Lecture 12

Digital Signal Processing

Chapter 9

Structures

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Difference equations

FIR

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) \tag{1}$$

+ Always stable.
+ Can be made with linear phase if $h(n)$ is symmetric.
- The order M is often large (more computationally demanding).
- Non-parametric (for example, difficult to describe resonance).

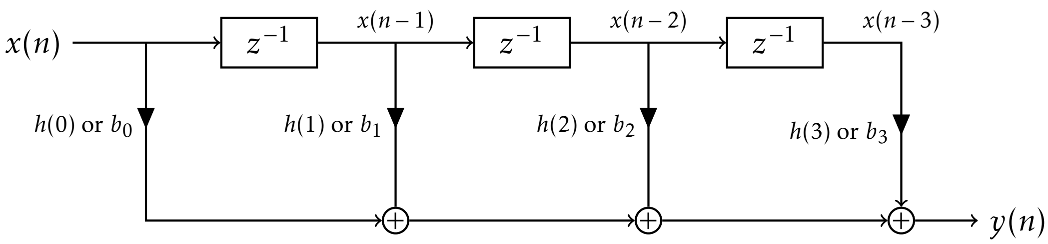
IIR

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{2}$$

+ The numerator and denominators orders M and N can be made small (less computationally demanding).
+ Parametric (for example, poles describe resonance).
- Can be unstable.
- Cannot have linear phase.

FIR filters

The following diagram is called direct form, transversal filter, or tapped delay filter.



From the figure we can immediately identify

$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + h(3)x(n-3) \tag{3}$$

$$= \sum_{k=0}^{3} h(k)x(n-k) = \sum_{k=0}^{3} b_k x(n-k) \tag{4}$$

and

$$Y(z) = h(0)X(z) + h(1)z^{-1}X(z) + h(2)z^{-2}X(z) + h(3)z^{-3}X(z) \tag{5}$$

$$= \sum_{k=0}^{3} h(k)z^{-k}X(z) = \sum_{k=0}^{3} b_k z^{-k}X(z) \tag{6}$$

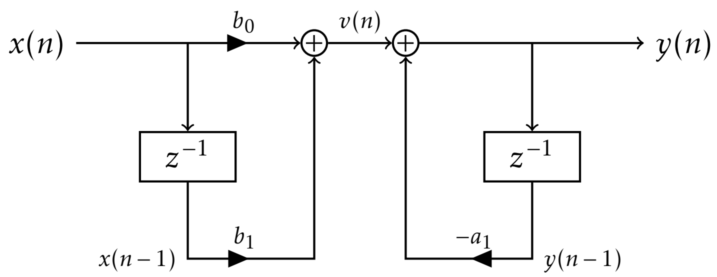
The impulse response for a linear phase FIR filter is symmetric. We can use this to reduce the number of multiplications.

IIR filters

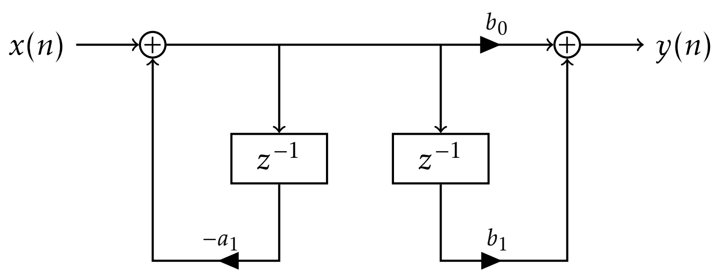
First order:

$$y(n) + a_1 y(n-1) = b_0 x(n) + b_1 x(n-1) \quad \Rightarrow \quad y(n) = -a_1 y(n-1) + v(n) \tag{7}$$

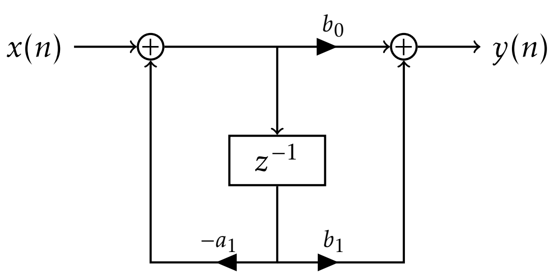
Can be drawn on direct form I.



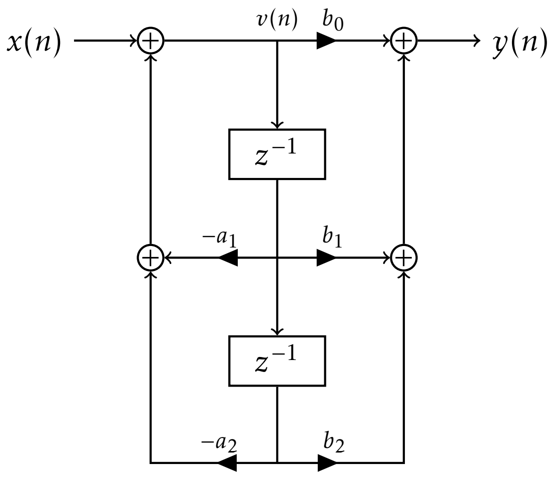
Since the system is linear we can change the order of the sub-systems.



The two delay lines can be merged to direct form II (normal form, canonical form).



Second order filter:



Introduce a helper variable $v(n)$ in order to find the transfer function.

$$V(z) = -z^{-1}a_1V(z) - z^{-2}a_2V(z) + X(z) \tag{8}$$

$$V(z) + z^{-1}a_1V(z) + z^{-2}a_2V(z) = X(z) \tag{9}$$

$$V(z) \cdot \left(1 + z^{-1}a_1 + z^{-2}a_2\right) = X(z) \tag{10}$$

$$V(z) = \frac{X(z)}{1 + z^{-1}a_1 + z^{-2}a_2} \tag{11}$$

Calculate the output signal from $v(n)$.

$$Y(z) = b_0V(z) + z^{-1}b_1V(z) + z^{-2}b_2V(z) \tag{12}$$

$$Y(z) = V(z) \cdot \left(b_0 + z^{-1}b_1 + z^{-2}b_2\right) \tag{13}$$

$$Y(z) = \frac{b_0 + z^{-1}b_1 + z^{-2}b_2}{1 + z^{-1}a_1 + z^{-2}a_2} \cdot X(z) \tag{14}$$

The same for the z-transform domain.

$$Y(z) + z^{-1}a_1Y(z) + z^{-2}a_2Y(z) = b_0X(z) + z^{-1}b_1X(z) + z^{-2}b_2X(z) \tag{15}$$

The same for the difference equation form.

$$y(n) + a_1y(n-1) + a_2y(n-2) = b_0x(n) + b_1x(n-1) + b_2x(n-2) \tag{16}$$

Parallel or cascade form

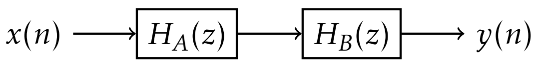
It is numerically better to implement IIR systems as cascaded or parallel first or second order sub-systems.

$$H(z) = \frac{1}{1 - \frac{1}{2}\cdot z^{-1} + \frac{1}{4}\cdot z^{-2} - \frac{1}{8}\cdot z^{-3}} \qquad \left[\text{poles in } p_1 = 0.5 \text{ och } p_{2,3} = \pm \mathrm{j}0.5\right] \tag{17}$$

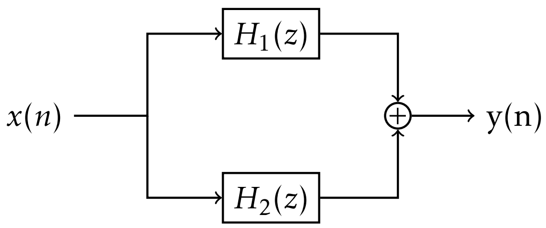
$$= \frac{1}{1 + \frac{1}{4}\cdot z^{-2}} \cdot \frac{1}{1 - \frac{1}{2}\cdot z^{-1}} \qquad [\text{cascade coupling of } H_A(z) \text{ and } H_B(z)] \tag{18}$$

$$= \frac{\frac{1}{2} + \frac{1}{4}\cdot z^{-1}}{1 + \frac{1}{4}\cdot z^{-2}} + \frac{\frac{1}{2}}{1 - \frac{1}{2}\cdot z^{-1}} \qquad [\text{parallel coupling of } H_1(z) \text{ and } H_2(z)] \tag{19}$$

Cascade (series) coupling:



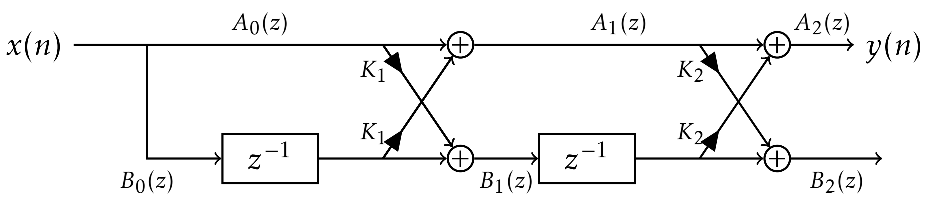
Parallel coupling:



Lattice filter

A very common structure when modelling signals, especially speech signals.

Second order lattice FIR



If all $|K_i| < 1$ then all roots (zeros) are inside the unit circle.

Analysis of lattice FIR

Step 0:

$$A_0(z) = B_0(z) = 1 \tag{20}$$

Step 1:

$$A_1(z) = 1 + K_1 z^{-1} \tag{21}$$

$$B_1(z) = K_1 + z^{-1} \tag{22}$$

Step 1:

$$A_2(z) = A_1(z) + K_2 z^{-1} B_1(z) \qquad = 1 + (K_1 + K_1 K_2) z^{-1} + K_2 z^{-2} \tag{23}$$

$$B_2(z) = K_2 A_1(z) + z^{-1} B_1(z) \qquad = K_2 + (K_1 + K_1 K_2) z^{-1} + z^{-2} \tag{24}$$

$B_2(z)$ can be obtained from $A_2(z)$ with the coefficients in reverse order.

Step m:

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z) \tag{25}$$

$$B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z) \tag{26}$$

In matrix form:

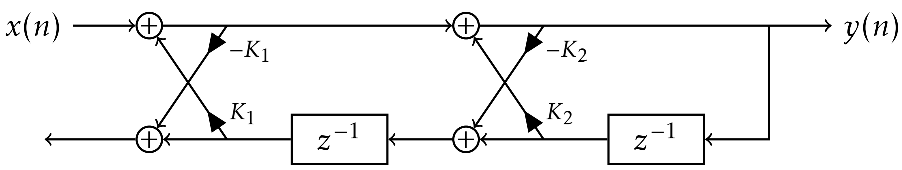
$$\begin{bmatrix} A_m(z) \\ B_m(z) \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} K_m \\ K_m & z^{-1} \end{bmatrix} \cdot \begin{bmatrix} A_{m-1}(z) \\ B_{m-1}(z) \end{bmatrix} \tag{27}$$

Reverse:

$$A_{m-1}(z) = \frac{1}{1 - K_m^2} \cdot [A_m(z) - K_m B_m(z)] \tag{28}$$

Second order lattice all-pole IIR

All-pole filters have poles only (all zeros at the origin).



Compare with lattice FIR:

$$H(z) = \frac{1}{A(z)} \tag{29}$$

If all $|K_i| < 1$ then all roots (poles) are inside the unit circle.

Example

Given:

$$H(z) = 1 - z^{-1} + \frac{1}{2} \cdot z^{-2} \qquad \left[\text{zeros in } z_{1,2} = \frac{1}{\sqrt{2}} \cdot e^{\pm j\pi \cdot \frac{1}{4}}\right] \tag{30}$$

Find: Calculate lattice FIR coefficients K_i.

Solution: Start with

$$A_2(z) = H(z) = 1 - z^{-1} + \frac{1}{2} \cdot z^{-2} \quad \Rightarrow \quad K_2 = \frac{1}{2} \tag{31}$$

$$B_2(z) = \frac{1}{2} - z^{-1} + z^{-2} \tag{32}$$

Calculate in reverse:

$$A_1(z) = \frac{1}{1 - K_2^2} \cdot [A_2(z) - K_2 B_2(z)] \tag{33}$$

$$= \frac{1}{1 - \frac{1}{4}} \cdot \left[\left(1 - z^{-1} + \frac{1}{2} \cdot z^{-2}\right) - \frac{1}{2} \cdot \left(\frac{1}{2} - z^{-1} + z^{-2}\right) \right] \tag{34}$$

$$= 1 - \frac{2}{3} \cdot z^{-1} \quad \Rightarrow \quad K_1 = -\frac{2}{3} \tag{35}$$

Finally, the answer is

$$K = \left\{ \begin{array}{cc} -\frac{2}{3} & \frac{1}{2} \end{array} \right\} \tag{36}$$

Algorithms

From lattice to system equation

Given: $K = \left\{ \begin{array}{cccc} K_1 & K_2 & \ldots & K_m \end{array} \right\}$

Find: $H(z)$

Solution:

$$A_0(z) = B_0(z) = 1 \tag{37}$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z) \tag{38}$$

$$B_m(z) = \text{coefficients in } A_m(z) \text{ in reverse order} \tag{39}$$

$$H(z) = A_{M-1}(z) \tag{40}$$

From system equation to lattice

Given: $H(z)$

Find: $K = \left\{ \begin{array}{cccc} K_1 & K_2 & \ldots & K_m \end{array} \right\}$

Solution:

$$A_{M-1}(z) = H(z) \tag{41}$$

$$K_m = \text{coefficients for the terms } z^{-m} \tag{42}$$

$$A_{m-1}(z) = \frac{1}{1 - K_m^2} \cdot [A_m(z) - K_m B_m(z)] \tag{43}$$

$$B_{m-1}(z) = \text{coefficients in } A_{m-1}(z) \text{ in reverse order} \tag{44}$$