5.1.4 The Symbol Error Probability for QPSK

$$r(t) = z_j(t) + N(t), \quad 0 \le t \le T_s, \quad j = 0, 1, \ldots, M-1 \tag{5.13}$$

$$r_1 = z_{j,1} + w_1 \tag{5.36}$$

$$r_2 = z_{j,2} + w_2 \tag{5.37}$$

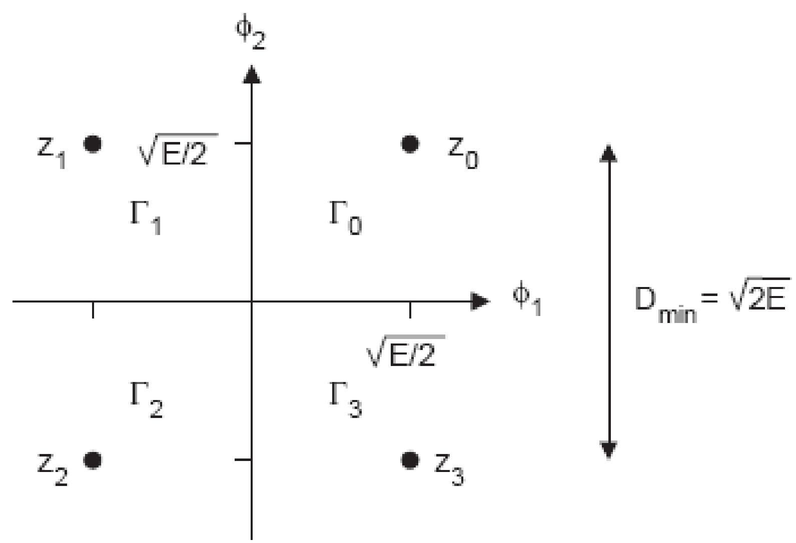


Figure 5.10: The signal space for QPSK if $\nu_\ell = (2\pi \frac{\ell}{M} + \pi/4)$ (see (5.4)).

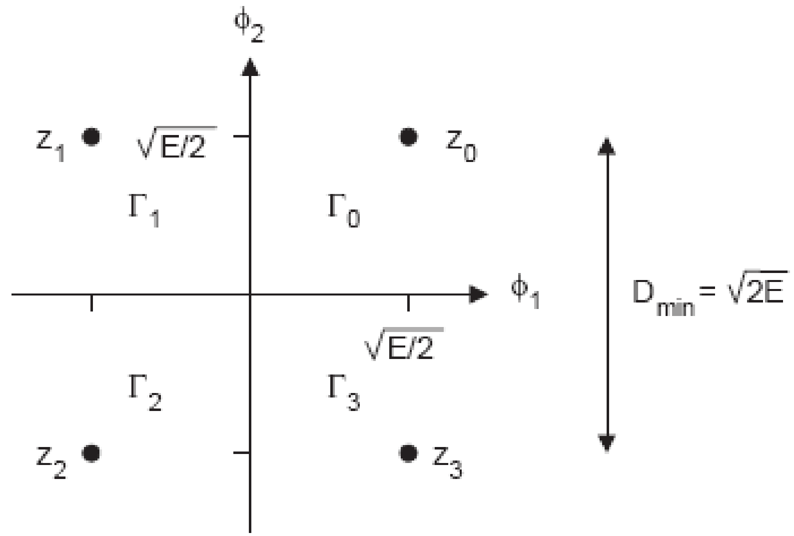


Figure 5.10: The signal space for QPSK if $\nu_\ell = (2\pi \frac{\ell}{M} + \pi/4)$ (see (5.4)).

$$
\begin{aligned}
&Prob\{\text{error}|m_0 \text{ sent}\} = 1 - Prob\{\text{correct decision}|m_0 \text{ sent}\} = \\
&= 1 - Prob\left\{w_1 \geq -\frac{D_{\min}}{2} \;,\; w_2 \geq -\frac{D_{\min}}{2}\right\} = \\
&= 1 - Prob\left\{w_1 \geq -\frac{D_{\min}}{2}\right\} \; Prob\left\{w_2 \geq -\frac{D_{\min}}{2}\right\} = \\
&= 1 - \left[1 - Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)\right]^2 = 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) \overset{\substack{\text{symmetry}\\ \downarrow}}{=} \\
&= Prob\{\text{error}|m_j \text{ sent}\} \;, j = 0, 1, 2, 3
\end{aligned}
\tag{5.38}
$$

$$\boxed{P_s = 2Q\left(\sqrt{\frac{D^2_{\min}}{2N_0}}\right) - Q^2\left(\sqrt{\frac{D^2_{\min}}{2N_0}}\right)}, \quad QPSK \tag{5.39}$$

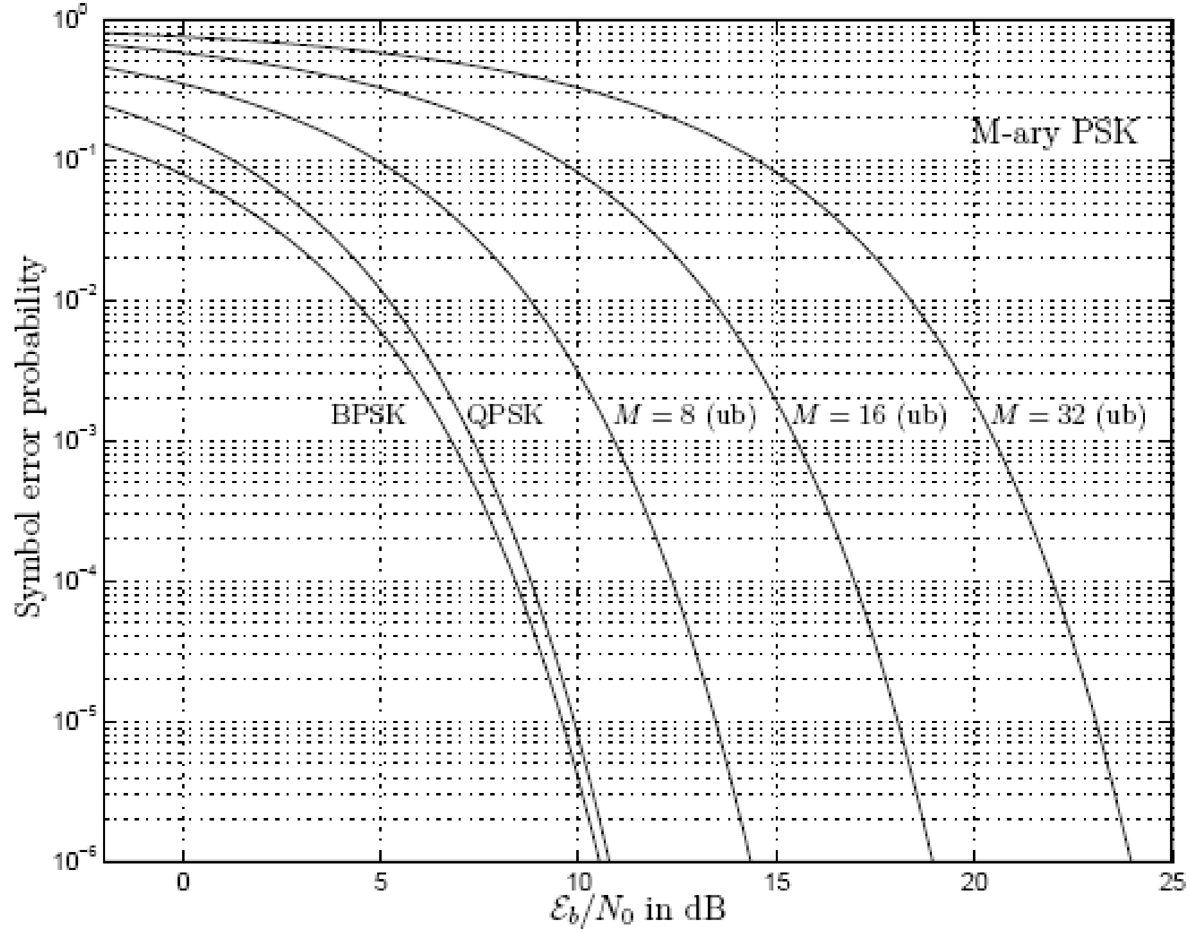


Figure 5.14: The symbol error probability for M-ary PSK, $M = 2, 4, 8, 16, 32$, see Table 5.1. In this figure upper bounds are denoted (ub). See also Subsection 5.1.5.

5.1.5 The Symbol Error Probability for M-ary PSK

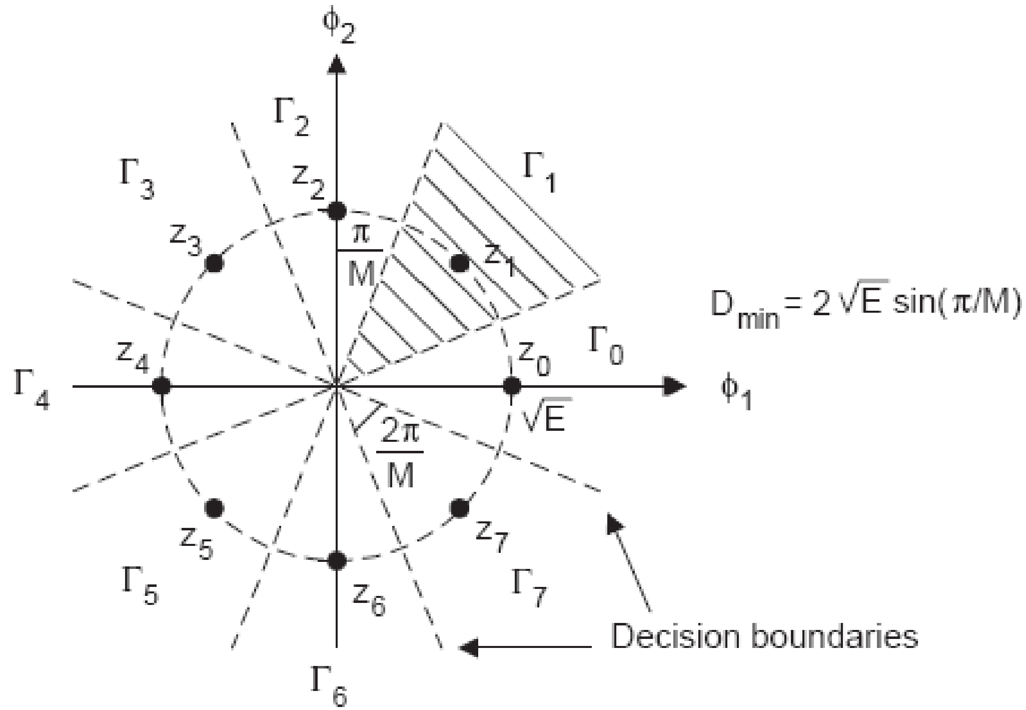


Figure 5.11: The signal space for M-ary PSK if $\nu_\ell = 2\pi\ell/M$ (see (5.4)). $M = 8$ in this figure.

$$\boxed{\begin{array}{l} Q\left(\sqrt{\frac{D^2_{\min}}{2N_0}}\right) \leq P_s < 2Q\left(\sqrt{\frac{D^2_{\min}}{2N_0}}\right) \\ D^2_{\min} = 4E\sin^2(\pi/M) \end{array}}\text{, M-ary PSK} \qquad (5.43)$$

5.1.6 The Symbol Error Probability for M-ary QAM

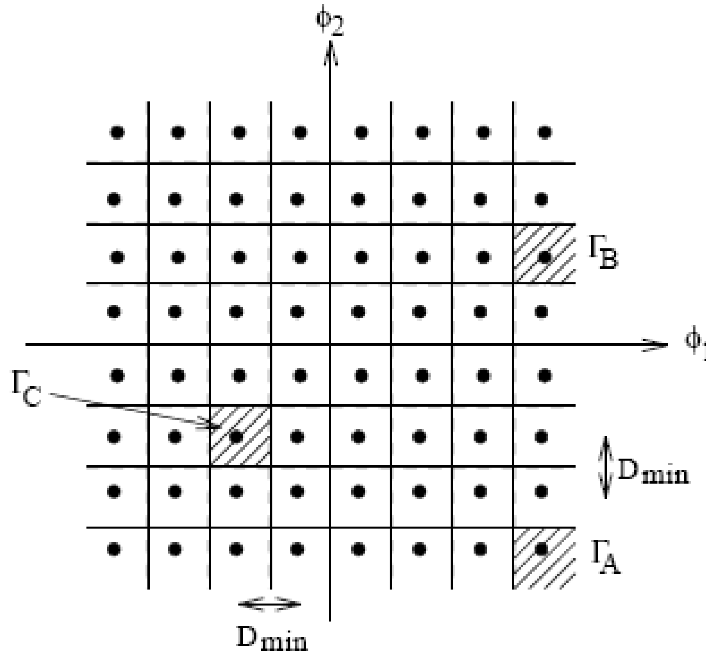


Figure 5.12: The signal space for M-ary QAM (compare with (5.4), see also Subsection 2.4.5.1). M=64 in this figure.

Γ_A: Compare with (5.39).

$$Prob\{\text{error}|m_A \text{ sent}\} = 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) \quad (5.46)$$

Γ_B:

$$\begin{aligned}
&Prob\{\text{error}|m_B \text{ sent}\} = \\
&= 1 - Prob\left\{w_1 > -\frac{D_{\min}}{2}, -\frac{D_{\min}}{2} \leq w_2 \leq \frac{D_{\min}}{2}\right\} = \\
&= 1 - \left(1 - Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)\right)\left(1 - 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)\right) = \\
&= 3Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - 2Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) \quad (5.47)
\end{aligned}$$

Γ_C:

$$\begin{aligned}
&Prob\{\text{error}|m_C \text{ sent}\} = \\
&= 1 - Prob\left\{-\frac{D_{\min}}{2} \leq w_1 \leq \frac{D_{\min}}{2}, -\frac{D_{\min}}{2} \leq w_2 \leq \frac{D_{\min}}{2}\right\} = \\
&= 1 - \left(1 - 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)\right)^2 = \\
&= 4Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - 4Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) \quad (5.48)
\end{aligned}$$

$$\boxed{P_s = \frac{4}{\sqrt{M}}\ (\sqrt{M}\text{-}1)\ Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - \frac{4}{M}\ (\sqrt{M}\text{-}1)^2\ Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)}, \quad \text{M-ary QAM} \tag{5.50}$$

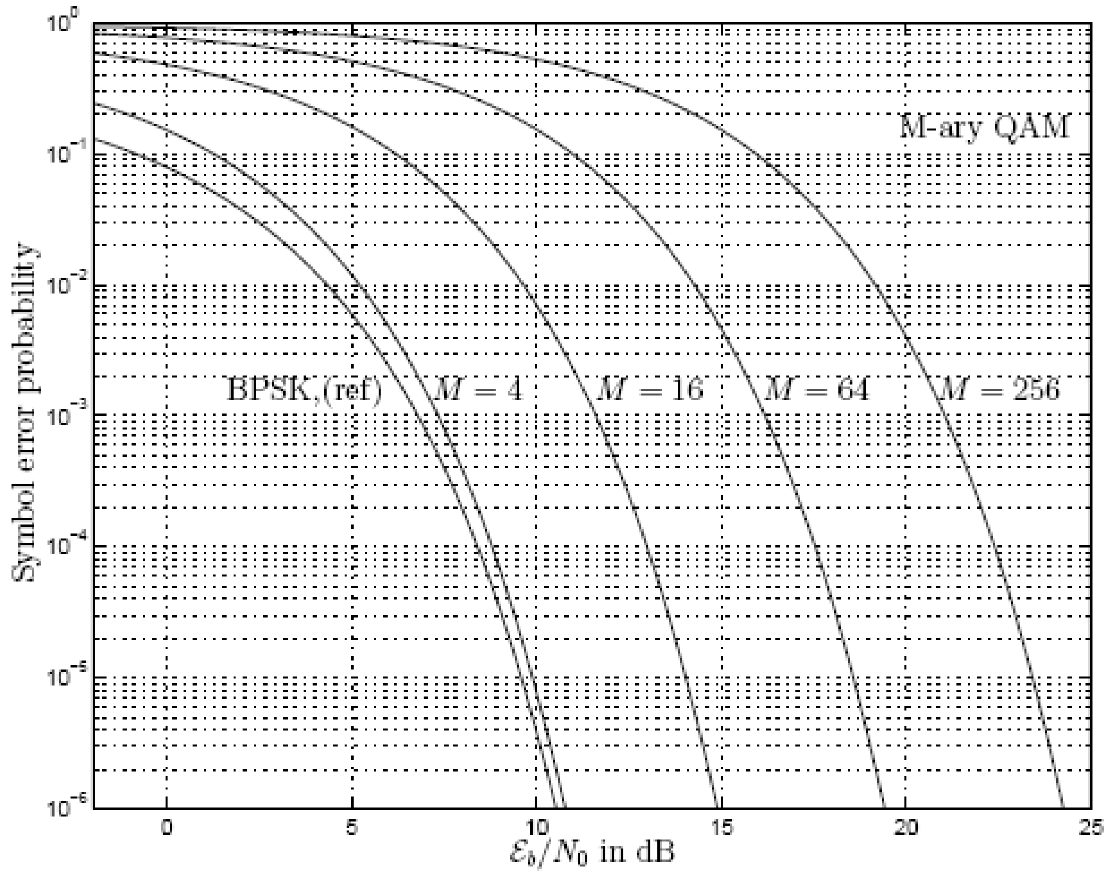


Figure 5.15: The symbol error probability for M-ary QAM, $M = 4, 16, 64, 256$, see Table 5.1. The specific assumptions are given in Subsection 2.4.5.1 and in Subsection 5.1.6. The bit error probability for BPSK is also given as a reference $(= Q(\sqrt{2\mathcal{E}_b/N_0}))$.

$M=2$	P_b	$Q\left(\sqrt{d_{\min}^2 \frac{\mathcal{E}_b}{N_0}}\right)$, (4.55)
	$d_{\min}^2$	$0 \le d_{\min}^2 \le 2$, (4.57)
	ρ	ρ_{bin}, (2.21)
M-ary PAM	P_s	$2\left(1-\frac{1}{M}\right) Q\left(\sqrt{d_{\min}^2 \frac{\mathcal{E}_b}{N_0}}\right)$, (5.35)
	$d_{\min}^2$	$\frac{6\log_2(M)}{M^2-1}$, Table 4.1 on page 281, (2.50)
	ρ	$\rho_{2-PAM} \cdot \log_2(M)$, (2.220)
M-ary PSK	P_s	$< 2Q\left(\sqrt{d_{\min}^2 \frac{\mathcal{E}_b}{N_0}}\right)$, (5.43)
	$d_{\min}^2$	$2\sin^2(\pi/M)\log_2(M)$, Table 4.1, Fig. 5.11
	ρ	$\rho_{BPSK} \cdot \log_2(M)$, (2.229)
M-ary QAM (rect., k even) (QPSK with $M=4$)	P_s	$4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{d_{\min}^2 \frac{\mathcal{E}_b}{N_0}}\right) -$ $-4\left(1-\frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{d_{\min}^2 \frac{\mathcal{E}_b}{N_0}}\right)$, (5.50)
	$d_{\min}^2$	$\frac{3\log_2(M)}{M-1}$, Table 4.1, Subsection 2.4.5.1
	ρ	$\rho_{BPSK} \cdot \log_2(M)$, (2.229)
M-ary FSK (orthogonal FSK)	P_s	$\le (M-1)Q\left(\sqrt{d_{\min}^2 \frac{\mathcal{E}_b}{N_0}}\right)$, Example 4.18c, Table 4.1
	$d_{\min}^2$	$\log_2(M)$, Table 4.1 on page 281
	ρ	See (2.245)
M-ary bi-orthogonal signals	P_s	$\le (M-2)Q\left(\sqrt{d_{\min}^2 \frac{\mathcal{E}_b}{N_0}}\right) +$ $+Q\left(\sqrt{2d_{\min}^2 \frac{\mathcal{E}_b}{N_0}}\right)$, (5.53)
	$d_{\min}^2$	$\log_2(M)$ if $M \ge 4$, (5.51)
	ρ	$\rho_{M\text{-bi-ort}} = \rho_{M/2\text{ -ort}} \cdot \frac{\log_2(M)}{\log_2(M/2)}$, (5.52)

Table 5.1: Symbol error probability and bandwidth efficiency results.

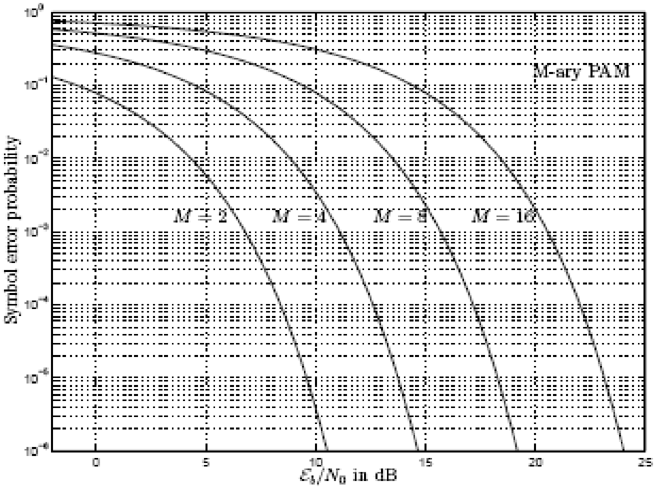


Figure 5.13: The symbol error probability for M-ary PAM, $M = 2, 4, 8, 16$, see Table 5.1. The specific assumptions are given in Subsection 2.4.1.1, and in Subsection 5.1.3.

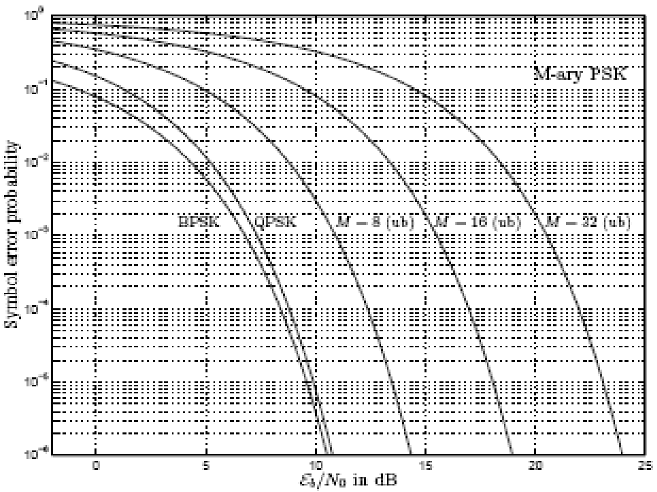


Figure 5.14: The symbol error probability for M-ary PSK, $M = 2, 4, 8, 16, 32$, see Table 5.1. In this figure upper bounds are denoted (ub). See also Subsection 5.1.5.

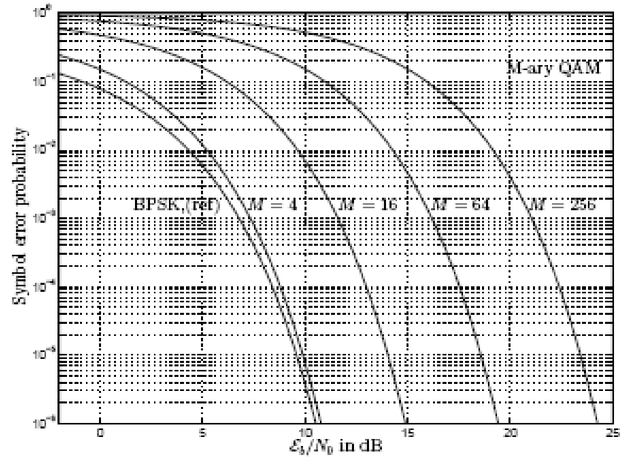


Figure 5.15: The symbol error probability for M-ary QAM, $M = 4, 16, 64, 256$, see Table 5.1. The specific assumptions are given in Subsection 2.4.5.1 and in Subsection 5.1.6. The bit error probability for BPSK is also given as a reference $(= Q(\sqrt{2\mathcal{E}_b/N_0}))$.

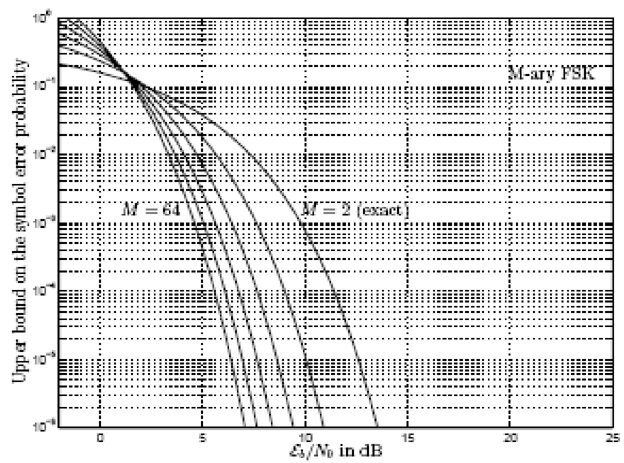


Figure 5.16: Upper bound (the union bound) on the symbol error probability for orthogonal equal energy M-ary FSK signal alternatives, $M = 2, 4, 8, 16, 32, 64$, see Table 5.1 and Example 4.18c. The result given for the binary case is exact $(= Q(\sqrt{\mathcal{E}_b/N_0}))$.

5.2.2 Power and Bandwidth Efficiency

We saw in (5.60) that the information bit rate R_b is limited by $d^2_{\min}$, c, P_z, N_0 and $P_{s,req}$. Let us divide both sides in (5.60) with the bandwidth W,

$$\boxed{\rho \leq \frac{d^2_{\min}}{\mathcal{X}} \cdot \frac{\mathcal{P}_z}{N_0 W} = \frac{d^2_{\min}}{\mathcal{X}} \cdot \mathcal{SNR}_r} \tag{5.61}$$

Note that the bandwidth efficiency ρ is limited by $d^2_{\min}$, c, $P_{s,req}$, and by the **received signal-to-noise power ratio** $\boldsymbol{\mathcal{SNR}_r = \mathcal{P}_z / N_0 W}$ within the signal bandwidth W. The bandwidth W is the physical bandwidth defined on the

5.2.3 Shannon's Capacity Theorem

In Shannons capacity theorem, [54], [68], [20], [43], for the bandlimited flat ($|H(f)|^2 = \alpha^2$ within the bandwidth W) AWGN channel, the capacity $\mathcal{C}$ for this channel is (in bits per second),

$$\boxed{\mathcal{C} = W \log_2\left(1 + \frac{\mathcal{P}_z}{N_0 W}\right)\ ,\ [b/s]} \tag{5.62}$$

where W is the physical bandwidth measured on the positive frequency axis containing **all** the signal power. This remarkable theorem states that ([43], [68]): There **exists** at least one signal construction method that achieves an arbitrary small error probability, if the bit rate $R_b < \mathcal{C}$. If $R_b > \mathcal{C}$, then the error probability P_s is high for every possible signal construction method.

$$\boxed{\boldsymbol{\mathcal{C} = W \log_2 \left(1 + \frac{\mathcal{P}_z}{N_0 W}\right) \ , \ [b/s]}} \tag{5.62}$$

$$\lim_{W\to\infty} \mathcal{C} = \lim_{W\to} \frac{W}{\ell n(2)} \ \ell n \left(1 + \frac{\mathcal{P}_z}{N_0 W}\right) = \frac{\mathcal{P}_z}{N_0 \ell n(2)} \tag{5.63}$$

$$\boxed{\begin{array}{l} \frac{\mathcal{C}}{W} = \log_2 \left(1 + \frac{P_z}{N_0 W}\right) = \log_2 \left(1 + \frac{\mathcal{C}}{W} \cdot \frac{\mathcal{E}_b}{N_0}\right) \ , \ [\text{bps/Hz}] \\ \text{or equivalently,} \\ \frac{\mathcal{E}_b}{N_0} = \frac{2^{\mathcal{C}/W} - 1}{\mathcal{C}/W} \end{array}} \tag{5.64}$$

Since $\mathcal{C}$ is the maximum bit rate, $\mathcal{E}_b$ here represents the minimum average received energy per information bit, for a given $\mathcal{P}_z$, $\mathcal{P}_z = \mathcal{C}\mathcal{E}_b$.

$$\frac{\mathcal{P}_z}{N_0 W} = \frac{\mathcal{C}}{W} \cdot \frac{\mathcal{E}_b}{N_0} = 2^{\mathcal{C}/W} - 1 \tag{5.65}$$

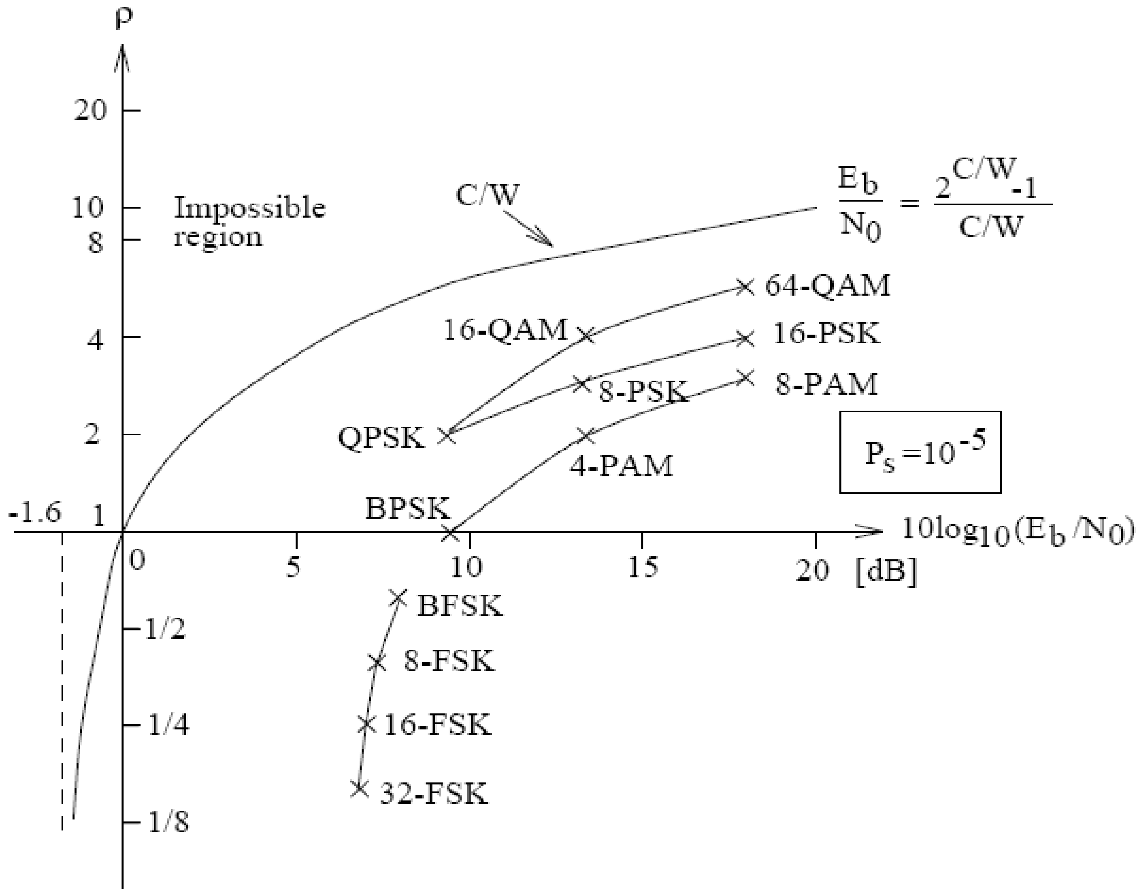


Figure 5.17: Sketch of the ρ versus $\mathcal{E}_b/N_0$ performance for some of the schemes studied in this section. Reliable communication is not possible above the capacity curve (see (5.64)).

5.2.3.1 Shannon Capacity for General $|H(f)|^2$ and $R_N(f)$

1. For a given average transmitted signal power P_{sent}, and channel quality function $q_{ch}(f) = |H(f)|^2/R_N(f)$, the parameter B below should first be determined,

$$P_{sent} = \int_{\Omega} \left(B - \frac{R_N(f)}{|H(f)|^2} \right) df \qquad (5.68)$$

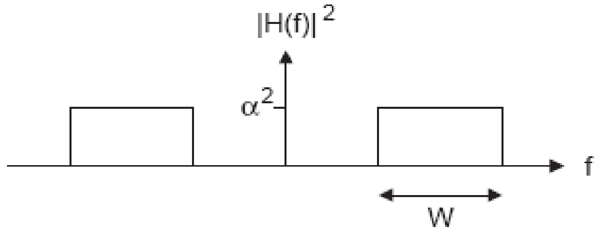
This is referred to as ”**waterfilling**”!

2. The capacity C is then found as,

$$C = \int_{\Omega} \frac{1}{2} \log_2 \left(\frac{|H(f)|^2}{R_N(f)} \cdot B \right) df \qquad (5.70)$$

EXAMPLE 5.20

Assume that $|H(f)|^2$ *is,*

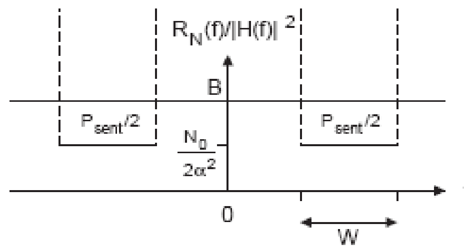


and that $R_N(f) = N_0/2$ *for all* f. *Calculate the capacity of this channel if the average transmitted signal power is* P_{sent}.

Solution:

The figure below shows $R_N(f)/|H(f)|^2$, *and the parameter* B.

Step 1:



From (5.68)–(5.69) we find that the value of B *is determined by the equality*

$$P_{sent} = \left(B - \frac{N_0}{2\alpha^2}\right) 2W$$

and B *is found to be*

$$B = \frac{P_{sent}}{2W} + \frac{N_0}{2\alpha^2}$$

Step 2:

$$C = \frac{W}{2} \log_2\left(\frac{\alpha^2}{N_0/2}\left(\frac{P_{sent}}{2W} + \frac{N_0}{2\alpha^2}\right)\right) \cdot 2 = W \log_2\left(1 + \frac{\alpha^2 P_{sent}}{N_0 W}\right)$$

$$r(t) = z_j(t) + N(t) = x_{I,j}(t)\cos(\omega_c t + \phi) - x_{Q,j}(t)\sin(\omega_c t + \phi) + N(t) \quad (5.75)$$

Technical problems to implement bandpass filters or bandpass correlators for different phase values (from the channel). Also very sensitive to synchronization errors!

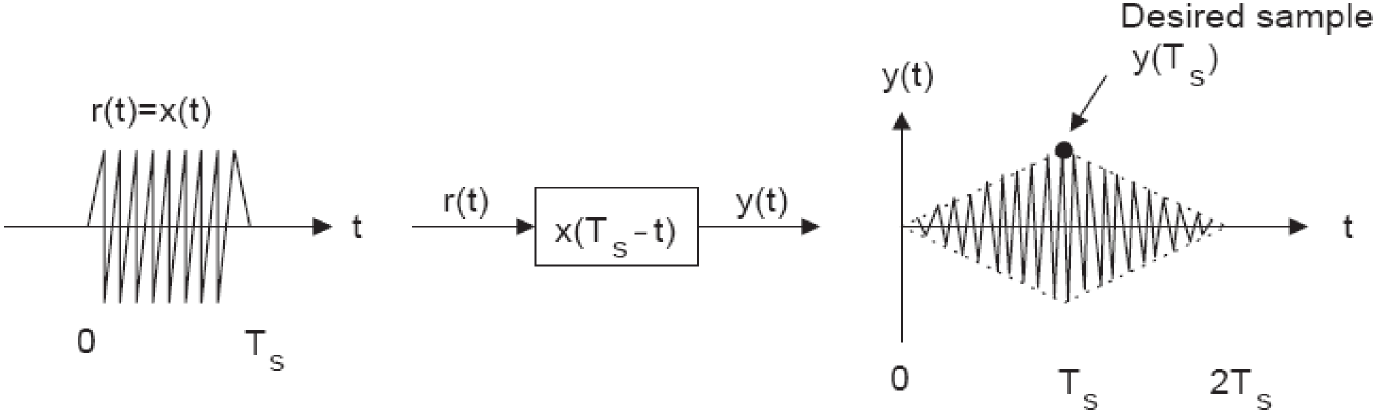


Figure 5.18: Illustrating bandpass filtering.

A better implementation is obtained by observing that the desired correlation can be obtained as :

$$C_\ell = \int_0^{T_s} r(t) z_\ell(t) dt =$$

$$r(t) = z_j(t) + N(t) = x_{I,j}(t)\cos(\omega_c t + \phi) - x_{Q,j}(t)\sin(\omega_c t + \phi) + N(t) \quad (5.75)$$

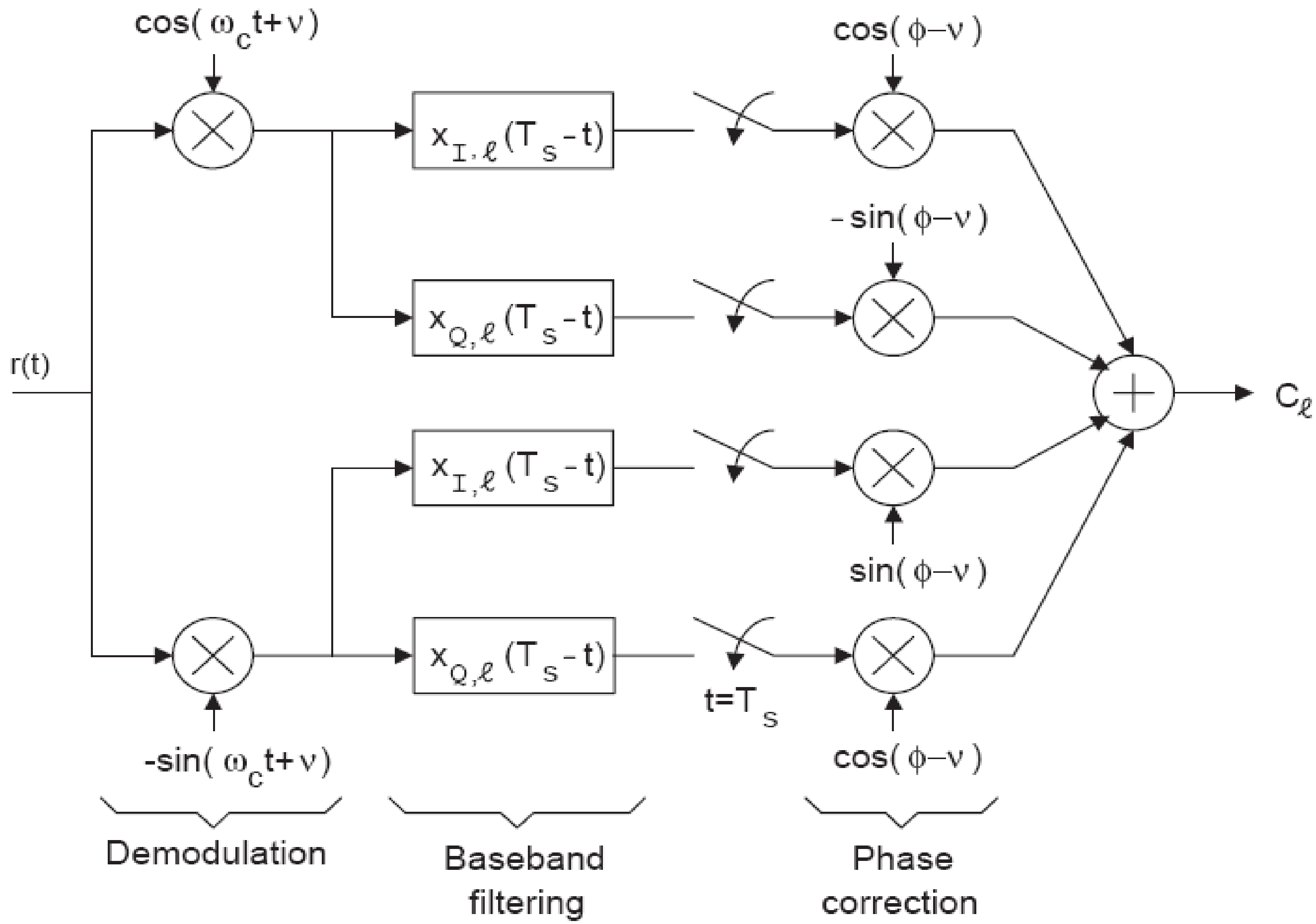


Figure 5.19: Structure of the receiver for bandpass signal alternatives. C_ℓ denotes the correlation between $r(t)$ and $z_\ell(t)$, see (5.76).

5.4.1 Diversity: Introductory Concepts

"Dont put all eggs in the same basket"

Assume that each message is sent in N dimensions (time/frequency/space etc)

$$s_j(t) = \sum_{n=1}^{N} s_{j,n}\phi_n(t) \ , \quad j = 0, 1, \ldots, M-1 \tag{5.79}$$

Assume independent attenuations in each dimension:

$$r(t) = z_j(t) + N(t) = \sum_{n=1}^{N} \alpha_n s_{j,n}\phi_n(t) + N(t) \tag{5.80}$$

$$\boldsymbol{z}_j = \begin{pmatrix} \alpha_1 & & & \mathbf{0} \\ & \alpha_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \alpha_N \end{pmatrix} \begin{pmatrix} s_{j,1} \\ \vdots \\ s_{j,N} \end{pmatrix} = \begin{pmatrix} \alpha_1 s_{j,1} \\ \vdots \\ \alpha_N s_{j,N} \end{pmatrix} \tag{5.81}$$

<u>Note</u>: It can be very "dangerous" to use only one (i.e. N=1) dimension!

We now introduce the concept of **diversity** in connection with Figure 5.21 and (5.80). Diversity is often used, e.g., for so-called **fading** channels (randomly varying signal levels, see Chapter 9), to improve the error probability. *Diversity can be obtained by spreading the same message over many dimensions.* Hence, in the receiver, message m_j has coordinates in, say L, dimensions. Let p denote the probability that a received signal is seriously distorted in any single dimension. The basic idea with diversity is that the probability for large distorsions in **all** dimensions ($\approx p^L$) is significantly lower than p. Observe that this requires that the distorsions in each dimension are essentially independent. So, intuitively speaking, there is a high probability that a few message carrying coordinates "survive" the channel without too much damage, and it is these coordinates that the receiver bases its decision on. Compare with Figure 5.21b,c assuming some of the α_n's are close to zero. It should also be mentioned here that there is a close relationship between the concept of diversity and the concept of **coding**.

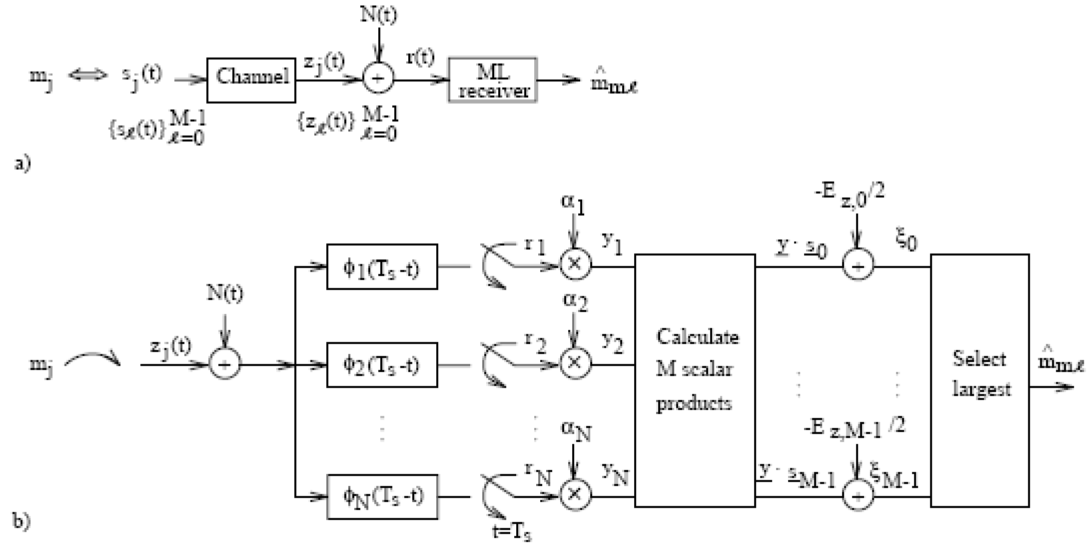


Figure 5.21:
a) The digital communication system; b) The ML receiver, assuming (5.80);

Observe that the channel attenuations are used as *multipliers in the receiver* according to the receiver structure in figure 5.8a on page 341!