Digital Signal Processing

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Digital Signal Processing, applications

PC:
- Image processing
- ECG, Ultrasound
- Compressing images, sound (MP3)
- Sound generation (synthesizers)
- Measurements
- Process Control (real time)

Signal processor:
- GSM (real time)
- ADSL (OFDM) (real time)
- Data network (real time)
- MP3-coding/decoding (real time)

Spec. circuits:
- CD player (real time)
- Modem equipment (real time)

Example: Echo processor

Sample rate: 8 kHz, D=500, (63 ms delay)
How does this sound?
We test this during the laboratory work using Matlab and DSP.

DSP starters kit from Texas Instruments DSK6713 used in laboratory work

Low-pass filter → A/D → Low-pass filter
Sampling → Digital sign. processing. → Reconstruction

Digital circuit

Delay D

Delay 2D

microphone

speaker

Laboratory work, Echo processor.
Example of analogous and digital circuits

Analogous RC-circuit

\[ y'(t) + a \ 1 \ y(t) = b \ 1 \ x(t) \]

Digital circuit

\[ x(n) \rightarrow \text{circuit} \rightarrow y(n) \]

\[ y(n) = -a \ 1 \ y(n-1) + b \ 1 \ x(n) \]

Program code executed when a new value is available at the A/D input (a=0.9, b=1)

\[
x = \text{ADinput};
y = 0.9 \times y_{\text{old}} + x;
y_{\text{old}} = y;
\text{DAoutput} = y;
\]

Example: MP3 coding of music

![Figure 9.12 Simplified block diagram of a ISO-MPEG1 coder.](image)

Example of reverberation (echoes)

![Figure 1.32: Various types of echoes generated by a single sound source in a room.](image)

Example of echo suppression

![Figure 1.46: Echo cancellation scheme.](image)
What is a time discrete signal?

Example of time discrete signals

Temperature

\[ x(n) = \{ 4.4, 7.8, 11.4, 10.5, 10.4, 12.2, 12.0 \ldots \} \]

Sinusoid

\[ x(n) = \sin(2 \pi \frac{1}{8} n) = \{ -0.7, -1, -0.7, 0, 0.7, 1, 0.7, 0, -0.7, -1 \ldots \} \]

Example of Digital Signal Processing

1a. \[ y(n) = \frac{1}{5} x(n) + \frac{1}{5} x(n-1) + \frac{1}{5} x(n-2) + \frac{1}{5} x(n-3) + \frac{1}{5} x(n-4) \]
   The circuit compute the average of the last five samples

1b. \[ y(n) = \frac{1}{5} x(n) - \frac{1}{5} x(n-1) + \frac{1}{5} x(n-2) - \frac{1}{5} x(n-3) + \frac{1}{5} x(n-4) \]
   What does these circuits (equations) do?

2a. \[ y(n) = 0.9 y(n-1) + x(n) \]

2b. \[ y(n) = 1.05 y(n-1) + x(n) \]

Goal in the course

To understand the properties of these, especially the relation between the properties in the time domain and the properties in the frequency domain

Sinusoids (notations)

\[ x(t) = \frac{10}{A \ \text{amplitude}} \cos(2 \pi \frac{440}{F_0} t - 0.4 \pi) \]

Period \[ T_0 = \frac{1}{F_0} \]

\[ \Omega = 2 \pi F \]

\[ x(t) = \frac{10}{A \ \text{amplitude}} \cos(2 \pi \frac{440}{F_0} (t - \frac{0.4 \pi}{2 \pi 440})) \]

Trigonometrically relations:

\[ \cos \Omega_0 = \frac{e^{j\Omega_0} + e^{-j\Omega_0}}{2} \]

Euler formula:

\[ \sin \Omega_0 = \frac{e^{j\Omega_0} - e^{-j\Omega_0}}{2j} \]

Example of speech and time-frequency analyses, spectrogram

Illustration of the tones (pitch) in Chinese

Time signal and spectrogram
Example of synthesized sounds:
Example 2
(upper: waveform, lower: frequency contents)

Sinusoid
\[ x(t) = \sin(2\pi \frac{F_0}{220} t) \]

Additive synthesis (sum of sinusoids)
\[ x(t) = \sum_k a_k \sin(2\pi k \frac{F_0}{220} t) \]

Trombone
Clarinet

Example of synthesized sounds:
Example 3
(upper: waveform, lower: frequency contents)

AM-synthesis
\[ x(t) = (1 + 0.8 \sin(2\pi \frac{F_0}{220} t)) \sin(2\pi 3\frac{F_0}{220} t) \]

FM-synthesis (Yamaha)
\[ x(t) = \sin \{ 2\pi \frac{F_0}{220} t + 3 \sin(2\pi \frac{F_0}{220} t) \} \]

Periodical signals,
signals based on harmonic sinusoids

Signals which are periodical, i.e. the same waveform is repeated with the period T can be written as a sum of harmonically sinusoids with frequencies

\[ F_0, 2F_0, 3F_0, 4F_0 \text{ and } \frac{F_0}{T} = \frac{1}{T} \text{ is called the fundamental frequency} \]

The signal can be written
\[ x(t) = A_0 + A_1 \sin(2\pi F_0 t + \phi_0) + A_2 \sin(2\pi 2F_0 t + \phi_2) + A_3 \sin(2\pi 3F_0 t + \phi_3) \text{ osv} \]

Example

Sampling page 21

\[ x(t) = 20 \cos(2\pi \frac{440}{F_0} t - 0.4\pi) \]

Sample the signal using the rate \( F_T = 1000 \text{ Hz} \)

or \( T = \frac{1}{F_T} = \frac{1}{1000} = 0.001 \text{ s} \) between the samples

\[ x(n) = x(t) \big|_{t=nT} = 20\cos(2\pi \frac{440}{1000} n - 0.4\pi) \]

\[ f_0 = \frac{440}{1000} = 0.44 \]

Notations:
\[ \Omega = 2\pi F \text{ frequency in Hertz or radians, respectively for time continuous signals.} \]
\[ \omega = 2\pi f \text{ frequency in Hertz or radians, respectively for time discrete signals.} \]
Time discrete sinusoid page 23

\[ x(n) = \cos(2\pi f_0 n) = \frac{1}{2}(e^{j2\pi f_0 n} + e^{-j2\pi f_0 n}) = \cos(2\pi (f_0 + 1)n) = \frac{1}{2}(e^{j2\pi (f_0 + 1)n} + e^{-j2\pi (f_0 + 1)n}) \]

\( n \) integer, \( f_0 = 1/8 = 0.125 \) (\( f_0 < 0.5 \) gives at least 2 sample/period)

\[ \text{How to plot the frequency contents?} \]

Listen to the signal by playing the signal using a D/A converter.

We select the period \(-0.5 < f < 0.5\)

and play using \( F_s=10000 \) Hz

\[ y(n) = \cos(2\pi 1/8 10000 t) = \cos(2\pi 1250 t) \]

Example of circuits pages 58, 59

A Delay (shift)

\[ x(n) \rightarrow z^{-1} \rightarrow y(n)=x(n-1) \]

B First order circuit

\[ x(n) \rightarrow z^{-1} \rightarrow y(n) \]

\[ y(n)=0.5y(n-1) + x(n-1) \]

C First order circuit

\[ x(n) \rightarrow z^{-1} \rightarrow y(n) \]

Here we need the Z-transform Chap 3.

More about structures in Chap. 8.
Discrete-Time Systems    (LTI systems)

FIR, IIR

FIR: Circuit with finite length memory

\[ y(n) = x(n) + x(n-1) \]

ex.

\[ y(n) = 0.5 \, y(n-1) + x(n) \]

IIR: Circuit with infinite length memory

\[ y(n) = \alpha x_1(n) + \beta x_2(n) \]

Linearity

if \[ x(n) = \alpha x_1(n) + \beta x_2(n) \]
gives \[ y(n) = \alpha y_1(n) + \beta y_2(n) \]

shift invariance

if \[ x(n) \Rightarrow y(n) \]
gives \[ x(n-1) \Rightarrow y(n-1) \]

BIBO-stability

Bounded input => bounded output

if \[ |x(n)| \leq M_x \]
then \[ |y(n)| \leq M_y < \infty \]

Some math’s used in the course

Complex numbers:

\[ z = a + j \, b = r \, e^{i \theta} \quad \text{with} \quad r = \sqrt{a^2 + b^2} \]

\[ \Phi = \arctan \left( \frac{b}{a} \right) \quad \text{if} \quad a > 0 \]

\[ r \, e^{i \theta} = r \cos \Phi + j \, r \sin \Phi \]

Euler formula:

\[ \cos \omega = \frac{1}{2} (e^{j \omega} + e^{-j \omega}) \]

\[ \sin \omega = \frac{1}{2j} (e^{j \omega} - e^{-j \omega}) \]

Rewriting using Euler formula:

\[ 1 + e^{-j \omega} = e^{-j \omega/2} (e^{j \omega/2} + e^{-j \omega/2}) = 2 \cos (\omega/2) \, e^{-j \omega/2} \]

\[ 1 - e^{-j \omega} = e^{-j \omega/2} (e^{j \omega/2} - e^{-j \omega/2}) = 2 \sin (\omega/2) \, e^{j \omega/2} \]

Integral:

\[ \int_{t=0}^{T} e^{-j2\pi ft} \, dt = \frac{e^{-j2\pi fT} - e^{-j2\pi f0}}{j2\pi f} = \frac{1 - e^{-j2\pi fT}}{j2\pi f} = \frac{e^{-j2\pi fT}}{j2\pi f} \left( e^{j2\pi fT} - e^{-j2\pi fT} \right) = \frac{\sin (2\pi f T/2) \, e^{-j\pi f T/2}}{2\pi f T/2} \]

The geometrical sum:

\[ S_1 = \sum_{n=0}^{\infty} \frac{1}{2}^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots = \frac{1}{1 - \frac{1}{2}} \quad \text{infinite sum} \]

\[ S_2 = \sum_{n=0}^{\infty} \frac{(1-\frac{1}{2})^n}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{(1-\frac{1}{2})^n}{1-\frac{1}{2}} \quad \text{finite sum} \]

Derivation of the finite geometrical sum:

\[ \text{Sum} = \sum_{n=0}^{\infty} a^n = 1 + a + a^2 + \ldots + a^k \]

Multiply with \( a \)

\[ a \cdot \text{Sum} = a + a^2 + \ldots + a^{k+1} \]

Form the difference

\[ \text{Sum} - a \cdot \text{Sum} = 1 - a^{k+1} \]

This gives the finite sum

\[ \text{Sum} = \frac{1 - a^{k+1}}{1 - a} \]

The infinite sum

\[ \sum_{n=0}^{\infty} a^n = 1 + a + a^2 + \ldots = \frac{1}{1-a} \quad \text{if} \quad |a| < 1 \]
Chapter 2  Convolution  pp 71-80

The most important relation between input signal and output signal is called convolution.

If we know the impulse response $h(n)$ can we determine the output for any other input signal. The only properties needed are linearity and time invariance (LTI).

Input signal  \[ x(n) \implies y(n) \]

Definition  

\[ \delta(n - k) \implies h(n - k) \]

\[ x(k) \ast \delta(n - k) \implies x(k)h(n - k) \]

\[ \sum_k x(k) \delta(n - k) \implies \sum_k x(k)h(n - k) \]

\[ y(n) = \sum_k x(k)h(n - k) = \sum_k h(n)h(n - k) = h(n) \ast x(n) \]

This is called convolution and is the most general relation given in this course.

Example of convolution

Given: Input signal och impulse response

\[
x(n) = \begin{bmatrix} \cdots & 0 & 0 & 2 & 4 & 6 & 4 & 2 & 0 & \cdots \end{bmatrix} \quad \text{and} \quad h(n) = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}
\]

Task: Determine the output signal (convolution)

\[
y(n) = \sum_k x(n - k)h(k) = \sum_k h(k)x(n - k) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) = 3x(n) + 2x(n-1) + x(n-2)
\]

Solution: We solve it 'graphically' by the method below.

$h(n)$ backwards:  \[
\begin{array}{cccccc}
& 1 & 2 & 3 & 4 & 6 & 4 & 2 & 0
\end{array}
\]

$x(k)$:  \[
\begin{array}{cccccc}
& \cdots & 0 & 0 & 2 & 4 & 6 & 4 & 2 & 0
\end{array}
\]

gives the result  \[
y(n) = \begin{bmatrix} 0 & 0 & 6 & 16 & 28 & 28 & 20 & 8 & 2 \end{bmatrix}
\]

Multiply and add. Then shift the impulse response to the right one step and repeat.

Properties of the convolution (normal properties), page 81..

\[
x_1(n) \ast x_2(n) = x_1(n) \ast x_2(n)
\]

\[
(x_1(n) \ast x_2(n)) \ast x_3(n) = x_1(n) \ast (x_2(n) \ast x_3(n))
\]

Input - output relations

Cascade circuits (serial)

\[
x(n) \rightarrow h_1(n) \rightarrow x(n) \ast h_1(n) \rightarrow h_2(n) \rightarrow y(n) = (x(n) \ast h_1(n)) \ast h_2(n)
\]

Parallel circuits

\[
x(n) \rightarrow h_1(n) \rightarrow x(n) \ast h_1(n) \rightarrow h_2(n) \rightarrow y(n) = x(n) \ast h_1(n) + x(n) \ast h_2(n)
\]
Definition of stability page 85.

A circuit is BIBO-stable (bounded input-bounded output) if
\[ x(n) \leq M_x \quad \text{yields} \quad |y(n)| \leq M_y \]

In the impulse response, we have the demand of stability
\[ |y(n)| = |\sum_{k=-\infty}^{\infty} h(n)x(n-k)| \leq \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \leq M_x \sum_{k=-\infty}^{\infty} |h(k)| \]

i.e. stable if
\[ \sum_{k=-\infty}^{\infty} |h(k)| \leq \infty \]

We solve the first order difference equation page 94

\[ y(n) = -a_1 y(n-1) + b_0 x(n) \]

Solve iteratively for \( n \geq 0 \)
\[ y(0) = -a_1 y(-1) + b_0 x(0) \]
\[ y(1) = -a_1 y(0) + b_0 x(1) = -a_1^2 y(-1) + b_0 x(1) + b_0 (-a_1) x(0) \]
\[ y(2) = -a_1 y(1) + b_0 x(2) = -a_1^3 y(-1) + b_0 x(2) + b_0 (-a_1) x(1) \]
\[ \quad + b_0 (-a_1)^2 x(0) \]
\[ y(n) = \sum_{k=0}^{\infty} b_0 (-a_1)^k x(n-k) + \underbrace{(-a_1)^n y(-1)}_{\text{depends on the initial state}} \]

Impulse response

If \( x(n) = \delta(n) \), then we have
\[ y(n) = h(n) = b_0 (-a_1)^n \quad \text{for} \quad n \geq 0 \]

For a general input signal we then determine the output using the convolution
\[ y(n) = \sum_{k} h(k) x(n-k) \]

For higher order systems, we will use the Z transform (chapter 3).

Difference equation pp 93-95.

General difference equation:
\[ y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k) \]

Example: FIR filter
\[ y(n) = 0.5 x(n) + 0.25 x(n-1) + 0.25 x(n-2) \]
gives the impulse response directly
\[ h(n) = [0.5 \quad 0.25 \quad 0.25] \]

Example: First order IIR filter
\[ y(n) = 0.5 y(n-1) + 2 x(n) \]

Example: Second order IIR filter
\[ y(n) = 0.5 y(n-1) + 0.5 y(n-2) + x(n) \]

We have to solve the difference equation to get the impulse response.
We end the chapter 2 by defining correlation functions.

**Autocorrelation function**

\[ r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n) x(n-l) \]

**Cross correlation function**

\[ r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n) x(n-l) \]

Write this using convolution notations.

\[ r_{xx}(l) = x(l) * x(-l) \]
\[ r_{yx}(l) = y(l) * x(-l) \]

**Input – Output Relations using correlation functions**

\[ y(n) = h(n) * x(n) \]

**Autocorrelation function for the input**

\[ r_{xx}(l) = x(l) * x(-l) \]

**Autocorrelation function for the output**

\[ r_{yy}(l) = y(l) * y(-l) = h(l) * x(l) * h(-l) * x(-l) = r_{hh}(l) * r_{xx}(l) \text{ if } r_{hh}(l) = h(l) * h(-l) \]

**Cross correlation function for input-output signal**

\[ r_{yx}(l) = y(l) * x(-l) = h(l) * x(l) * x(-l) = h(l) * r_{xx}(l) \]

We can use this to estimate an unknown system \( h(n) \) by choosing the input \( x(n) \). With the autocorrelation function \( r_{xx}(l) = \delta(l) \). Then we direct has \( h(l) = r_{yx}(l) \).

**Example of correlation functions for the delay in GSM systems.**

![Signal before GSM](image1.png)

![Signal after GSM](image2.png)

![Cross correlation](image3.png)
Chapter 3

The z-transform

We assume a causal impulse response \( h(n) \).
Causal means that
\[
h(n) = 0 \quad \text{for} \quad n < 0
\]

We now define the z-transform of the impulse response as
\[
H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}
\]
with
\[
z = re^{j\omega}
\]
a complex number often written in polar form.

\( H(z) \) is a complex function of a complex variable \( z \).

Some examples of z-transforms using the definition.

Time function (FIR) The Z-transform

\[
h(n) = H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = h(0) + h(1) z^{-1} + h(2) z^{-2} + ...
\]

\[
\delta(n) = \{...0 \uparrow 1 0...\}
\]

\[
\delta(n-1) = z^{-1}
\]

\[
h(n-1) = z^{-1} H(z)
\]

\[
h(n) = \{3 2 1\}
\]

\[
H(z) = 3 + 2z^{-1} + z^{-2}
\]

Proof of the transform of the delayed signal:

\[
y(n) = x(n-1) \quad \leftrightarrow \quad Y(z) = \sum_{n} y(n) z^{-n} = \sum_{n} x(n-1) z^{-n} = \sum_{n} x(n-1) z^{-n-1} z^{-1} X(z)
\]

More examples

Time function (IIR) The Z-transform

\[
h(n) = a^n u(n) \quad \leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{1}{1-a} \quad \text{if} \quad |z| > a \quad (\text{ROC})
\]

\[
h(n) = u(n) \quad \leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{1}{1-z} \quad \text{if} \quad |z| > 1 \quad (\text{ROC})
\]

ROC means Region of Convergence, i.e. the values of \( z \) for which the sum converge.

For a causal signal (the signal =0 for negative \( n \)), we have the ROC defined by a plane \( |z| \geq R_{\text{max}} \).
This is the normal situation in this course.

For a non-causal signal we must check the ROC in every case as we show with next example.
Example of z-transform of non-causal, page 154

Given:

\[ x(n) = \left( \frac{1}{2} \right)^n \text{ for all } n \]

Task: Determine \( X(z) \)

Solution: \( x(n) \) is non-zero for negative \( n \).

\[
X(z) = \sum_{n=-\infty}^{0} x(n) z^{-n} = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^{n} z^{-n} = \\
= \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^{n} z^{-n} \left( 1 - \left( \frac{1}{2} \right)^{n+1} \right) + \left( 1 - \left( \frac{1}{2} \right)^{1+1} \right) = \\
= 1 + \frac{1}{1 - \frac{1}{2} z^{-1}} - 1 = \frac{1}{1 - (1/2)^2} (1 - \frac{1}{2} z^{-1} \left( 1 - \frac{1}{2} z^{-1} \right))
\]

if \( |z| < 2 \) and \( |z| > 1/2 \)

Convolution transfers to multiplication

\[
y(n) = h(n) \ast x(n) \quad \Leftrightarrow \quad Y(z) = H(z)X(z)
\]

Solution of second order difference equation

First order system we solved direct in chapter 2. Using the z-transform we now solve higher order systems. We start with a second order system.

We solve for \( n \geq 0 \) and assume that \( y(n) \) and \( x(n) \) are zero for negative \( n \) (initial condition are zero).

Given a second order difference equation

\[
y(n) - 1.27 y(n-1) + 0.81 y(n-2) = x(n-1) - x(n-2)
\]

We now z-transform each part (both \( x(n) \) and \( y(n) \) are causal)

\[
Y(z) - 1.27 z^{-1} Y(z) + 0.81 z^{-2} Y(z) = z^{-1} X(z) - z^{-2} X(z)
\]

Then, we have

\[
Y(z) = \left( \frac{z^{-1} - 0 \cdot z^{-2}}{1 - 1.27 z^{-1} + 0.81 \cdot z^{-2}} \right) X(z) = H(z)X(z)
\]

Using formula tables, we can find \( y(n) \) and \( h(n) \).

In most cases, we want to find properties of \( H(z) \).
Example: Fibonacci sequence page 210

Fibonacci sequence is the sequence

\{1, 1, 2, 3, 5, 8, 13, …\}

Can we find a closed solution for this? Yes

Difference equation:

\[ y(n) = y(n-1) + y(n-2) \]

with \( y(1) = 1, \ y(2) = 1 \)

A: Use impulse response

\[ y(n) = y(n-1) + y(n-2) + \delta(n) \]

Solution:

\[ Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + 1 \]

\[ Y(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{A}{1 - p_1 z^{-1}} + \frac{B}{1 - p_2 z^{-1}} \]

\[ Y(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} \]

\[ p_1 = 0.5(1 + \sqrt{5}), \quad p_2 = 0.5(1 - \sqrt{5}), \]

with \( A_1 = (1 + \sqrt{5})/(2\sqrt{5}), \quad A_2 = -(1 - \sqrt{5})/(2\sqrt{5}) \)

Answer: \( y(n) = A_1 p_1^n + A_2 p_2^n \) for \( n \geq 0 \)

B: Use initial value. (see the textbook page 210)

\[ y(n) = y(n-1) + y(n-2) \]

med \( y(-1) = 0, \ y(-2) = 1 \)

Inverse z-transform: Use formula tables

A: Definition page 181

B: Polynomial division Page 183

C: Formula tables

Solution of difference equation with non-zero initial conditions

We solve for \( n \geq 0 \) and even that \( y(n) \) is nonzero for negative \( n \).

As before \( x(n) \) are zero for negative \( n \).

We define one-sided Z-transform

\[ Y^+(z) = \sum_{n=0}^{\infty} y(n)z^{-n} \quad \text{but now} \quad y(n) \neq 0 \quad \text{for} \quad n < 0 \]

With this definition the transform of shift is now

Let

\[ y_0(n) = y(n-1) \] then the \( Z^- \) – transform (using the def)

\[ Y_0^-(z) = \sum_{n=0}^{\infty} y_0(n)z^{-n} = \sum_{n=0}^{\infty} y(n-1)z^{-n} = \]

\[ y(-1) + \sum_{n=1}^{\infty} y(n-1)z^{-(n-1)}z^{-1} = z^{-1}Y^r(z) + y(-1) \]

This means that we must add the initial value \( y(-1) \) to have the correct answer.
Example: First order system with nonzero initial conditions.

Given:
\[ y(n) = ay(n-1) + x(n) \quad \text{with initial value} \quad y(-1) \]

Task: Determine \( y(n) \) for \( n \geq 0 \)

Solution:
\[ y(n) = ay(n-1) + x(n) \quad Z^+ \text{-transform} \]

\[ Y^+(z) = az^{-1}Y^+(z) + y(-1) + X(z) \quad \text{gives} \]
\[ Y^+(z) = \frac{1}{1-az^{-1}}X(z) + \frac{1}{1-az^{-1}}y(-1) \]

Graphical representation, Direct form II, page 265

Given: Given a system of the form

Task: Determine the relation between \( x[n] \) and \( y[n] \)

Solution: Introduce the extra variable \( w[n] \) and the notations \( X(z), W(z) \) and \( Y(z) \).

Determine the signal at point A.

\[ W(z) = a_1z^{-1}W(z) + X(z) \Rightarrow W(z) = \frac{1}{1-a_1z^{-1}}X(z) \]

\[ Y(z) = b_0W(z) + b_1z^{-1}W(z) \Rightarrow Y(z) = \frac{b_0 + b_1z^{-1}}{1-a_1z^{-1}}X(z) \]

i.e.
\[ Y(z) = H(z) \cdot X(z) \]
Chapter 3    The z-transform, continue

We define the Z-transform of the impulse response as $h(n)$

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

with $h(n) = 0$ för $n < 0$

and $z = r e^{j\omega}$

$H(z)$ is a complex function of a complex variable.

We applied the z-transform to the second order difference equation

$$y(n) - 1.27 y(n-1) + 0.81 y(n-2) = x(n-1) - x(n-2)$$

with the z-transform

$$Y(z) - 1.27 z^{-1} Y(z) + 0.81 z^{-2} Y(z) = z^{-1} X(z) - z^{-2} X(z)$$

Then, we got

$$Y(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27 z^{-1} + 0.81 z^{-2}} X(z) = H(z) X(z)$$

Now, we continue using this example.

Poles and zeros

From the difference equation we always got a system function $H(z)$ which is a rational function in $z^{-1}$.

We factorize $H(z)$.

$$H(z) = \frac{B(z)}{A(z)} = \frac{z^{-1} - z^{-2}}{1 - 1.27 z^{-1} + 0.81 z^{-2}} = \frac{z^{-1} - z^{-2}}{z^{2} - 1.27 z + 0.81}$$

Determine the roots to the denominator

$$z^2 - 1.27 z + 0.81 = 0 \text{ ger}$$

$$z = \frac{1.27 \pm \sqrt{(1.27)^2 - 0.81}}{2} = 0.64 \pm j 0.64 = 0.9 e^{j \pi/4}$$

Determine the roots to the numerator

$$z - 1 = 0 \text{ gives}$$

$$z = 1$$

Poles and zeros, continue 1

$H(z)$ can now be written

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27 z^{-1} + 0.81 z^{-2}} = \frac{z^{-1} - z^{-2}}{z^{2} - 1.27 z + 0.81} = \frac{z^{-1} - z^{-2}}{(z - 0.9 e^{j \pi/4})(z - 0.9 e^{-j \pi/4})}$$

The roots to the numerator are called zeros and the roots to the denominator are called poles. We plot them in a complex diagram, a pole-zero plot. The zeros are marked with a ring and the poles with a cross.

Poles: $p_{1,2} = 0.9 e^{\pm j \pi/4}$

Zeros: $z = 1$

z-plane (complex diagram)

Note: $H(z) = 0$ at a zero

$H(z) = \infty$ at a pole

$H(z)$ small closed to a zero (the magnitude)

$H(z)$ large closed to a pole (the magnitude)
Poles and zeros, continue 2

The description with poles and zeros is very useful and we called this the pole-zero plot.

Plots of $H(z)$ (Complex function of a complex variable)

We know try to plot the magnitude of $H(z)$

$$|H(z)|$$

Plot in MATLAB. Scales $-1 < \text{Re}(z) < 1$, $-1 < \text{Im}(z) < 1$

Poles and zeros, continue 3

We have complex poles.

The $Z$-transform of 2 order circuit with complex poles are (see the textbook or formula table)

$$h(n) = r^n \sin(\omega_0 n) u(n) =$$

$$= r^n \left(\frac{e^{j\omega_0 n}}{2} - \frac{e^{-j\omega_0 n}}{2}\right) u(n)$$

$$H(z) = \frac{1}{2} \left(\frac{1}{1 - r e^{j\omega_0} z^{-1}} - \frac{1}{1 - r e^{-j\omega_0} z^{-1}}\right) =$$

$$= \ldots = \frac{r \sin(\omega_0) z^{-1}}{1 - 2r \cos(\omega_0) z^{-1} + r^2}$$

Stability

A system is stable if a bounded input signal gives a bounded output signal. This is called BIBO-stability (Bounded input - Bounded output).

A sufficient condition for stability

$$\sum_n |h(n)| < \infty$$

This is equivalent with all poles inside the unit circle.

FIR-filters

FIR-filters have all poles in the origin and are always stable.

$$h(n) \text{ finite length gives } \sum_n |h(n)| \text{ always bounded}$$

Stability of IIR-filters

First order filters:

$$H(z) = \frac{1}{1 - az^{-1}} \quad h(n) = a^n u(n)$$

Pole in

$$p = a$$

Stable if $|a| < 1$, i.e. pole inside the unit circle.

Second order:

$$H(z) = \frac{1}{(1-p_1 z^{-1})(1-p_2 z^{-1})} = \frac{z^2}{(z-p_1)(z-p_2)} =$$

$$= \frac{1}{(1+p_1)(z^{-1} + p_2 z^{-2})}$$

We look at the two cases

A Real poles
B Complex poles
Real roots (poles), example

\[ H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} = \frac{A}{1 - p_1 z^{-1}} + \frac{B}{1 - p_2 z^{-1}} \]

\[ h(n) = (A p_1^n + B p_2^n) u(n) \]

Stable if \( |p_1| < 1, |p_2| < 1 \) i.e. poles inside the unit circle.

Complex roots (poles) (Complex conjugate), example

\[ p_{1,2} = r e^{j\omega_0} \]

\[ H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} = \frac{1}{(1 -(p_1 + p_2)z^{-1} + p_1 p_2 z^{-2})} = \frac{1}{(1 - (r e^{j\omega_0} + r e^{-j\omega_0})z^{-1} + r^2 z^{-2})} = \frac{1}{1 - 2r \cos \omega_0 z^{-1} + r^2 z^{-2}} \]

Stable if \( |p_1| < 1, |p_2| < 1 \) i.e. poles inside the unit circle.

Some comments

Stability

FIR-filters have all poles in the origin
FIR-filters are always stable

IIR-filters have not all poles in the origin
IIR-filters are stable if and only if all poles are inside the unit circle

Comparison of the analogous \( H(s) \) and the time discrete \( H(z) \)

Analogous: Stable if poles in the left half plane

\[ H(s) = \int h(t) e^{-st} dt \]

Time discrete: Stable if poles inside the unit circle

\[ H(z) = \sum_{n} h(n) z^{-n} \]

with \( z = e^s \)

Example:

\[ P_{\text{analogus}} = -0.5 \pm 0.5 j \]

\[ P_{\text{discrete}} \sim e^{-0.5 j 0.5} = e^{-j \frac{\pi}{2}} \]

\[ = 0.6 e^{j 0.16 \pi} \]

\[ s = \sigma + j \Omega = \sigma + j 2 \pi F \]

\[ z = r e^{j \Omega} = r e^{j 2 \pi F} \]
Solving the general difference equation.

\[ y(n) + \sum_{k=1}^{N} a_k y(n - k) = \sum_{k=0}^{M} b_k x(n - k) \]

\[ Y(z) + a_1 z^{-1} Y(z) + \ldots + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \ldots + b_M z^{-M} X(z) \]

\[ Y(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} X(z) = H(z) X(z) \]

\[ H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{z^{-M}}{z^{-N}} \frac{(z - z_1)(z - z_2) \ldots (z - z_N)}{(z - p_1)(z - p_2) \ldots (z - p_N)} \]

\( z_i \) zeros \hspace{1cm} (roots of the numerator)

\( p_i \) poles \hspace{1cm} (roots of the denominator)
Lecture 5

Chapter 4

The Fourier transform of a analogous signal, FT
The Fourier transform of a digital signal, DTFT

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Transforms in Digital Signal Processing

Analogous signals:

The Fourier transform of analogous signals, FT
The Fourier serial expansion of periodic signals (later)
(The Laplace transform)

Time discrete signals (sampled, digital signals)

The Fourier transform of a digital signal, DTFT, chapter 4
The Fourier serial expansion of periodic signal (later)
The Discrete Fourier transform (DFT, FFT), chapter 7
The Z-transform of a digital signal chapter 3.

Chapter 4 The Fourier transform pp 236-238

Definition: The Fourier transform of an analogous signal FT

\[ X(F) = \int \limits_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt \]

\[ x(t) = \int \limits_{F=\omega}^{\infty} X(F) e^{j2\pi Ft} dF = \]

\[ = \frac{1}{2\pi} \int \limits_{\Omega=2\pi}^{\infty} X(F) e^{j2\pi Ft} d\Omega \]

Convergence: If \( x(t) \) stable, i.e. \( \int |x(t)| < \infty \)

Weaker convergence

\[ \int |x(t)| \to \infty \]

\[ \int |x(t)|^2 < \infty \quad \text{bounded energy} \]
Fourier transform of a rectangular pulse pages 257-258

Time space

Analogous rectangular pulse (rectangular time window)

\[ x(t) = \begin{cases} 
1 & 0 \leq t \leq T \\
0 & \text{elsewhere}
\end{cases} \]

Frequency space

\[ X(F) = \frac{T}{2\pi F} e^{-j2\pi FT/2} \]

Determining the \( X(F) \)

\[
X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt = \int_{0}^{T} x(t) e^{-j2\pi Ft} dt = \frac{e^{-j2\pi FT} - 1}{-j2\pi F} + \frac{e^{j2\pi FT/2} - e^{-j2\pi FT/2}}{j2\pi F} = \frac{T}{2\pi F} e^{-j2\pi FT/2}
\]

Plot in MATLAB of the Fourier transform of a rectangular pulse

Analogous rectangular pulse (rectangular time window) \( T=4 \)

Time function

\[ x(n) = \{3 \ 2 \ 1\} \]

Fourier transform

\[ X(f) = 3 + 2e^{-j2\pi f} + e^{-j2\pi f} = 3 + 2e^{-j\omega} + e^{-j2\omega} \]

The delta function

\[ x(n) = \delta(n) = \{1\} \]

\[ X(f) = 1 \]

Time shift (delay)

\[ x(n) = \delta(n-1) = \{0 \ 1\} \]

\[ X(f) = e^{-j2\pi f} = e^{-j\omega} \]

The convolution

\[ y[n] = \delta[n] * h[n] \]

\[ Y(\omega) = H(\omega)X(\omega) \] (proof, see next page)
Convolution in the time domain corresponds to multiplication in the frequency domain.

Given
\[ y(n) = x(n) * h(n) = \sum_l x(l) h(n-l) \]

Solution
\[ Y(\omega) = \sum_n y(n) e^{-j\omega n} = \sum_n \sum_l x(l) h(n-l) e^{-j\omega(n-l)} = \sum_l x(l) e^{-j\omega l} \sum_n h(n-l)e^{-j\omega(n-l)} = X(\omega)H(\omega) \]

Relation to the z-transform

We start with a causal impulse response, \( h(n) \). Causal means that \( h(n) = 0 \) for \( n < 0 \)

The Fourier transform DTFT of the impulse response is
\[ H(\omega) = \sum_{n=0}^{\infty} h(n)e^{-j\omega n} = \sum_{n=0}^{\infty} h(n)e^{-j2\pi fn} \]

The Z-transform of impulse response is
\[ H(z) = \sum_{n=0}^{\infty} h(n)z^{-n} \]
with \( z = r e^{j\omega} \)

\( z \) is complex variable and \( H(z) \) is a complex function of a complex variable often written as magnitude and phase.

Note: If \( h(n) \) is causal and stable, then we have
\[ H(\omega) = H(z) \bigg|_{z=e^{j\omega}} \]

Expansion of the Fourier transform (see math’s)

A Sine, cosine
\[ x[n] = \cos(2\pi f_0 n) = \frac{1}{2}(e^{j2\pi f_0 n} + e^{-j2\pi f_0 n}) \]
\[ X(f) = \frac{1}{2}(\delta(f-f_0) + \delta(f+f_0)) = 2\pi \delta(f-f_0) + \delta(f+f_0) \]
\[ X(f) = \frac{1}{2j}(\delta(f-f_0) - \delta(f+f_0)) \]

B Unit step
\[ x[n] = u(n) \] unit step
\[ X(f) = \frac{1}{1-e^{-2\pi f}} + \frac{1}{2} \delta(f) \]

Note. Here \( X(f) \) is given for the interval \( 0 \leq f \leq \frac{1}{2} \)

Then, \( X(f) \) must be periodical.

Inverse Fourier transform of an rectangle pulse

Ideal low pass filter
Ideal low pass filter (non-causal) is defined by
\[ H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c, \quad |f| \leq f_c \\ 0 & \text{elsewhere} \end{cases} \]

The corresponding impulse response
\[ h(n) = \frac{1}{2\pi} \int_0^{\omega_c} H(\omega)e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi} \left[ \frac{e^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c} = 2f_c \sin \omega_c n \]

A causal low pass FIR filter can be achieved by selecting \( N \) samples around origin and then delay the impulse response \((N-1)/2 \) (N odd)
\[ h(n) = \frac{1}{\pi} \sin \omega_c \left( \frac{n-1}{2} \right), \quad 0 \leq n \leq N-1 \]

\[ h(n) = \frac{\sin \frac{\omega_c (n-\frac{N-1}{2})}{\omega_c (n-\frac{N-1}{2})}}{\omega_c (n-\frac{N-1}{2})} \]