Lecture 3

Digital Signal Processing

Chapter 3

z-transforms

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z-transforms

We define the z-transform of an impulse response $h(n)$ as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \tag{1}$$

We assume a causal impulse response $h(n)$. Causal means that $h(n) = 0$ for $n < 0$. The sum is therefore limited to

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \tag{2}$$

where $z = r \cdot \mathrm{e}^{\mathrm{j}\omega}$ is a complex number. Complex numbers are often written as a magnitude and a phase. The transform $H(z)$ is therefore a complex valued function of a complex valued variable.

Example

Some examples of z-transforms directly from the definition:

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \left\{ \ \underline{1} \quad 0 \quad \ldots \ \right\}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}
$h(n-k)$	$\Leftrightarrow$	$z^{-k} H(z)$
$h_1(n) = \left\{ \ \underline{3} \quad 2 \quad 1 \ \right\}$	$\Leftrightarrow$	$H_1(z) = 3 + 2z^{-1} + z^{-2}$
$h_2(n) = \left\{ \ \underline{0} \quad 3 \quad 2 \quad 1 \ \right\}$	$\Leftrightarrow$	$H_2(z) = 0 + 3z^{-1} + 2z^{-2} + z^{-3} = z^{-1}\left(3 + 2z^{-1} + z^{-2}\right)$

Proof for the time delay.

$$y(n) = x(n-1) \quad \Leftrightarrow \quad Y(z) = \sum_n y(n) z^{-n} \tag{3}$$

$$= \sum_n x(n-1) z^{-n} \tag{4}$$

$$= z^{-1} \sum_n x(n-1) z^{-(n-1)} \tag{5}$$

$$= z^{-1} \sum_m x(m) z^{-m} \tag{6}$$

$$= z^{-1} X(z) \tag{7}$$

Example

An IIR-system and its z-transform.

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} z^{-n} \tag{8}$$

$$= \frac{1-(z^{-1})^{\infty+1}}{1-z^{-1}} \tag{9}$$

$$= \frac{1}{1-z^{-1}} \qquad \text{if } |z| > 1 \text{ (ROC)} \tag{10}$$

$$h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n} \tag{11}$$

$$= \sum_{n=0}^{\infty} \left(a \cdot z^{-1}\right)^n \tag{12}$$

$$= \frac{1-(a \cdot z^{-1})^{\infty+1}}{1-z^{-1}} \tag{13}$$

$$= \frac{1}{1-a \cdot z^{-1}} \qquad \text{if } |z| > |a| \text{ (ROC)} \tag{14}$$

ROC means *region of convergence*: for which z the sum converges. For a causal signal the ROC becomes a region $|z| \geq R_{\text{min}}$. This is the normal case in this course.

Example of z-transform of non-causal signal (page 154)

Given:

$$x(n) = \left(\frac{1}{2}\right)^{|n|} \qquad \text{for all } n \tag{15}$$

Find: The z-transform $X(z)$ of $x(n)$.

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \tag{16}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} \cdot z^{-n} \tag{17}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} \cdot z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cdot z^{-n} - 1 \tag{18}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cdot z^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cdot z^{-n} - 1 \tag{19}$$

$$= \frac{1 - \left(\frac{1}{2} \cdot z\right)^{\infty+1}}{1 - \frac{1}{2} \cdot z} + \frac{1 - \left(\frac{1}{2} \cdot z^{-1}\right)^{\infty+1}}{1 - \frac{1}{2} \cdot z^{-1}} - 1 \tag{20}$$

$$= \frac{1}{1 - \frac{1}{2} \cdot z} + \frac{1}{1 - \frac{1}{2} \cdot z^{-1}} - 1 \tag{21}$$

$$= \frac{1 - \left(\frac{1}{2}\right)^2}{\left(1 - \frac{1}{2} \cdot z\right)\left(1 - \frac{1}{2} \cdot z^{-1}\right)} \qquad \text{if } |z| < 2 \text{ and } |z| > \frac{1}{2} \text{ (ROC)} \tag{22}$$

We are forced to check the ROC for each case individually.

Convolution becomes multiplication

Convolution of the sequences in time-domain becomes a multiplication of the polynomials in z-domain.

$$y(n) = h(n) * x(n) \quad \Leftrightarrow \quad Y(z) = H(z)X(z) \tag{23}$$

Proof:

$$Y(z) = \sum_{n} y(n) z^{-n} \tag{24}$$

$$= \sum_{n} \sum_{k} h(k) x(n-k) z^{-n} \tag{25}$$

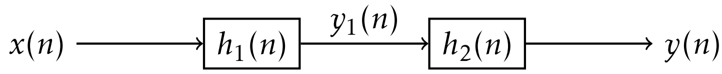
$$= \sum_{n} \sum_{k} h(k) x(n-k) z^{-(n-k)} z^{-k} \tag{26}$$

$$= \sum_{k} h(k) z^{-k} \cdot \sum_{n} x(n-k) z^{-(n-k)} \tag{27}$$

$$= H(z)X(z) \tag{28}$$

Cascade or Serial coupling of systems

Cascading two systems gives:



The impulse response of the whole system is the convolution of the impulse responses for the individual systems.

$$h_{tot}(n) = h_1(n) * h_2(n) \tag{29}$$

The system equation of the whole system is the product of the system equations for the individual systems.

$$H_{tot}(z) = H_1(z)H_2(z) \tag{30}$$

Convolution in the time-domain is multiplication if the z-domain. See earlier proof for convolution of system and signal.

Example

Calculate interests using the z-transform, continued from chapter 2. We can now determine a formula for the balance on the account. We had:

$$y(n) = 1.05 \cdot y(n-1) + x(n) \tag{31}$$

$$x(n) = 100 \cdot u(n) \tag{32}$$

Apply the z-transform.

$$Y(z) = 1.05 \cdot z^{-1} Y(z) + X(z) \tag{33}$$

$$X(z) = 100 \cdot \frac{1}{1 - z^{-1}} \tag{34}$$

Solve for $Y(z)$.

$$Y(z) = \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot X(z) \tag{35}$$

$$= \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot \frac{100}{1 - z^{-1}} \tag{36}$$

$$= 100 \cdot \left(\frac{21}{1 - 1.05 \cdot z^{-1}} - \frac{20}{1 - z^{-1}} \right) \tag{37}$$

Apply inverse-transforms to obtain the time-sequence. The balance at years 1, 2, 3, 5, 10 and 2000 is

$$y(n) = 100 \cdot (21 \cdot 1.05^n - 20)\, u(n) \tag{38}$$

$$= \left\{ \; 100 \quad 205 \quad 315 \quad \ldots \quad 680 \quad \ldots \quad 3572 \quad \ldots \quad 10^{45} \; \right\} \tag{39}$$

Solution to second order difference equations

First order difference equations were solved in chapter 2. Using the z-transform we can now easily solve higher order difference equations. We solve for $n \geq 0$ and assume that both $y(n)$ and $x(n)$ are zero for negative n (system in rest).

Given a second order difference equation:

$$y(n) - 1.27y(n-1) + 0.81y(n-2) = x(n-1) - x(n-2) \tag{40}$$

We can z-transform each term in the equation and we get (assuming both $x(n)$ and $y(n)$ are causal).

$$Y(z) - 1.27z^{-1}Y(z) + 0.81z^{-2}Y(z) = z^{-1}X(z) - z^{-2}X(z) \tag{41}$$

Solve for $Y(z)$.

$$Y(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} \cdot X(z) = H(z)X(z) \tag{42}$$

Using tables of formulas for z-transforms we can also easily determine $y(n)$ and $h(n)$. In most cases we are only looking for the system properties from $H(z)$.

Example: Fibonacci sequence (page 210)

The Fibonacci sequence is a sequence where a value is the sum of the two previous values.

$$x(n) = \left\{ \underline{1} \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad \ldots \right\} \tag{43}$$

Can we find a closed form equation for this sequence?

$$y(n) = y(n-1) + y(n-2) \qquad \text{where } y(0) = 1 \text{ and } y(1) = 1 \tag{44}$$

Solution using impulse response.

$$y(n) = y(n-1) + y(n-2) + \delta(n) \tag{45}$$

Apply the z-transform.

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + 1 \tag{46}$$

$$= \frac{1}{1 - z^{-1} - z^{-2}} \tag{47}$$

Partial fraction expansion gives

$$Y(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} \tag{48}$$

where

$$p_1 = \frac{1}{2}\left(1 + \sqrt{5}\right) \qquad A_1 = \frac{1 + \sqrt{5}}{2\sqrt{5}} \tag{49}$$

$$p_2 = \frac{1}{2}\left(1 - \sqrt{5}\right) \qquad A_2 = -\frac{1 - \sqrt{5}}{2\sqrt{5}} \tag{50}$$

After inverse z-transform the closed form equation for the sequence becomes

$$y(n) = A_1 p_1^n + A_2 p_2^n \qquad \text{for } n \geq 0 \tag{51}$$

This can also be solved as a difference equation with initial values (later in the course, see page 210).

Inverse z-transform

For most of the cases, use tables of formulas.

Definition (page 181)

The definition of the inverse z-transform is

$$y(n) = \frac{1}{2\pi \mathrm{j}} \cdot \oint_C Y(z) z^{n-1} \, \mathrm{d}z \tag{52}$$

where C is a closed path around the origin and inside the ROC. Solved using residual calculus.

Polynomial division (page 183)

$$Y(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \tag{53}$$

$$= 1 + \frac{3}{2} \cdot z^{-1} + \frac{7}{4} \cdot z^{-2} + \frac{15}{8} \cdot z^{-3} + \cdots \tag{54}$$

$$y(n) = \left\{ \; \underline{1} \quad \tfrac{3}{2} \quad \tfrac{7}{4} \quad \tfrac{15}{8} \quad \ldots \; \right\} \tag{55}$$

Use table of known transforms

This is the method we will use the most.

First order

$$Y(z) = \frac{1}{1 - 0.9z^{-1}} \tag{56}$$

$$y(n) = 0.9^n \cdot u(n) \tag{57}$$

Second order with real poles by partial fraction expansion

$$Y(z) = \frac{1}{1 - \frac{3}{2} \cdot z^{-1} + \frac{1}{2} \cdot z^{-2}} = \frac{2}{1 - z^{-1}} - \frac{1}{1 - \frac{1}{2} \cdot z^{-1}} \tag{58}$$

$$y(n) = 2u(n) - \left(\frac{1}{2}\right)^n \cdot u(n) \tag{59}$$

Second order with complex poles by table of formulas

$$Y(z) = \frac{\frac{1}{2} \cdot \sin\left(\frac{\pi}{4}\right) \cdot z^{-1}}{1 - 2\cos\left(\frac{\pi}{4}\right) \cdot z^{-1} + \frac{1}{4} \cdot z^{-2}} \tag{60}$$

$$y(n) = \left(\frac{1}{2}\right)^n \cdot \sin\left(\frac{\pi}{4}n\right) u(n) \tag{61}$$

Split the system into first and second order polynomials by partial fraction expansion. For second order polynomials, check first and foremost if the poles are real or complex valued.

More on this in the next lecture

Solving difference equations with initial values

First order difference equations were solved earlier and we assumed that the initial values were zero for $n < 0$. We can use the z-transform even if $y(n) \neq 0$ for $n < 0$. We solve for $n \geq 0$ and assume $x(n) = 0$ for $n < 0$ but $y(n) \neq 0$ (system not in rest).

We define the *one sided* z-transform as

$$Y^+(z) = \sum_{n=0}^{\infty} y(n) z^{-n} \tag{62}$$

even if $y(n) \neq 0$ for $n < 0$. Using this definition the z-transform of time shift is different. Assuming $y_0(n) = y(n-1)$ the z^+-transform becomes

$$Y_0^+(z) = \sum_{n=0}^{\infty} y_0(n) z^{-n} \tag{63}$$

$$= \sum_{n=0}^{\infty} y(n-1) z^{-n} \tag{64}$$

$$= y(-1) + \sum_{n=1}^{\infty} y(n-1) z^{-(n-1)} z^{-1} \tag{65}$$

$$= z^{-1} Y^+(z) + y(-1) \tag{66}$$

We now have to apply the initial values $y(n)$ for $n < 0$ to get the correct answer.

Using the one sided z-transform we can now solve difference equations with initial values. See the example in the book for a solution of the Fibonacci-sequence (page 207).

Common:

$$x(n) \quad \Leftrightarrow \quad X^+(z) \tag{67}$$

$$\tag{68}$$

$$x(n-k) \quad \Leftrightarrow \quad z^{-k} \cdot \left[X^+(z) + \sum_{n=1}^{k} x(-n) z^n \right] \tag{69}$$

$$= \left[x(-k) + x(-k+1)z^{-1} + \cdots + x(-1)z^{-k+1} + z^{-k} X^+(z) \right] \tag{70}$$

Example

First order difference equation with initial values.

Given: $y(n) = ay(n-1) + x(n)$ with a given initial value for $y(-1)$.

Find: $y(n)$ for $n \geq 0$.

Solution:

$$Y^+(z) = a \cdot \left[z^{-1} Y^+(z) + y(-1) \right] + X(z) \tag{71}$$

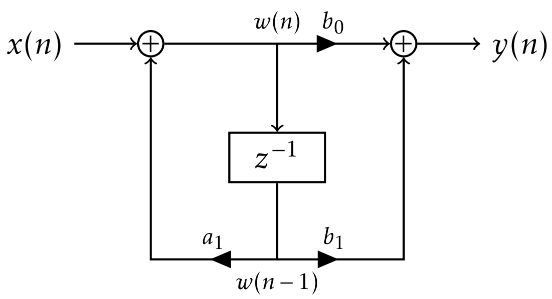
$$= \frac{1}{1 - az^{-1}} \cdot X(z) + \frac{a}{1 - az^{-1}} \cdot y(-1) \tag{72}$$

Inverse transform with the given values of $X(z)$ and $y(-1)$.

Direct form II (page 265)

We can illustrate difference equations graphically. A simple diagram is shown below, called Direct form II or canonical form. More about this in chapter 9.

Given: System drawn on the form:



Find: The connection between $x(n)$ and $y(n)$.

Solution: Introduce the helper variable $w(n)$ and the notations $X(z)$, $W(z)$ and $Y(z)$. Determine the signal $w(n)$.

$$W(z) = a_1 z^{-1} W(z) + X(z) \tag{73}$$

$$= \frac{1}{1 - a_1 z^{-1}} \cdot X(z) \tag{74}$$

Determine the output signal $y(n)$.

$$Y(z) = b_1 z^{-1} W(z) + b_0 W(z) \tag{75}$$

$$= \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \cdot X(z) \tag{76}$$

$$= H(z)X(z) \tag{77}$$

Therefore is $Y(z) = H(z)X(z)$ where

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \tag{78}$$