Lecture 10: Time variant channels

Frequency-non-selective, slowly fading channel

Significantly simplifed modelling. For complex basesband, signals are mutltiplied with a complex constant

Underspread channel: $B_{\mathcal{D}} T_m \ll 1$

Conditions for Frequency-non-selective, slowly fading channel

$$k_w T_m \approx \frac{k_w}{f_{\text{coh}}} \ll T_s \ll t_{\text{coh}} \approx \frac{1}{B_{\text{D}}}$$

$$\frac{t}{t_{\text{coh}}} \approx B_{\mathcal{D}} \ll R_s \ll \frac{f_{\text{coh}}}{k_w} \approx \frac{1}{k_w T_m}$$

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Underspread channel: $B_{\mathcal{D}} T_m \ll 1$

Can we have Frequency-non-selective, slowly fading channels if $B_{\mathcal{D}} T_m \approx 1$

NO. Note that $B_{\mathcal{D}} T_m$ is a channel parameter, out of our control

Conditions for Frequency-non-selective, slowly fading channel

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Lecture 10: Time variant channels

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Assume an underspread channel. Complex baseband model becomes

$$z(t) = a e_s(t) \cos(\omega_c t + \theta_s(t) + \phi)$$

$e_s(t)$ and $\theta_s(t)$ describe signal

a and ϕ describe channel

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a Rayleigh ϕ Uniform

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Lecture 10: Time variant channels

Frequency-non-selective, slowly fading channel

a **Rayleigh** $p_a(x) = \frac{2x}{b} \exp\left(-\frac{x^2}{b}\right), \; x \geq 0$ $E\{a\} = \frac{\sqrt{\pi b}}{2}$

$E\{a^2\} = b$

ϕ **Uniform** $p_\phi(y) = \frac{1}{2\pi}, \; -\pi \leq y \leq \pi$

$$z(t) = a e_s(t) \cos(\omega_c t + \theta_s(t) + \phi)$$

$e_s(t)$ and $\theta_s(t)$ describe signal

a and ϕ describe channel

Conditions for Frequency-non-selective, slowly fading channel

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Lecture 10: Time variant channels

Frequency-non-selective, slowly fading channel

$$z(t) = a e_s(t) \cos(\omega_c t + \theta_s(t) + \phi)$$

Bit error rate?

Lecture 10: Time variant channels

Bit error rate?

$$z(t) = a e_s(t) \cos(\omega_c t + \theta_s(t) + \phi)$$

$$P_b \approx Q\left(\sqrt{a^2 \frac{E_b d_{\min}^2}{N_0}}\right)$$

Normal formula for error probability
Constant in front of $Q(\ )$ is unimportant

Lecture 10: Time variant channels

Bit error rate?

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This is instantaneous error rate !!!
We don't really care about that

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Example

- Generate a random channel a
- Simulate 10^6 BPSK symbols
- Get 100 bit errors

Is BER $100/10^6 = 10^{-4}$?

Lecture 10: Time variant channels

Bit error rate?

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Example

- Generate a random channel a
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Is BER $100/10^6 = 10^{-4}$?

No, only for the channel we got

Lecture 10: Time variant channels

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First way

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Example

Better way
Repeat 10^6 times (FOR)
- Generate a channel a
- Send and receive a BPSK

END

Measure a total of 1000 bit errors

What does this mean?

Lecture 10: Time variant channels

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$$z(t) = a e_s(t) \cos(\omega_c t + \theta_s(t) + \phi)$$

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First way

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That the channel we randomly generated was "better than average"

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Lecture 10: Time variant channels

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First way

- Generate a random channel a
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And for that channel, the BER is 10^{-3} when averaged over noise distribution

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Example

Better way
Repeat 10^6 times (FOR)
- Generate a channel a
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END

Measure a total of 1000 bit errors

The average BER is 10^{-3} when averaged over the channel and noise

Lecture 10: Time variant channels

Bit error rate? $z(t) = a e_s(t) \cos(\omega_c t + \theta_s(t) + \phi)$

$$P_b \approx Q\left(\sqrt{a^2 \frac{E_b d_{\min}^2}{N_0}}\right)$$

Channel dependent

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Channel dependent

We should take an expectation,
But of which variable, a or P_b ?

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Over this one, since

E(f(x)) ≠ f(E(x))

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Putting the average here would mean:
"Simulate only the performance at the average SNR value"

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Example

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Repeat 10^6 times (FOR)

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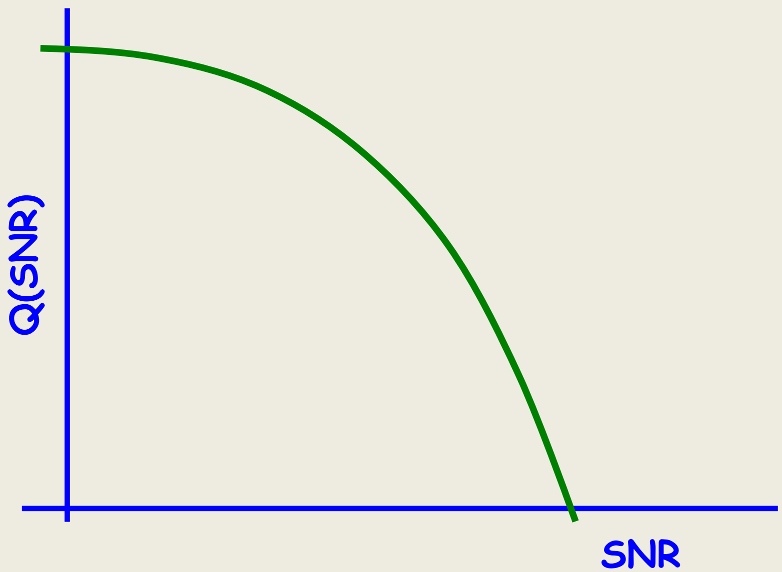
Not good, since Q( ) is not linear



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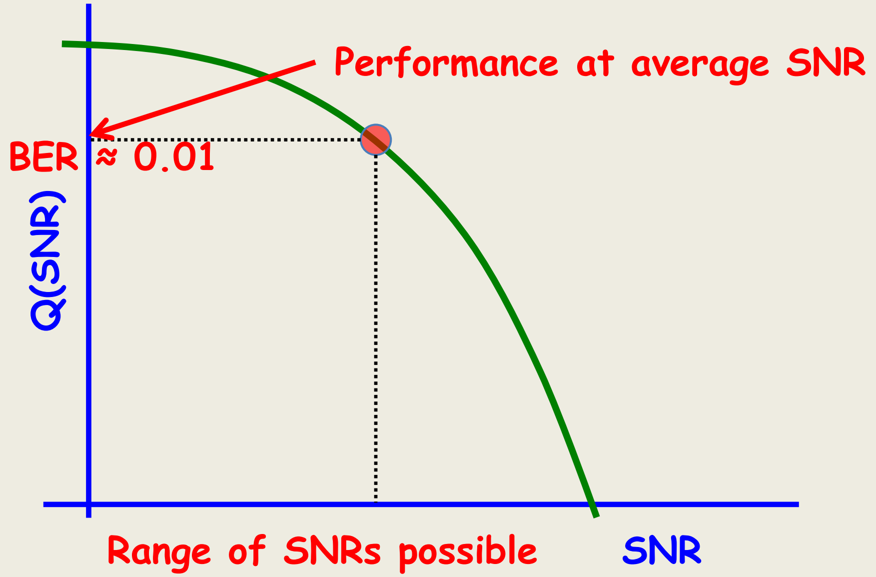

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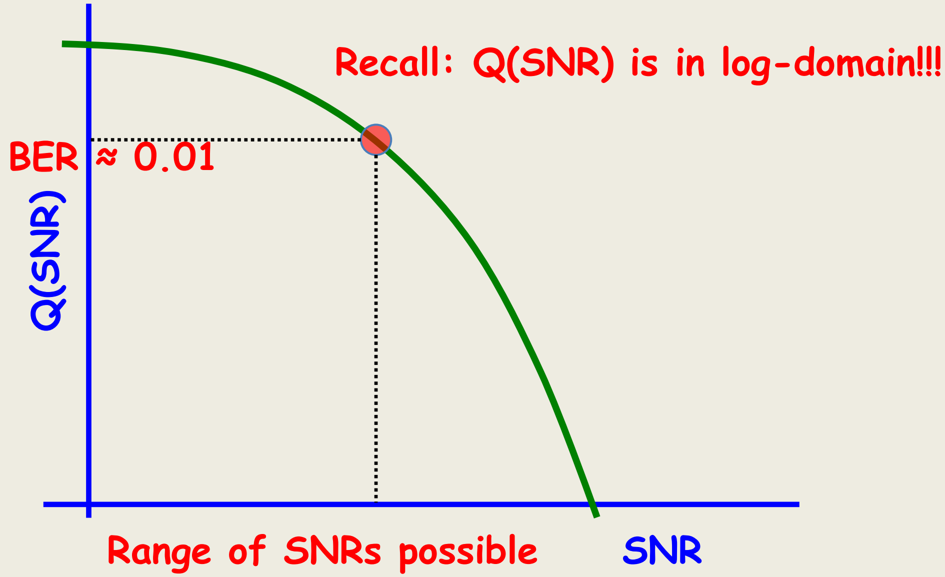

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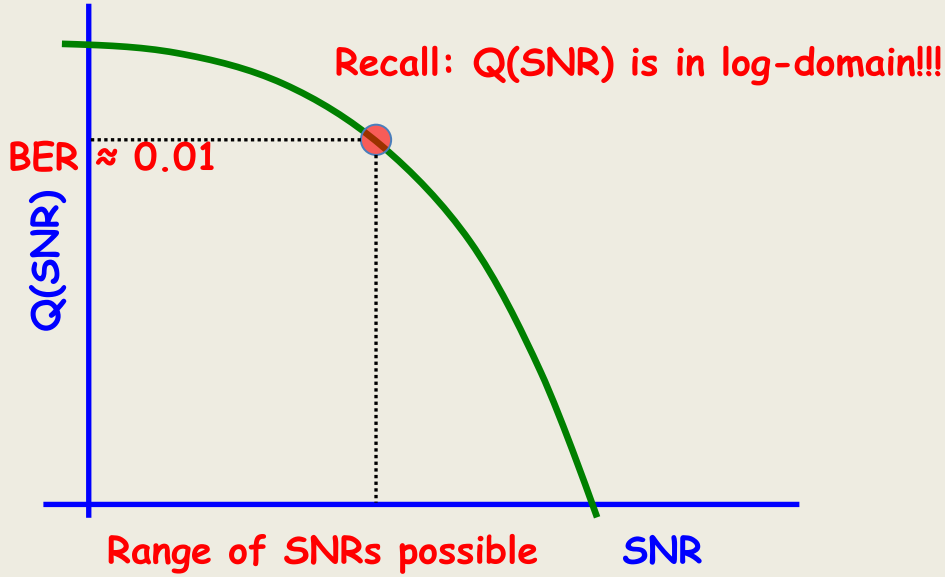

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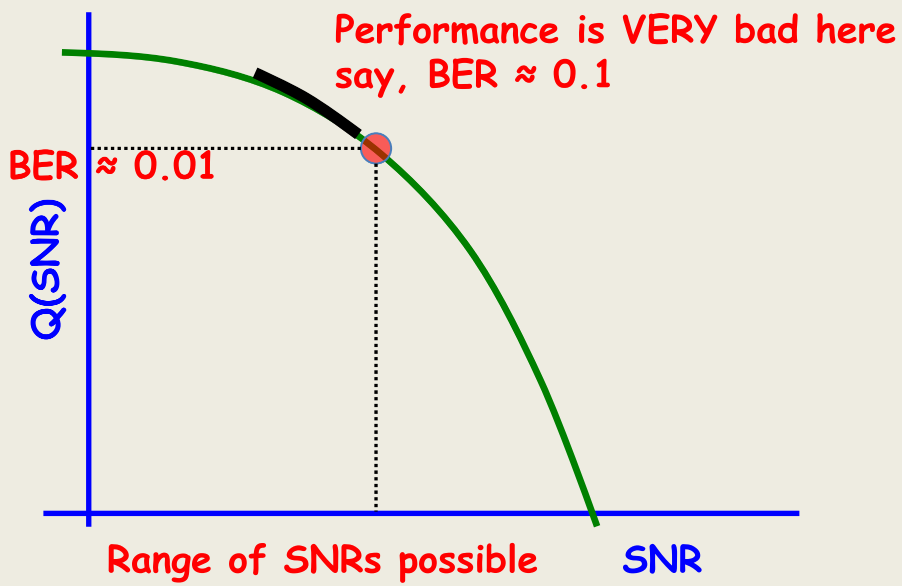

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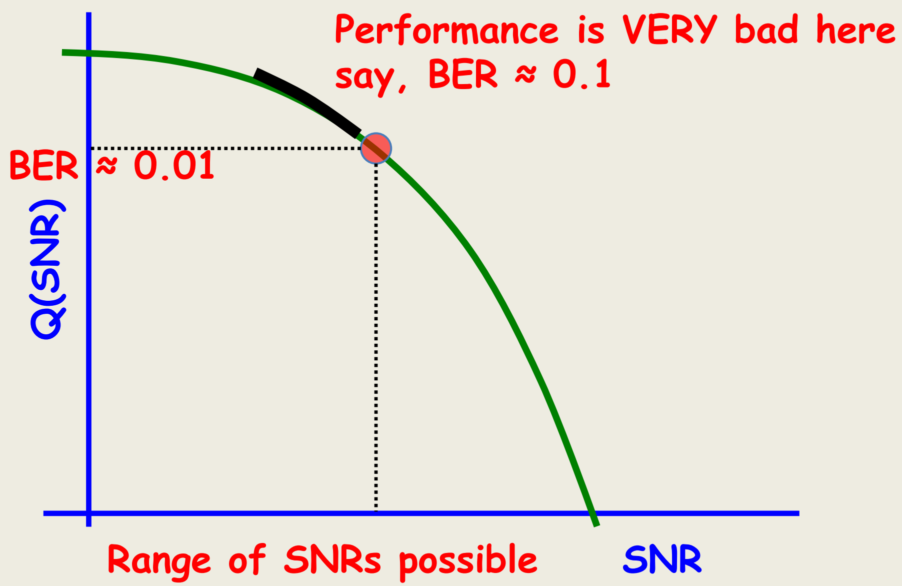

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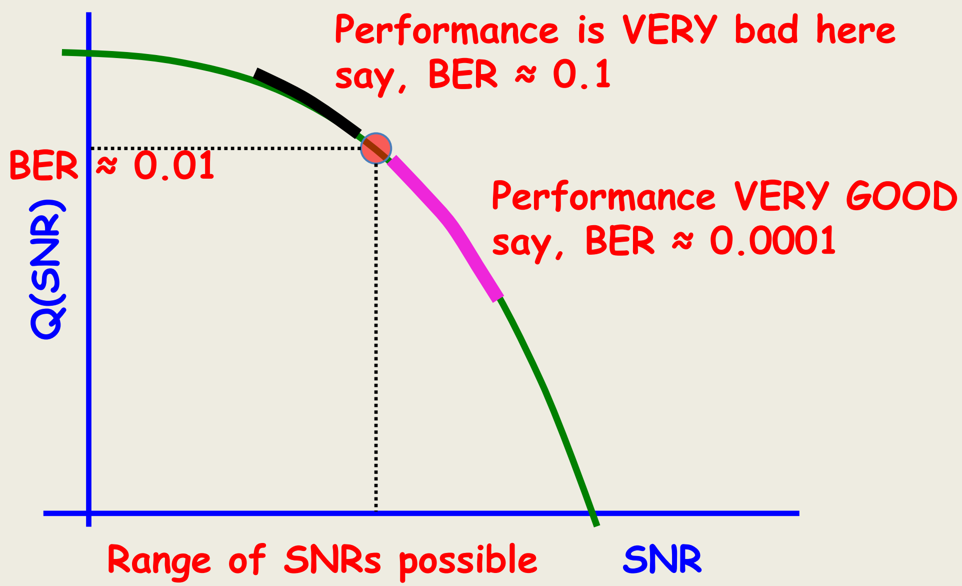

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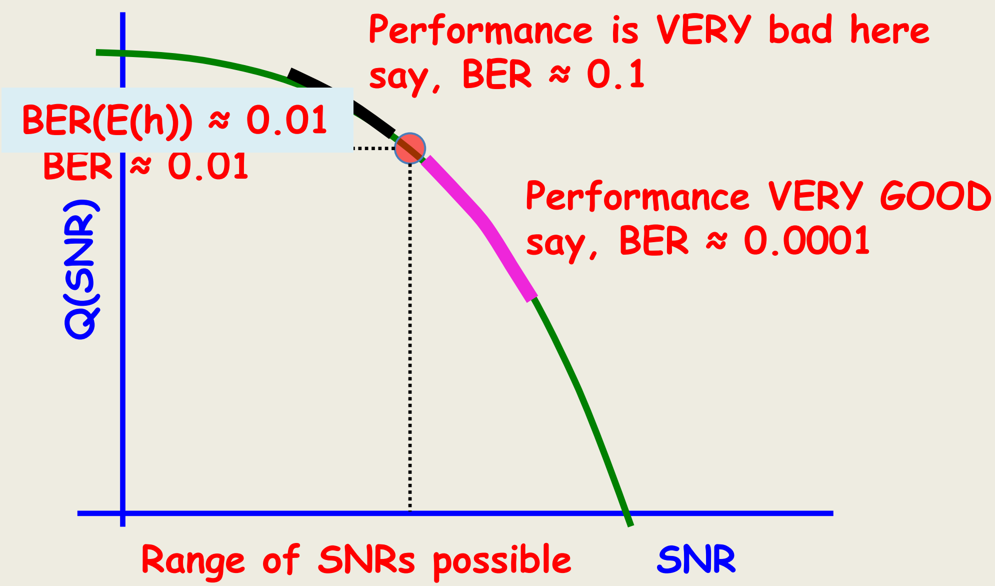

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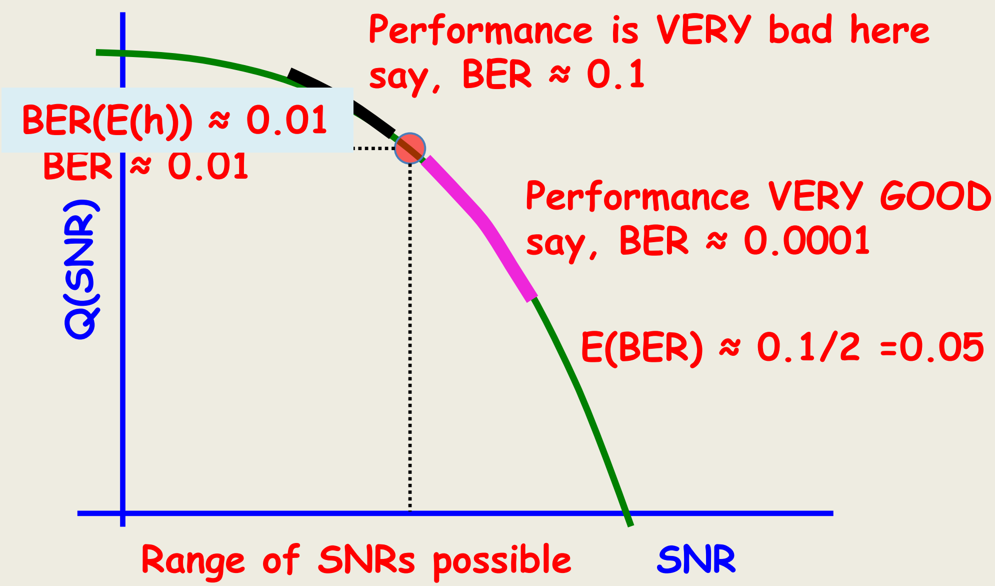

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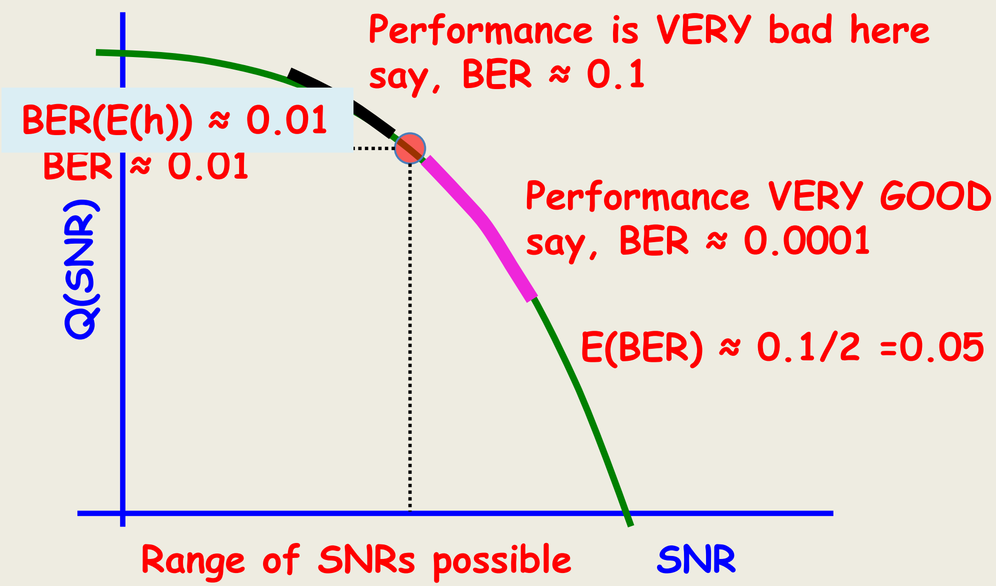

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Conclusion

We need $E\{P_b\}$

This point is super important

Lecture 10: Time variant channels

Frequency-non-selective, slowly fading channel

$$z(t) = a e_s(t) \cos(\omega_c t + \theta_s(t) + \phi)$$

Bit error rate?

$$\bar{P}_b = \int_0^\infty \Pr(\mathrm{error}|a = x) p_a(x) \mathrm{d}x$$

Lecture 10: Time variant channels

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$$z(t) = a e_s(t) \cos(\omega_c t + \theta_s(t) + \phi)$$

Bit error rate?

$$\bar{P}_b = \int_0^\infty \Pr(\text{error}|a = x) p_a(x) \mathrm{d}x = E\left\{\Pr(\text{error}|a)\right\}$$

Lecture 10: Time variant channels

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$$\bar{P}_b = \int_0^\infty \Pr(\text{error}|a = x) p_a(x) \mathrm{d}x = E\left\{\Pr(\text{error}|a)\right\}$$

$$= \text{long derivation}$$

Lecture 10: Time variant channels

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$$= \text{long derivation}$$

$$= \frac{1}{2 + d_{\min}^2\mathcal{E}_b/N_0 + \sqrt{2 + d_{\min}^2\mathcal{E}_b/N_0}\sqrt{d_{\min}^2\mathcal{E}_b/N_0}}$$

$$\mathcal{E}_b = E\{a^2\}E_{b,sent} = bE_{b,sent}$$

Lecture 10: Time variant channels

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$\mathcal{E}_b/N_0$ large

$$= \frac{1}{2d_{\min}^2\mathcal{E}_b/N_0}$$

$$\mathcal{E}_b = E\{a^2\}E_{b,sent} = bE_{b,sent}$$

Lecture 10: Time variant channels

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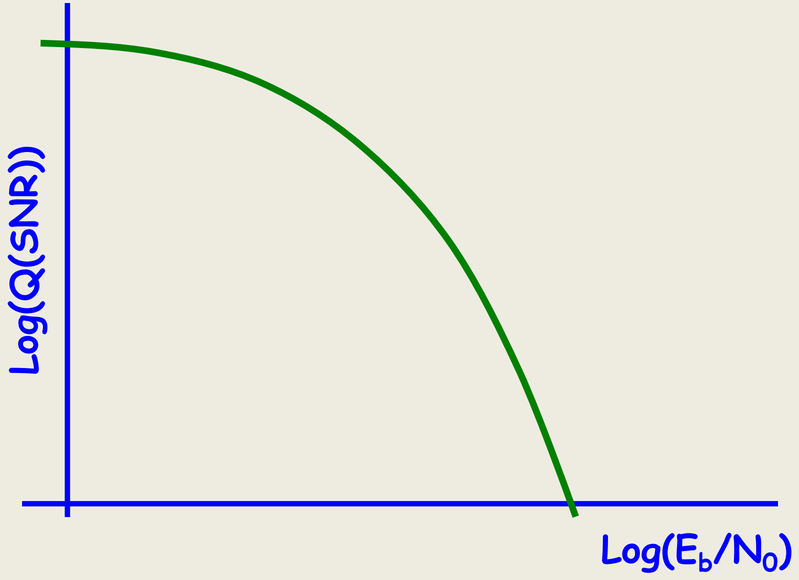
$\mathcal{E}_b/N_0$ large

Super important consequence



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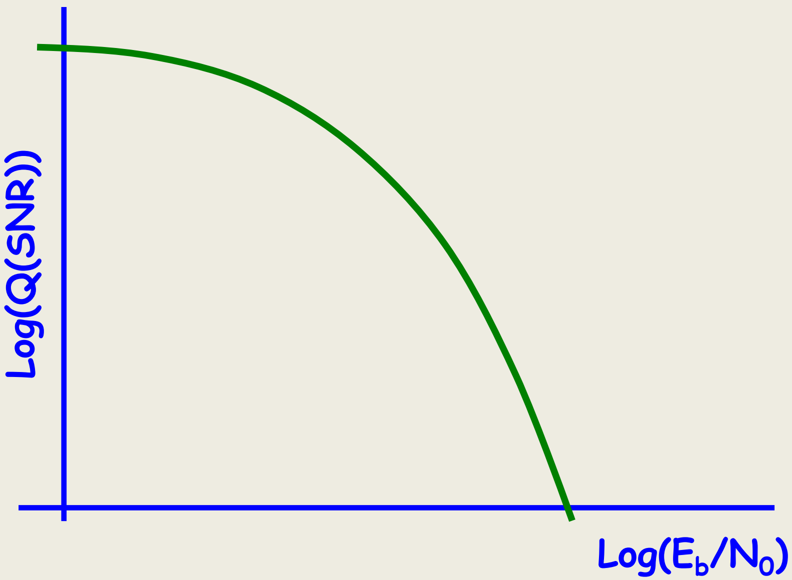
Bit error rate?

$$\bar{P}_b = \frac{1}{2d_{\min}^2 \mathcal{E}_b/N_0}$$

$\mathcal{E}_b/N_0$ large

$$\log(\bar{P}_b) = -\log(2d_{\min}^2) - \log(\mathcal{E}_b/N_0)$$

Super important consequence



Lecture 10: Time variant channels

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Bit error rate?

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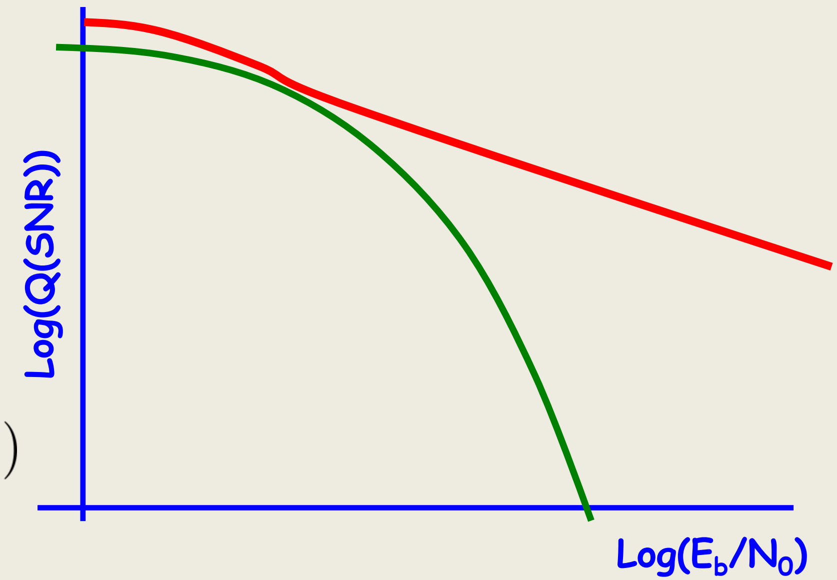
$\mathcal{E}_b/N_0$ large

$$\log(\bar{P}_b) = -\log(2 d_{\min}^2) - \log(\mathcal{E}_b/N_0)$$

Straight line. Very poor BER

Super important consequence



Lecture 10: Time variant channels

Example 9.1, p. 592

Assume that equally likely, binary orthogonal FSK signals, with equal energy, are sent from the transmitter. Hence, $s_i(t) = \sqrt{2E_{b,sent}/T_b}\cos(2\pi f_i t)$ *in* $0 \leq t \leq T_b$, $i = 0, 1$.

These signals are communicated over a Rayleigh fading channel, i.e. the received signal is (see (9.38)),

$$r(t) = a\sqrt{2E_{b,sent}/T_b}\cos(2\pi f_i t + \phi) + N(t)$$

Assume that the incoherent receiver in Figure 5.28 on page 397 is used. From (5.109) it is known that for a given value of a,

$$P_b = \frac{1}{2}\, e^{-a^2 E_{b,sent}/2N_0}$$

since $a^2 E_{b,sent}$ *then is the average received energy per bit.*

For the Rayleigh fading channel, and the same receiver, P_b *can be calculated by using (9.43),*

$$\boxed{P_b = \int_0^\infty \Pr\{error|a = x\} p_a(x) = E\{\Pr\{error|a\}\}}$$

Lecture 10: Time variant channels

Example 9.1, p. 592

$$E\{\Pr\{error|a\}\} = E\left\{\frac{1}{2}\,e^{-a^2 E_{b,sent}/2N_0}\right\} =$$

$$E\left\{\frac{1}{2}\,e^{-a_1^2 E_{b,sent}/2N_0}\right\} \cdot E\left\{e^{-a_2^2 E_{b,sent}/2N_0}\right\}$$

See page 184 $$P_b = \frac{1/2}{1 + \frac{E_{b,sent}}{N_0} \cdot \frac{E\{a^2\}}{2}} = \frac{1}{2 + \mathcal{E}_b/N_0}$$

Observe the dramatic increase in P_b due to the Rayleigh fading channel. P_b is no longer exponentially decaying in $\mathcal{E}_b/N_0$, it now decays essentially as $(\mathcal{E}_b/N_0)^{-1}$! *As an example, assuming $\mathcal{E}_b/N_0 = 1000$ (30 dB), we obtain*

$$P_b = \begin{cases} 0.5e^{-500} \approx 3.6 \cdot 10^{-218} & , \quad AWGN \\ (1002)^{-1} \approx 10^{-3} & , \quad Rayleigh{+}AWGN \end{cases}$$

Lecture 10: Trellis coded modulation

Slides to be updated next year…..

Lecture 4: Capacity

Shannon Capacity

Before going on, we go through what the term capacity means

Given a scalar channel of form $y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$

We know that the capacity is $C = \log_2 \left(1 + \frac{A}{N_0} \right)$

But what does this mean?

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Build a codebook of all information sequences possible to send of length K

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Build a codebook of all information sequences possible to send of length K

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Sending K bits of information means:

pick one of the rows, and tell the receiver which row it is

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n,\ n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Build a codebook of codewords to send for each information word, length N

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 ….. 00
000000 ….. 01
000000 ….. 10

1111111 ….. 10
1111111 ….. 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ….. $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ….. $x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ….. $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n,\ n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

If this is my data

1111111 10
1111111 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

If this is my data

1111111 10
1111111 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

I send this one

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Shannon Capacity

As x over this channel used N times

$$y = \sqrt{A}x + n, \quad n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

If this is my data

1111111 10
1111111 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ..... $x_{1(N-1)}x_{1N}$

$x_{21}x_{22}x_{23}x_{24}$ ..... $x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ..... $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Shannon Capacity

$$y = \sqrt{A}x + n, \ \ n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Clearly, bit rate is K/N bits/channel use

Information book

000000 ….. 00
000000 ….. 01
000000 ….. 10

1111111 ….. 10
1111111 ….. 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14}$ ….. $x_{1(N-1)}x_{1N}$
$x_{21}x_{22}x_{23}x_{24}$ ….. $x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4}$ ….. $x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \cdots\cdots y_{(N-1)} y_N$

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \cdots\cdots x_{1(N-1)} x_{1N}$

$x_{21} x_{22} x_{23} x_{24} \cdots\cdots x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \cdots\cdots x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

$$y = \sqrt{A}x + n,\ \ n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \ldots\ldots y_{(N-1)} y_N$

Compare with this one

$$d_1 = \sum_{n=1}^{N} |y_n - x_{1n}|^2$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \ldots\ldots x_{1(N-1)} x_{1N}$
$x_{21} x_{22} x_{23} x_{24} \ldots\ldots x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \ldots\ldots x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \ldots\ldots y_{(N-1)} y_N$

Compare with this one

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \ldots\ldots x_{1(N-1)} x_{1N}$

$x_{21} x_{22} x_{23} x_{24} \ldots\ldots x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \ldots\ldots x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \;.....\; y_{(N-1)} y_N$

Compare with this one

$$d_{2^K} = \sum_{n=1}^{N} |y_n - x_{2^K n}|^2$$

Information book

000000 00
000000 01
000000 10

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \;.....\; x_{1(N-1)} x_{1N}$
$x_{21} x_{22} x_{23} x_{24} \;.....\; x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \;.....\; x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

$$y = \sqrt{A}x + n, \ n \sim CN(0, N_0)$$

$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \ldots\ldots y_{(N-1)} y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

Information book

000000	.....	00
000000	.....	01
000000	.....	10
1111111	.....	10
1111111	.....	11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \ldots\ldots x_{1(N-1)} x_{1N}$

$x_{21} x_{22} x_{23} x_{24} \ldots\ldots x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \ldots\ldots x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

$$y = \sqrt{A}x + n, \;\; n \sim CN(0, N_0)$$
$$C = \log_2\left(1 + \frac{A}{N_0}\right)$$

Receiver observes

$y_1 y_2 y_3 y_4 \;\cdots\; y_{(N-1)} y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

Information book

000000 00
000000 01
000000 10

So data is this one

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \;\cdots\; x_{1(N-1)} x_{1N}$

$x_{21} x_{22} x_{23} x_{24} \;\cdots\; x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \;\cdots\; x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

Receiver observes

$y_1y_2y_3y_4 \cdots y_{(N-1)}y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

This is ML decoding and is optimal

Capacity means the following

Information book

000000 00
000000 01
000000 10

So data is this one

1111111 10
1111111 11

K

Codebook

$x_{11}x_{12}x_{13}x_{14} \cdots x_{1(N-1)}x_{1N}$

$x_{21}x_{22}x_{23}x_{24} \cdots x_{2(N-1)}x_{2N}$

$x_{2^K1}x_{2^K2}x_{2^K3}x_{2^K4} \cdots x_{2^K(N-1)}x_{2^KN}$

N

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \cdots y_{(N-1)} y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

Information book

000000 00
000000 01
000000 10

So data is this one

1111111 10
1111111 11

K

This is ML decoding and is optimal

Capacity means the following

1. If $K/N \leq C$, and $K \to \infty$ then Prob(Correct detection)=1

Codebook

$x_{11} x_{12} x_{13} x_{14} \cdots x_{1(N-1)} x_{1N}$

$x_{21} x_{22} x_{23} x_{24} \cdots x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \cdots x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \;\cdots\; y_{(N-1)} y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

This is ML decoding and is optimal

Capacity means the following

1. If $K/N \leq C$, and $K \to \infty$ then
   Prob(Correct detection)=1
2. If $K/N > C$, then
   Prob(Incorrect detection)=1

Information book

000000 00
000000 01
000000 10

So data is this one

1111111 10
1111111 11

K

Codebook

$x_{11} x_{12} x_{13} x_{14} \;\cdots\; x_{1(N-1)} x_{1N}$

$x_{21} x_{22} x_{23} x_{24} \;\cdots\; x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \;\cdots\; x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

Receiver observes

$y_1 y_2 y_3 y_4 \,.....\, y_{(N-1)} y_N$

Take smallest

$$d_2 = \sum_{n=1}^{N} |y_n - x_{2n}|^2$$

Information book

000000 00
000000 01
000000 10

So data is this one

1111111 10
1111111 11

K

To reach C, code-symbols must be
Random complex Gaussian variables
That is, generate codebook randomly

If it is generated with, say, 16QAM
C cannot be reached

Codebook

$x_{11} x_{12} x_{13} x_{14} \,.....\, x_{1(N-1)} x_{1N}$

$x_{21} x_{22} x_{23} x_{24} \,.....\, x_{2(N-1)} x_{2N}$

$x_{2^K 1} x_{2^K 2} x_{2^K 3} x_{2^K 4} \,.....\, x_{2^K (N-1)} x_{2^K N}$

N

Lecture 4: Capacity

Lessons learned:

- Good signals are random
- Not all signals can be sent
- Hard to decode
- We need the two first bullets,
  But in a controlled way

Chapter 8

Trellis-coded Signals

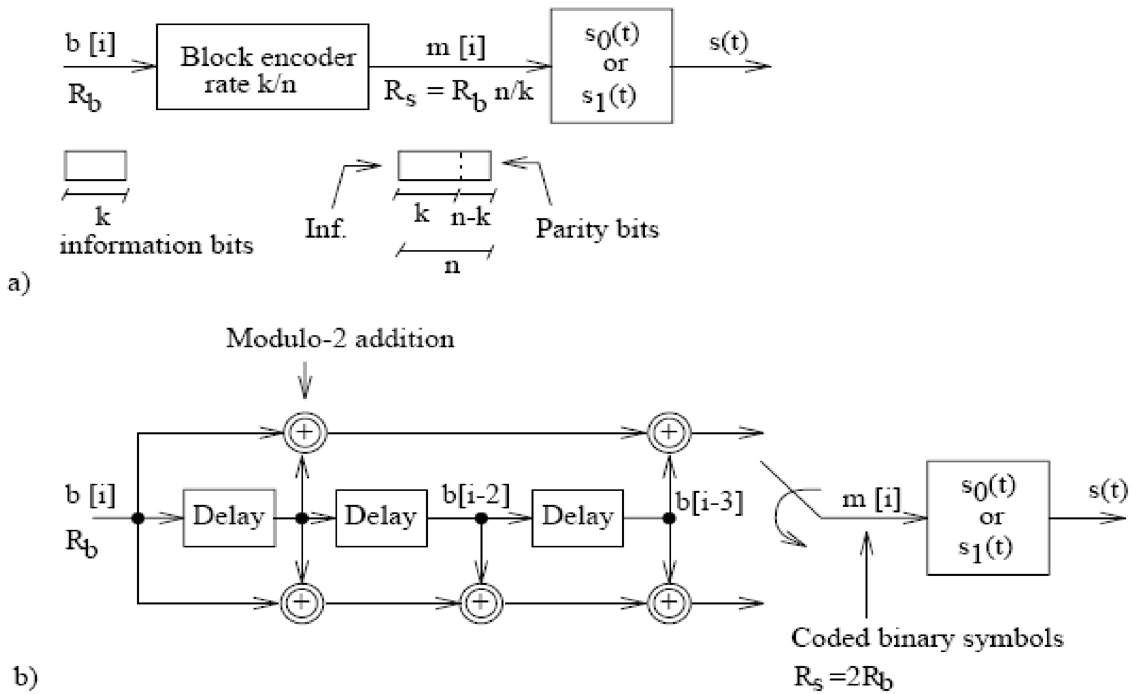


Figure 8.1: a) Block coding, $r_c = k/n$. b) Convolutional coding, $r_c = 1/2$.

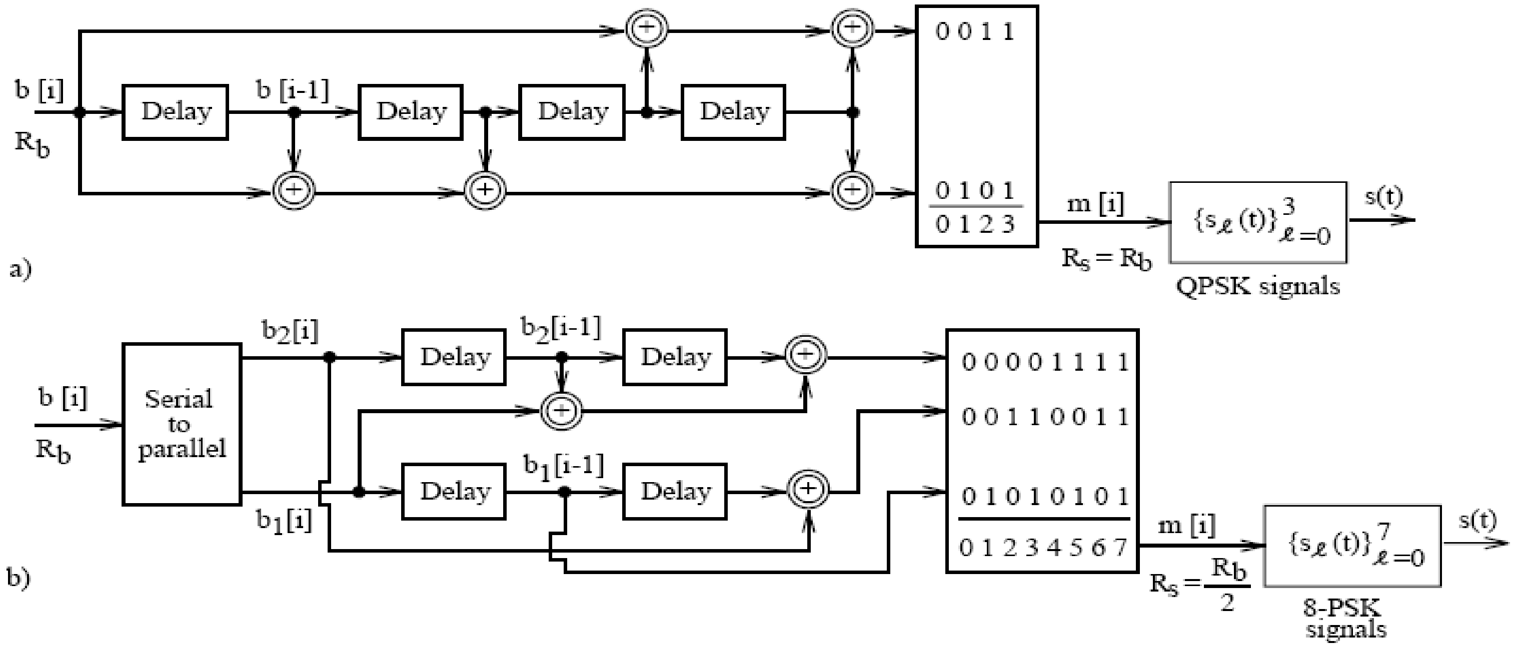
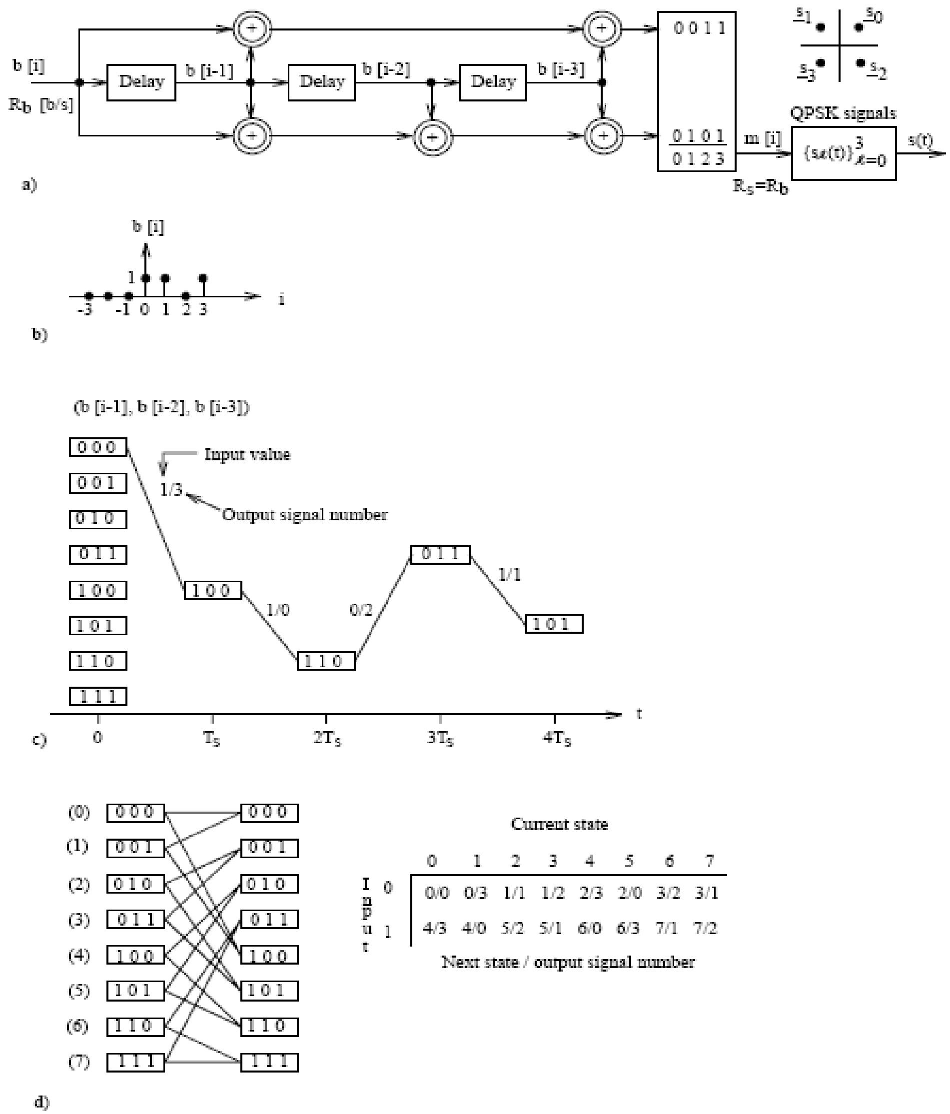


Figure 8.2: a) Rate $r_c = 1/2$ convolutional encoder combined with QPSK; b) Rate $r_c = 2/3$ convolutional encoder combined with 8-PSK, from [63], [64].

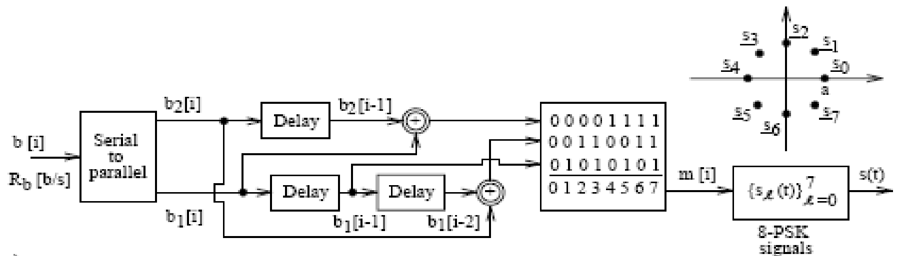


Current state

Input	0	1	2	3	4	5	6	7
0	0/0	0/3	1/1	1/2	2/3	2/0	3/2	3/1
1	4/3	4/0	5/2	5/1	6/0	6/3	7/1	7/2

Next state / output signal number

Figure 8.4: a) A rate 1/2 convolutional encoder combined with QPSK signal alternatives; b) A specific input sequence $b[i]$; c) The corresponding path in the trellis; d) A trellis section, and a table containing all relevant parameters.



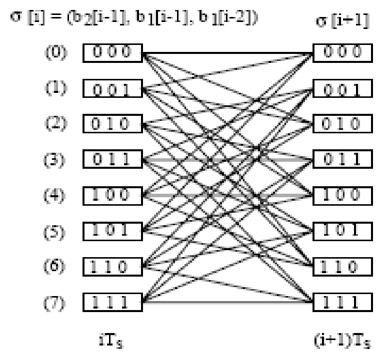
a)

Current state σ [i]

INPUT $\binom{b_2[i]}{b_1[i]}$	(000) 0	(001) 1	(010) 2	(011) 3	(100) 4	(101) 5	(110) 6	(111) 7
$\binom{0}{0}$	0/0	0/2	1/1	1/3	0/4	0/6	1/5	1/7
$\binom{0}{1}$	2/4	2/6	3/5	3/7	2/0	2/2	3/1	3/3
$\binom{1}{0}$	4/2	4/0	5/3	5/1	4/6	4/4	5/7	5/5
$\binom{1}{1}$	6/6	6/4	7/7	7/5	6/2	6/0	7/3	7/1

σ [i+1] / m [i]

b)



c)

Figure 8.6: a) An example of TCM, from [63]–[64]; b) The mappings $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$; c) A trellis section.

<u>Memory (redundancy, dependancy) is introduced among the sent signal alternatives!</u>

This gives us some new properties like, e.g.,:

Which of the following signal sequences are impossible?

1. $s_3(t), s_2(t - T_b), s_1(t - 2T_b), s_1(t - 3T_b)$
2. $s_3(t), s_2(t - T_b), s_2(t - 2T_b), s_1(t - 3T_b)$
3. $s_3(t), s_1(t - T_b), s_0(t - 2T_b), s_2(t - 3T_b)$
4. $s_3(t), s_1(t - T_b), s_3(t - 2T_b), s_1(t - 3T_b)$

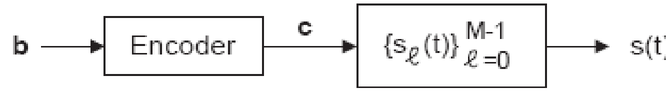
Note: In the uncoded case all signal sequences are possible.

Find the "missing" signal, in the sequence below,

$$s_1(t), s_3(t - T_b),\ ?\ , s_2(t - 3T_b), s_3(t - 4T_b), s_0(t - 5T_b)$$

Note: This is not possible to do in the uncoded case!

2.32 Let us here study adaptive coding and modulation according to the block diagram below.



$$\bar{E}_{sent} = r_c \log_2(M) E_{b,sent} = \frac{k}{n} \log_2(M) E_{b,sent} \tag{8.4}$$

$$R_s = 1/T_s = \frac{1}{r_c} \cdot \frac{1}{\log_2(M)} \cdot R_b = \frac{1}{k/n} \cdot \frac{1}{\log_2(M)} \cdot R_b \tag{8.5}$$

$$W = c \cdot R_s \tag{8.6}$$

Typically, the bandwidth W is fixed and given but:
the rate of the encoder
the number of signal alternatives
and the bit rate can be **ADAPTIVE**, see (8.5)-(8.6)!

We have memory in the sequence of sent signal alternatives!

Some sequences are impossible, see problem!

Only ”good” sequences are sent!