s}_\ell\|^2 = 4$$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

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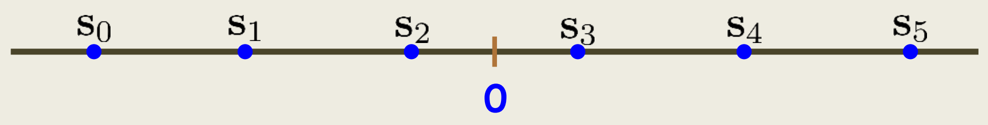


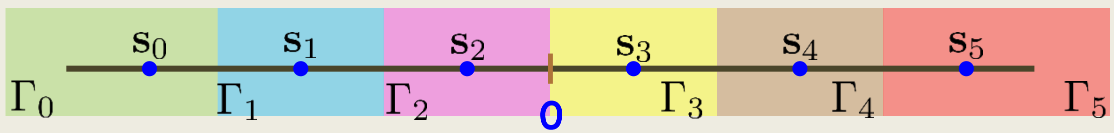
Assume ML. Where are decision regions?

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Assume ML. Where are decision regions?

Lecture 3: Error Probabilities

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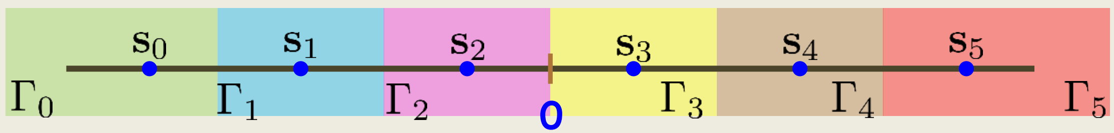


Symbol error probability

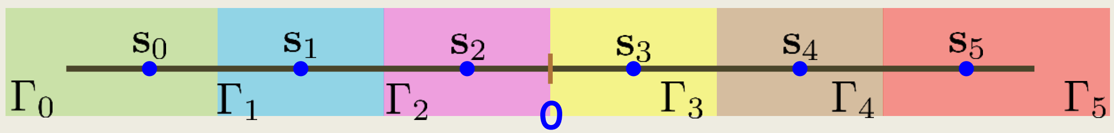
$$P_\mathrm{s} = \Pr(\hat{m} \neq m)$$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Symbol error probability

$$P_{\mathrm{s}} = \Pr(\hat{m} \neq m) = \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) \Pr(m_\ell \text{ sent})$$

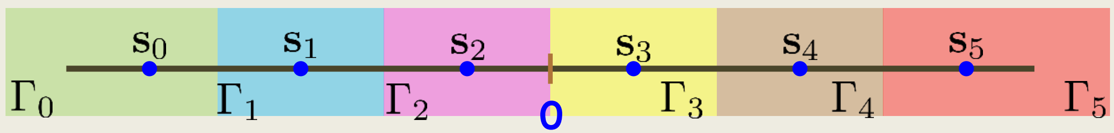
By law of total probability

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



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$$= \frac{1}{M} \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent})$$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

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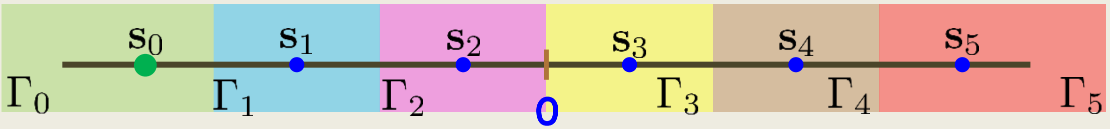

Symbol error probability

$$P_s = \Pr(\hat{m} \neq m) = \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) \Pr(m_\ell \text{ sent})$$

$$= \frac{1}{M} \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent})$$

Case I-a

$$= \frac{1}{M} \boxed{\Pr(\hat{m} \neq m_0 | m_0 \text{ sent})} + \frac{1}{M} \sum_{\ell=1}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent})$$

Lecture 3: Error Probabilities

Case I: M-PAM

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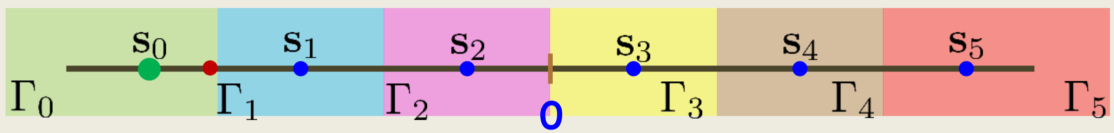

Symbol error probability

$$r = s_0 + w_1$$

Case I-a

$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) =$$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

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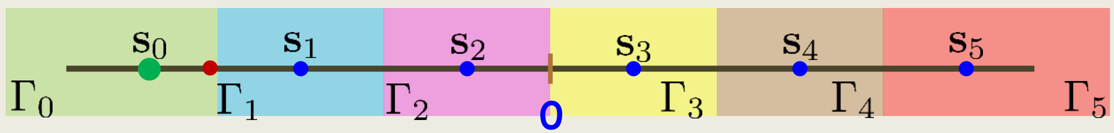

Symbol error probability

When do we get an error?

$$r = s_0 + w_1$$

Case I-a

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Lecture 3: Error Probabilities

Case I: M-PAM

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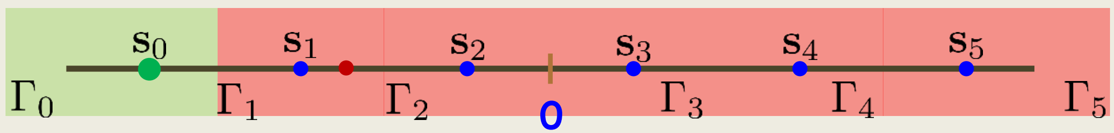

Symbol error probability

When do we get an error? Exactly when $r \notin \Gamma_0$

$$r = s_0 + w_1$$

Case I-a

$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) =$$

Lecture 3: Error Probabilities

Case I: M-PAM

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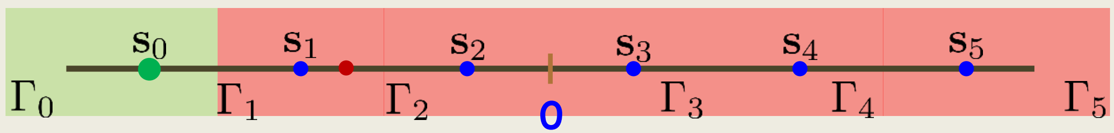

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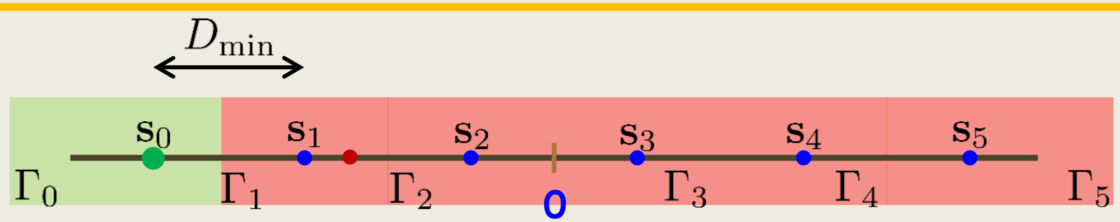

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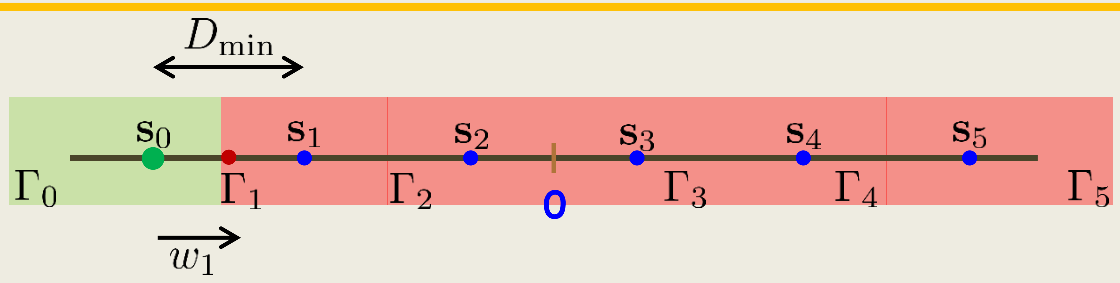

Symbol error probability

$$r = s_0 + w_1$$

When do we get an error? Exactly when $r \notin \Gamma_0$

What is required for that to happen? $w_1 > \frac{D_{\min}}{2}$

Case I-a

$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

Lecture 3: Error Probabilities

Case I: M-PAM

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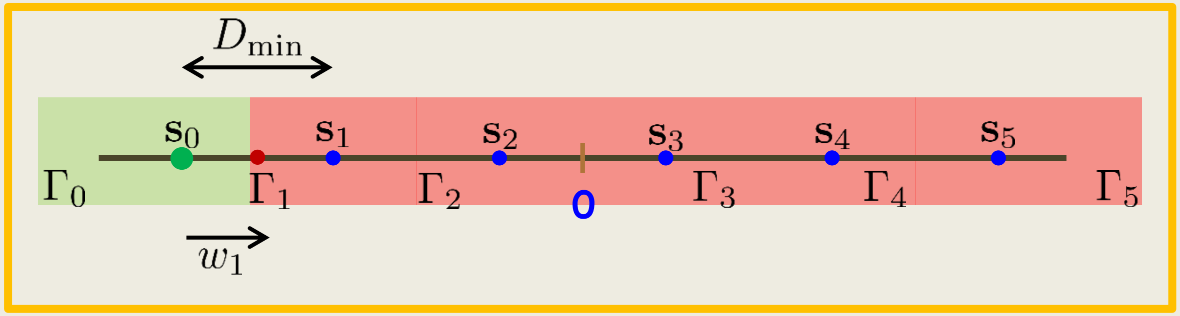

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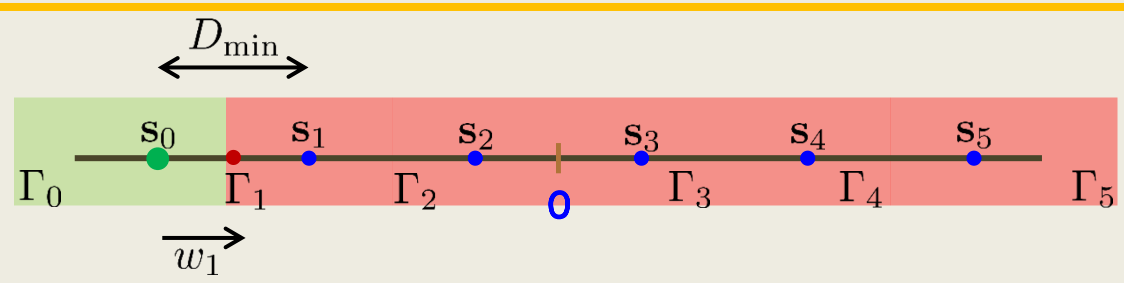

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Zero-mean Gaussian, variance $N_0/2$

Case I-a

$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right)$$

Lecture 3: Error Probabilities

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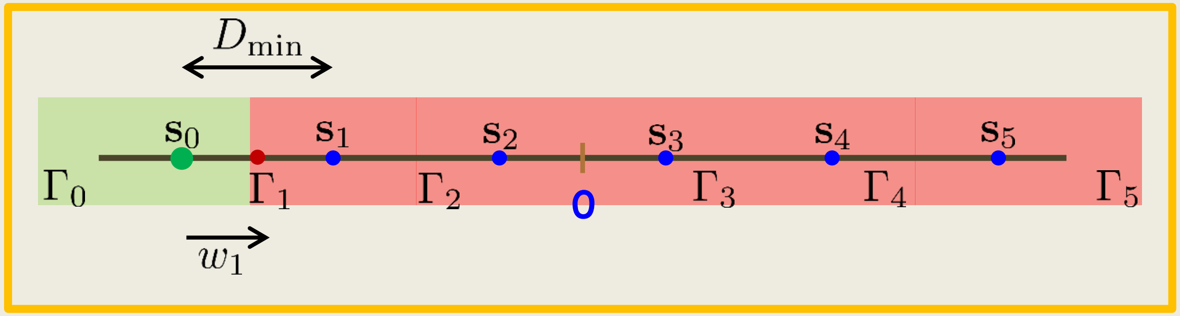

Symbol error probability

$$r = s_0 + w_1$$

When do we get an error? Exactly when $r \notin \Gamma_0$

What is required for that to happen? $w_1 > \frac{D_{\min}}{2}$

Always true (unless multiplying with negative number)

Case I-a

$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) = \Pr\left(\frac{w_1}{\sqrt{N_0/2}} > \frac{D_{\min}}{2\sqrt{N_0/2}}\right)$$

Lecture 3: Error Probabilities

Case I: M-PAM

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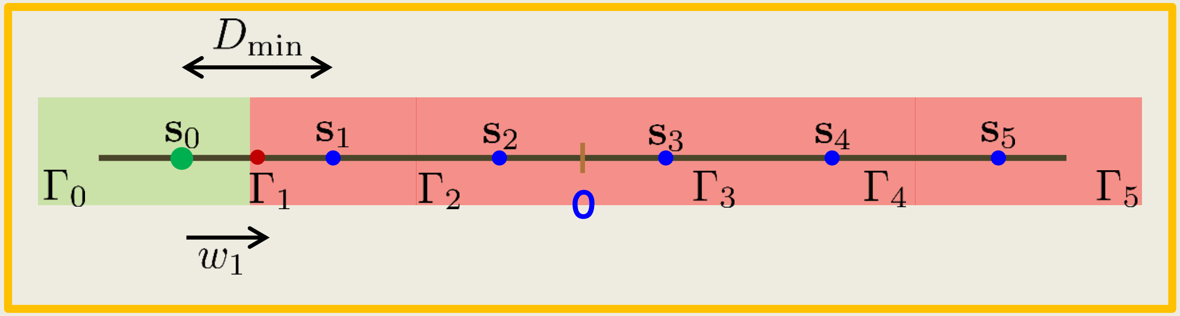

Symbol error probability

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When do we get an error? Exactly when $r \notin \Gamma_0$

What is required for that to happen? $w_1 > \frac{D_{\min}}{2}$

Zero-mean Gaussian, variance 1

Var(aX)=a^{2}Var(X)

Case I-a

$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) = \Pr\left(\frac{w_1}{\sqrt{N_0/2}} > \frac{D_{\min}}{2\sqrt{N_0/2}}\right)$$

Lecture 3: Error Probabilities

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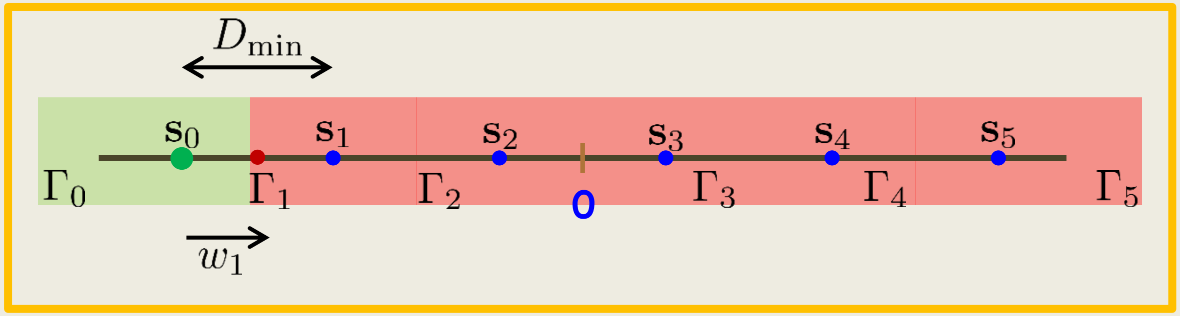

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What is required for that to happen? $w_1 > \frac{D_{\min}}{2}$

Zero-mean Gaussian, variance 1

Case I-a

$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) = \Pr\left(z > \frac{D_{\min}}{\sqrt{2N_0}}\right)$$

Lecture 3: Error Probabilities

Case I: M-PAM

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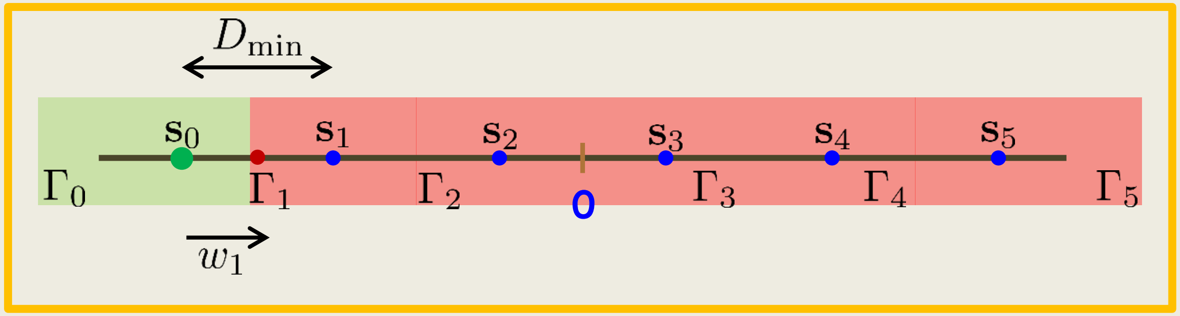

Symbol error probability

$$r = s_0 + w_1$$

When do we get an error? Exactly when $r \notin \Gamma_0$

What is required for that to happen? $w_1 > \frac{D_{\min}}{2}$

By definition

Case I-a

$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) = \Pr\left(z > \frac{D_{\min}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

Lecture 3: Error Probabilities

Case I: M-PAM

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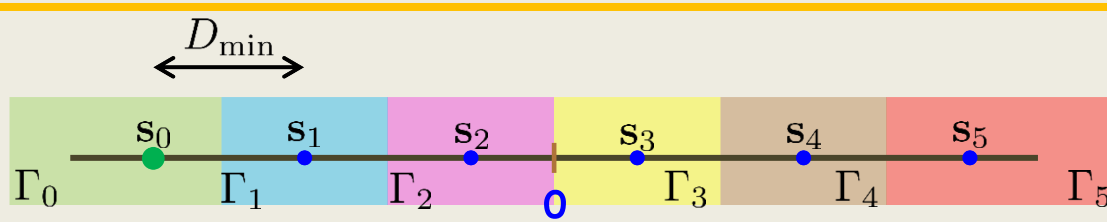

Symbol error probability

$$P_{\mathrm{s}} = \Pr(\hat{m} \neq m) = \frac{1}{M}\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) + \frac{1}{M}\sum_{\ell=1}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent})$$

Case I-a

$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) = \Pr\left(z > \frac{D_{\min}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

Lecture 3: Error Probabilities

Case I: M-PAM

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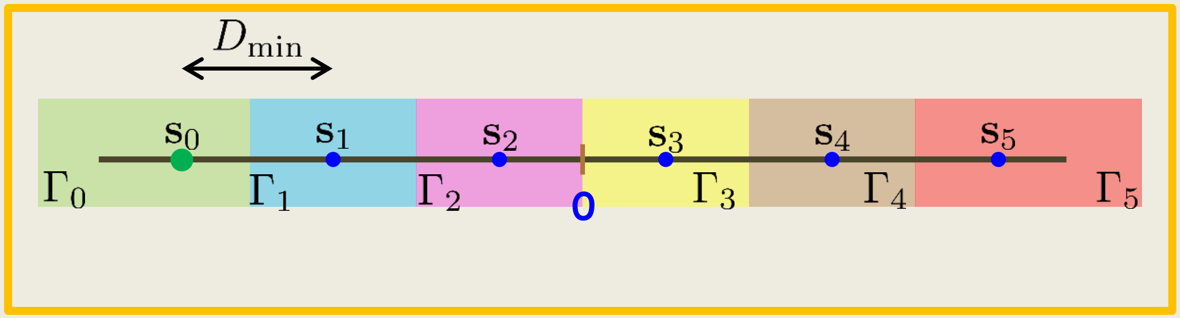


Symbol error probability

$$P_s = \Pr(\hat{m} \neq m) = \frac{1}{M} Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) + \frac{1}{M}\sum_{\ell=1}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent})$$

Case I-a

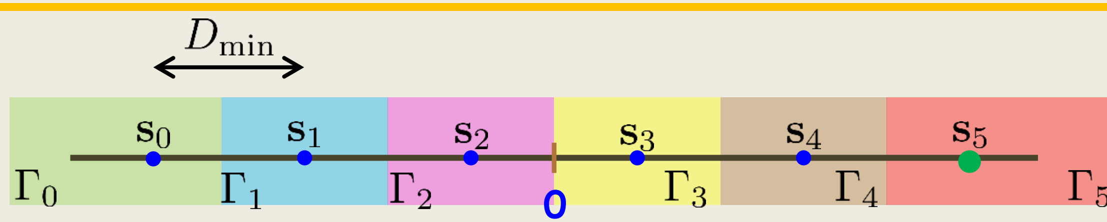
$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) = \Pr\left(z > \frac{D_{\min}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Symbol error probability

$$P_{\mathrm{s}} = \Pr(\hat{m} \neq m) = \frac{1}{M} Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) + \frac{1}{M}\Pr(\hat{m} \neq m_{M-1} | m_{M-1} \text{ sent})$$

$$+ \frac{1}{M}\sum_{\ell=1}^{M-2} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent})$$

Case I-a

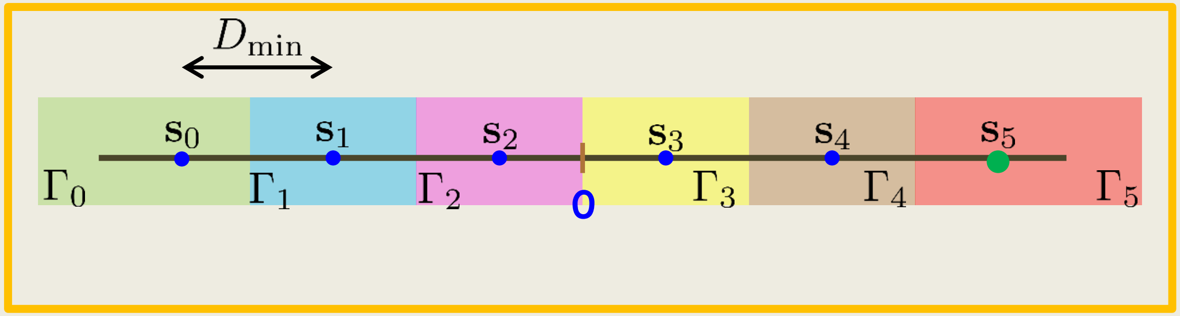
$$\Pr(\hat{m} \neq m_5 | m_5 \text{ sent}) =$$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Symbol error probability

$$P_{\mathrm{s}} = \Pr(\hat{m} \neq m) = \frac{1}{M} Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) + \frac{1}{M} \Pr(\hat{m} \neq m_{M-1} | m_{M-1} \text{ sent})$$

$$+ \frac{1}{M} \sum_{\ell=1}^{M-2} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent})$$

Case I-a

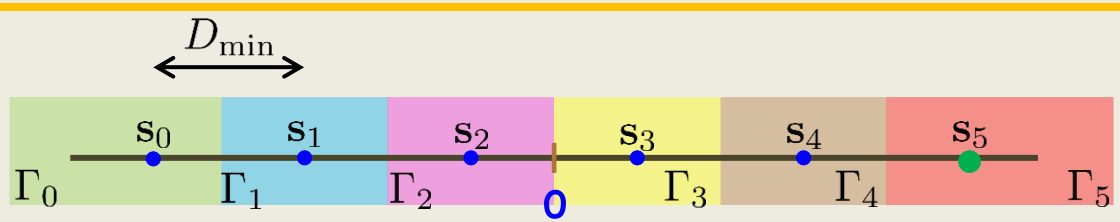
$$\boxed{\Pr(\hat{m} \neq m_5 | m_5 \text{ sent}) = Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)}$$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Symbol error probability

$$P_{\mathrm{s}} = \Pr(\hat{m} \neq m) = \frac{2}{M} Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) + \frac{1}{M}\sum_{\ell=1}^{M-2} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent})$$

Case I-a

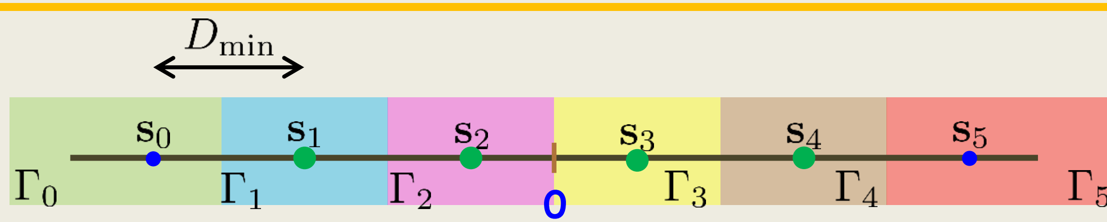
$$\Pr(\hat{m} \neq m_5 | m_5 \text{ sent}) = Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Symbol error probability

When do we get an error? Exactly when $r \notin \Gamma_m$

Case I-b

$\Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) =$

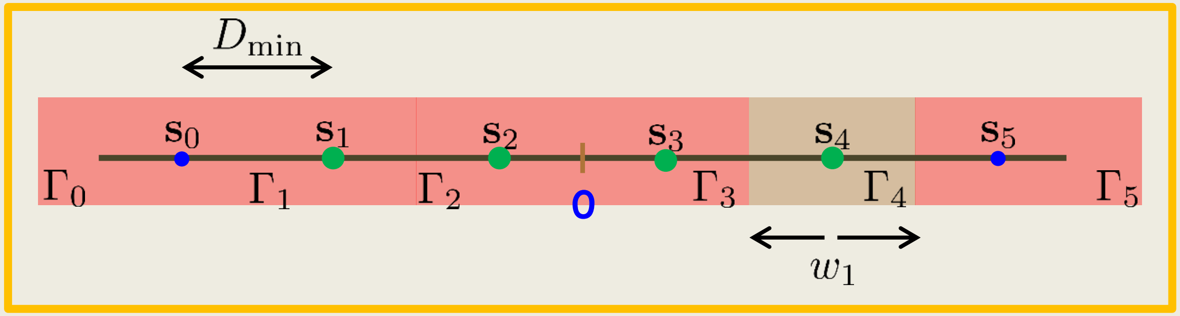
$1 \leq \ell \leq M-2$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Symbol error probability

When do we get an error? Exactly when $r \notin \Gamma_m$

What is required for that to happen?

$$w_1 > \frac{D_{\min}}{2} \text{ or } w_1 < -\frac{D_{\min}}{2}$$

Case I-b

$$\Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) =$$

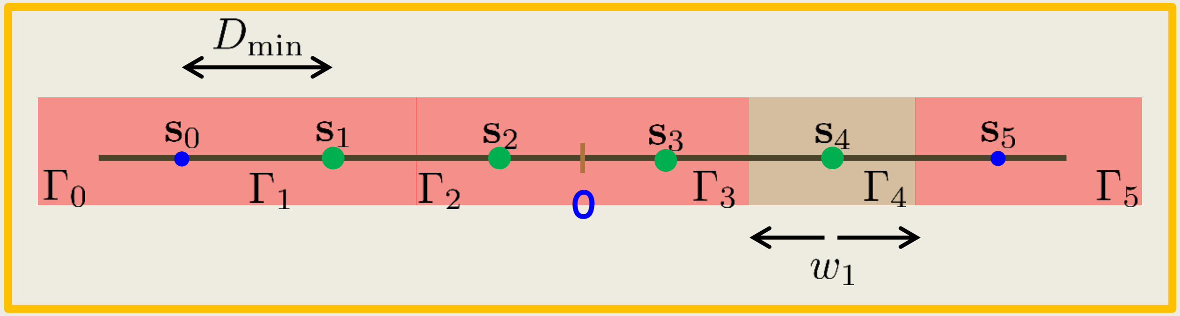
$1 \leq \ell \leq M-2$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Symbol error probability

When do we get an error? Exactly when $r \notin \Gamma_m$

What is required for that to happen?

When is P(A or B) = P(A) + P(B) ?

$$w_1 > \frac{D_{\min}}{2} \text{ or } w_1 < -\frac{D_{\min}}{2}$$

Case I-b

$\Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) =$

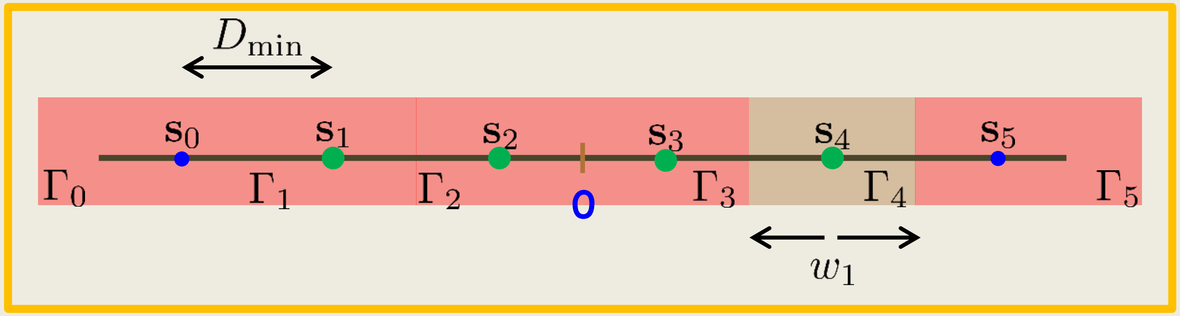
$1 \leq \ell \leq M-2$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Symbol error probability

When do we get an error? Exactly when $r \notin \Gamma_m$

What is required for that to happen?

When is P(A or B) = P(A) + P(B) ?
When A and B cannot happen at the same time

$$w_1 > \frac{D_{\min}}{2} \text{ or } w_1 < -\frac{D_{\min}}{2}$$

Case I-b

$$\Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) =$$

$1 \leq \ell \leq M-2$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

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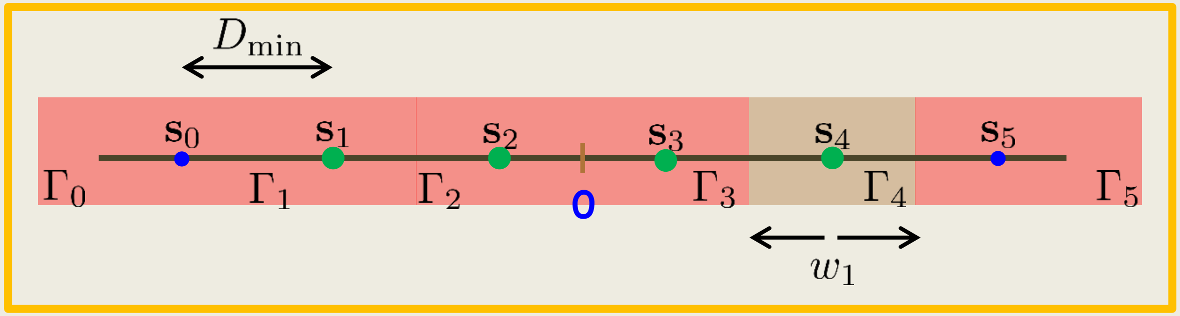

Symbol error probability

When do we get an error? Exactly when $r \notin \Gamma_m$

What is required for that to happen?

$$w_1 > \frac{D_{\min}}{2} \text{ or } w_1 < -\frac{D_{\min}}{2}$$

These two cannot happen simultaneously

When is P(A or B) = P(A) + P(B) ?
When A and B cannot happen at the same time

Case I-b

$$\Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) =$$

$$1 \leq \ell \leq M-2$$

Lecture 3: Error Probabilities

Case I: M-PAM

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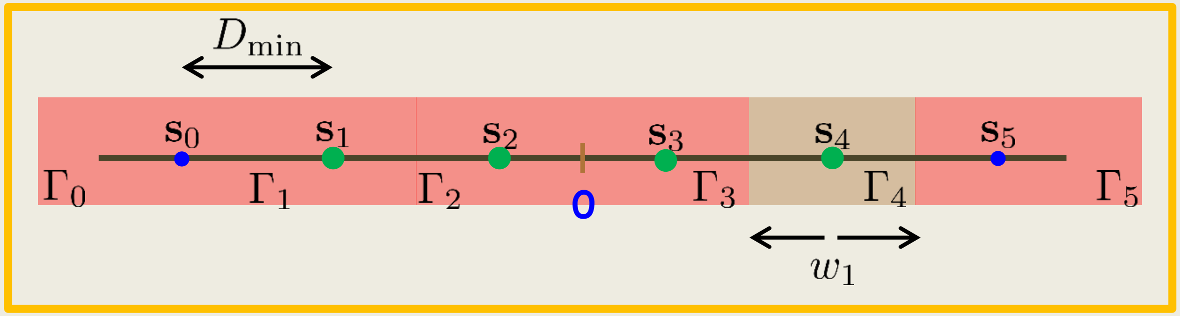

Symbol error probability

When do we get an error? Exactly when $r \notin \Gamma_m$

What is required for that to happen?

$$w_1 > \frac{D_{\min}}{2} \quad \text{or} \quad w_1 < -\frac{D_{\min}}{2}$$

Case I-b

$$\Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) + \Pr\left(w_1 < -\frac{D_{\min}}{2}\right)$$

$1 \leq \ell \leq M-2$

Lecture 3: Error Probabilities

Case I: M-PAM

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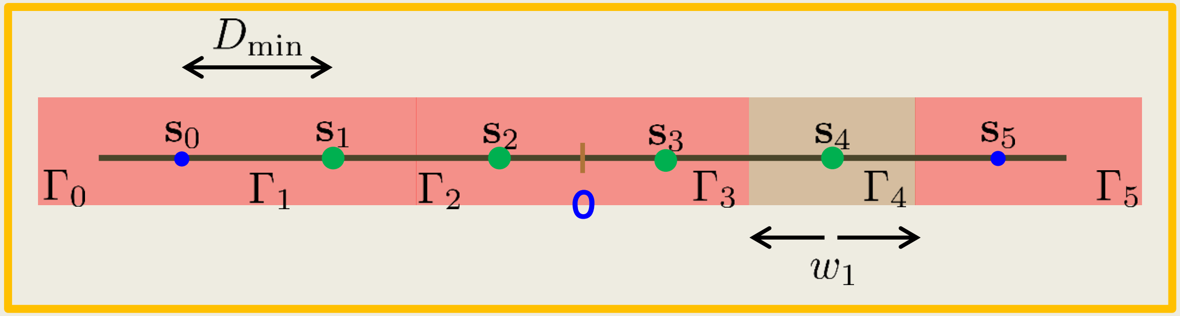


Symbol error probability

When do we get an error? Exactly when $r \notin \Gamma_m$

What is required for that to happen?

$$w_1 > \frac{D_{\min}}{2} \text{ or } w_1 < -\frac{D_{\min}}{2}$$

Are these equal ?

Case I-b

$$\Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) + \Pr\left(w_1 < -\frac{D_{\min}}{2}\right)$$

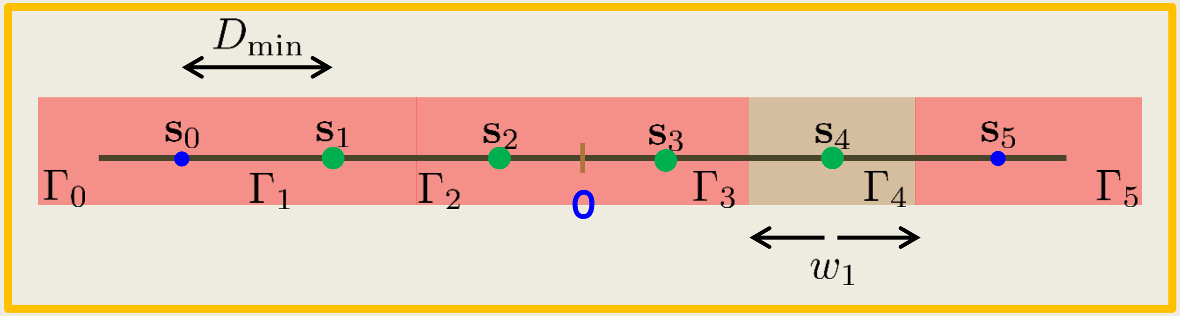
$1 \leq \ell \leq M-2$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Symbol error probability

When do we get an error? Exactly when $r \notin \Gamma_m$

What is required for that to happen?

$$w_1 > \frac{D_{\min}}{2} \quad \text{or} \quad w_1 < -\frac{D_{\min}}{2}$$

Are these equal ? Yes, since Gaussian pdf is symmetric

Case I-b

$$\Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) + \Pr\left(w_1 < -\frac{D_{\min}}{2}\right)$$

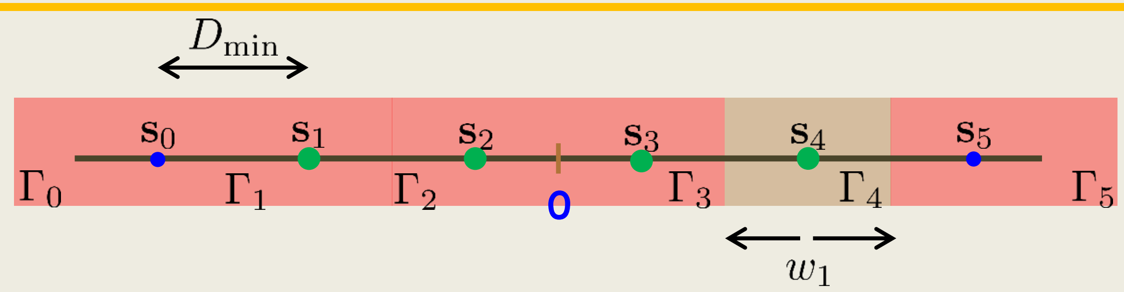
$1 \leq \ell \leq M - 2$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Symbol error probability

When do we get an error? Exactly when $r \notin \Gamma_m$

What is required for that to happen?

$$w_1 > \frac{D_{\min}}{2} \text{ or } w_1 < -\frac{D_{\min}}{2}$$

Case I-b

$$\Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) + \Pr\left(w_1 < -\frac{D_{\min}}{2}\right) = 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

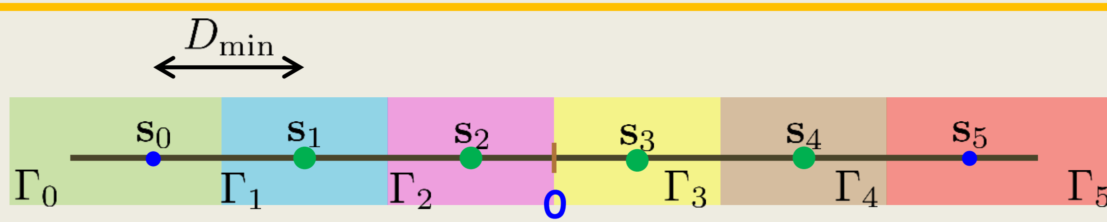
$1 \leq \ell \leq M-2$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

What does signal set look like? Regular spacing



Symbol error probability

$$P_{\rm s} = \Pr(\hat{m} \neq m) = \frac{2}{M} Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) + \frac{1}{M}\sum_{\ell=1}^{M-2} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent})$$

Case I-b

$$\Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) + \Pr\left(w_1 < -\frac{D_{\min}}{2}\right) = 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

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Lecture 3: Error Probabilities

Case I: M-PAM

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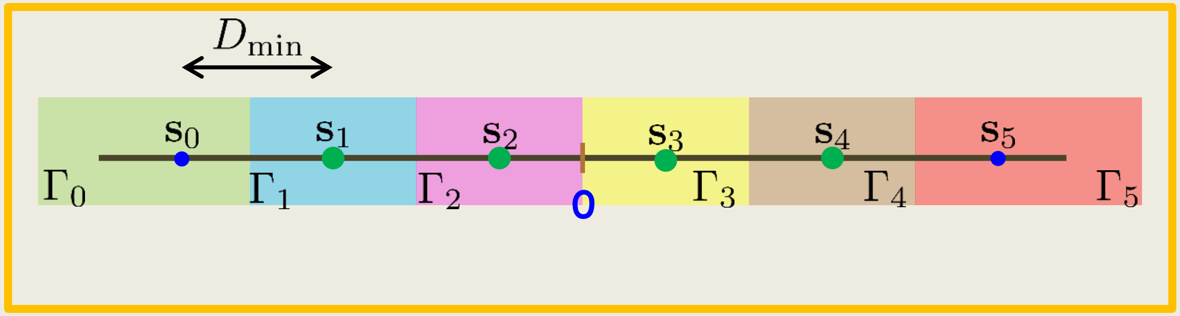

Symbol error probability

$$P_{\mathrm{s}} = \Pr(\hat{m} \neq m) = \frac{2}{M} Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) + \frac{M-2}{M} 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

Case I-b

$$\Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) = \Pr\left(w_1 > \frac{D_{\min}}{2}\right) + \Pr\left(w_1 < -\frac{D_{\min}}{2}\right) = 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

$1 \leq \ell \leq M-2$

Lecture 3: Error Probabilities

Case I: M-PAM

What is dimensionality, N? 1

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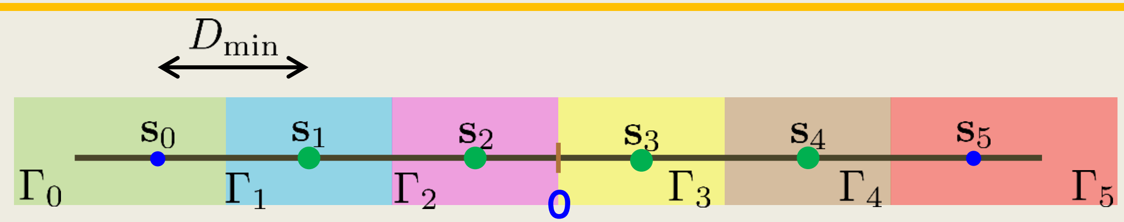


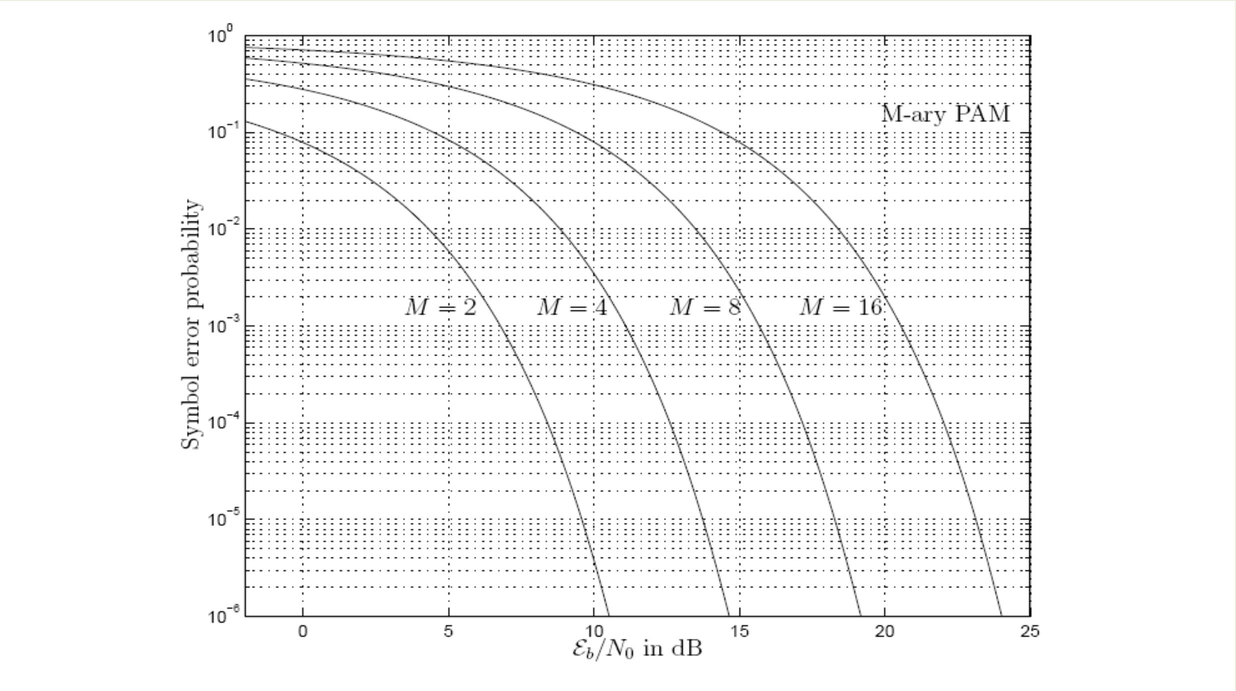
Symbol error probability

$$P_{\mathrm{s}} = \Pr(\hat{m} \neq m) = \frac{2}{M} Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) + \frac{M-2}{M} 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

$$\boxed{= 2\frac{M-1}{M} Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)}$$

Lecture 3: Error Probabilities

Case I: M-PAM



$$P_s = 2\frac{M-1}{M}Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

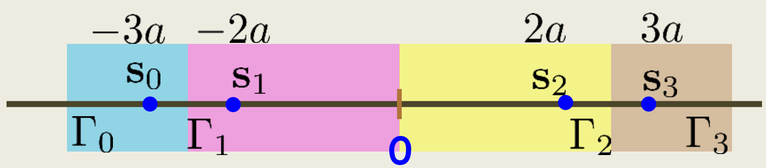
Illustration in signal space?

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Illustration in signal space?

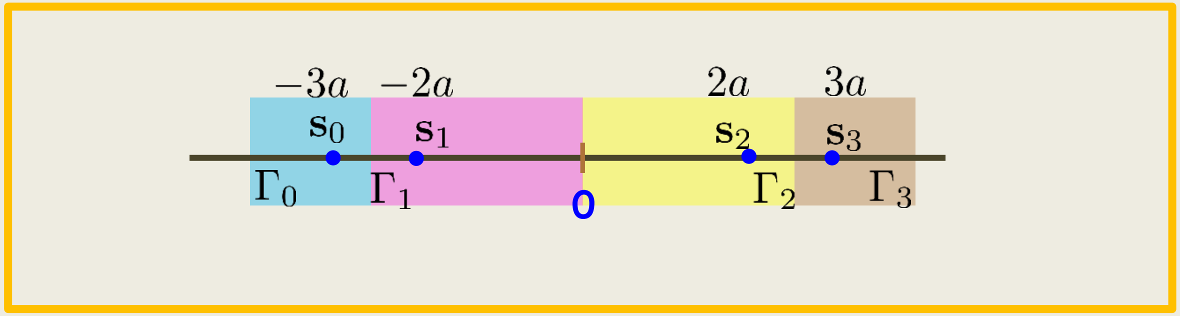


Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Illustration in signal space?



$$P_s = \Pr(\hat{m} \neq m) = \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) \Pr(m_\ell \text{ sent})$$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

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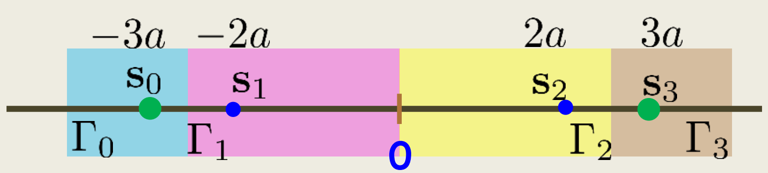


$$P_\mathrm{s} = \Pr(\hat{m} \neq m) = \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) \Pr(m_\ell \text{ sent})$$

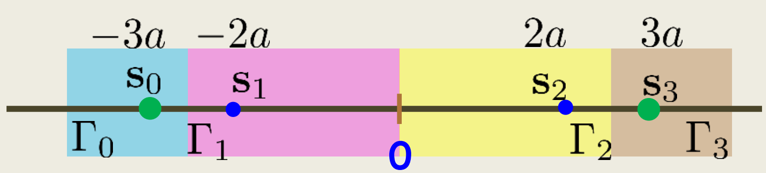
Case II-a $\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) =$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Illustration in signal space?



$$P_s = \Pr(\hat{m} \neq m) = \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) \Pr(m_\ell \text{ sent})$$

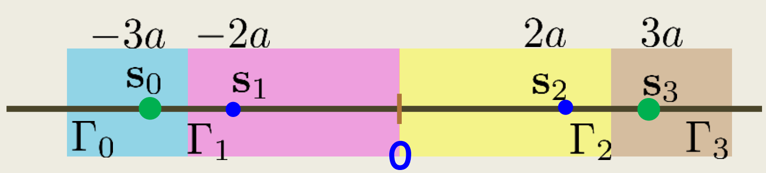
Case II-a $\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = \Pr\left(w_1 > \frac{a}{2}\right)$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Illustration in signal space?



$$P_s = \Pr(\hat{m} \neq m) = \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) \Pr(m_\ell \text{ sent})$$

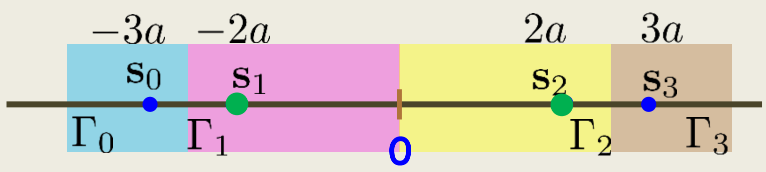
Case II-a $\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = \Pr\left(w_1 > \frac{a}{2}\right) = Q\left(\sqrt{\frac{a^2}{2N_0}}\right)$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Illustration in signal space?



$$P_s = \Pr(\hat{m} \neq m) = \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) \Pr(m_\ell \text{ sent})$$

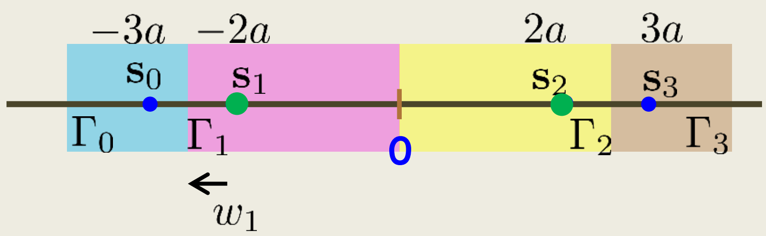
Case II-b $\Pr(\hat{m} \neq m_1 | m_1 \text{ sent}) =$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Illustration in signal space?



$$P_s = \Pr(\hat{m} \neq m) = \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) \Pr(m_\ell \text{ sent})$$

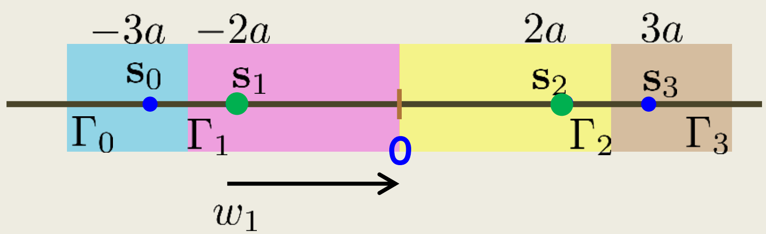
Case II-b $\Pr(\hat{m} \neq m_1 | m_1 \text{ sent}) = \Pr\left(w_1 < -\frac{a}{2}\right) +$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Illustration in signal space?



$$P_s = \Pr(\hat{m} \neq m) = \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) \Pr(m_\ell \text{ sent})$$

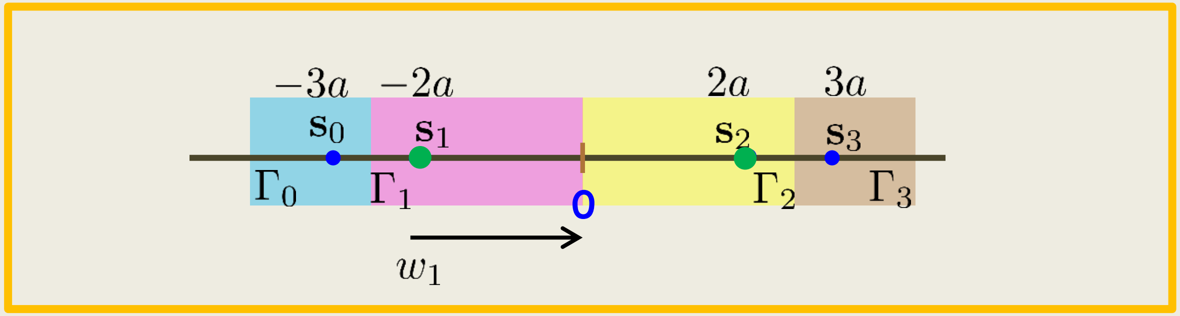
Case II-b $\Pr(\hat{m} \neq m_1 | m_1 \text{ sent}) = \Pr\left(w_1 < -\frac{a}{2}\right) + \Pr\left(w_1 > 2a\right)$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Illustration in signal space?



$$P_s = \Pr(\hat{m} \neq m) = \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) \Pr(m_\ell \text{ sent})$$

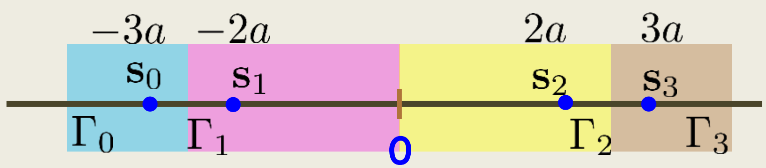
Case II-b $\Pr(\hat{m} \neq m_1 | m_1 \text{ sent}) = Q\left(\sqrt{\dfrac{a^2}{2N_0}}\right) + Q\left(\sqrt{\dfrac{8a^2}{N_0}}\right)$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Illustration in signal space?



$$P_s = \Pr(\hat{m} \neq m) = \frac{1}{2}Q\left(\sqrt{\frac{a^2}{2N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{a^2}{2N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{8a^2}{N_0}}\right)$$

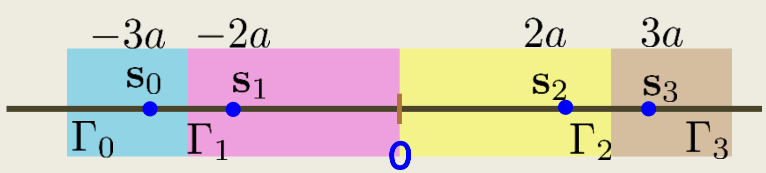
Case II-b $\Pr(\hat{m} \neq m_1 | m_1 \text{ sent}) = Q\left(\sqrt{\frac{a^2}{2N_0}}\right) + Q\left(\sqrt{\frac{8a^2}{N_0}}\right)$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Illustration in signal space?



$$P_s = \Pr(\hat{m} \neq m) = \frac{1}{2}Q\left(\sqrt{\frac{a^2}{2N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{a^2}{2N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{8a^2}{N_0}}\right)$$

Case II-a

Case II-b

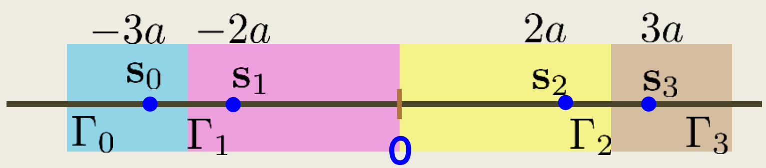
Case II-b $\Pr(\hat{m} \neq m_1 | m_1 \text{ sent}) = Q\left(\sqrt{\frac{a^2}{2N_0}}\right) + Q\left(\sqrt{\frac{8a^2}{N_0}}\right)$

Lecture 3: Error Probabilities

Case II: non-regular 4-PAM

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\} \quad E_g = 1$$

Illustration in signal space?



$$P_s = \Pr(\hat{m} \neq m) = Q\left(\sqrt{\frac{a^2}{2N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{8a^2}{N_0}}\right)$$

Lecture 3: Error Probabilities

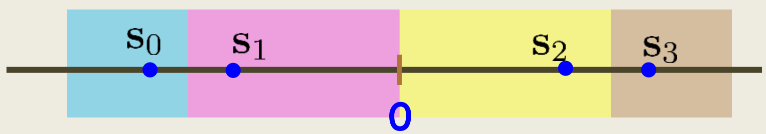
Checkpoint

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\}$$

~~$E_g = 1$~~

$E_g = 0.5$

Illustration in signal space?



Lecture 3: Error Probabilities

Checkpoint

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\ -2ag(t),\ 2ag(t),\ 3ag(t)\}$$

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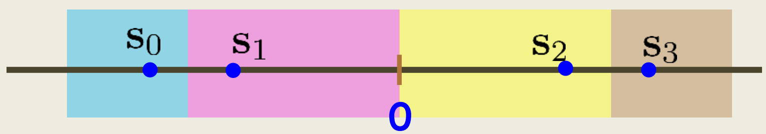

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \left\{-\frac{3}{\sqrt{2}}a\tilde{g}(t),\ -\sqrt{2}a\tilde{g}(t),\ \sqrt{2}a\tilde{g}(t),\ \frac{3}{\sqrt{2}}a\tilde{g}(t)\right\} \quad E_{\tilde{g}} = 1$$

Lecture 3: Error Probabilities

Checkpoint

$$\{s_\ell(t)\}_{\ell=0}^{M-1} = \{-3ag(t),\, -2ag(t),\, 2ag(t),\, 3ag(t)\}$$

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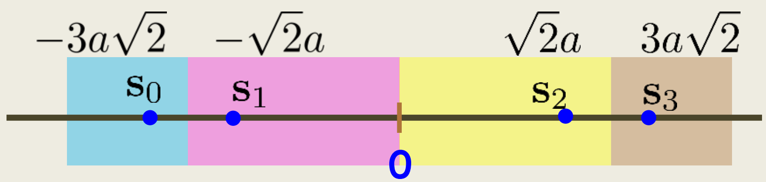

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Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N?

What does signal set look like in signal space?

Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

What does signal set look like in signal space?

Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

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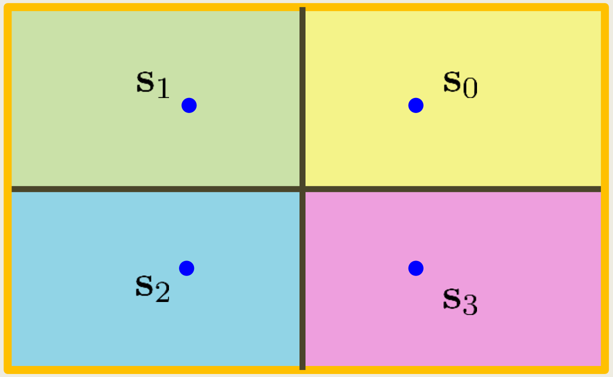

Lecture 3: Error Probabilities

Case II: QPSK

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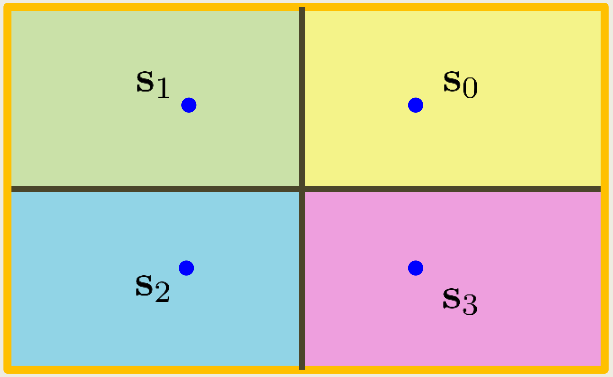

What are cordinates if E_s=E
(E_s is average energy per symbol)

Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

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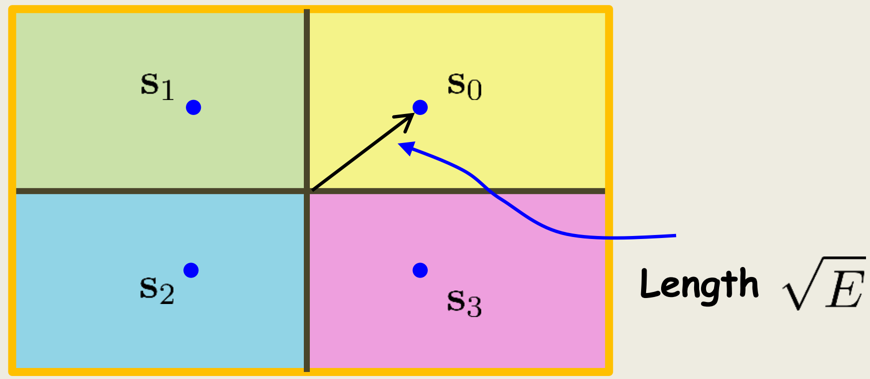

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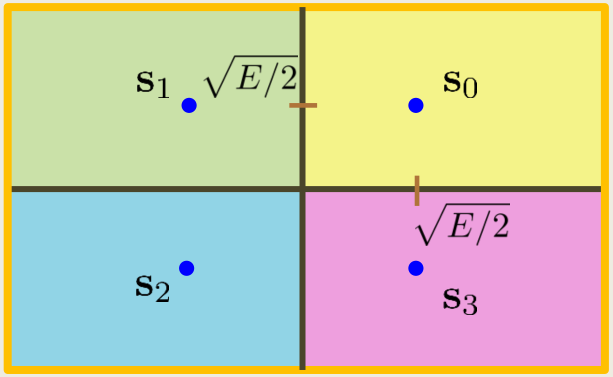

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Lecture 3: Error Probabilities

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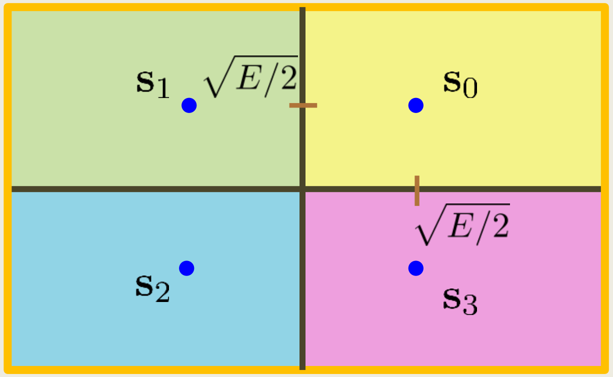

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Math model of received signal

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Lecture 3: Error Probabilities

Case II: QPSK

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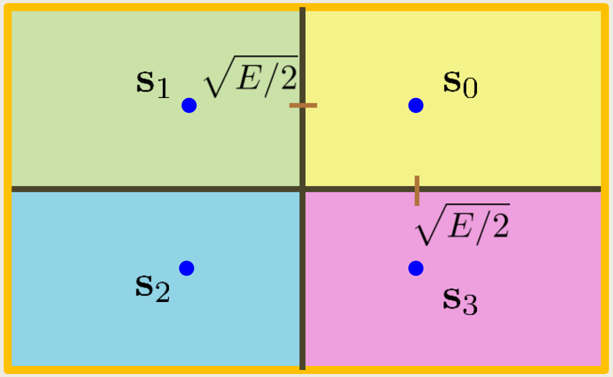

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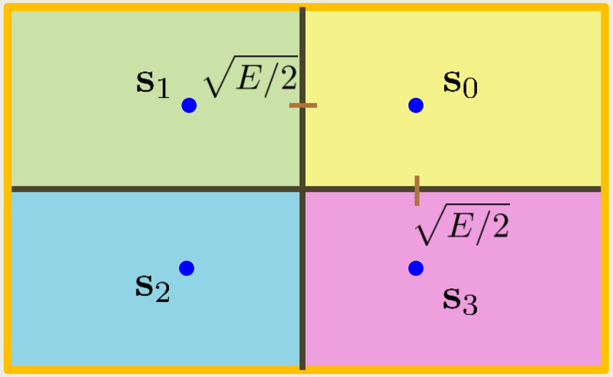
What does the signals look like in the time-domain?

Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

What does signal set look like in signal space?



What are cordinates if E_s=E
(E_s is average energy per symbol)

Math model of received signal

$$r_1 = s_{\ell,1} + w_1$$
$$r_2 = s_{\ell,2} + w_2$$

What does the signals look like in the time-domain? We don't know, could be

1. $s_\ell(t) = A_\ell g(t)\cos(2\pi f_c t) - B_\ell g(t)\sin(2\pi f_c t)$

2. Send one PAM signal today. Send one tomorrow.

Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

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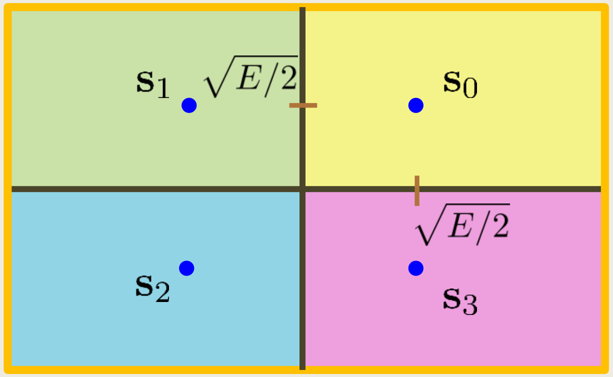

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Lecture 3: Error Probabilities

Case II: QPSK

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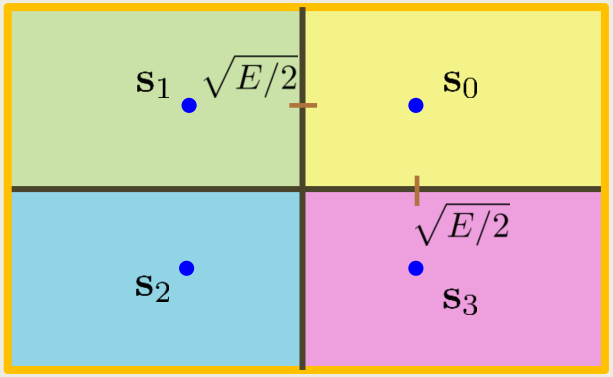

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By symmetry

Lecture 3: Error Probabilities

Case II: QPSK

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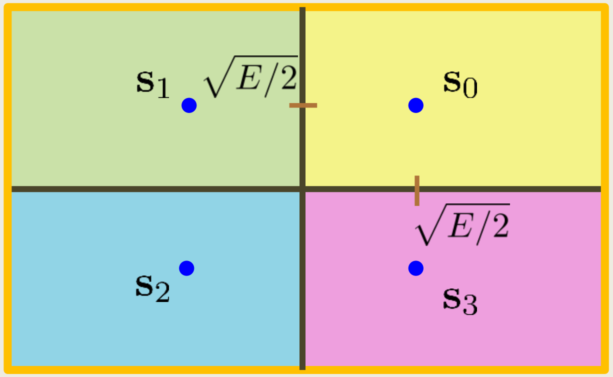

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$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = 1 - \Pr(\hat{m} = m_0 | m_0 \text{ sent})$$

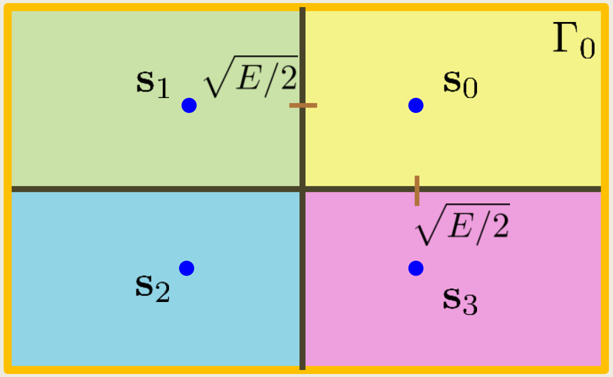
Standard trick

Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

What does signal set look like in signal space?



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Lecture 3: Error Probabilities

Case II: QPSK

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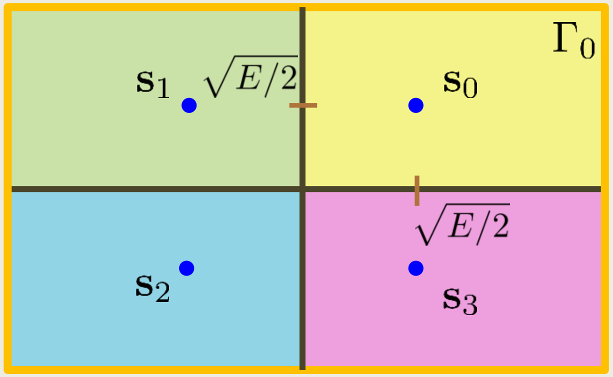

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$$= 1 - \Pr\left(w_1 > -\sqrt{\frac{E}{2}} \text{ and } w_2 > -\sqrt{\frac{E}{2}}\right)$$

Lecture 3: Error Probabilities

Case II: QPSK

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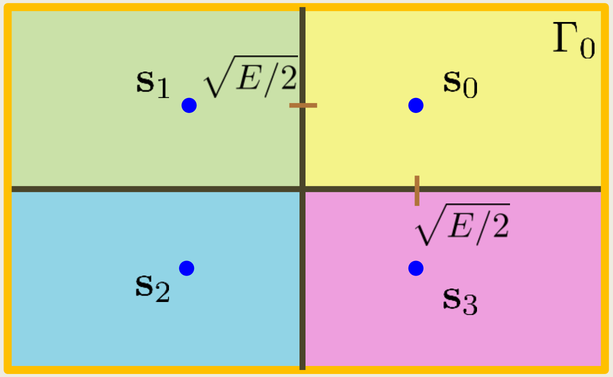

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Pr(A and B) = Pr(A)Pr(B)
iff A and B are independent

Lecture 3: Error Probabilities

Case II: QPSK

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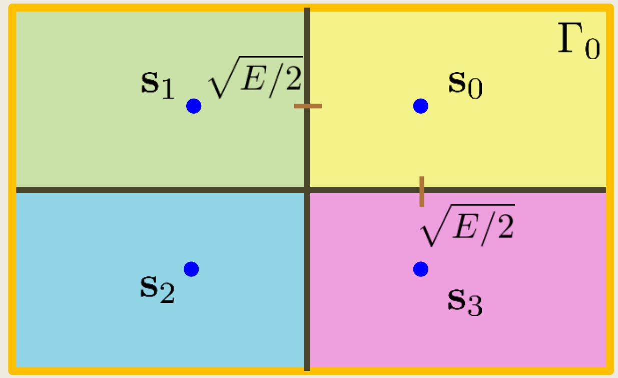

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Independent ?

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Lecture 3: Error Probabilities

Case II: QPSK

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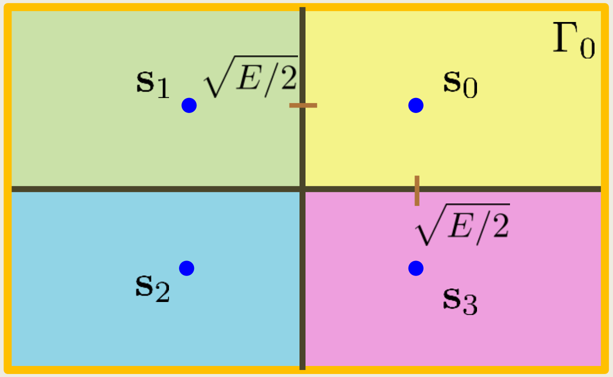

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Independent ? Yes, we proved that last lecture

Lecture 3: Error Probabilities

Case II: QPSK

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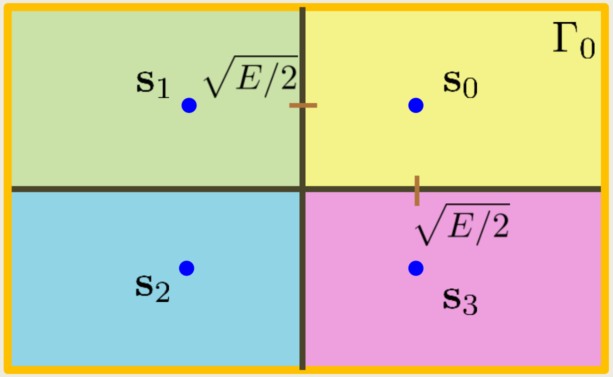

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Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

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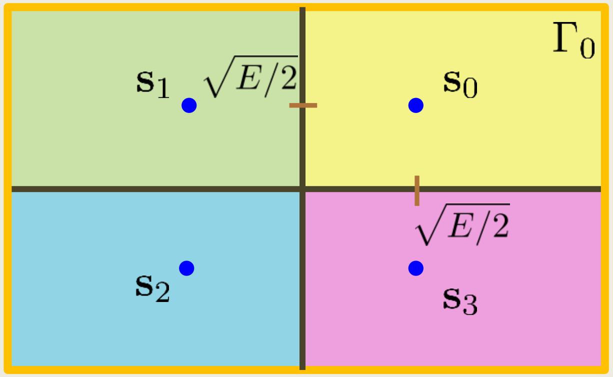

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Pr(X>a) = 1-Pr(X<a)

Lecture 3: Error Probabilities

Case II: QPSK

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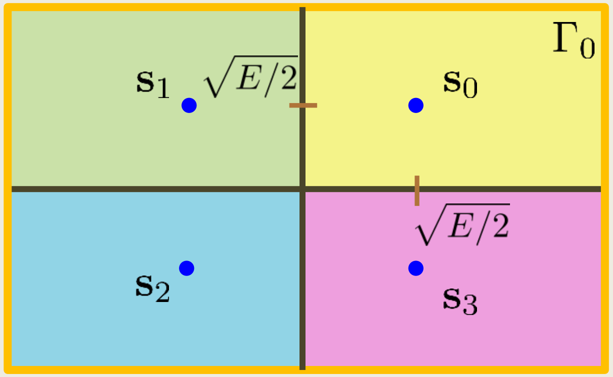

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Lecture 3: Error Probabilities

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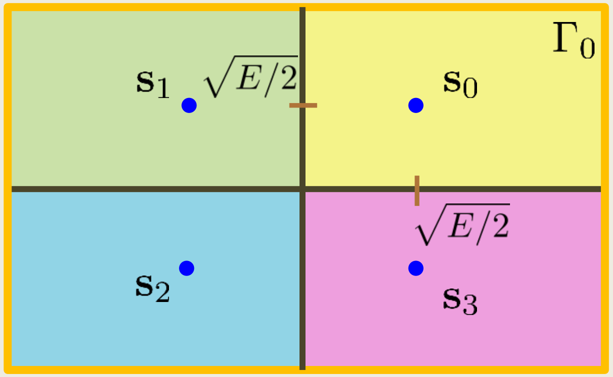

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Lecture 3: Error Probabilities

Case II: QPSK

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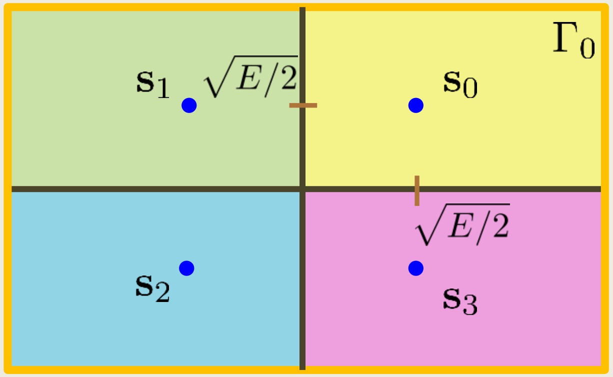

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Lecture 3: Error Probabilities

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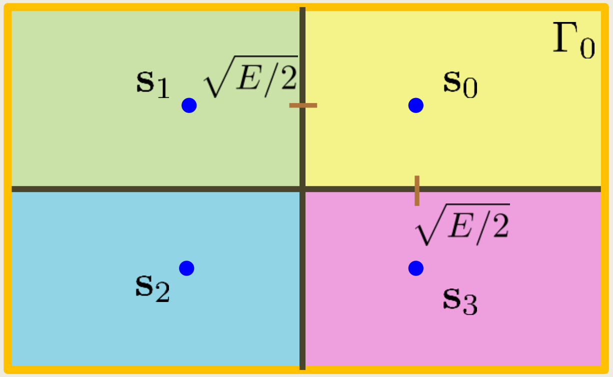

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$$r_2 = s_{\ell,2} + w_2$$

$$P_s = \Pr(\hat{m} \neq m) = \sum_{\ell=0}^{M-1} \Pr(\hat{m} \neq m_\ell | m_\ell \text{ sent}) \Pr(m_\ell \text{ sent}) = \Pr(\hat{m} \neq m_0 | m_0 \text{ sent})$$

$$\Pr(\hat{m} \neq m_0 | m_0 \text{ sent}) = 1 - \Pr(\hat{m} = m_0 | m_0 \text{ sent}) = 1 - \Pr(\mathbf{r} \in \Gamma_0)$$

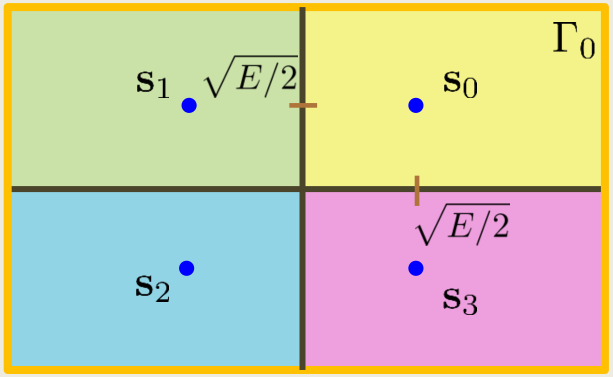
$$= 1 - \left[1 - Q\left(\sqrt{\frac{E}{N_0}}\right)\right]^2 = \boxed{2Q\left(\sqrt{\frac{E}{N_0}}\right) - Q^2\left(\sqrt{\frac{E}{N_0}}\right)}$$

Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

What does signal set look like in signal space?



What are cordinates if E_s=E
(E_s is average energy per symbol)

Math model of received signal

$$r_1 = s_{\ell,1} + w_1$$
$$r_2 = s_{\ell,2} + w_2$$

Bit error probability ?

Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

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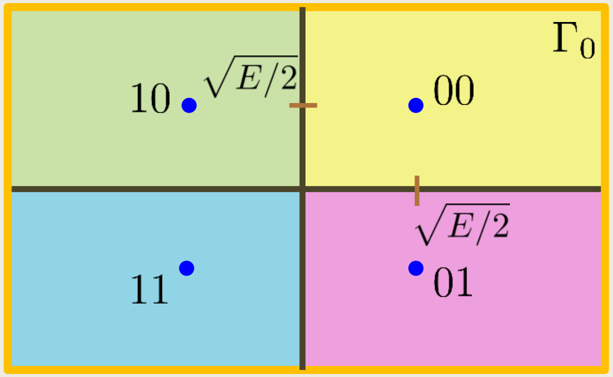

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Math model of received signal

$$r_1 = s_{\ell,1} + w_1$$
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Bit error probability ? Depends on bit-mapping. Assume the above mapping

Lecture 3: Error Probabilities

Case II: QPSK

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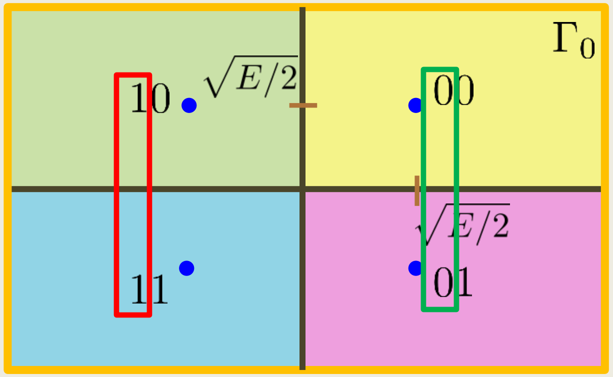

What are cordinates if E_s=E
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Math model of received signal

$$r_1 = s_{\ell,1} + w_1$$
$$r_2 = s_{\ell,2} + w_2$$

Bit error probability ? Depends on bit-mapping. Assume the above mapping

Observe (important): The left bit decides if we are to the left or to the right

Lecture 3: Error Probabilities

Case II: QPSK

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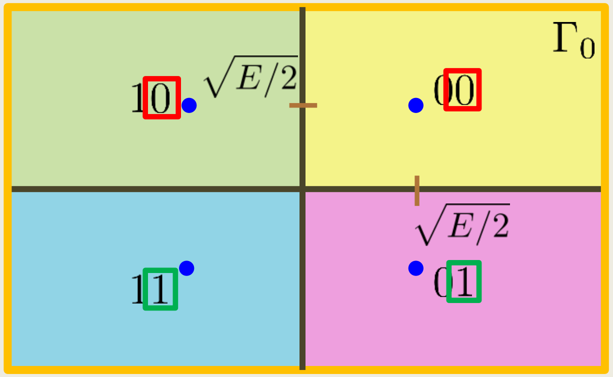


What are cordinates if E_s=E
(E_s is average energy per symbol)

Math model of received signal

$$r_1 = s_{\ell,1} + w_1$$
$$r_2 = s_{\ell,2} + w_2$$

Bit error probability ? Depends on bit-mapping. Assume the above mapping

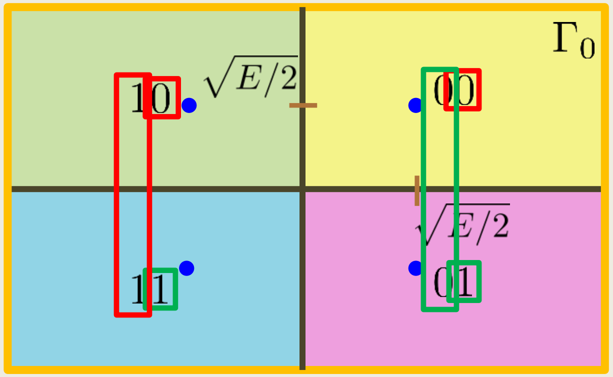
Observe (important): The left bit decides if we are to the left or to the right
The right bit decides if we are up or down

Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

What does signal set look like in signal space?



What are cordinates if E_s=E
(E_s is average energy per symbol)

Math model of received signal

$$r_1 = s_{\ell,1} + w_1$$
$$r_2 = s_{\ell,2} + w_2$$

Bit error probability ? Depends on bit-mapping. Assume the above mapping

Observe (important): The left bit decides if we are to the left or to the right
The right bit decides if we are up or down

We make a mistake in the left bit iff (assume we are down) $w_2 > \sqrt{E/2}$

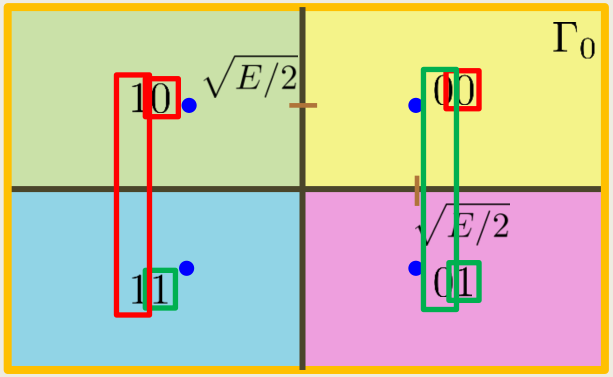
Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

What does signal set look like in signal space?

What are cordinates if E_s=E
(E_s is average energy per symbol)



Math model of received signal

$$r_1 = s_{\ell,1} + w_1$$
$$r_2 = s_{\ell,2} + w_2$$

Bit error probability ? Depends on bit-mapping. Assume the above mapping

Observe (important): The left bit decides if we are to the left or to the right
The right bit decides if we are up or down

We make a mistake in the left bit iff (assume we are down) $w_2 > \sqrt{E/2}$

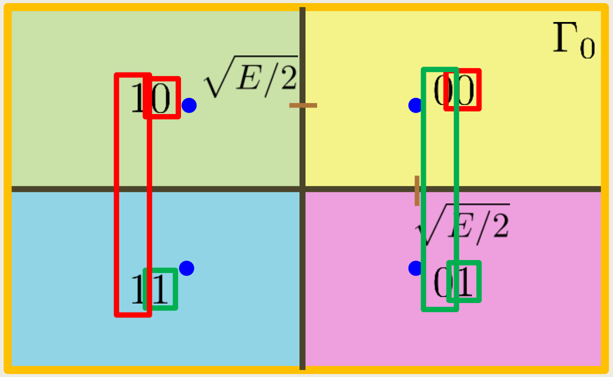
We make a mistake in the right bit iff (assume we are to the left) $w_1 > \sqrt{E/2}$

Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

What does signal set look like in signal space?



What are cordinates if E_s=E
(E_s is average energy per symbol)

Math model of received signal

$$r_1 = s_{\ell,1} + w_1$$
$$r_2 = s_{\ell,2} + w_2$$

Bit error probability ? Depends on bit-mapping. Assume the above mapping

Observe (important): The left bit decides if we are to the left or to the right
The right bit decides if we are up or down

We make a mistake in the left bit iff with probability $Q\left(\sqrt{\frac{E}{N_0}}\right)$

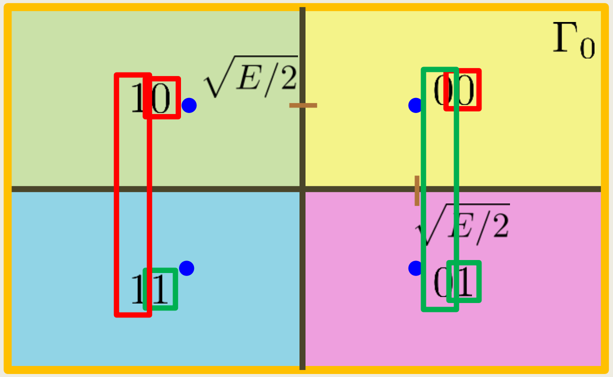
We make a mistake in the right bit iff with probability $Q\left(\sqrt{\frac{E}{N_0}}\right)$

Lecture 3: Error Probabilities

Case II: QPSK

What is dimensionality, N? 2

What does signal set look like in signal space?



What are cordinates if E_s=E
(E_s is average energy per symbol)

Math model of received signal

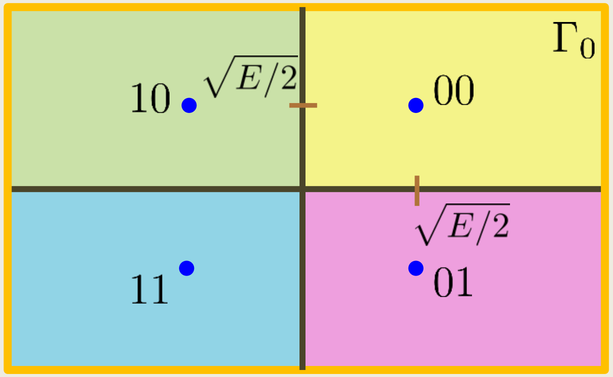
$$r_1 = s_{\ell,1} + w_1$$
$$r_2 = s_{\ell,2} + w_2$$

Bit error probability ? Depends on bit-mapping. Assume the above mapping

Observe (important): The left bit decides if we are to the left or to the right
The right bit decides if we are up or down

$$P_\text{b} = Q\left(\sqrt{\frac{E}{N_0}}\right)$$

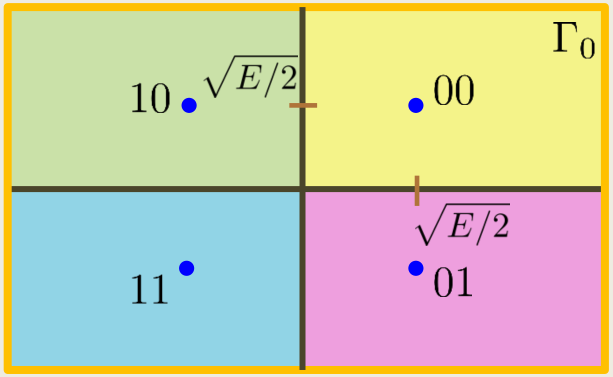
Case II: QPSK



Ps from Pb $P_{\mathrm{s}} = \Pr(\text{bit 1 or bit 2 in error})$

Lecture 3: Error Probabilities

Case II: QPSK



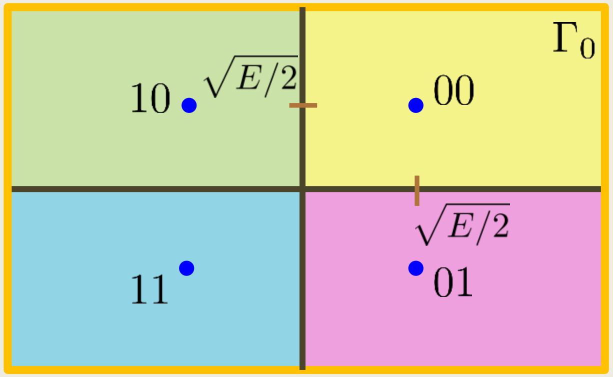
Ps from Pb $P_s = \Pr(\text{bit 1 or bit 2 in error})$

When is P(A or B) = P(A) + P(B) ?

When A and B cannot happen at the same time

Lecture 3: Error Probabilities

Case II: QPSK



Ps from Pb $P_s = \Pr(\text{bit 1 or bit 2 in error})$

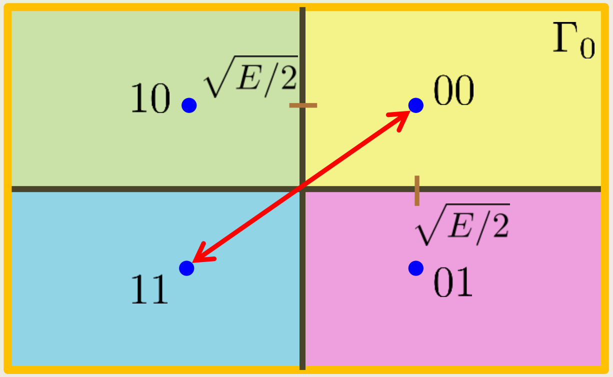
When is P(A or B) = P(A) + P(B) ?

When A and B cannot happen at the same time

Can both bits be in error at the same time?

Lecture 3: Error Probabilities

Case II: QPSK



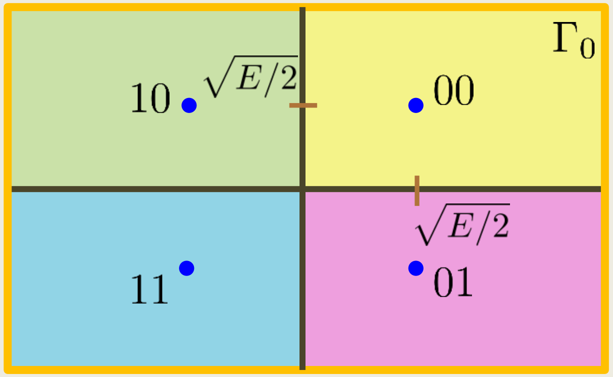
Ps from Pb $P_s = \Pr(\text{bit 1 or bit 2 in error})$

When is P(A or B) = P(A) + P(B) ?
When A and B cannot happen at the same time

Can both bits be in error at the same time? YES

Lecture 3: Error Probabilities

Case II: QPSK

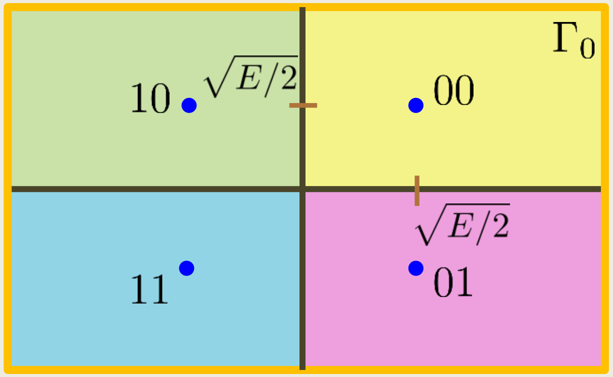


Ps from Pb $P_\mathrm{s} = \Pr(\text{bit 1 or bit 2 in error})$

P(A or B) = P(A) + P(B) - P(A and B)

Lecture 3: Error Probabilities

Case II: QPSK

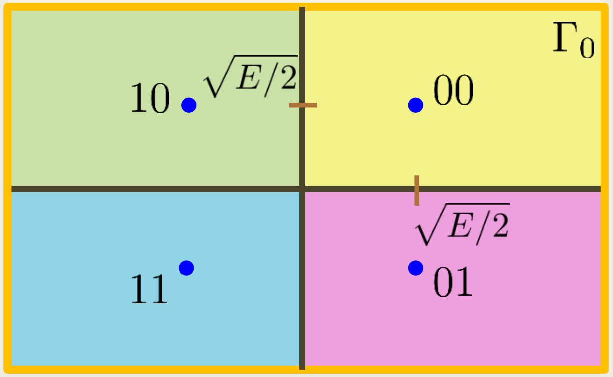


Ps from Pb $$P_s = \Pr(\text{bit 1 or bit 2 in error})$$

$$= \Pr(\text{bit 1 in error}) + \Pr(\text{bit 2 in error}) - \Pr(\text{bit 1 and 2 in error})$$

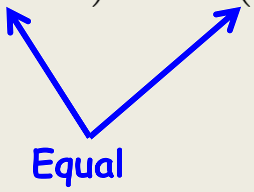
Lecture 3: Error Probabilities

Case II: QPSK



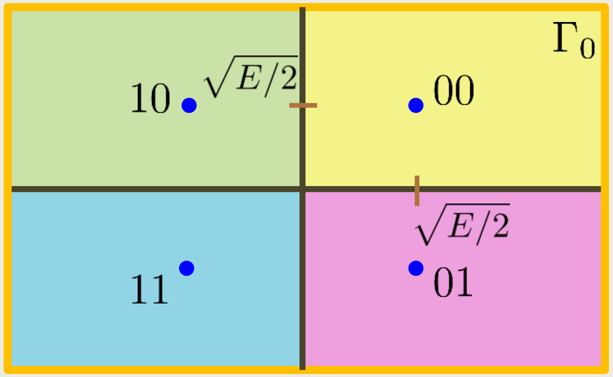
Ps from Pb $P_s = \Pr(\text{bit 1 or bit 2 in error})$

$$= \Pr(\text{bit 1 in error}) + \Pr(\text{bit 2 in error}) - \Pr(\text{bit 1 and 2 in error})$$



Lecture 3: Error Probabilities

Case II: QPSK



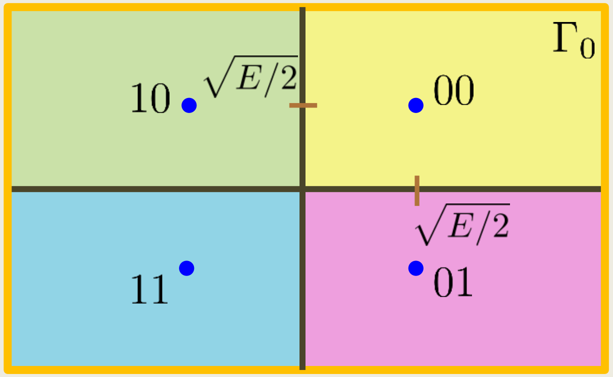
Ps from Pb
$$P_s = \Pr(\text{bit 1 or bit 2 in error})$$
$$= \Pr(\text{bit 1 in error}) + \Pr(\text{bit 2 in error}) - \Pr(\text{bit 1 and 2 in error})$$

Equal

Independent

Lecture 3: Error Probabilities

Case II: QPSK



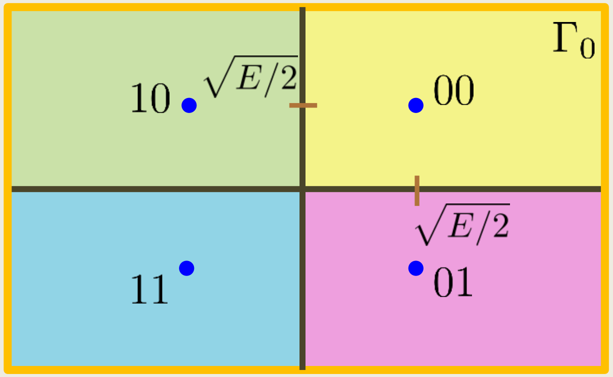
Ps from Pb $P_s = \Pr(\text{bit 1 or bit 2 in error})$

$$= \Pr(\text{bit 1 in error}) + \Pr(\text{bit 2 in error}) - \Pr(\text{bit 1 and 2 in error})$$

$$= 2\Pr(\text{bit 1 in error}) - [\Pr(\text{bit 1 in error})]^2$$

Lecture 3: Error Probabilities

Case II: QPSK



Ps from Pb

$$P_s = \Pr(\text{bit 1 or bit 2 in error})$$

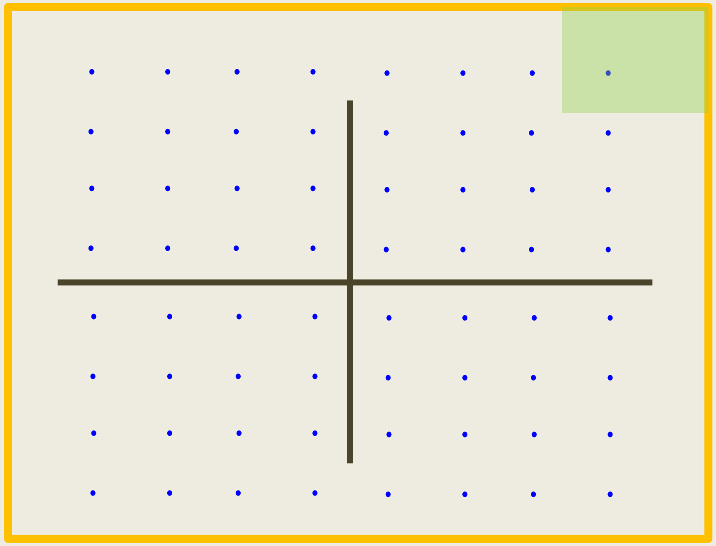
$$= \Pr(\text{bit 1 in error}) + \Pr(\text{bit 2 in error}) - \Pr(\text{bit 1 and 2 in error})$$

$$= 2\Pr(\text{bit 1 in error}) - [\Pr(\text{bit 1 in error})]^2$$

$$= 2Q\left(\sqrt{\frac{E}{N_0}}\right) - Q^2\left(\sqrt{\frac{E}{N_0}}\right)$$

Lecture 3: Error Probabilities

Case III: M-QAM

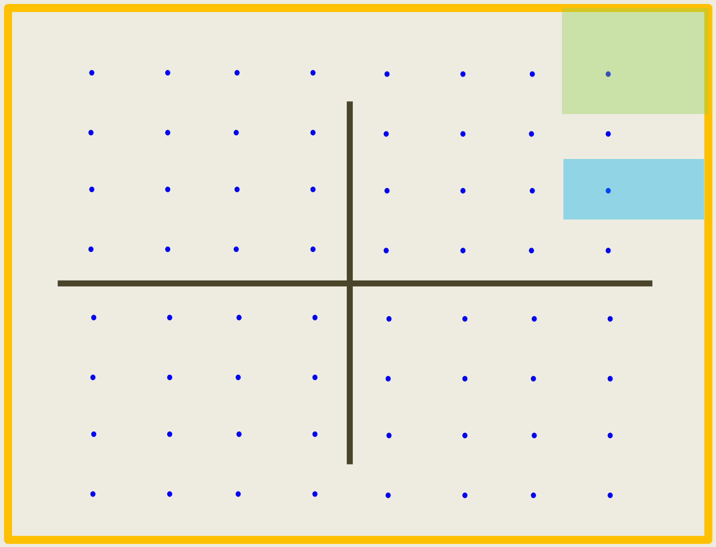


3 cases:

1. 4 Corner points

Lecture 3: Error Probabilities

Case III: M-QAM

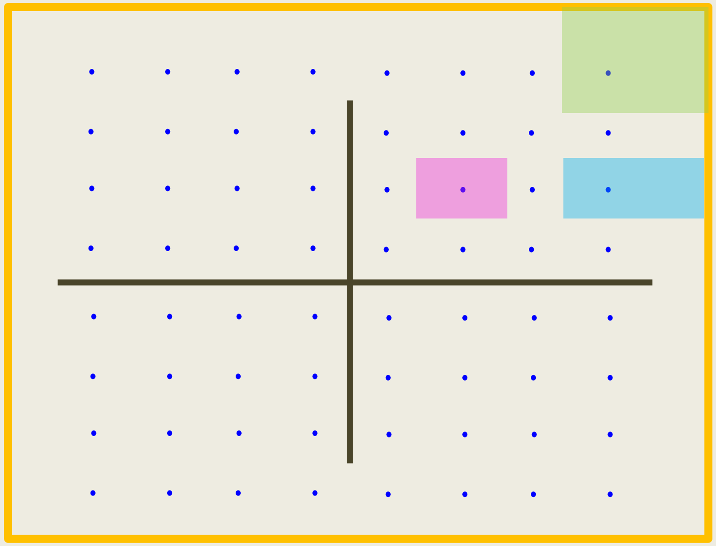


3 cases:

1. 4 Corner points
2. $4\sqrt{M} - 8$ Edge points

Lecture 3: Error Probabilities

Case III: M-QAM

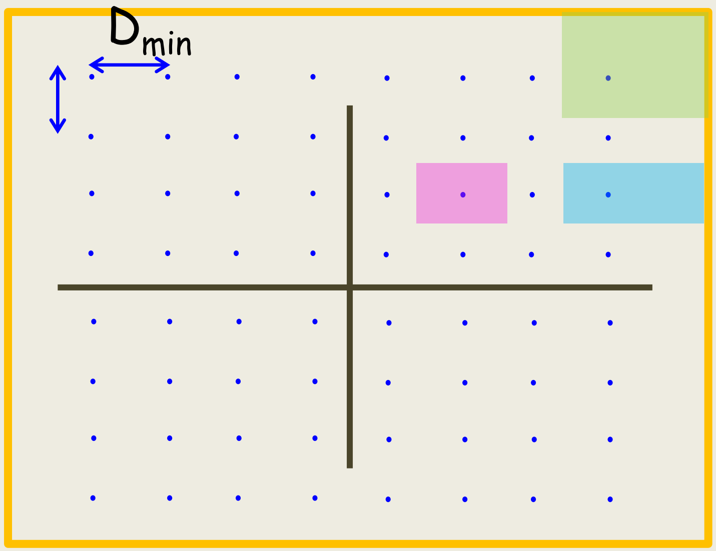


3 cases:

1. 4 Corner points
2. $4\sqrt{M}-8$ Edge points
3. $M-4\sqrt{M}+4$ Interior points

Lecture 3: Error Probabilities

Case III: M-QAM



3 cases:

1. 4 Corner points
2. $4\sqrt{M}-8$ Edge points
3. $M-4\sqrt{M}+4$ Interior points

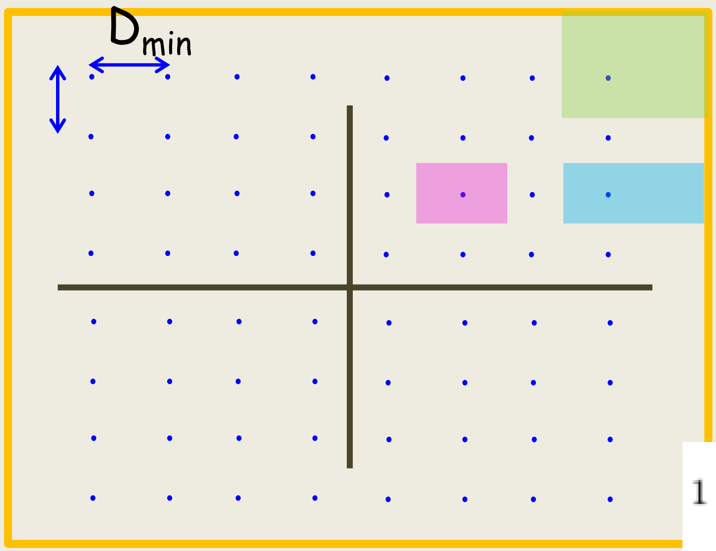
$$1 - Prob\left\{w_1 > -\frac{D_{\min}}{2}, -\frac{D_{\min}}{2} \le w_2 \le \frac{D_{\min}}{2}\right\} =$$

$$1 - \left(1 - Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)\right)\left(1 - 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)\right) =$$

$$3Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - 2Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

Lecture 3: Error Probabilities

Case III: M-QAM



3 cases:

1. 4 Corner points
2. $4\sqrt{M}-8$ Edge points
3. $M-4\sqrt{M}+4$ Interior points

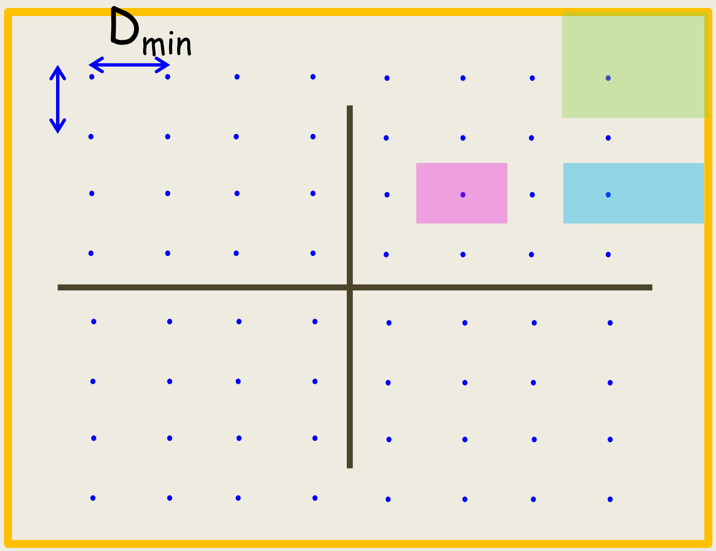
$$1 - Prob\left\{-\frac{D_{\min}}{2} \leq w_1 \leq \frac{D_{\min}}{2}, -\frac{D_{\min}}{2} \leq w_2 \leq \frac{D_{\min}}{2}\right\} =$$

$$1 - \left(1 - 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)\right)^2 =$$

$$4Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - 4Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

Lecture 3: Error Probabilities

Case III: M-QAM



3 cases:

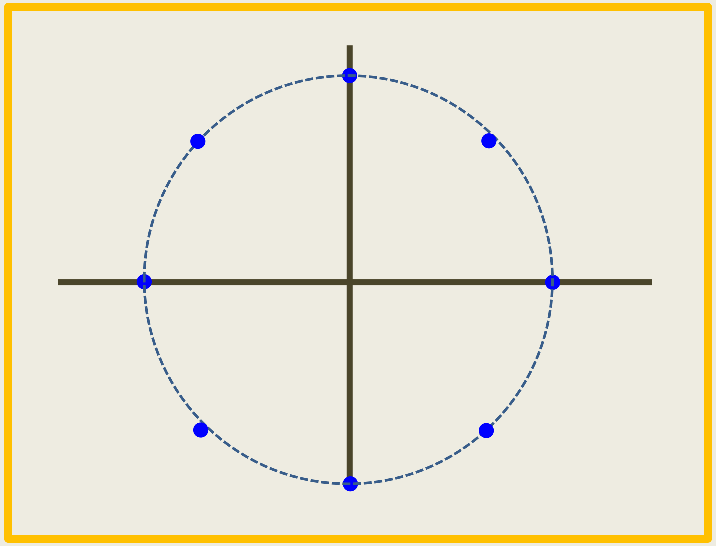
1. 4 Corner points
2. $4\sqrt{M}-8$ Edge points
3. $M-4\sqrt{M}+4$ Interior points

Verify at home

$$P_s = \frac{4}{\sqrt{M}}\,(\sqrt{M}\text{-}1)\;Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - \frac{4}{M}\,(\sqrt{M}\text{-}1)^2\;Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right), \quad \text{M-ary QAM} \tag{5.50}$$

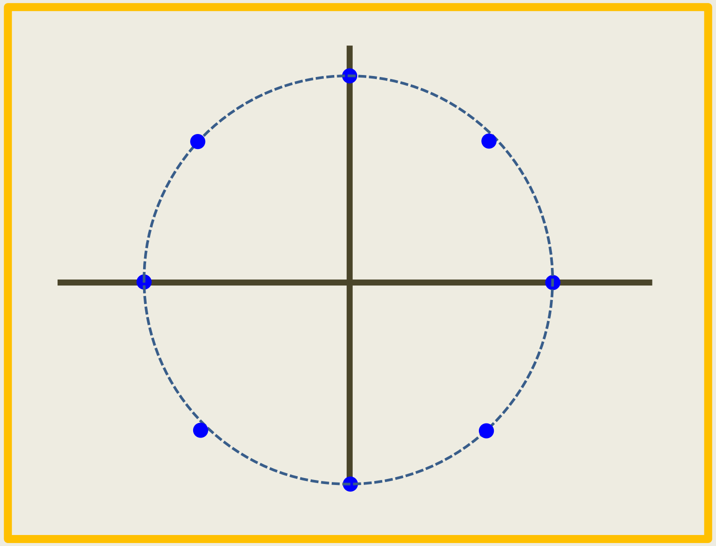
Lecture 3: Error Probabilities

Case IV: M-PSK



Lecture 3: Error Probabilities

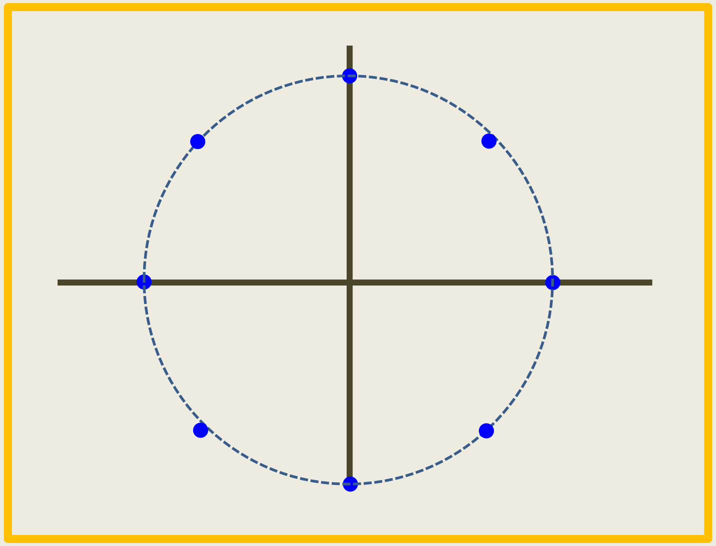
Case IV: M-PSK



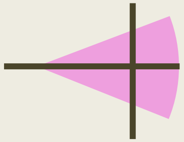
Decision region

Lecture 3: Error Probabilities

Case IV: M-PSK

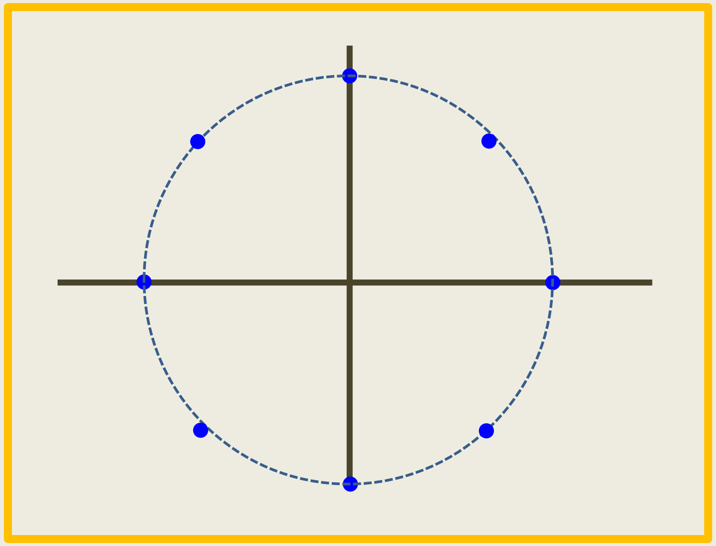


Decision region $$\Pr(\text{correct decision}) = \int_{\mathcal{S}_0} \mathrm{P}(w_1)\mathrm{P}(w_2)\mathrm{d}w_1\mathrm{d}w_2$$

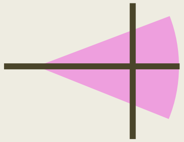
$\mathcal{S}_0 =$ 

Lecture 3: Error Probabilities

Case IV: M-PSK



Decision region $$\Pr(\text{correct decision}) = \int_{\mathcal{S}_0} \mathrm{P}(w_1)\mathrm{P}(w_2)\mathrm{d}w_1\mathrm{d}w_2$$

$\mathcal{S}_0 =$ 

Closed form complicated