Lecture 2

Digital Signal Processing

Chapter 2

Convolution
Impulse response
Difference equations
Correlation functions

Mikael Swartling   Nedelko Grbic   Bengt Mandersson

rev. 2016

Department of Electrical and Information Technology
Lund University

Convolution (page 71–80)

The most important connection between input signal and output signal is called *convolution*. If we know the impulse response $h(n)$ of a system, we can calculate the output signal for any input signal. We are only assuming the properties of linearity and time invariance (LTI).

Input signal	$\rightarrow$	Output signal
$x(n)$	$\rightarrow$	$y(n)$
$\delta(n)$	$\rightarrow$	$h(n)$
$\delta(n-k)$	$\rightarrow$	$h(n-k)$
$x(k)\delta(n-k)$	$\rightarrow$	$x(k)h(n-k)$
$\sum_k x(k)\delta(n-k)$	$\rightarrow$	$\sum_k x(k)h(n-k)$

$$y(n) = \sum_k x(k)h(n-k) = \sum_k h(k)x(n-k) = h(n) * x(n) \tag{1}$$

This dependence is called is called convolution and is the most common and diverse formula in the course.

Example of convolution

Given: Input signal $x(n)$ and impulse response $h(n)$.

$$x(n) = \left\{ \underline{2} \quad 4 \quad 6 \quad 4 \quad 2 \right\} \tag{2}$$

$$h(n) = \left\{ \underline{3} \quad 2 \quad 1 \right\} \tag{3}$$

Find: Output signal $y(n)$.

$$y(n) = \sum_k h(n-k)x(k) = \sum_k h(k)x(n-k) \tag{4}$$

$$= h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) \tag{5}$$

$$= 3x(n) + 2x(n-1) + x(n-2) \tag{6}$$

Solution: We solve the convolution graphically with the following visual procedure

For $n = 0$:

$h(0-k)$	1	2	$\underline{3}$					
$x(k)$			$\underline{2}$	4	6	4	2	
$h(0-k)x(k)$			$\underline{6}$					$\sum = \underline{6} = y(0)$

For $n = 1$:

$h(1-k)$	1	2	$\underline{3}$				
$x(k)$		$\underline{2}$	4	6	4	2	
$h(1-k)x(k)$		4	12				$\sum = 16 = y(1)$

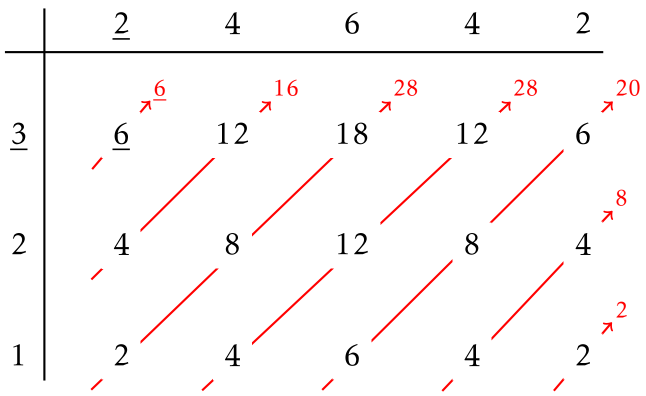
For $n = 2$:

$h(2-k)$	1	2	$\underline{3}$				
$x(k)$	$\underline{2}$	4	6	4	2		
$h(2-k)x(k)$	2	8	18				$\sum = 28 = y(2)$

Multiply the components of each rows and add the results. Shift the impulse response one step to the right and repeat. Repeat as long as $h(n-k)$ covers the signal $x(k)$. The output is

$$y(n) = \left\{ \; \underline{6} \quad 16 \quad 28 \quad 28 \quad 20 \quad 8 \quad 2 \; \right\} \tag{7}$$

Equivalent solution with a table.



Multiply rows and columns in the matrix. Sum along the anti-diagonals and read the result in the direction of the diagonal.

```
>> x = [2, 4, 6, 4, 2];
>> h = [3, 2, 1];
>> y = conv(x,h)
y =
      6     16     28     28     20      8      2
```

Properties of convolution (page 81)

The usual properties apply.

Commutativity

$$x_1(n) * x_2(n) = x_2(n) * x_1(n) \tag{8}$$

Associativity

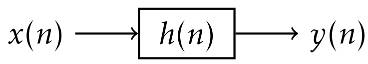
$$x_1(n) * [x_2(n) * x_3(n)] = [x_1(n) * x_2(n)] * x_3(n) \tag{9}$$

Distributivity

$$x_1(n) * [x_2(n) + x_3(n)] = x_1(n) * x_2(n) + x_1(n) * x_3(n) \tag{10}$$

Input-output

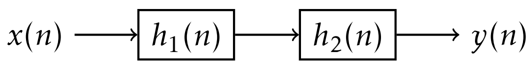
$$y(n) = x(n) * h(n) \tag{11}$$



Cascade or Serial coupling

$$y(n) = x(n) * h_1(n) * h_2(n) \tag{12}$$

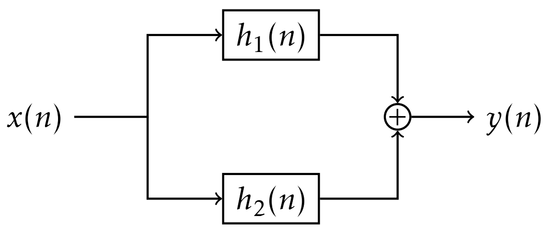
$$h(n) = h_1(n) * h_2(n) \tag{13}$$



Parallel coupling

$$y(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)] = x(n) * [h_1(n) + h_2(n)] \tag{14}$$

$$h(n) = h_1(n) + h_2(n) \tag{15}$$



Stability (sid 85)

A system is BIBO-stable (bounded input-bounded output) if

$$|x(n)| \le M_x \quad \Rightarrow \quad \left|y(n)\right| \le M_y \tag{16}$$

or equivalently

$$\left|y(n)\right| = \left|\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right| \tag{17}$$

$$\le \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)| \tag{18}$$

$$\le M_x \cdot \sum_{k=-\infty}^{\infty} |h(k)| \tag{19}$$

The system is therefore stable if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \tag{20}$$

Difference equations (page 93–95)

General:

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k) \tag{21}$$

Example

The FIR-filter

$$y(n) = 0.5x(n) + 0.25x(n-1) + 0.15x(n-2) \tag{22}$$

immediately gives us the impulse response

$$h(n) = \left\{ \begin{array}{ccc} 0.5 & 0.25 & 0.15 \end{array} \right\} \tag{23}$$

A first order IIR-filter:

$$y(n) = 0.5y(n-1) + 2x(n) \tag{24}$$

A second order IIR-filter:

$$y(n) = 0.5y(n-1) + 0.5y(n-2) + x(n) \tag{25}$$

For IIR-filters we have to solve the difference equation in order to determine the impulse response $h(n)$. We will solve a first order difference equation (page 94).

$$y(n) = -a_1 y(n-1) + b_0 x(n) \tag{26}$$

Solve iteratively for $n \geq 0$.

$$y(0) = -a_1 y(-1) + b_0 x(0) \tag{27}$$

$$y(1) = -a_1 y(0) + b_0 x(1) = (-a_1)^2 y(-1) + b_0 x(1) + (-a_1) b_0 x(0) \tag{28}$$

$$y(2) = -a_1 y(1) + b_0 x(2) = (-a_1)^3 y(-1) + b_0 x(2) + (-a_1) b_0 x(1) + (-a_1)^2 b_0 x(0) \tag{29}$$

$$y(n) = \sum_{k=0}^{n} (-a_1)^k \cdot b_0 x(n-k) + \underbrace{(-a_1)^{n+1} \cdot y(-1)}_{\text{often } 0} \tag{30}$$

We will wait until chapter 3 and the z-transform to solve higher order difference equations.

Example

Given:

$$h(n) = \left(\frac{1}{2}\right)^n \cdot u(n) \tag{31}$$

$$x(n) = u(n) \tag{32}$$

Find:

$$y(n) = h(n) * x(n) \tag{33}$$

Solution: Convolution gives

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k) \qquad [h(k) = 0 \text{ if } k < 0 \text{ and } x(n-k) = 0 \text{ if } k > n] \tag{34}$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k \cdot u(k) \cdot u(n-k) = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \tag{35}$$

$$= \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^n \qquad n \geq 0 \tag{36}$$

The solution is therefore

$$y(n) = \left[2 - \left(\frac{1}{2}\right)^n\right] \cdot u(n) \tag{37}$$

Correlation functions (sid 118)

How similar are two signals?

Auto correlation function

$$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n)x(n-k) = x(k) * x(-k) \tag{38}$$

Cross correlation function

$$r_{yx}(k) = \sum_{n=-\infty}^{\infty} y(n)x(n-k) = y(k) * x(-k) \tag{39}$$

Cross correlation for input and output signals

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

The auto correlation for the input signal:

$$r_{xx}(k) = x(k) * x(-k) \tag{40}$$

The cross correlation between the input signal and the output signal:

$$r_{yx}(k) = y(k) * x(-k) \tag{41}$$

$$= h(k) * x(k) * x(-k) \tag{42}$$

$$= h(k) * r_{xx}(k) \tag{43}$$

The auto correlation for the output signal:

$$r_{yy}(k) = y(k) * y(-k) \tag{44}$$

$$= h(k) * x(k) * h(-k) * x(-k) \tag{45}$$

$$= r_{hh}(k) * r_{xx}(k) \tag{46}$$

We can determine an unknown system $h(n)$ by using an input signal $x(n)$. For example, if $x(n)$ is white noise, then

$$r_{xx}(k) = \delta(k) \tag{47}$$

and therefore the impulse response becomes

$$h(k) = r_{yx}(k) \tag{48}$$

Example of IIR-filter

Determine the balance of a bank account with interest.

Given: Deposit is 100 every year with 5% interest.

$$x(n) = 100 \cdot u(n) \tag{49}$$

$$y(n) = \text{balance at year } n \tag{50}$$

Find: Balance after 1, 2, 5 and 20 years.

Solution: The current balance is the balance from last year plus 5% interest and the deposit for the current year.

$$y(n) = 1.05y(n-1) + x(n) \tag{51}$$

We have a recursive system where the new balance depends on both the previous balance (old output signal) and the deposit (input signal). This is an IIR-filter.

Iterative solution gives:

$$y(0) = 1.05y(-1) + x(0) = 100 \tag{52}$$

$y(n) = 0$ for $n < 0$ before the saving started.

$$y(1) = 1.05y(0) + x(1) = 1.05 \cdot 100 + 100 \tag{53}$$

$$y(2) = 1.05y(1) + x(2) = 1.05 \cdot (1.05 \cdot 100 + 100) + 100 \tag{54}$$

$$y(3) = \dots \tag{55}$$

Using the z-transform we can determine a formula for $y(n)$ (more on that later).

$$Y(z) \cdot (1 - 1.05z^{-1}) = X(z) \tag{56}$$

$$X(z) = \frac{100}{1 - z^{-1}} \tag{57}$$

$$Y(z) = \frac{100}{(1 - 1.05z^{-1})(1 - z^{-1})} \tag{58}$$