Lecture 10

Digital Signal Processing

Chapter 7

Discrete Fourier transform
DFT

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The Discrete-Time Fourier Transform (DTFT)

The Fourier transform of a discrete signal:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-\mathrm{j}\omega n} \tag{1}$$

$$\begin{aligned} x(n) &= \int_{-\pi}^{\pi} X(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega && (2) \\ &= \int_{0}^{2\pi} X(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega && (3) \end{aligned}$$

Convergence if $x(n)$ is stable:

$$\sum_n |x(n)| < \infty \tag{4}$$

Weaker convergence if

$$\sum_n |x(n)| \to \infty \tag{5}$$

but

$$\sum_n |x(n)|^2 < \infty \tag{6}$$

The z-transform

Let $h(n)$ be a causal impulse response. Causal means that $h(n) = 0$ for $n < 0$.

The z-transform of the impulse response is defined as

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \tag{7}$$

where $z = r \cdot \mathrm{e}^{\mathrm{j}\omega}$ is a complex number that we often write as a magnitude and a phase. $H(z)$ is thus a complex function of a complex variable.

If $h(n)$ is causal and stable then

$$H(\omega) = H(z \mid z = \mathrm{e}^{\mathrm{j}\omega}) \tag{8}$$

The Discrete Fourier Transform (DFT)

Let

$$x(n) = \left\{ \begin{matrix} \ldots & 0 & \underline{1} & 1 & 1 & 1 & 0 & 0 & 0 & \ldots \end{matrix} \right\} \tag{9}$$

Choose a length N and calculate the normal DTFT at the frequencies

$$\omega = 2\pi \cdot \left\{ \begin{array}{cccccc} 0 & \frac{1}{N} & \frac{2}{N} & \frac{3}{N} & \dots & \frac{N-1}{N} \end{array} \right\} \tag{10}$$

This yields the Discrete Fourier Transform (DFT)

$$X_{\text{DFT}}(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \dots, N-1 \tag{11}$$

and the inverse transform (IDFT)

$$x_{\text{IDFT}}(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) e^{j2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } n = 0, 1, \dots, N-1 \tag{12}$$

Periodicity

The Fourier transform (DTFT)

$X(\omega)$ is periodic in ω since $e^{j\omega n} = e^{j(\omega+2\pi)n}$.

The Discrete Fourier transform (DFT)

Both $x(n)$ and $X(k)$ are periodic with period N since $n' = n + pN$ and $k' = k + pN$ for p integers and yields the same numerical values.

$$e^{j2\pi \cdot \frac{k}{N} \cdot (n+pN)} = e^{j2\pi \cdot \frac{k}{N} \cdot n} \cdot e^{j2\pi kp} \tag{13}$$

$$e^{j2\pi \cdot \frac{k+pN}{N} \cdot n} = e^{j2\pi \cdot \frac{k}{N} \cdot n} \cdot e^{j2\pi np} \tag{14}$$

Indices are calculated modulo-N.

If $x(n)$ is defined only for $0 \le n < N$ (length N) we get

$$X_{\text{DFT}}(k) = X(\omega \mid \omega = 2\pi \cdot \frac{k}{N}) \tag{15}$$

which is the same as sampling $X(\omega)$ in N uniformly distributed frequencies.

If N is an even power-of-2, the calculations can be made efficiently, on the order of $O(NlogN)$ instead of $O(N^2)$ for the direct DFT implementation. The algorithm is caleld the Fast Fourier Transform (FFT) and is described in chapter 8 but is not a part of the course.

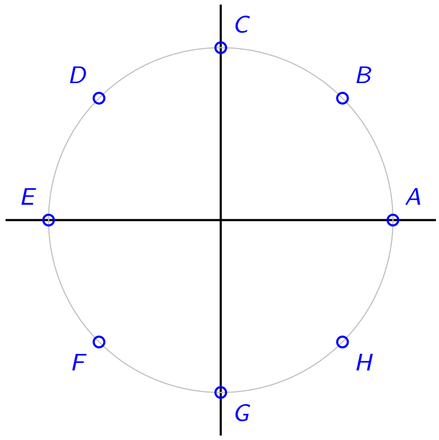
Roots of unity

$$\sum_{k=0}^{N-1} e^{j2\pi \cdot \frac{k}{N} \cdot (n-l)} = \begin{cases} N & \text{if } n - l = 0 + pN \\ 0 & \text{if } n - l \neq 0 + pN \end{cases} \tag{16}$$

$$= N \cdot \delta(n - l \mod N) \tag{17}$$

$$\equiv N \cdot \delta((n-l))_N \tag{18}$$

The sum of the points evenly distributed around the unit circle is 0.



The sum of the values at the points are:

$$\begin{aligned} A+A+A+A+A+A+A+A &= 8 \quad \text{for } n-l=0 \\ A+B+C+D+E+F+G+H &= 0 \quad \text{for } n-l=1 \\ A+C+E+G+A+C+E+G &= 0 \quad \text{for } n-l=2\text{, and so on} \end{aligned}$$

Compare to the integral of $\cos(\Omega t)$ over a whole number of periods that is 0 except when $\Omega = 0$.

Special properties for the DFT

Both $x(n)$ and $X(k)$ are periodic, which imposes the property that all indices are calculated modulo N.

$$x(n) = \left\{ \begin{array}{cccccccccc} \cdots & 3 & 4 & \underline{1} & 2 & 3 & 4 & 1 & 2 & \cdots \end{array} \right\} \tag{19}$$

$$x(n-1) = \left\{ \begin{array}{cccccccccc} \cdots & 2 & 3 & \underline{4} & 1 & 2 & 3 & 4 & 1 & \cdots \end{array} \right\} \tag{20}$$

Circular shift becomes

$$x(n-n_0 \mod N) \quad \Rightarrow \quad X(k)\cdot e^{-j2\pi\cdot\frac{k}{N}\cdot n_0} \tag{21}$$

Example of shift for the DFT:

$$x(n) = \left\{ \begin{array}{cccc} \underline{1} & 2 & 3 & 4 \end{array} \right\} \quad \Rightarrow \quad x(n-1) = \left\{ \begin{array}{cccc} \underline{4} & 1 & 2 & 3 \end{array} \right\} \tag{22}$$

Circular convolution and the DFT, length N (page 476–477)

Multiplication in the DFT-domain is circular convolution in the time domain.

$$X(k) = X_1(k)\cdot X_2(k) \quad \Rightarrow \quad x(n) = x_1(n) \otimes x_2(n) = \sum_{k=0}^{N-1} x_1(k)\cdot x_2(n-k \mod N) \tag{23}$$

Example

Given:

$$x(n) = \left\{ \ \underline{1} \quad 2 \quad 3 \quad 4 \ \right\} \tag{24}$$

$$h(n) = \left\{ \ \underline{2} \quad 2 \quad 1 \quad 1 \ \right\} \tag{25}$$

Find:

$$y_C(n) = x(n) \otimes_4 h(n) \tag{26}$$

Graphical solution (with $x(n)$ repeated and $h(n)$ time reversed):

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$					
$x(k)$	2	3	4	$\underline{1}$	2	3	4	1	2	3
$y_C(k)$				$\underline{15}$	13	15	17			

Equivalent solution with a convolution table.

	2	3	4	$\underline{1}$	2	3	4
				$\underline{15}$	13	15	17
$\underline{2}$	4	6	8	$\underline{2}$	4	6	8
2	4	6	8	2	4	6	8
1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4

For linear convolution the length of this convolution is 7, but in circlular convolution the indices wrap around modulo-4 instead.

```
>> x = [1 2 3 4];
>> h = [2 2 1 1];
>> yc = ifft(fft(x).*fft(h))
yc =
    15     13     15     17
```

Linear convolution and the DFT

The convolution between $x(n)$ and $h(n)$ yields $y(n)$ of length $4+4-1$. Choose a DFT length of $N = 8$.

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$									
$x(k)$	0	0	0	$\underline{1}$	2	3	4	0	0	0	0	1	2	3
$y_L(k)$				$\underline{2}$	6	11	17	13	7	4	0			

```
>> x = [1 2 3 4];
>> h = [2 2 1 1];
>> yl = ifft(fft(x,8).*fft(h,8))
yl =
     2      6      11      17      13       7       4       0
```

The linear convolution is closely related to the circular convolution.

$$y_C(n) = \left\{\; y_L(0)+y_L(4) \quad y_L(1)+y_L(5) \quad y_L(2)+y_L(6) \quad y_L(3)+y_L(7) \;\right\} \tag{27}$$

$$= \left\{\; 2{+}13 \quad 6{+}7 \quad 11{+}4 \quad 17{+}0 \;\right\} \tag{28}$$

$$= \left\{\; 15 \quad 13 \quad 15 \quad 17 \;\right\} \tag{29}$$

The linear convolution wrap around modulo-4 and sum to form the circular convolution.

Sampling the spectrum

Let

$$x(n) = a^n \cdot u(n) \quad \Rightarrow \quad X(\omega) = \frac{1}{1 - ae^{-\mathrm{j}\omega}} \tag{30}$$

Read $X(\omega)$ at N frequencies and form $X(k) = X(\omega \mid \omega = 2\pi k/N)$.

The signal $x(n)$ is an infinitely long sequence, but the invers-DFT of $X(k)$ yields a finitely long sequence of length N.

What is $x_{\mathrm{DFT}}(n) = \mathrm{IDFT}\{X(k)\}$?

$$X(k) = \frac{1}{1 - ae^{-\mathrm{j}2\pi \cdot \frac{k}{N}}} \tag{31}$$

Inverse-DFT yields:

$$x_{\mathrm{DFT}}(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n}{N} \cdot k} \tag{32}$$

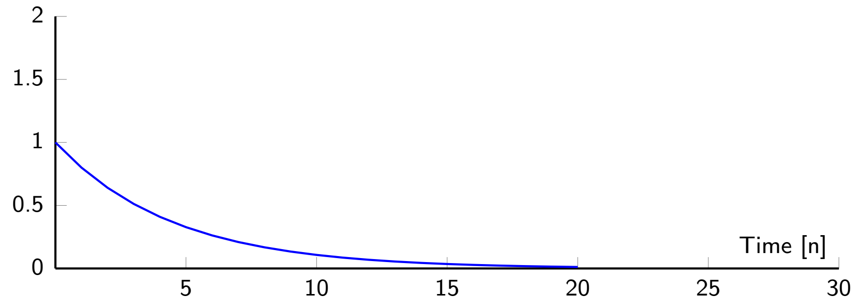
$$= \frac{1}{N} \cdot \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} x(m) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{m}{N} \cdot k} \cdot \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n}{N} \cdot k} \tag{33}$$

$$= \frac{1}{N} \cdot \sum_{m=-\infty}^{\infty} x(m) \cdot \sum_{k=0}^{N-1} \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n-m}{N} \cdot k} \tag{34}$$

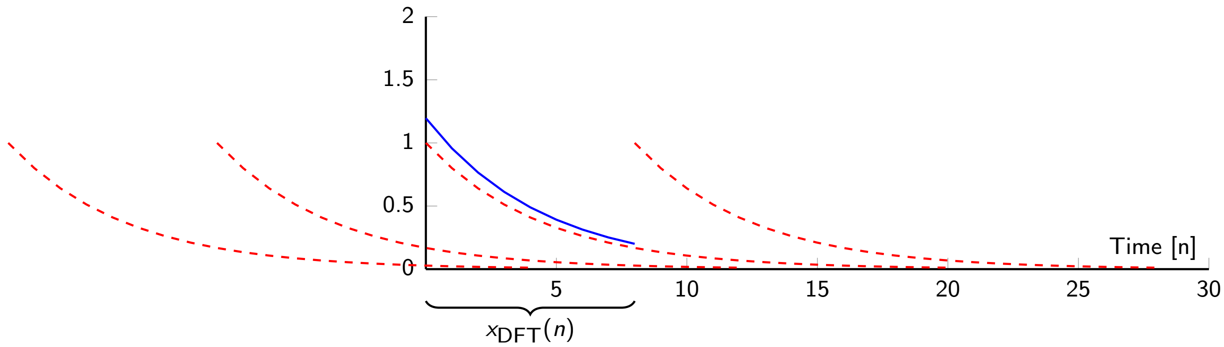
$$= \frac{1}{N} \cdot \sum_{m=-\infty}^{\infty} x(m) \cdot N \cdot \delta(n-m \mod N) \tag{35}$$

$$= \sum_{m=-\infty}^{\infty} x(n-mN) \tag{36}$$

The signal $x(n)$ extends to $+\infty$.



The signal $x_{\mathrm{DFT}}(n)$ is the sum of all shifted and repeated $x(n)$. The DFT size is $N = 8$.



Periodicity in time

A square pulse of length L is

$$x(n) = \left\{ \begin{array}{ccccc} \underline{1} & 1 & 1 & \ldots & 1 \end{array} \right\} \tag{37}$$

and its DTFT and DFT are

$$X(\omega) = \sum_{n=0}^{L-1} x(n) \mathrm{e}^{-\mathrm{j}\omega n} = \frac{\sin\left(\omega \cdot \frac{L}{2}\right)}{\sin\left(\omega \cdot \frac{1}{2}\right)} \cdot \mathrm{e}^{-\mathrm{j}\omega \cdot \frac{L-1}{2}} \tag{38}$$

and

$$X(k) = \sum_{n=0}^{L-1} x(n) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{N} \cdot n} = \frac{\sin\left(2\pi \cdot \frac{L}{2N} \cdot k\right)}{\sin\left(2\pi \cdot \frac{1}{2N} \cdot k\right)} \cdot \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{L-1}{2} \cdot k} \tag{39}$$

Let

$$Y_1(k) = X(k) \quad \Rightarrow \quad y1(n) = \mathrm{IDFT}\{Y_1(k)\} \tag{40}$$

and

$$Y_2(k) = X(k) \cdot X(k) \quad \Rightarrow \quad y1(n) = \mathrm{IDFT}\{Y_2(k)\} = x(n) \otimes x(n) \tag{41}$$

```
>> N = 16;
>> L = 6; % or L=10
>> k = 0:N-1;
>> k(k==0) = sqrt(eps);
>> X = sin(2*pi*L*k/(2*N))./sin(2*pi*k/(2*N)).*exp(-j*2*pi*(L-1)*k/(2*N));
>> y1 = real(ifft(X));
>> y2 = real(ifft(X.*X));
```

What does $y_1(n)$ and $y_2(n)$ look like for $N = 16$ and for $L = 6$ or $L = 10$?

```
>> subplot(2,1,1); stem(k, y1);
>> subplot(2,1,2); stem(k, y2);
```

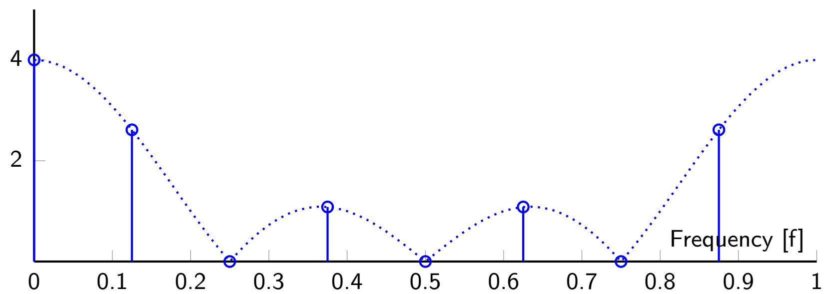
A practical application

Increase the resolution in frequency by *zero padding* or *trailing zeros*.

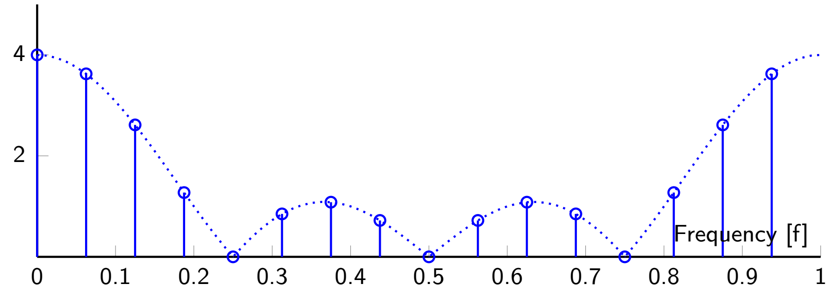
Let

$$x(n) = \left\{ \underline{1} \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad \ldots \right\} \tag{42}$$

Calculate the DFT of length $N = 8$ of $x(n)$.



Calculate the DFT of length $N = 16$ of $x(n)$.



The DFT on matrix form (page 459–460)

Define

$$W_N = \mathrm{e}^{-\mathrm{j}2\pi/N} = W \tag{43}$$

The DFT and the IDFT becomes

$$X(k) = \sum_{n=0}^{N-1} x(n) W^{kn} \tag{44}$$

$$x(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) W^{-kn} \tag{45}$$

Let

$$\mathbf{x} = \begin{bmatrix} x(0) & x(1) & \cdots & x(N-1) \end{bmatrix}^{\mathrm{T}} \tag{46}$$

and

$$\mathbf{X} = \begin{bmatrix} X(0) & X(1) & \cdots & X(N-1) \end{bmatrix}^{\mathrm{T}} \tag{47}$$

and

$$\mathbf{D} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)(N-1)} \end{bmatrix} \tag{48}$$

We can now describe the DFT and the IDFT as

$$\mathbf{X} = \mathbf{D}\mathbf{x} \tag{49}$$

and

$$\mathbf{x} = \mathbf{D}^{-1}\mathbf{X} \quad \text{or} \quad \mathbf{x} = \frac{1}{N} \cdot \mathbf{D}^{*}\mathbf{X} \tag{50}$$

Some identities:

$$\mathbf{D}^{-1} = \frac{1}{N} \cdot \mathbf{D} \tag{51}$$

$$\mathbf{D}\mathbf{D}^{*} = N \cdot \mathbf{I} \tag{52}$$